Filomat 40:3 (2026), 881–890
https://doi.org/10.2298/FIL2603881A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Projections on weak*-closed subspaces

Iman Amani[a], Ali Ghaffari[a,*], Feridoun Habibian[a]

[a]*Department of Mathematics, University of Semnan, P. O. Box 35195-363, Semnan, Iran*

Abstract. Let G be a locally compact group and S a weak *-closed translation invariant subspace of $L^\infty(G)$. M.E.B. Bekka proved that S is the range of a projection on $L^\infty(G)$ which commutes with translation if and only if S is the range of a projection on $L^\infty(G)$ which commutes with convolution. Our first purpose in this paper is to generalize Bekka's results for a certain class of left Banach G-module. This result is used to show that G is amenable if and only if whenever X is a left Banach G-module and S is a weak*-closed right invariant subspace of X^* which is complemented in X^*, then S is the range of a projection on X^* which commutes with convolution. Finally, we explore the link between the projections properties and amenability of group algebras.

1. Introduction and Notations

Let G be a locally compact group and $C_b(G)$ be the space of bounded continuous complex-valued functions on G with supremum norm. Let $LUC(G)$ denote the space of bounded left uniformly continuous complex-valued functions on G. Let $M(G)$ be the space of complex-valued, regular Borel measures on G. Recall that $L^1(G)$ is a Banach subalgebra and an ideal in $M(G)$ with a bounded approximate identity. We denote by $P^1(G)$ the convex set formed by the probability measures in $L^1(G)$. Let $L^\infty(G)$ denote the Banach space of essentially bounded complex-valued functions on G with the essential supremum norm as defined in [5]. For each $t \in G$, define the left and right translation operators on $L^\infty(G)$ by $l_t f(s) = f(t^{-1}s)$ and $r_t f(s) = f(st)$ for all $s \in G$ and $f \in L^\infty(G)$.

If X is a Banach space, then X^* denotes its continuous dual. Also if $f \in X^*$ and $x \in X$, then the value of f at x will be written as $f(x)$ or $\langle f, x\rangle$. Suppose M is a subspace of X, and N is a subspace of X^*. Their annihilators $M^\perp$ and $^\perp N$ are defined as follows:

$$M^\perp = \{f \in X^*; \langle f, x\rangle = 0 \text{ for all } x \in M\},$$

$$^\perp N = \{x \in X; \langle f, x\rangle = 0 \text{ for all } f \in N\}.$$

If Y is another Banach space, then $\mathcal{B}(X, Y)$ will denote the space of bounded continuous linear operators from X into Y. As far as possible, we follow [3] in our notation and refer to [19] for basic functional analysis and to [5] for basic harmonic analysis.

2020 *Mathematics Subject Classification*. Primary 43A10; Secondary 43A60.
Keywords. keywords, Banach algebras, complemented subspace, bounded linear operators, group algebras, weak* topology.
Received: 08 July 2025; Accepted: 13 October 2025
Communicated by Dragan S. Djordjević
* Corresponding author: Ali Ghaffari
Email addresses: imanamani7070@gmail.com (Iman Amani), aghaffari@semnan.ac.ir (Ali Ghaffari), fhabibian@semnan.ac.ir (Feridoun Habibian)
ORCID iDs: https://orcid.org/0009-0008-1880-3879 (Iman Amani), https://orcid.org/0000-0003-2143-6607 (Ali Ghaffari), https://orcid.org/0000-0002-5296-9185 (Feridoun Habibian)

The bounded projections on $L^\infty(G)$ which commute with convolutions and translations have been studied by Lau in [14] and by Lau and Losert in [12], see also [13] and [16]. They also went further, and for several subspaces S of $L^\infty(G)$, they have obtained a number of interesting and nice results. A closed, left translation invariant subspace S of $L^\infty(G)$ is said to be invariantly complemented in $L^\infty(G)$ if S is the range of a continuous projection on $L^\infty(G)$, which commutes with all left translation operators on $L^\infty(G)$. This concept was introduced by Lau [14] for locally compact groups and was studied in [12], see also [6] and [7].

For a locally compact abelian group G, Gilbert [10] characterized weak* closed translation invariant complemented subspaces of $L^\infty(G)$ by their spectra. In [24], Wood investigated the ideals in the Fourier algebra of a locally compact group G which are complemented by an invariant projection, see also [25]. In [23] Takahashi proved that if G is a compact group, then any weak* closed complemented left translation invariant subspace S of $L^\infty(G)$ is invariantly complemented, i.e., S admits a left translation invariant closed complement. Note that if $\mathbb{T}$ is the circle group, then the Hardy space H_∞ is a weak* closed translation invariant subalgebra of $L^\infty(H)$ and not complemented. Tahmasebi [21] initiated the study of projections on hypergroups, extending the definition from groups, see also [9] and [22].

Bekka proved that if S is a weak* closed subspace of $L^\infty(G)$, then S is the range of a continuous projection on $L^\infty(G)$, which commutes with all left translation operators if and only if S is the range of a continuous projection on $L^\infty(G)$, which commutes with all convolution operators. Our first result is a generalization of this fact to weak* closed subspaces of X^* where X is a left G-module Banach. Forrest in [8] proved that the analogue of Bekka's theorem holds for $A(G)$ when G is an amenable group.

It was shown by Lau and Losert [12] that if G is an amenable locally compact group and X is a left Banach G-module, then any weak* closed translation invariant subspace of $L^\infty(G)$ is the range of a continuous projection commuting with translations. We will show that if S is a weak*-closed right translation invariant subspace of X^*, then S is the range of a continuous projection commuting with convolution.

In this paper, among the other things, we shall study projections on dual of a left Banach G-module and we also extend a result on group proved by Bekka in [2]. This result is used to show that G is amenable if and only if whenever X is a left Banach G-module and S is a weak* closed right translation invariant subspace of X^*, then S is the range of a continuous projection commuting with convolution. Finally, we give sufficient conditions and some necessary conditions for G to have a left invariant mean.

