Filomat 40:3 (2026), 891–915
https://doi.org/10.2298/FIL2603891F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Numerical inversion of the multiplicative Laplace transform via multiplicative Laguerre polynomials

Edinson Fuentes[a,*], Luis E. Garza[b], Martha L. Saiz[c]

[a] *Universidad Pedagógica y Tecnológica de Colombia, Facultad de Ciencias de la Educación, Tunja, Colombia*
[b] *Universidad de Colima, Facultad de Ciencias, Colima, México*
[c] *Universidad Pedagógica y Tecnológica de Colombia, Facultad de Estudios a Distancia, Tunja, Colombia*

Abstract. In this manuscript, we introduce the multiplicative Laguerre polynomials (MLPs) that arise as one of the solutions of the multiplicative Sturm-Liouville equation

$$\frac{d^*}{dx}\left(e^{x\omega(x)} \odot \frac{d^*y}{dx}\right) \oplus \left(e^{n\omega(x)} \odot y\right) = 1, \quad x > 0,$$

where $\omega(x) = x^\alpha e^{-x}$ with $\alpha > -1$. Here, $\frac{d^*}{dx} f(x)$ denotes the multiplicative derivative of the function f at x, defined by

$$\lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h},$$

whenever this limit exists. We compute the multiplicative Laplace transform of the multiplicative Laguerre polynomials and establish the multiplicative version of Tricomi's formula. Furthermore, we introduce two numerical methods for approximating the inverse multiplicative Laplace transform, based on properties of the multiplicative Laguerre polynomials. We illustrate the obtained results with some examples related to the solution of nonlinear classical second-order differential equations.

1. Introduction

Multiplicative calculus (MC) is a type of Non-Newtonian calculus that is associated with the definition of multiplicative (or geometric) derivative (4) (see [20]). It is commonly referred to as geometric calculus, and it has been successfully applied in various contexts, including population growth modeling [33], modeling the radius of certain human body cells [2], efficient approximation of linear and nonlinear signal representations [5], and other natural phenomena described by multiplicative differential equations [3]. Moreover, problems

2020 *Mathematics Subject Classification*. Primary 42C05; Secondary 42C10, 33C45, 44A10.
Keywords. Multiplicative calculus, Laguerre polynomials, inverse multiplicative Laplace transform.
Received: 17 August 2025; Revised: 22 November 2025; Accepted: 02 December 2025
Communicated by Dragan S. Djordjević
* Corresponding author: Edinson Fuentes
Email addresses: edinson.fuentes@uptc.edu.co (Edinson Fuentes), luis_garza1@ucol.mx (Luis E. Garza), martha.saiz@uptc.edu.co (Martha L. Saiz)
ORCID iDs: https://orcid.org/0000-0002-5430-5016 (Edinson Fuentes), https://orcid.org/0000-0002-2569-186X (Luis E. Garza), https://orcid.org/0000-0002-6906-3953 (Martha L. Saiz)

that are complicated to address using classical calculus can be solved in a more efficient way using MC, such as the problem of approximating exponentially varying weights or the analysis of nonlinear differential equations whose solution is an exponential function, as will be shown in this contribution. Although this definition of multiplicative calculus is usually the standard definition considered in the literature, it is worth noticing that there are other kinds of multiplicative calculus that have been studied in other contexts. Among others, bigeometric calculus [49] is useful in economics, fractal like scaling, and power laws; projective multiplicative calculus [15] appears in more advanced functional analysis and abstract differential geometry; and time-scale multiplicative calculus [6] is defined on arbitrary time scales (discrete, continuous, hybrid) and it is used in hybrid dynamical systems and economic models that mix continuous and discrete growth.

Moreover, numerical methods in MC have recently been addressed. For instance, an iterative method based on MC similar to Newton's method to solve nonlinear equations was constructed in [32] , numerical methods to solve multiplicative partial differential equations were introduced in [48], classical methods for solving differential equations, such as the Runge-Kutta method, were discussed in [30], methods for approximating positive functions through multiplicative series were studied in [10], and methods for multiplicative dynamical systems describing the position of a particle were considered in [28]. These works highlight the growing relevance of multiplicative approaches and justify the need for specialized numerical methods such as the numerical method to approximate the inverse multiplicative Laplace transform.

On the other hand, in classical calculus, the generalized Laguerre polynomials $L_n^{(\alpha)}$ are the polynomials defined by the following Rodrigues' formula

$$L_n^{(\alpha)}(x) = \frac{1}{n!\omega(x)}\frac{d^n}{dx^n}\left[x^n\omega(x)\right], \tag{1}$$

where $\omega(x) = x^\alpha e^{-x}$ is the Laguerre weight function, $\alpha > -1$, and n is a non-negative integer. The case $\alpha = 0$ was originally studied by Laguerre [24], and it is common to denote these polynomials by $L_n(x) = L_n^{(0)}(x)$. The general case, where $\alpha > -1$, was studied by Sonin [34]. For this reason, $L_n^{(\alpha)}$ is sometimes referred to as the Sonin-Laguerre or the generalized Laguerre polynomial. However, in this context, it will simply be referred to as the Laguerre polynomial. A nice summary of properties, as well as many references about Laguerre polynomials, can be found in [7, 36].

These polynomials have a wide range of applications, including their use in approximating the inverse Laplace transform [39]. The Laplace transform is an important tool in engineering [12], and its numerical inversion has historically been a significant challenge due to its inherently ill-posed nature, representing an important problem in applied mathematics and engineering. Numerous references on numerical techniques for this problem can be found in [8]. Among these techniques, the Tricomi's method and the Weeks' method are two of the most well-known approaches; see [9, 16, 21, 38–40] among many others.