2. Main results

We introduce the following crucial concept which is a general property for Banach spaces.

Definition 2.1. *Let X be a complex Banach space and let G be a topological group. We say that the complex Banach space X constitutes a left Banach G-module if there exists a mapping $(t, x) \mapsto t.x$ $(G \times X \to X)$ having the following properties:*

(i) $t.(x + y) = t.x + t.y$, $\alpha(t.x) = t.\alpha x$, $(st).x = s.(t.x)$ *and* $e.x = x$, *where* $\alpha \in \mathbb{C}$, $s, t \in G$ *and* $x, y \in X$;

(ii) for all $x \in X$, the map $t \mapsto t.x$ is continuous from G into X;

(iii) there exists $k \in \mathbb{R}$ such that $\|t.x\| \le k\|x\|$ for every $t \in G$ and $x \in X$.

Let X be a left Banach G-module. We may define for X a Banach $L^1(G)$-module structure via a vector-valued integral. We put $\mu.x = \int t.x d\mu(t)$. We define for each $f \in X^*$, $t \in G$ and $x \in X$,

$$\langle f.\delta_t, x\rangle = \langle f, t.x\rangle.$$

Define also

$$\langle f.\mu, x\rangle = \int \langle f.\delta_t, x\rangle d\mu(t).$$

Then $f.\mu \in X^*$, $f.\mu = f.\delta_t$ if $\mu = \delta_t$ and $(f.\mu_1).\mu_2 = f.(\mu_1 * \mu_2)$ for all $\mu_1, \mu_2 \in M(G)$.

If U runs over a basis of open relatively compact neighborhoods of e in G, we put

$$\varphi_U.x = \frac{1}{|U|}\int_U t.xdt = \frac{1}{|U|}\int t.x1_U(t)dt.$$

By the continuity of the mapping $t \mapsto t.x$ for any $x \in X$, we obtain $\lim_U \varphi_U.x = x$. It follows that $\{\varphi_U\}$ is a bounded approximate identity for X in $L^1(G)$, see [18]. A subspace $S \subseteq X^*$ is called right invariant if $S.\delta_t \subseteq S$ for all $t \in G$. We say that S is topologically right invariant if $f.\varphi \in S$ whenever $f \in S$ and $\varphi \in L^1(G)$.

Let $S \subseteq X^*$ be a subspace of X^* that is right invariant [resp. topologically right invariant]. A projection $P : X^* \to S$ is called right invariant [resp. topologically right invariant] if $P(f.\delta_t) = P(f).\delta_t$ for all $f \in S$ and $t \in G$ [resp. $P(f.\varphi) = P(f).\varphi$ for all $f \in S$ and $\varphi \in L^1(G)$].

Lemma 2.2. *Let X be a left Banach G-module.*

(i) $X^*L^1(G)$ is weak-dense in X^*;*

(ii) If S is a weak-closed subspace of X^*, then S is right invariant if and only if S is topologically right invariant;*

(iii) Let S be a weak-closed subspace of X^*. If P is a weak*-weak* continuous projection from X^* into S and P commutes with right translations, then P also commutes with convolutions from the right, and vice versa.*

Proof. (*i*) Note that $X^*L^1(G)$ is a closed linear subpace of X^*, see Theorem 32.22 in [11]. Let $\{\varphi_\alpha\}$ be a bounded approximate identity for X in $L^1(G)$ and $f \in X^*$. Then the net $\{f.\varphi_\alpha\}$ is in $X^*L^1(G)$ and converges in the weak*-topology to f.

(*ii*) Let $f \in S$ and $\varphi \in P^1(G)$. We will show that $f.\varphi \in S$. We assume to the contrary that $f.\varphi$ is not in S. Part (b) of the separation Theorem 3.4 in [19] shows that there exist $x \in X$ and $\gamma \in \mathbb{R}$ such that

$$Re\langle f.\delta_t, x\rangle < \gamma < Re\langle f.\varphi, x\rangle$$

for all $t \in G$. The mapping $t \mapsto \langle f.\delta_t, x\rangle$ is obviousely continuous. Lemma 2.1 in [19] implies that

$$Re\langle f.\varphi, x\rangle = \int Re\langle f.\delta_t, x\rangle d\varphi(t) \leq \gamma < Re\langle f.\varphi, x\rangle.$$

This is contradiction. It follows that S is topologically right invariant.

Conversely, let $f \in S$ and $t \in G$. Let $\{\varphi_\alpha\}$ be a bounded approximate identity for X in $L^1(G)$ and $f \in X^*$. Obviousely the net $\{f.\delta_t * \varphi_\alpha\}$ converges in the weak*-topology to $f.\delta_t$, and so $f.\delta_t \in S$.

(*iii*) Let $P^* : X^{**} \to X^{**}$ be the adjoint operator of P, i.e., P^* is the bounded linear operator of X^{**} into X^{**} which satisfies $\langle P^*(F), f\rangle = \langle F, P(f)\rangle$ for all $F \in X^{**}$ and $f \in X^*$. We next show that for each $x \in X$, $P^*(x) \in X$. Let $x \in X$ and $\{f_\alpha\}$ be a net in X^* such that $f_\alpha \to f$ in the weak*-topology. We have

$$\lim_\alpha \langle P^*(x), f_\alpha\rangle = \lim_\alpha \langle P(f_\alpha), x\rangle = \langle P(f), x\rangle,$$

since P is weak*-weak* continuous. Therefore $P^*(x) \in X^{**}$ is weak* continuous. By ([19], Chapter 3), $P^*(x) \in X$.