The main reason for the popularity of the Weeks' method is that it provides an analytic formula for the domain function of a smooth function $f(x)$ with bounded exponential growth, expressed as an expansion in Laguerre polynomials:

$$f(x) = \sum_{n=0}^{\infty} f_n L_n(2bx)e^{(\sigma-b)x}, \; b > 0, \; \sigma > \sigma_0, \tag{2}$$

where σ_0 denotes the abscissa of convergence of the Laplace transform $F(s) = \int_0^\infty f(x)e^{-sx}dx$ of $f(x)$. That is, all the singularities of $F(s)$ are located in the half-plane $\mathfrak{Re}(s) < \sigma_0$. In particular, if $\sigma = b$ we obtain the Tricomi's method.

Furthermore, within the framework of MC, the multiplicative Laguerre polynomials (MLPs) with $\alpha = 0$ were studied in [23]. In that work, the authors obtained the MLPs in terms of the classical Laguerre polynomials, given by the formula:

$$\tilde{L}_n^{(0)}(x) = e^1\left(\prod_{k=1}^{n} e^{(-1)^k\frac{n!}{(n-k)!(k!)^2}}\right)^{x^k} = e^{L_n^{(0)}(x)}.$$

The MLPs arise as solutions of a particular case of a Sturm-Liouville multiplicative differential equation introduced in [17]. This equation is given by

$$\frac{d^*}{dx}\left(e^{p(x)} \odot \frac{d^*y}{dx}\right) \oplus (e^{q(x)} \odot y) \oplus \left(e^{\gamma\omega(x)} \odot y\right) = 1, \ x \in (a,b), \tag{3}$$

where $d^*y/dx = y^*$ denotes the multiplicative derivative of a positive function y (see [4]). Here $p(x)$, $q(x)$, and $\omega(x)$ are real-valued continuous functions, and γ is a constant known as the spectral parameter. The operators $\oplus$ and $\odot$ are binary operations defined in the MC framework in the domain $\mathbb{R}^+$, given by $a \oplus b := ab$ and $a \odot b := a^{\ln b} = b^{\ln a}$. With these definitions, (3) can be rewritten in the equivalent form:

$$\left((y^*)^{p(x)}\right)^* \oplus \left(y^{q(x)}\right) \oplus \left(y^{\gamma\omega(x)}\right) = 1, \ x \in (a,b).$$

This equation reveals a close connection with classical differential equations, which will be further explored in Section 3.

Moreover, the study of the Sturm-Liouville theory has contributed to the development of several mathematical theories, such as Fourier analysis, orthogonal polynomials, and the Laplace transform, among others [25]. Furthermore, these theories have been linked to solutions of problems that arise in physics and engineering. Thus, to study properties of the theory of Sturm-Liouville from the framework of MC can offer new perspectives to address classical problems where the Sturm-Liouville theory is applied.

Several studies have explored particular cases of the functions $p(x)$, $q(x)$, and $\omega(x)$, which are similar to those of classical calculus. In each case, a family of multiplicative orthogonal polynomials arises. The particular cases where $q(x) = 0$ are summarized in the following table.

Polynomials	$p(x)$	$\omega(x)$	λ	(a,b)	**Ref.**
Legendre	$1-x^2$	1	$n(n+1)$	$(-1,1)$	[18]
Chebyshev (first kind)	$(1-x^2)\omega(x)$	$\frac{1}{\sqrt{1-x^2}}$	n^2	$(-1,1)$	[41]
Chebyshev (second kind)	$(1-x^2)\omega(x)$	$\sqrt{1-x^2}$	$n(n+2)$	$(-1,1)$	[41]
Jacobi ($\alpha,\beta > -1$)	$(1-x^2)\omega(x)$	$(1-x)^\alpha(1+x)^\beta$	$n(n+\alpha+\beta+1)$	$(-1,1)$	[10]
Hermite	e^{-x^2}	e^{-x^2}	$2n$	$(-\infty,\infty)$	[19, 42]
Laguerre	xe^{-x}	e^{-x}	n	$(0,\infty)$	[23]

Table 1: Multiplicative orthogonal polynomials with $q(x) = 0$.

Furthermore, the multiplicative Bessel polynomials arise when $p(x) = x$, $q(x) = -\frac{v^2}{x}$, and $\omega(x) = x$ for $x \in (0,b]$, where b is a positive real number, and v is a real number (see [26, 50]).

On the other hand, the theory of orthogonal polynomials, in particular the Laguerre polynomials, plays a fundamental role in mathematical physics, as it provides practical tools for solving a wide range of problems related to Sturm–Liouville differential equations [7], function approximations [37], quadrature formulas [37], spectral methods for partial differential equations [31], and, as previously mentioned, the approximation of the inverse Laplace transform, among many others (see [7, 36]).

As a consequence, due to the important role that Laguerre polynomials and the inverse Laplace transform play in various fields of pure and applied mathematics, one of the main aims of this contribution is to introduce the multiplicative generalized Laguerre polynomials (for $\alpha > -1$), examine some of their properties, and explore their application in the approximation of the inverse multiplicative Laplace transform (MLT) via the Tricomi's and Weeks' methods, all within the framework of MC.

The structure of the manuscript is as follows. Section 2 provides definitions and properties related to both MC and MLT using the same notation given in [45]. Then, in Section 3, we establish and prove the connection between classical and multiplicative homogeneous differential equations when they share the same solution. We also introduce the reverse relationship. These results lead to the conclusion that MC can be applied to solve classical differential equations, whose solutions are not easily found using traditional methods, by solving their corresponding multiplicative differential equations.

In Section 4, we introduce the MLPs that arise as solutions of the multiplicative Sturm–Liouville equation (20). We deduce some of their properties, including the orthogonality relation. Subsequently, we present the MLT of the MLPs and the multiplicative version of Tricomi's formula. This formula allows for the computation of the MLT of positive functions that can be expressed as a multiplicative Laguerre series. In classical calculus, this result was originally established by Tricomi in [38, 39], and has been widely used in many areas of mathematics and applied sciences. It is commonly referred to as Tricomi's formula.

In the final section, we introduce two methods for the numerical inversion of the MLT, based on specific properties of the MLPs. These methods are known as the multiplicative Tricomi's method and the multiplicative Weeks' method, respectively. Finally, we apply the previously introduced techniques to solve a nonlinear classical second-order differential equation within the framework of MC. All plots, as well as the numerical data in the tables presented in the manuscript, were generated using Matlab R2024a.

2. Multiplicative calculus and Laplace transform

In this section, we present the definition and properties of the multiplicative Laplace transform (MLT) [45]. To achieve this, we first present the fundamental definitions and properties of MC. These concepts are analogous to those of classical calculus, and their proofs can be found in [4, 17, 35].

2.1. An overview of multiplicative calculus

Henceforth, $\mathbb{R}^+$ denotes the set of all positive real numbers.

Definition 2.1. *Let $f : \mathbf{A} \subseteq \mathbb{R} \to \mathbb{R}^+$ be a function. The multiplicative (or geometric) derivative, or *-derivative, of f at $x \in \mathbf{A}$ is defined by*

$$f^*(x) := \frac{d^*}{dx} f(x) = \lim_{h\to 0} \left(\frac{f(x+h)}{f(x)} \right)^{1/h}, \tag{4}$$

if the above limit exists and it is positive.

In particular, if f is differentiable in the classical sense at x, there exists a relationship between the classical derivative and the multiplicative derivative (see [35, Theorem 1]):

$$f^*(x) = e^{(\ln \circ f)'(x)} = e^{\frac{f'(x)}{f(x)}}, \tag{5}$$

where $(\ln \circ f)(x) = \ln(f(x))$. Furthermore, the n-th multiplicative derivative of the positive function f is defined by $f^{*(n)}(x) := e^{(\ln \circ f)^{(n)}(x)}$ (see [35]). Additionally, if f is multiplicative differentiable at the point x, then (see [42, Theorem 2.1])

$$f'(x) = f(x) \ln(f^*(x)). \tag{6}$$

The multiplicative derivative satisfies the following properties.

Proposition 2.2. *(see [4]) Let f, g be multiplicative differentiable at x, and let ϕ be classical differentiable at x, then*

1. $(kf)^*(x) = f^*(x)$, $k \in \mathbb{R}^+$,
2. $(fg)^*(x) = f^*(x)g^*(x)$,
3. $(f/g)^*(x) = f^*(x)/g^*(x)$,
4. $(f^\phi)^*(x) = f^*(x)^{\phi(x)} f(x)^{\phi'(x)}$. *In particular, if f is a constant function*

$$(f^\phi)^{*(n)}(x) = f(x)^{\phi^{(n)}(x)}, \quad n = 0, \dots. \tag{7}$$

A multiplicative integral is also defined in [4] for positive bounded functions.

Definition 2.3. *Let $f : [a,b] \to \mathbb{R}^+$ be a positive function that is Riemann integrable on $[a,b]$. The multiplicative integral of f, denoted by $\int_a^b f(x)^{dx}$, is defined as*

$$\int_a^b f(x)^{dx} = \exp\left(\int_a^b (\ln f(x))dx\right) = e^{\int_a^b (\ln f(x))dx}. \tag{8}$$

*In such a case, f is said to be multiplicative integrable or *-integrable.*

In the same way, if f is *-integrable on $[a,b]$, then (see [4]) $\int_a^b f(x)dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}$. This multiplicative integral satisfies the following properties.

Proposition 2.4. *(See [4]) Let $f, g : [a,b] \to \mathbb{R}^+$ be bounded and *-integrable functions, and let $\phi : [a,b] \to \mathbb{R}^+$ be classical differentiable at $x \in [a,b]$, then*

1. $\int_a^b \left(f(x)^k\right)^{dx} = \left(\int_a^b f(x)^{dx}\right)^k, \; k \in \mathbb{R}$,
2. $\int_a^b (f(x)g(x))^{dx} = \left(\int_a^b f(x)^{dx}\right)\left(\int_a^b g(x)^{dx}\right)$,
3. $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$,
4. *Multiplicative integration by parts formula:* $\int_a^b \left(f^*(x)^{\phi(x)}\right)^{dx} = \frac{f(b)^{\phi(b)}}{f(a)^{\phi(a)}} \left[\int_a^b \left(f(x)^{\phi'(x)}\right)^{dx}\right]^{-1}$.

Finally, a set of real and positive functions $\{f_1(x), \ldots, f_n(x)\}$ is said to be *multiplicatively linearly independent* on an interval $\mathbf{I}$ if and only if the equation $\prod_{k=1}^n f_k^{c_k}(x) = 1$ admits only the trivial solution $c_1 = \cdots = c_n = 0$, where c_k are real constants, for every $x \in \mathbf{I}$.

Proposition 2.5. *([10, Proposition 8]) The set of real and positive functions $\{f_1(x), \ldots, f_n(x)\}$ is multiplicatively linearly independent on the interval $\mathbf{I}$ if and only if the set $\{\ln(f_1(x)), \ldots, \ln(f_n(x))\}$ is linearly independent on $\mathbf{I}$.*

2.2. Multiplicative Laplace transform

Now, we consider the multiplicative Laplace transform (MLT) and introduce some of its properties, whose proofs can be found in [45].

Definition 2.6. *[45, Definition 3.1 and 3.3] Let f be a positive function on $[0,\infty)$. The MLT of f is defined as*

$$\mathcal{L}_m\{f(x)\}(s) := \int_0^\infty \left(f(x) \odot e^{e^{-sx}}\right)^{dx} = e^{\int_0^\infty \ln(f(x))e^{-sx}dx} = e^{\mathcal{L}\{\ln(f(x))\}(s)},$$

where $\mathcal{L}\{\cdot\}$ is the classical Laplace transform. Moreover, if $F_m(s)$ is the MLT of a continuous function $f(x)$, the function $\mathcal{L}_m^{-1}\{F_m(s)\}(x)$ is called the inverse MLT of $F_m(s)$.