Now, let $f \in X^*$ and $\psi \in P^1(G)$. Let $\{\psi_\alpha\}$ be a net of convex combinations of point measures on G such that $f.\psi_\alpha \to f.\psi$ and $P(f).\psi_\alpha \to P(f).\psi$ in the weak*-topology. For every $x \in X$,

$$\begin{aligned} \langle P(f).\psi, x\rangle &= \lim_\alpha \langle P(f).\psi_\alpha, x\rangle = \lim_\alpha \langle P(f.\psi_\alpha), x\rangle \\ &= \langle f.\psi_\alpha, P^*(x)\rangle = \langle P(f.\psi), x\rangle. \end{aligned}$$

As $x \in X$ is chosen arbitrary, we have $P(f.\psi) = P(f).\psi$ for all $f \in X^*$ and $\psi \in P^1(G)$.

Conversely, let P be a topologically right invariant. Let $\mathcal{U}$ denote the family of symmetric compact neighborhoods of e and regard $\mathcal{U}$ as a directed set in the usual way: $U \succ V$ if $U \subseteq V$. For each $U \in \mathcal{U}$, choose a function $\varphi_U \in P^1(G)$ such that $\varphi_U(G \setminus U) = \{0\}$. Let f be in X^* and t in G. For every $U \in \mathcal{U}$ and $x \in X$, $\langle P(f.\varphi_U * \delta_t), x\rangle = \langle P(f).\varphi_U * \delta_t, x\rangle$. Since P is weak*-weak* continuous, we have $\langle P(f.\delta_t), x\rangle = \langle P(f).\delta_t, x\rangle$. It follows that $P(f.\delta_t) = P(f).\delta_t$. This completes our proof. □

If f is a complex-valued function defined locally almost everywhere on G, and if $s, t \in G$, then $l_s f(t) = f(s^{-1}t)$ whenever this is defined. Notice that, for $\varphi \in L^1(G)$, $s \mapsto \delta_s * \varphi = l_s\varphi$ is a continuous mapping of G into $L^1(G)$ [5]. Clearly $L^1(G)$ is a left Banach G-module.

Recall that a linear functional $m \in L^\infty(G)^*$ is called a mean if $m \geq 0$, $\|m\| = 1$. A mean m is topologically left invariant (left invariant mean) $\langle m, f.\varphi\rangle = 1$ ($\langle m, l_t f\rangle = \langle m, f\rangle$) for all $f \in L^\infty(G)$, $\varphi \in P^1(G)$ and $t \in G$. G is called amenable if it admits a left invariant mean on $L^\infty(G)$. Amenable locally compact groups include all compact groups and all solvable groups. However, the free group on two generators is not amenable. More information on this problem can be found in [18] and [17].

Example 2.3. *(i) Let G be a locally compact group. As known (see [18]) if G is a nondiscrete compact abelian group (or more generally, G is amenable as discrete), there exists a left invariant mean m on $L^\infty(G)$ which is not a topologically left invariant mean on $L^\infty(G)$. Put $S = \{c1_G; c \in \mathbb{C}\}$. It is easy to see that S is a right translation invariant weak*-closed subspace of $L^\infty(G)$. Now define $P(f) = \langle m, f\rangle 1_G$. Then P is a projection of $L^\infty(G)$ into S commuting with right translations. Choose $\varphi \in P^1(G)$ and $f \in L^\infty(G)$ such that $\langle m, \hat{\varphi} * f\rangle \neq \langle m, f\rangle$ where $\hat{\varphi}(s) = \Delta(s^{-1})\varphi(s^{-1})$ for $s \in G$ and Δ is the Haar modulus function on G. It is easy to see that $f.\varphi = \hat{\varphi} * f$. We can write*

$$P(f.\varphi) = \langle m, f.\varphi\rangle 1_G = \langle m, \hat{\varphi} * f\rangle 1_G \neq \langle m, f\rangle 1_G = P(f).\varphi.$$

This shows that (in Lemma 2.2) weak-weak* continuity of P is necessary.*

(ii) Let G be a locally compact group. Baker, Lau and Pym [1] proved that $Hom(L^\infty(G), L^\infty(G))$ (where $T \in Hom(L^\infty(G), L^\infty(G))$ means $T(f.\varphi) = T(f).\varphi$ for every $f \in L^\infty(G)$ and $\varphi \in L^1(G)$) can be identified isometrically isomorphic with $LUC(G)^*$. Indeed, for every $T \in Hom(L^\infty(G), L^\infty(G))$ there exists a unique element $F \in LUC(G)^*$ such that $T(f) = Ff$ for all $f \in L^\infty(G)$.

Now, let G be a compact group and let P be a bounded projection from $L^\infty(G)$ onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Therefore $P(f) = Ff$ for some $F \in LUC(G)^* = C(G)^* = M(G)$.

Now, let $\{f_\alpha\}$ converges to f in the weak-topology of $L^\infty(G)$, and $\varphi \in L^1(G)$. It is known that $L^1(G)$ is a two-sided ideal in $M(G)$ [5], and so $\varphi * F \in L^1(G)$. We can write*

$$\begin{aligned} \lim_\alpha \langle P(f_\alpha), \varphi\rangle &= \lim_\alpha \langle Ff_\alpha, \varphi\rangle = \lim_\alpha \langle f_\alpha, \varphi * F\rangle \\ &= \langle f, \varphi * F\rangle = \langle P(f), \varphi\rangle. \end{aligned}$$

This shows that P is weak-weak* continuous.*

(iii) Let G be a locally compact group. The space $L^\infty(G)$ may be embedded into $\mathcal{B}(L^1(G), L^\infty(G))$ by the linear map T such that $T(f)(\varphi) = f.\varphi$ where $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Since $\mathcal{B}(L^1(G), L^\infty(G))$ carries naturally the strong operator topology, T allows us to consider the induced topology on $L^\infty(G)$, which we denote by τ_s. Let S be the range of a τ_s-continuous projection P on X^* such that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$. Let $\varphi, \psi \in L^1(G)$. Since the mapping $t \mapsto f.\delta_t$ is τ_s-continuous, we have

$$\begin{aligned} \langle P(f.\varphi), \psi\rangle &= \int \langle P(f.\delta_t), \psi\rangle d\varphi(t) = \int \langle P(f).\delta_t, \psi\rangle d\varphi(t) \\ &= \langle P(f).\varphi, \psi\rangle. \end{aligned}$$

Since this relation holds for all $\psi \in L^1(G)$, we conclude that $P(f.\varphi) = P(f.\varphi)$. Conversely, let $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Let $t \in G$ and $\{\varphi_\alpha\}$ be a bounded approximate identity for $L^1(G)$. Obviousely $\{f.\varphi * \delta_t\}$ converges to $f.\delta_t$ in the τ_s-topology. On the other hand, the projection P is τ_s-continuous. Therefore

$$\begin{aligned} \langle P(f.\delta_t), \varphi\rangle &= \lim_\alpha \langle P(f.\varphi_\alpha * \delta_t), \varphi\rangle = \lim_\alpha \langle P(f).\varphi_\alpha * \delta_t, \varphi\rangle \\ &= \lim_\alpha \langle P(f), \varphi_\alpha * \delta_t * \varphi\rangle = \langle P(f).\delta_t, \varphi\rangle \end{aligned}$$

for all $\varphi \in L^1(G)$. It follows that P commutes with right translations.

Theorem 2.4. *Let X be a left Banach G-module. Suppose that S is a weak*-closed right invariant subspace of X^*. The following statetments are equivalent:*

(i) *S is topologically invariantly complemented in X^*, i.e., S is the range of a continuous projection P on X^* such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$;*

(ii) *S is invariantly complemented in X^*, i.e., S is the range of a continuous projection P on X^* such that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$;*

(iii) *$S \cap X^*L^1(G)$ is topologically invariantly complemented in $X^*L^1(G)$.*

Proof. $(i) \Rightarrow (ii)$ Recall that, by Lemma 2.2, S is right invariant if and only if it is topologically right invariant. Let $P : X^* \to S$ be a bounded projection such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$. Let $\{\varphi_\alpha\}$ be a bounded approximate identity for X in $L^1(G)$. For $f \in X^*$, $x \in X$ and $t \in G$,

$$\begin{aligned} \langle P(f.\delta_t), x\rangle &= \lim_\alpha \langle P(f.\delta_t), \varphi_\alpha.x\rangle = \lim_\alpha \langle P(f.\delta_t).\varphi_\alpha, x\rangle \\ &= \lim_\alpha \langle P(f.\delta_t * \varphi_\alpha), x\rangle = \lim_\alpha \langle P(f).\delta_t * \varphi_\alpha, x\rangle \\ &= \lim_\alpha \langle P(f).\delta_t, \varphi_\alpha.x\rangle = \langle P(f).\delta_t, x\rangle. \end{aligned}$$

Hence we conclude that P is a bounded projection of X^* onto S such that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$.

$(ii) \Rightarrow (iii)$ Let $P : X^* \to S$ be a bounded projection such that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$. Suppose that $f \in X^*$ and $\varphi, \psi \in P^1(G)$. We claim that $P(f.\varphi * \psi) = P(f.\varphi).\psi$. Let $x \in X$, and $\epsilon > 0$ be given. By density we may suppose that $\psi \in P^1(G) \subseteq M(G)$ has compact support, say K. As the mapping $t \mapsto \varphi * \delta_t$ is continuous [5], for every $t \in G$, there exists an open neighborhood U_t of t in G such that for all $s \in U_t$, $\|\varphi * \delta_s - \varphi * \delta_t\|_1 < \epsilon$ and also $\|s.x - t.x\| < \epsilon$. Since K is compact, the cover $\{U_t\}$ contains a finite subcover $U_{t_1}, ..., U_{t_m}$. We can find a finite sequence $\{A_1, ..., A_m\}$ of measurable sets which are disjoint and such that

$$K \subseteq \bigcup_{i=1}^m A_i, \|\varphi * \delta_t - \varphi * \delta_{t_i}\|_1 < \epsilon, \|t.x - t_i.x\| < \epsilon \text{ whenever } t \in A_i.$$

If $i \in \{1, ..., m\}$, we also put $\alpha_i = \int_{A_i} \psi(t)dt$. Then $\alpha_1 + \cdots + \alpha_m = 1$. We can write

$$\begin{aligned} \epsilon\|P^*(x)\|\|f\| &> \|P^*(x)\|\|f\| \sum_{i=1}^m \int_{A_i} \|\varphi * \delta_t - \varphi * \delta_{t_i}\|_1 d\psi(t) \\ &\geq \Big|\sum_{i=1}^m \int_{A_i} \langle P^*(x), f.\varphi * \delta_t - f.\varphi * \delta_{t_i}\rangle d\psi(t)\Big| \\ &= \Big|\langle P^*(x), f.\varphi * \psi - \sum_{i=1}^m \alpha_i f.\varphi * \delta_{t_i}\rangle\Big| \\ &= \Big|\langle P(f.\varphi * \psi) - \sum_{i=1}^m \alpha_i P(f.\varphi * \delta_{t_i}), x\rangle\Big|, \end{aligned}$$

and also

$$\begin{aligned} \epsilon\|P(f.\varphi)\| &> \sum_{i=1}^m \|P(f.\varphi)\| \int_{A_i} \|t.x - t_i.x\| d\psi(t) \\ &\geq \Big|\sum_{i=1}^m \int_{A_i} \langle P(f.\varphi), t.x\rangle - \langle P(f.\varphi * \delta_{t_i}), x\rangle d\psi(t)\Big| \\ &= \Big|\langle P(f.\varphi).\psi - \sum_{i=1}^m \alpha_i P(f.\varphi * \delta_{t_i}), x\rangle\Big|. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we conclude that $P(f.\varphi * \psi) = P(f.\varphi).\psi$.