A positive function f on $[0,\infty)$ is said to be of β−double exponential order if and only if there exist positive constants x_0, K, and β such that $|f(x)| \leq Ke^{e^{\beta x}}$, for $x > x_0$.

The following theorem guarantees the existence of the MLT.

Theorem 2.7. *[45, Theorem 3.7] Let f be a positive function of β−double exponential order for $x > x_0$ on $[0,\infty)$, and let $f(x)$ be a piecewise continuous function given on $[0,\infty)$. Then, for $s > \beta$, $\mathcal{L}_m\{f(x)\}$ exists.*

The MLT satisfies the following properties.

Proposition 2.8. *(See[45]) Let $\mathcal{L}_m\{f(x)\} = F_m(s)$ be the MLT of f. The following properties hold.*

1. Multiplicative linearity property. *Let k_1, k_2 be arbitrary constants and f_1, f_2 functions which have MLT, then*

$$\mathcal{L}_m\left\{f_1^{k_1}(x) \oplus f_2^{k_2}(x)\right\} = \left(\mathcal{L}_m\{f_1(x)\}\right)^{k_1} \oplus \left(\mathcal{L}_m\{f_2(x)\}\right)^{k_2}.$$

2. Multiplicative change of scale property. *For any non-negative real number b*

$$\mathcal{L}_m\{f(bx)\} = F_m\left(\frac{s}{b}\right)^{\frac{1}{b}}.$$

3. Multiplicative derivatives of Laplace transforms. *For any non-negative integer number n*

$$\mathcal{L}_m\{f(x)^{x^n}\} = \left(F_m^{*(n)}(s)\right)^{(-1)^n}.$$

4. Multiplicative first shifting property.

$$\mathcal{L}_m\{f(x) \odot e^{e^{ax}}\} = \mathcal{L}_m\{f(x)^{e^{ax}}\} = \mathcal{L}_m\{f(x)\}(s)\Big|_{s\to s-a} = F_m(s-a).$$

5. Transform of multiplicative derivatives. *Let $f, f^*, f^{**}, \ldots, f^{*(n-1)}$ be continuous functions and $f^{*(n)}$ be a piecewise continuous function on the interval $0 \le x \le A$. Also, suppose that there exist positive real numbers K, β and x_0 such that*

$$|f(x)| \le Ke^{e^{\beta x}},\ |f^*(x)| \le Ke^{e^{\beta x}},\ \ldots,\ |f^{*(n-1)}(x)| \le Ke^{e^{\beta x}}$$

for $x \ge x_0$. Then for $s > \beta$ MLT $\mathcal{L}_m\{f^{*(n)}(x)\}$ exists and can be calculated by the formula

$$\mathcal{L}_m\{f^{*(n)}(x)\} = \frac{1}{f(0)^{s^{n-1}} f^*(0)^{s^{n-2}} f^{**}(0)^{s^{n-3}} \cdots f^{*(n-1)}(0)} F_m(s)^{s^n}.$$

6. *Let f_1 and f_2 be positive definite continuous functions.*

$$f_1 = f_2 \text{ if and only if } \mathcal{L}_m\{f_1\} = \mathcal{L}_m\{f_2\}. \tag{9}$$

3. Homogeneous second-order multiplicative differential equations

In this section, we first introduce some properties of multiplicative series along with the multiplicative version of Frobenius' theorem (see [10, Theorem 7]), which will be used to solve the multiplicative Laguerre differential equation in the following section. We then investigate the relationship between homogeneous linear second-order multiplicative differential equations and classical second-order differential equations when they share the same solution, and conversely.

A point x_0 is said to be a multiplicative ordinary point of the homogeneous linear second-order multiplicative differential equation

$$(y^{**}) \oplus (y^*)^{f(x)} \oplus (y)^{g(x)} = 1, \tag{10}$$

where f, g are functions of x and $y^{*(n)}(x) = e^{(\ln y)^{(n)}}$ $(n = 1, 2)$, if and only if both f and g are analytic in the classical sense at x_0. Otherwise, x_0 is called a multiplicative singular point.

Similarly, a point x_0 is said to be a multiplicative regular singular point of the homogeneous linear second-order multiplicative differential equation

$$(y^{**})^{A(x)} \oplus (y^*)^{B(x)} \oplus (y)^{C(x)} = 1, \tag{11}$$

where A, B and C are functions of x, if and only if the functions

$$(x-x_0)\frac{B(x)}{A(x)} \text{ and } (x-x_0)^2\frac{C(x)}{A(x)}$$

are analytic in the classical sense at x_0.

A multiplicative power series centered at the fixed point $x_0 \in \mathbb{R}$ is an infinite product of the form

$$\prod_{n=0}^{\infty}(a_n)^{(x-x_0)^n}, \ \ a_n \in \mathbb{R}^+. \tag{12}$$

The multiplicative series (12) is convergent if and only if the classical series $\sum_{n=0}^{\infty}\ln(a_n)(x-x_0)^k$ is convergent (see [46, Lemma 4]).

Finally, the following result, known as the multiplicative Frobenius theorem, guarantees the existence of solutions around a multiplicative regular singular point, and its proof can be found in [10, Theorem 7].

Theorem 3.1. *(Multiplicative Frobenius' theorem) Let $x = x_0$ be a multiplicative regular singular point of* (11)*. Then, there exists at least one solution of the form*

$$y(x) = \prod_{k=0}^{\infty}(a_k)^{(x-x_0)^{k+L}}, \ \ a_k \in \mathbb{R}^+, \ a_0 \neq 1, \tag{13}$$

where L is a constant to be determined. Moreover, there exists $R > 0$ such that the multiplicative series (13) *converges at least on the interval $0 < x - x_0 < R$.*

On the other hand, the following results describe the relation between classical and multiplicative second-order differential equations when they share the same solution. A first approach to these kind of results can be found in [17].

Theorem 3.2. *Let f and g be continuous functions on $[0, \infty)$. The positive function $y(x) > 0$ is a solution of the homogeneous linear second-order classical differential equation with initial conditions*

$$\begin{cases} y'' + f(x)y' + g(x)y = 0, \\ y(0) = y_0, \ y'(0) = y_1, \end{cases} \tag{14}$$

if and only if $y(x)$ is also a solution of the nonlinear second-order multiplicative differential equation with initial conditions

$$\begin{cases} y^{**} \oplus \left[(e^{f(x)} \oplus y^*) \odot y^*\right] \oplus e^{g(x)} = 1, \\ y(0) = y_0, \ y^*(0) = e^{\frac{y_1}{y_0}}. \end{cases} \tag{15}$$

Proof. Suppose that $y(x) > 0$ is a solution of the classical differential equation (14). Since y is positive, there exists a classical differentiable function y_m such that $y = e^{y_m}$, which implies that $y_m = \ln y$, $y_m' = \frac{y'}{y}$, and $y_m'' = \frac{y''y-(y')^2}{y^2}$. Substituting these expressions into the following differential equation, we obtain $y_m'' + (f(x) + y_m')y_m' + g(x) = \frac{1}{y}(y'' + f(x)y' + g(x)y) = 0$. Thus, we conclude that y_m satisfies the classical differential equation

$$y_m'' + (f(x) + y_m')y_m' + g(x) = 0. \tag{16}$$

As a consequence, considering the multiplicative form of the above equation, it follows that $y = e^{y_m}$ is a solution of (15). Since, from (7) $y^* = e^{y_m'}$, $y^{**} = e^{y_m''}$, we have

$$y^{**} \oplus \left[(e^{f(x)} \oplus y^*) \odot y^*\right] \oplus e^{g(x)} = e^{y_m''+(f(x)+y_m')y_m'+g(x)} = e^0 = 1.$$

Moreover, the initial conditions are immediate from (5) and (6). The converse follows in a similar way. □

The above theorem states that if $y > 0$ is a solution of (14), then it is also a solution of (15), and the converse is also true. In particular, if $y_m = -\int f(x)dx$, then (16) becomes $y_m'' + g(x) = 0$. As a consequence, we obtain the following corollary.

Corollary 3.3. *Let f and g be continuous functions on $[0,\infty)$, and let $y = e^{y_m} = e^{-\int f(x)dx}$. The positive function $y(x) > 0$ is a solution of the homogeneous linear second-order classical differential equation with initial conditions*

$$\begin{cases} y'' + f(x)y' + g(x)y = 0, \\ y(0) = y_0, \ y'(0) = y_1, \end{cases}$$

if and only if y is a solution of the second-order multiplicative differential equation with initial conditions

$$\begin{cases} y^{**} \oplus e^{g(x)} = 1, \\ y(0) = y_0, \ y^{*}(0) = e^{\frac{y_1}{y_0}}. \end{cases}$$

The following example shows that solving a homogeneous linear multiplicative differential equation is equivalent to solving a more complex classical differential equation.

Example 3.4. *We consider the second-order multiplicative differential equation with initial conditions*

$$\begin{cases} y^{**} \oplus y^{-a^2} = 1, \\ y(0) = e, \ y^{*}(0) = e^{-a}, \ a \in \mathbb{R}. \end{cases}$$

Applying the MLT and its properties (Proposition 2.8), we obtain

$$\mathcal{L}_m\{y^{**}\} \oplus (\mathcal{L}_m\{y\})^{-a^2} = 1 \leftrightarrow \mathcal{L}_m\{y\} = e^{\frac{1}{s+a}},$$

and its inverse transform is given by $y(x) = e^{e^{-ax}}$. As a consequence, we have solved equation $y^{**} \oplus e^{-a^2 e^{-ax}} = 1$. Setting $g(x) = -a^2 e^{-ax}$, $f(x) = (y_m)' = -(e^{-ax})' = ae^{-ax}$ and using Corollary 3.3, we find that $y = e^{e^{-ax}}$ is also solution of the classical differential equation with initial conditions

$$\begin{cases} y'' + ae^{-ax}y' - a^2 e^{-ax} y = 0, \\ y(0) = e, \ y'(0) = -ae. \end{cases}$$

In a similar way, we have the following theorem.

Theorem 3.5. *Let f and g be continuous functions on $[0,\infty)$. The positive function $y(x) > 0$ is a solution of the homogeneous second-order multiplicative differential equation with initial conditions*

$$\begin{cases} y^{**} \oplus (y^{*})^{f(x)} \oplus (y)^{g(x)} = 1, \\ y(0) = y_0, \ y^{*}(0) = y_1, \end{cases} \tag{17}$$

if and only if y is also a solution of the second-order classical differential equation with initial conditions

$$\begin{cases} y'' + \left(f(x) - \frac{y'}{y}\right) y' + g(x) y \ln y = 0, \\ y(0) = y_0, \ y'(0) = y_0 \ln(y_1). \end{cases} \tag{18}$$

Proof. Suppose that $y(x) > 0$ is a solution of the multiplicative differential equation (17). Since y is positive, there exists a differentiable function y_m such that $y = e^{y_m}$. From (7) we have $y^{*} = e^{y'_m}$ and $y^{**} = e^{y''_m}$, then

$$e^{y''_m + f(x)y'_m + g(x)y_m} = e^{y''_m}\left(e^{y'_m}\right)^{f(x)} (e^{y_m})^{g(x)} = y^{**} \oplus (y^{*})^{f(x)} \oplus (y)^{g(x)} = 1,$$

which implies that y_m satisfies the classical differential equation $y''_m + f(x)y'_m + g(x)y_m = 0$. Now, we will prove that $y = e^{y_m}$ is also a solution of (18):

$$\begin{aligned} y'' + \left(f(x) - \frac{y'}{y}\right) y' + g(x) y \ln y &= y''_m e^{y_m} + (y'_m)^2 e^{y_m} + \left(f(x) - \frac{y'_m e^{y_m}}{e^{y_m}}\right) y'_m e^{y_m} + g(x) e^{y_m} y_m \\ &= e^{y_m}(y''_m + f(x)y'_m + g(x)y_m) = 0. \end{aligned}$$

Thus the required result follows. Moreover, the initial conditions are immediate from (5) and (6). The converse follows in a similar way. □

Now, we apply the above theorem to solve a nonlinear homogeneous classical differential equation whose solution is not easily found using traditional methods.