Now, let $f.\psi \in X^*L^1(G)$. Since $L^1(G)$ has a bounded approximate identity, Cohen's factorization theorem [11] implies that each $\psi \in L^1(G)$ has the form $\psi_1 * \psi_2$ for $\psi_1, \psi_2 \in L^1(G)$. Since

$$P(f.\psi) = P(f.\psi_1 * \psi_2) = P(f.\psi_1).\psi_2,$$

we have $P(X^*L^1(G)) \subseteq X^*L^1(G)$. Thus $P\Big|_{X^*L^1(G)}$ is a projection on $X^*L^1(G) \cap S$ commuting with convolutions.

(*iii*) $\Rightarrow$ (*i*) Let $P : X^*L^1(G) \to S \cap X^*L^1(G)$ be a bounded projection such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$. Let $\{\varphi_\alpha\}$ be a bounded approximate identity for X in $L^1(G)$. Define $P' : X^* \to X^*$ by $\langle P'(f), x\rangle = \lim_\alpha \langle P(f.\varphi_\alpha), x\rangle$. It is not hard to see that the limit exists. Indeed, $x = \psi.y$ for some $\psi \in L^1(G)$ and $y \in X$ (see Theorem 32.22 in [11]). We can write

$$\begin{aligned}\langle P(f.\psi), y\rangle &= \lim_\alpha \langle P(f.\varphi_\alpha * \psi), y\rangle = \lim_\alpha \langle P(f.\varphi_\alpha), \psi.y\rangle \\ &= \lim_\alpha \langle P(f.\varphi_\alpha), x\rangle = \langle P'(f), x\rangle.\end{aligned}$$

We claim that P' is a bounded projection of X^* onto S and that $P'(f.\varphi) = P'(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$. The operator P' is obviously linear. For $f \in S$ and $x \in X$, we have

$$\langle P'(f), x\rangle = \lim_\alpha \langle P(f.\varphi_\alpha), x\rangle = \lim_\alpha \langle f.\varphi_\alpha, x\rangle = \langle f, x\rangle,$$

and so P' is the identity map on S. If $f \in X^*$, then

$$\langle P'(f), x\rangle = \lim_\alpha \langle P(f.\varphi_\alpha), x\rangle = 0,$$

for all $x \in {}^\perp S$. Since S is a weak*-closed subspace of X^*, $({}^\perp S)^\perp = S$ and so $P'(f) \in S$ [19]. Consequently P' is an extension of P to X^* as a bounded projection.

Next, let $f \in X^*$ and $\varphi \in L^1(G)$. We can write

$$\begin{aligned}\langle P'(f.\varphi), x\rangle &= \lim_\alpha \langle P(f.\varphi * \varphi_\alpha), x\rangle = \lim_\alpha \langle P(f.\varphi_\alpha * \varphi), x\rangle \\ &= \lim_\alpha \langle P(f.\varphi_\alpha).\varphi, x\rangle = \langle P'(f).\varphi, x\rangle.\end{aligned}$$

This completes the proof. □

Theorem 2.4 is proved by Bekka [2] for the case $X = L^1(G)$. His proof is completely different.

Corollary 2.5. *Let G be a locally compact group. Let S be a weak*-closed right translation invariant subspace of $L^\infty(G)$. The following statements are equivalent:*

- *(i) S is topologically invariantly complemented in $L^\infty(G)$;*
- *(ii) S is invariantly complemented in $L^\infty(G)$;*
- *(iii) $S \cap LUC(G)$ is topologically invariantly complemented in $LUC(G)$;*
- *(iv) The left ideal ${}^\perp S$ has bounded right approximate identity.*

Proof. By Theorem 2.4, (*i*), (*ii*) and (*iii*) are equivalent. Now if (*i*) holds, there exists a bounded projection $P : L^\infty(G) \to S$ such that $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. By Theorem in [1], there exists $F \in LUC(G)^*$ such that $Ff = f$ for all $f \in L^\infty(G)$. Let $f \in ({}^\perp S)^*$, and let f' be any Hahn-Banach extension of f to a continuous functional on $L^1(G)$. Now, let $\{\varphi_\alpha\}$ be a bounded approximate identity bounded by 1 [5]. By Theorem 3.15 in [19], it has a converging subnet $\{\varphi_\beta\}$. We consider $E : ({}^\perp S)^* \to \mathbb{C}$ defined by $\langle E, f\rangle = \lim_\beta \langle Ff' - f', \varphi_\beta\rangle$. It is easy to see that $xE = E$ for all $x \in {}^\perp S$. This shows that ${}^\perp S$ has a bounded right approximate identity, see Proposition 2.2.1 and its proof in [20].

Now, let $\{\varphi_\alpha\}$ be a bounded approximate identity in ${}^\perp S$. Without loss of generality, we may assume that $\varphi_\alpha \to E$ in the weak*-topology. Define $P : L^\infty(G) \to L^\infty(G)$ by $\langle P(f), \varphi\rangle = \langle Ef - f, \varphi\rangle$, where $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Clearly, P is a bounded projection from $L^\infty(G)$ onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. □

Let X be a left Banach G-module. It was shown by Lau and Losert [13] that if G is an amenable locally compact group, then any weak* closed left translation invariant subspace of X^* is the range of a continuous projection commuting with left translations. We will show that if S is a weak*-closed left translation invariant subspace of X^*, then S is the range of a continuous projection commuting with convolution.

Corollary 2.6. *Let G be a locally compact group. Then G is amenable if and only if whenever X is a left Banach G-module and S is a weak*-closed right invariant subspace of X^* which is complemented in X^*, then there exists a projection P of X^* onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$.*

Proof. Let G be amenable. Applying the Theorem 1 of Lau and Losert [13], there exists a continuous projection Q from X^* onto S such that Q commutes with the right translations $\{\delta_t; t \in G\}$. By Theorem 2.4, we can find a projection P of X^* onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$.