Example 3.6. *We consider the nonlinear second-order classical differential equation with initial conditions*

$$\begin{cases} yy'' - (y')^2 + y^2 \ln y = 0, \\ y(0) = 2,\ y'(0) = 0. \end{cases} \tag{19}$$

It is difficult to solve this differential equation using classical calculus. However, applying the above Theorem (3.5)*, the problem reduces to solving the multiplicative differential equation with initial conditions*

$$\begin{cases} y^{**} \oplus y = 1, \\ y(0) = 2,\ y^{*}(0) = 1. \end{cases}$$

Using the MLT and its properties (Proposition 2.8), we obtain

$$\mathcal{L}_m\{y^{**}\} \oplus \mathcal{L}_m\{y\} = 1 \Leftrightarrow \mathcal{L}_m\{y\} = 2^{\frac{s}{s^2+1}} = \left(e^{\frac{s}{s^2+1}}\right)^{\ln 2},$$

and its inverse transform is given by $y(x) = (e^{\cos x})^{\ln 2} = 2^{\cos x}$. *As a consequence, the function* $y(x) = 2^{\cos x}$ *is a solution of* (19).

More information on multiplicative differential equations can be found in [14, 43, 44, 46, 47], among many others.

4. The multiplicative Laguerre polynomials and their Laplace Transform

In this section, we first introduce the MLPs that arise as one of the solutions of the multiplicative Laguerre differential equation:

$$\frac{d^*}{dx}\left(e^{x\omega(x)} \odot \frac{d^*y}{dx}\right) \oplus \left(e^{n\omega(x)} \odot y\right) = 1, \quad x > 0, \tag{20}$$

where $\omega(x) = x^{\alpha}e^{-x}$ is the Laguerre weight function, $\alpha > -1$, and n is a non-negative integer. We also state the orthogonality property satisfied by these polynomials and present some properties required to obtain their MLT. Finally, we explicitly compute the associated MLT.

4.1. *The multiplicative Laguerre polynomials*

The multiplicative differential equation (20) is a particular case of the multiplicative Sturm-Liouville equation (3), as stated above. Notice that (20) is equivalent to

$$(y^{**})^{x} \oplus (y^{*})^{1+\alpha-x} \oplus y^{n} = 1, \quad x > 0, \tag{21}$$

and $x = 0$ is a multiplicative regular singular point of equation (21).

On the other hand, according to Theorem 3.5, the equation (21) has the same positive solution than the second-order classical differential equation

$$xy'' + \left(1 + \alpha - x - \frac{y'}{y}\right)y' + ny\ln y = 0,\ x > 0.$$

The following theorem states that one of the solutions of (21) is a multiplicative polynomial and that it is unique.

Theorem 4.1. *Let $\alpha > -1$. Consider the multiplicative differential equation*

$$(y^{**})^x \oplus (y^*)^{1+\alpha-x} \oplus y^\gamma = 1, \;\; x > 0, \tag{22}$$

where $\gamma \in \mathbb{R}$ is a spectral parameter. The equation (22) has a multiplicative polynomial solution, not identically one, of the form

$$y_n(x) = \prod_{k=0}^{n} (a_k)^{x^k}, \;\; a_k \in \mathbb{R}^+, \;\; a_0 \neq 1, \tag{23}$$

if and only if $\gamma = n$ for every non-negative integer n. Moreover, the solution y^c (c constant) is the only multiplicative polynomial solution, i.e. every other solution of (22) that is multiplicatively linearly independent to y on $(0, \infty)$ is not a multiplicative polynomial.

Proof. Since equation (22) has a multiplicative regular singular point at $x = 0$, the multiplicative Frobenius' theorem ensures the existence of at least one solution of (22) around $x = 0$ of the form

$$y(x) = \prod_{k=0}^{\infty} (a_k)^{x^{k+L}}, \;\; a_0 \neq 1. \tag{24}$$

In particular, $y(x) = (a_0)^{x^L}$ is solution. Taking its multiplicative derivatives and substituting into (22), we obtain

$$\left((a_0)^{L(L-1)x^{L-2}}\right)^x \left((a_0)^{Lx^{L-1}}\right)^{1+\alpha-x} \left((a_0)^{x^L}\right)^\gamma = 1.$$

This expression is equivalent to $(a_0)^{(\gamma-L)x}(a_0)^{(L(L+\alpha))}$. As a consequence, we deduce that $L = 0$ or $L = -\alpha$.