To prove the converse, let X be a left Banach G-module and S be a weak*-closed right invariant subspace of X^* which is complemented in X^*. Let P be a projection of X^* onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(G)$. It is not hard to see that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$. Indeed, let $x \in X$ be given. Then $x = \psi.y$ for some $\psi \in L^1(G)$ and $y \in X$ (see Theorem 32.22 in [11]). We can write

$$\begin{aligned}\langle P(f.\delta_t), x\rangle &= \langle P(f.\delta_t), \psi.y\rangle = \langle P(f.\delta_t).\psi, y\rangle = \langle P(f.\delta_t * \psi), y\rangle \\ &= \langle P(f).\delta_t * \psi, y\rangle = \langle P(f).\delta_t, \psi.y\rangle = \langle P(f).\delta_t, , x\rangle.\end{aligned}$$

Since this holds for all $x \in X$, we conclude that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$. Clearly G is amenable by Theorem 1 in [13]. □

Corollary 2.7. *Let G be a locally compact group. Then G is compact if and only if whenever S is a weak* closed complemented right invariant subspace of $L^\infty(G/H)$, H a closed normal subgroup of G, then there exists a weak*-weak* continuous projection P of $L^\infty(G/H)$ onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G/H)$ and $\varphi \in L^1(G/H)$.*

Proof. If G is compact, then G/H is amenable. Therefore S is topologically invariantly complemented, see corollary 2.6. Let S be the range of a continuous projection commuting with convolution, say P. Since G/H is compact, part (*ii*) of Example 2.3 shows that P is weak*-weak* continuous.

To prove the converse, take $H = \{e\}$ and $S = \mathbb{C}1$. It is not hard to see that S is a weak* closed complemented right invariant subspace of $L^\infty(G)$. Since $\dim S = 1 < \infty$, by Lemma 4.21 in [19], S is complemented in $L^\infty(G)$. By assumption, there exists a weak*-weak* continuous projection P of $L^\infty(G)$ onto S such that $P(f.\varphi) = P(f).\varphi$ for all $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, and so $P(f.\delta_t) = P(f).\delta_t$ for all $f \in L^\infty(G)$ and $t \in G$. On the other hand, the members of $L^1(G)$ are exactly those linear functionals on $L^\infty(G)$ that are continuous relative to its weak*-topology. Therefore $P = \varphi$ for some $\varphi \in L^1(G)$. If $f \in L^\infty(G)$, then $\langle f.\delta_t, \varphi\rangle = \langle f, \varphi\rangle$ for every $t \in G$. So that

$$\langle f, \psi * \varphi\rangle = \int \langle f, \delta_t * \varphi\rangle d\psi(t) = \int \langle f, \varphi\rangle d\psi(t) = \langle f, \varphi\rangle \int d\psi(t).$$

Define $\rho_\varphi : L^1(G) \to L^1(G)$ by setting $\rho_\varphi(\psi) = \psi * \varphi$. Since the range of ρ_φ is finite dimensiona, ρ_φ is compact. Filali in [8] proved that finite-dimensional left ideals exist in $L^1(G)$ if and only if G is compact. Consequently G is compact. □

Theorem 2.8. *Let G be an amenable locally compact group. If X is a reflexive left Banach G-module and S is a right invariant subspace of X^* which is complemented in X^*, then there exists a projection P of X^* onto S such that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$ and $t \in G$.*

Proof. The assumptions imply that there is a projection P' from X^* onto S of norm $\|P'\|$. So let $\mathcal{P}$ be the set of all projections P from X^* onto S such that $\|P\| \le \|P'\|$. $\mathcal{P}$ is thus non-empty and convex, and norm bounded. Now let $\mathcal{B}(X^*)$ be the space of all bounded linear operators on X^*, under the weak operator topology. It is not hard to see that $\mathcal{P}$ is a compact convex subset of $\mathcal{B}(X^*)$. If $t \in G$ and $P \in \mathcal{P}$, we define $P_t \in \mathcal{B}(X^*)$ by $P_t(f) = P(f.\delta_t).\delta_{t^{-1}}$. Obviousely $P_t \in \mathcal{P}$ for all $P \in \mathcal{P}$ and $t \in G$. For a given $t \in G$, the mapping $P \mapsto P_t$ is a homeomorphism from $\mathcal{P}$ onto $\mathcal{P}$. It is not hard to see that the multiplication $(t, P) \mapsto P_t : G \times \mathcal{P} \to \mathcal{P}$ is separately continuous. Indeed, let $\{t_\alpha\}$ be a net in G converging to t. For every $f \in X^*$ and $x \in X$,

$$\begin{aligned}\lim_\alpha \langle P_{t_\alpha}(f), x\rangle &= \lim_\alpha \langle P(f.\delta_{t_\alpha}).\delta_{t_\alpha^{-1}}, x\rangle = \lim_\alpha \langle P(f.\delta_{t_\alpha}), t_\alpha^{-1}.x\rangle \\ &= \lim_\alpha \langle f.\delta_{t_\alpha}, P^*(t_\alpha^{-1}.x)\rangle = \lim_\alpha \langle f, t_\alpha.P^*(t_\alpha^{-1}.x)\rangle \\ &= \langle f, t.P^*(t^{-1}.x)\rangle = \langle P(f.\delta_t).\delta_{t^{-1}}, x\rangle.\end{aligned}$$

It follows that the map $(t, P) \to P_t$ is separately continuous. The jointly continuity of $(t, P) \to P_t$ is equivalent to the separate continuity in each variable, a classical result due to Ellis. Consequently $(G, \mathcal{P})$ is an affine flow, for more information see [18]. By Theorem 5.4 in [18], there exists a point P in $\mathcal{P}$ that is invariant under the action of G, that is, $P_t = P$ whenever $t \in G$. It follows that $P(f.\delta_t) = P(f).\delta_t$ for all $f \in X^*$. This completes our proof. □

Let X be a left Banach G-module. Let $\mathcal{B}(X^*)$ be the space of bounded linear operators from X^* into X^*. By the weak* operator topology on $\mathcal{B}(X^*)$, we shall mean the locally convex topology determined by the family of seminorms $\{\rho_{f,x}; f \in X^* \text{ and } x \in X\}$ where $\rho_{f,x}(T) = |\langle T(f), x\rangle|$. It is known that the unit ball of $\mathcal{B}(X^*)$ is compact in the weak* operator topology [3]. For each $t \in G$, define $\delta_t \in \mathcal{B}(X^*)$ by $\delta_t(f) = f.\delta_t$. Let $\overline{co\{\delta_t; t \in G\}}$ (here $co(A)$ will denote the convex hull of a subset A of a linear space) denote the closure of $co\{\delta_t; t \in G\}$ in the weak* operator topology. Then $\overline{co\{\delta_t; t \in G\}}$ is a semigroup and a compact subset of $\mathcal{B}(X^*)$.

Theorem 2.9. *Let G be a locally compact group. Then G is amenable if and only if whenever X is a left Banach G-module and H is a closed subgroup of G, then there exists a projection P of X^* onto $X^*_H = \{f \in X^*; f.\delta_t = f \text{ for all } t \in H\}$ such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(H)$.*

Proof. If H is a closed subgroup of G, it is easy to see that

$$X^*_H = \{f \in X^*; f.\delta_t = f \text{ for all } t \in H\}$$

is a right translation invariant weak*-closed subspace of X^*. If G is amenable, then H is also amenable [17]. Let m be a right invariant mean on $C_b(H)$. Then m is a positive functional on $C_b(H)$ with norm one. Hence there exists a net $\{\mu_\alpha\}$ in $C_b(H)^*$ such that each μ_α is a convex combination of point evaluations and $\{\mu_\alpha\}$ converges to m in the weak*-topology of $C_b(H)^*$. Let $f \in X^*$ be given. Since $\{f.\mu_\alpha\}$ is a bounded subset of X^*, the net $\{f.\mu_\alpha\}$ admits a subnet $\{f.\mu_\beta\}$ converging to an element h in X^* in the weak*-topology [3]. In order to show that $h \in X^*_H$, let $s \in H$ and $x \in X$ be given. Define $f' : H \to \mathbb{C}$ by $f'(t) = \langle f.\delta_t, x\rangle$. Clearly $f' \in C_b(H)$. We have

$$\begin{aligned}\langle h.\delta_s, x\rangle &= \lim_\beta \langle f.\mu_\beta * \delta_s, x\rangle = \lim_\beta \int \langle f.\delta_t, x\rangle d\mu_\beta * \delta_s(t) \\ &= \lim_\beta \int f'(t) d\mu_\beta * \delta_s(t) = \lim_\beta \int f'(ts) d\mu_\beta(t) \\ &= \langle M, R_s f'\rangle = \langle M, f'\rangle = \lim_\beta \int f'(t) d\mu_\beta(t) \\ &= \lim_\beta \langle f.\mu_\beta, x\rangle = \langle h, x\rangle.\end{aligned}$$

Since this holds for all $x \in X$, we conclude that $\overline{co\{f.\delta_t; t \in H\}} \cap X^*_H \neq \emptyset$ for all $f \in X^*$.

For each $f \in X^*$ let $K(f) = \{T \in \overline{co\{\delta_t; t \in H\}}; T(f) \in X^*_H\}$. The sets $K(f)$ are obviously compact in the weak* operator topology. We shall show that the family $\{K(f); f \in X^*\}$ has the finite intersection property. Since $K(f)$ is compact, it will follow that

$$\bigcap \{K(f); f \in X^*\} \neq \emptyset,$$

and if T is any member of this intersection, then $T(f) \in X^*_H$ for all $f \in X^*$. We proceed by induction. For $f \in X^*$, $K(f) \neq \emptyset$. Let $n \in \mathbb{N}$, $f_1, f_2, ..., f_n \in X^*$, and assume that $\bigcap\{K(f_i); 1 \leq i \leq n-1\} \neq \emptyset$. If S is a member of this intersection and if $R \in K(S(f_n))$, then $RS \in \bigcap\{K(f_i); 1 \leq i \leq n\}$. Thus $\{K(f); f \in X^*\}$ has the finite intersection property, as required.

Since $T \in \overline{co\{\delta_t; t \in H\}}$, there exists a net $\{T_\alpha\}$ such that each T_α is a convex combination of point evaluations and $\{T_\alpha\}$ converges to T in the weak* operator topology of $\mathcal{B}(X^*)$. For $f \in X^*_H$ and $x \in X$, we have

$$\langle T(f), x\rangle = \lim_\alpha \langle T_\alpha(f), x\rangle = \langle f, x\rangle,$$

and so T is the identity map on X^*_H. It follows that T is a continuous projection of X^* onto X^*_H. By Corollary 2.6, we can find a projection P of X^* onto X^*_H such that $P(f.\varphi) = P(f).\varphi$ for all $f \in X^*$ and $\varphi \in L^1(H)$.

To prove the converse, we consider $X = L^1(G)$. Then $L^1(G)$ is a left G-module with respect to the action $(t, \varphi) \to \delta_t * \varphi$. Let

$$X^*_G = \{f \in L^\infty(G); f.\delta_t = f \text{ for all } t \in G\}.$$

It is easy to see that the subalgebra X^*_G of $L^\infty(G)$ consisting of constant functions. Indeed, if φ and ψ are in $P^1(G)$, then

$$\begin{aligned}\langle f.\varphi, \psi\rangle &= \langle f, \varphi * \psi\rangle = \int \langle f, \delta_t * \psi\rangle d\varphi(t) = \int \langle f.\delta_t, \psi\rangle d\varphi(t) \\ &= \int \langle f, \psi\rangle d\varphi(t) = \langle f, \psi\rangle\end{aligned}$$

whenever $f \in X^*_G$. This shows that $f.\psi = f$, and so $f \in LUC(G)$. On the other hand,

$$\begin{aligned}\langle l_{s^{-1}} f, \varphi\rangle &= \int l_{s^{-1}} f(t) d\varphi(t) = \int f(st) d\varphi(t) = \int f(t) d\delta_s * \varphi(t) \\ &= \langle f, \delta_s * \varphi\rangle = \langle f.\delta_s, \varphi\rangle = \langle f, \varphi\rangle,\end{aligned}$$

for all $s \in G$ and $\varphi \in L^1(G)$. Consequently $l_s f = f$ for all $s \in G$, and so $f(s) = f(e)$ for all $s \in G$. By assumption, there exists a continuous projection P from $L^\infty(G)$ onto X^*_G such that P commutes with convolutions. In particular, P commutes with the right translations. Since $L^\infty(G)$ is a Banach lattice, we consider the modulus $|P|$ of P. Note that

$$|P|(f) = \sup\{|T(h)|; h \in L^\infty(G) \text{ and } |h| \leq f\} \text{ for all } f \in L^\infty(G)^+.$$