We now analyze the solution corresponding to $L = 0$. From (24), a solution to (22) has the form $y(x) = \prod_{k=0}^{\infty}(a_k)^{x^k}$. Substituting in (22), we get

$$\begin{aligned} 1 &= \left(\prod_{k=2}^{\infty}(a_k)^{k(k-1)x^{k-2}}\right)^x \left(\prod_{k=1}^{\infty}(a_k)^{kx^{k-1}}\right)^{1+\alpha-x} \left(\prod_{k=0}^{\infty}(a_k)^{x^k}\right)^\gamma \\ &= \left(\prod_{k=1}^{\infty}(a_{k+1})^{k(k+1)x^k}\right)\left(\prod_{k=0}^{\infty}(a_{k+1})^{(k+1)(1+\alpha)x^k}\right)\left(\prod_{k=1}^{\infty}(a_k)^{-kx^k}\right)\left(\prod_{k=0}^{\infty}(a_k)^{\gamma x^k}\right) \\ &= \left(a_1^{(1+\alpha)}a_0^\gamma\right)\left(\prod_{k=1}^{\infty}(a_{k+1})^{k(k+1)+(k+1)(1+\alpha)}a_k^{\gamma-k}\right)^{x^k}. \end{aligned}$$

Since the set $\{(a)^{x^n}\}_{n\geq 0}$, with $a \in \mathbb{R}^+$, is multiplicatively linearly independent, it follows that

$$(a_{k+1})^{(k+1)(k+1+\alpha)}(a_k)^{\gamma-k} = 1, \;\; k = 0, 1, 2, \ldots. \tag{25}$$

Assuming that $y(x) = \prod_{k=0}^{n}(a_k)^{x^k}$, then a_n is the last non-one coefficient, i.e., $a_{k+1} = 1$ for $k \geq n$. Taking $k = n$ in (25) we deduce that the exponent of a_n must be $\gamma = n$. Conversely, if $\gamma = n$, then from (25) we get $a_{k+1} = 1$ for $k \geq n$.

Moreover, assume that $y(x) = \prod_{k=0}^{n}(a_k)^{x^k}$ and $z(x) = \prod_{k=0}^{\infty}(b_k)^{x^{k-\alpha}}$, $b_k \in \mathbb{R}^+$, are both multiplicative positive solutions of (22) with $\gamma = n$. In such a case,

$$\left[(y^*)^{x\omega(x)}\right]^* \oplus y^{n\omega(x)} = 1, \; \left[(z^*)^{x\omega(x)}\right]^* \oplus z^{n\omega(x)} = 1.$$

Since $1 \odot z = 1 \odot y = 1$, using the distributive property of $\odot$, we obtain $1 = \frac{\left(\left[(y^*)^{x\omega(x)}\right]^* \odot z\right) \oplus \left(y^{n\omega(x)} \odot z\right)}{\left(\left[(z^*)^{x\omega(x)}\right]^* \odot y\right) \oplus \left(z^{n\omega(x)} \odot y\right)}$. Since, $y^{n\omega(x)} \odot z = z^{n\omega(x)} \odot y$, we obtain

$$1 = \frac{\left(\left[(y^*)^{x\omega(x)}\right]^* \odot z\right)}{\left(\left[(z^*)^{x\omega(x)}\right]^* \odot y\right)}. \tag{26}$$

On the other hand, since $\frac{\left((y^*)^{x\omega(x)}\odot z\right)^*}{\left((z^*)^{x\omega(x)}\odot y\right)^*} = \frac{\left(\left[(y^*)^{x\omega(x)}\right]^*\odot z\right)\oplus\left((y^*)^{x\omega(x)}\odot z^*\right)}{\left(\left[(z^*)^{x\omega(x)}\right]^*\odot y\right)\oplus\left((z^*)^{x\omega(x)}\odot y^*\right)}$, and $(y^*)^{x\omega(x)} \odot z^* = (z^*)^{x\omega(x)} \odot y^*$, then by using (26), we conclude

$$\left(\frac{(y^* \odot z)^{x\omega(x)}}{(z^* \odot y)^{x\omega(x)}}\right)^* = \left(\frac{(y^*)^{x\omega(x)} \odot z}{(z^*)^{x\omega(x)} \odot y}\right)^* = 1.$$

As a consequence, from [10, Proposition 3], there exists $C \in \mathbb{R}^+$ such that $\left(\frac{y^*\odot z}{z^*\odot y}\right)^{x\omega(x)} = C$, which is equivalent to

$$x\omega(x)\left[(\ln y)' \ln z - (\ln z)' \ln y\right] = x\omega(x)\begin{vmatrix} \ln z & \ln y \\ (\ln z)' & (\ln y)' \end{vmatrix} = \ln C. \tag{27}$$

- If $\ln C = 0$, it follows that $\ln y$ and $\ln z$ are linearly dependent on $(0, \infty)$, i.e., $\ln z$ is a constant multiple of $\ln y = \sum_{k=0}^{n} \ln(a_k) x^k$, i.e. $\ln z$ must be a polynomial. Thus, z and y are multiplicative polynomials such that $z = y^c$.
- If $\ln C \neq 0$, we conclude that $\ln y$ and $\ln z$ are linearly independent on $(0, \infty)$, then y is a multiplicative polynomial and z is a multiplicative power series convergent in $(0, \infty)$.

Finally, following [36, Theorem 4.2.2], taking $x \to 0$ in (27) we find that $\ln y$ and $\ln z$ cannot be both polynomials unless one of them is a multiple of the other. Using Proposition 2.5, it follows that y and z are not both multiplicative polynomials unless one of them is a multiple of the other. In fact, $z(x) = \prod_{k=0}^{\infty} (b_k)^{x^{k-\alpha}}$ is a multiplicative power series convergent in $(0, \infty)$. □

As a consequence of the previous theorem, the multiplicative Laguerre differential equation (21) admits a multiplicative polynomial solution of the form (23) for every non-negative integer n. From (25) we get for $k = 1, 2 \ldots, n$ with $\gamma = n$

$$\begin{aligned} a_k &= (a_{k-1})^{\frac{k-1-n}{k(k+\alpha)}} \\ &= (a_{k-2})^{\frac{(k-2-n)(k-1-n)}{k(k-1)(k+\alpha)(k-1+\alpha)}} \\ &\ \ \vdots \\ &= (a_0)^{\frac{(k-1-n)(k-2-n)\cdots(-n)}{[k(k-1)\cdots 1][(k+\alpha)(k-1+\alpha)\cdots(1+\alpha)]}} = (a_0)^{\frac{n!}{(n-k)!(\alpha+1)_k}\frac{(-1)^k}{k!}}. \end{aligned}$$