It is known that $|P|$ commutes with the right translations, see [15]. Since $P \neq 0$, $|P| \neq 0$. If $h \geq 0$, then $|P(h)| \leq \|h\||P|(1)$, and it follows that $|P|(1) > 0$. Hence $\frac{|P|}{|P|(1)}$ commutes with right translations on $L^\infty(G)$ also. Without loss of generality we may assume that P is a positive operator and $P(1) = 1$. Let $\{\varphi_\beta\}$ be an approximate identity in $L^1(G)$ such that each φ_β belongs to $P^1(G)$ [5]. Then we may suppose that $\{\varphi_\beta\}$ converges in the weak*-topology on $L^\infty(G)^*$, say to E. Define $m \in L^\infty(G)^*$ by setting $\langle m, f\rangle = \langle E, P(f)\rangle$. Then $\langle m, 1\rangle = \|m\| = 1$ and $\langle m, f.\delta_t\rangle = \langle m, f\rangle$ for each $f \in L^\infty(G)$ and $t \in G$. Hence G is amenable. This completes our proof. □

References

[1] J. Baker, A. T. Lau, J. Pym, *Module homomorphisms and topological centres associated with weakly sequentially complete Banach algebras,* J. Funct. Anal. **158** (1998), 186–208.

[2] M. E. B. Bekka, *Complemented subspace of $L^\infty(G)$, ideals of $L^1(G)$ and amenability,* Monatsh Math. **109** (1990), 195–203.

[3] H. G. Dales, *Banach algebra and automatic continuity,* London Math. Soc. Monogr. Ser. Clarendon Press, 2000.

[4] M. Filali, *Finite-dimensional left ideals in some algebras associated with a locally compact group,* Proc. Amer. Math. Soc. **127** (1999), 2325–2333 .

[5] G. B. Folland, *A course in abstract harmonic analysis,* CRC Press, Boca Raton, FL., 1995.

[6] B. E. Forrest, *Amenability and ideals in $A(G)$,* Austral. J. Math. Ser. A **53** (1992), 143–155.

[7] B. E. Forrest, E. Kaniuth, A. T. Lau, N. Spronk, *Ideals with bounded approximate identities in Fourier algebras,* J. Funct. Anal. **203** (2003), 286–304.

[8] B. E. Forrest, *Amenability and bounded approximate identitys in ideals of $A(G)$*, Illinois J. Math. **34** (1990), 1–25.

[9] A. Ghaffari, T. Haddadi, S. Javadi, M. Sheibani, *On the structure of hypergroups with respect to the induced topology*, Rocky Mountain J. Math. **52** (2022), 519–533.

[10] J. E. Gilbert, *On projections of $L^{\infty}(G)$ onto translation invariant subspaces* Proc. London Math. Soc. **19** (1969), 69–88.
[11] E. Hewitt, K. A. Ross, *Abstract harmonic analysis,* Vol. I, Springer Verlage, Berlin, 1963; Vol. II, Springer Verlage, Berlin, 1970.
[12] A. T. Lau, V. Losert, *Complementation of certain subspaces of $L^{\infty}(G)$ of a locally compact group*, Pacific J. Math. **141** (1990), 295–310.
[13] A. T. Lau, V. Losert, *Weak*-closed complemented invariant subspaces of $L^{\infty}(G)$ and amenable locally compact groups*, Pacific J. Math. **123** (1986), 149–159.
[14] A. T. Lau, *Invariantly complemented subspaces of $L^{\infty}(G)$ and amenable locally group,* Illinois J. Math. **26** (1982), 226–235.
[15] A. T. Lau, F. Ghahramani, *Multipliers and modulus on Banach algebras related to locally compact groups*, J. Funct. Anal. **150** (1997), 478–497.
[16] A.T. Lau, A. Ulger, *Characterizations of closed ideals with bounded approximate identities in commutative Banach algebras, complemented subspaces of the group von Neumann algebras and applications,* Trans. Amer. Math. Soc. **366** (2014), 4151–4171.
[17] A. L. T. Paterson, *Amenability*, Amer. Math. Soc. Math. Survey and Monographs 29, Providence, Rhode Island, 1988.
[18] J. P. Pier, Amenable locally compact groups, John Wiley And Sons, New York, 1984.
[19] W. Rudin, *Functional analysis*, McGraw Hill, New York, 1991.
[20] V. Runde,*Lectures on amenability*, Lecture Notes in Mathematics 1774, Springer Verlag, 2002.
[21] N. Tahmasebi, *Hypergroups and invariant complemented subspaces*, J. Math. Anal. Appl. **414** (2014), 641–655.
[22] N. Tahmasebi, *Fixed point properties, invariant means and invariant projections related to hypergroups*, J. Math. Anal. Appl. **437** (2016), 526–544.
[23] Y. Takahashi, *A characterization of certain weak* closed subalgebras of $L^{\infty}(G)$,* Hokhaido Math. J. **11** (1982), 116–124.
[24] P. J. Wood, *Invariant complementation and projectivity in the fourier algebra,* Proc. Amer. Math. Soc. **131** (2002), 1881–1890.
[25] P. J. Wood, *Complemented ideals in the fourier algebra of a locally compact group,* Proc. Amer. Math. Soc. **128** (1999), 445–451.