Substituting $y_n(x) = \prod_{k=0}^{n} (a_k)^{x^k}$, for $n \geq 1$ we obtain

$$y_n(x) = (a_0)^{1+\sum_{k=1}^{n} \frac{n!}{(n-k)!(\alpha+1)_k}\frac{(-x)^k}{k!}} = (a_0)^{\frac{n!}{(\alpha+1)_n}\left(\sum_{k=0}^{n} \frac{(\alpha+1)_n}{(n-k)!(\alpha+1)_k}\frac{(-x)^k}{k!}\right)}.$$

Taking into account (1), we derive the following explicit formula for classical Laguerre polynomials:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} = \sum_{k=0}^{n} \frac{(\alpha+1)_n}{(n-k)!(1+\alpha)_k}\frac{(-x)^k}{k!}. \tag{28}$$

where $(\alpha)_n$ is the Pochhammer symbol or shifted factorial, defined by $(\alpha+1)_0 := 1$ and $(\alpha+1)_n := (\alpha+1)(\alpha+2)\cdots(\alpha+n)$ for $n > 0$.

Therefore, from (28), we obtain $y_n(x) = a_0^{\frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)}$. Since y_n being a solution of (21) implies that y^c ($c \in \mathbb{R}$) is also a solution, it is customary to choose an appropriate value for a_0. In particular, setting $a_0 = e^{\frac{(\alpha+1)_n}{n!}}$ defines the solution $y_n(x)$ as the *multiplicative Laguerre polynomial* of degree n, denoted by

$$\tilde{L}_n^{(\alpha)}(x) = e^{L_n^{(\alpha)}(x)} \Leftrightarrow L_n^{(\alpha)}(x) = \ln\left(\tilde{L}_n^{(\alpha)}(x)\right), \quad n \geq 0. \tag{29}$$

In particular, if $\alpha = 0$ we simply write $\tilde{L}_n(x) = \tilde{L}_n^{(0)}(x)$.

The main property of the Laguerre polynomials is their orthogonality, which in the multiplicative version is given as follows.

Proposition 4.2. *The sequence* $\{\tilde{L}_n^{(\alpha)}\}_{n\geq 0}$ *of MLPs is multiplicatively orthogonal on* $[0,\infty)$ *with respect to the weight* $\omega(x) = x^\alpha e^{-x}$ $(\alpha > -1)$*. Moreover, for any non-negative integers* n *and* m*, we have*

$$\int_0^\infty \left(\tilde{L}_n^{(\alpha)}(x) \odot \tilde{L}_m^{(\alpha)}(x)\right)^{\omega(x)dx} = \exp\left(\frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}\right),$$

where $\delta_{n,m}$ *is the Kronecker delta.*

Proof. Suppose n and m are non-negative integers. From (8) and (29)

$$\int_0^\infty \left(\tilde{L}_n^{(\alpha)}(x) \odot \tilde{L}_m^{(\alpha)}(x)\right)^{\omega(x)dx} = e^{\int_0^\infty \ln\left[\left(\tilde{L}_n^{(\alpha)}(x)\right)^{L_m^{(\alpha)}(x)\omega(x)}\right]dx} = e^{\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\omega(x)dx},$$

the required result follows from the orthogonality of classical Laguerre polynomials

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\omega(x)dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}. \tag{30}$$

□

An alternative proof of the previous result that uses only MC properties can be found in [23]. The MLPs satisfy the following properties.

Proposition 4.3. *Let* $\{\tilde{L}_n^{(\alpha)}\}_{n\geq 0}$ *be the sequence of MLPs. Then,*

1. *For* $n \geq 0$

$$\tilde{L}_n^{(\alpha)}(x) = \prod_{k=0}^{n} e^{\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}}. \tag{31}$$

In particular,

$$\tilde{L}_n^{(\alpha)}(0) = e^{\binom{n+\alpha}{n}} \text{ and } \frac{d^*}{dx}\left[\tilde{L}_n^{(\alpha)}(x)\right]_{x=0} = e^{-\binom{n+\alpha}{n-1}}. \tag{32}$$

2. *For* $n \geq 0$

$$\tilde{L}_n^{(\alpha)}(x) = \prod_{k=0}^{n} \tilde{L}_{n-k}(x) \odot e^{\binom{\alpha+k-1}{k}} = \prod_{k=0}^{n}\left(\tilde{L}_{n-k}(x)\right)^{\binom{\alpha+k-1}{k}}, \tag{33}$$

where $\binom{-1}{0} = 1$ *and* $\binom{\alpha+k-1}{k} = 0$ *for any* $k > 0$ *with* $\alpha + k - 1 < k$.

Proof. The first equality follows directly from (28), while (32) is immediate from (31). For (33), we use the following known relation (see [1]):

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha+1}{n}L_{n-k}(x).$$

Substituting this into (29), we obtain $\tilde{L}_n^{(\alpha)}(x) = \left(e^{L_n(x)}\right)^{\binom{\alpha-1}{0}}\left(e^{L_{n-1}(x)}\right)^{\binom{\alpha+1-1}{1}}\cdots\left(e^{L_0(x)}\right)^{\binom{\alpha+n-1}{n}}$. This implies (33). □

4.2. Multiplicative Laplace transform of multiplicative Laguerre polynomials and Tricomi's formula

Finally, we present the MLT of the MLPs and establish the multiplicative version of Tricomi's formula.

Proposition 4.4. *The MLT of the n-th multiplicative Laguerre polynomial for $\alpha = 0$ is given by*

$$\mathcal{L}_m\{\tilde{L}_n(x)\}(s) = e^{\frac{1}{s}\left(1-\frac{1}{s}\right)^n}. \tag{34}$$

Moreover, for $\alpha > -1$, it is given by

$$\mathcal{L}_m\{\tilde{L}_n^{(\alpha)}(x)\}(s) = \prod_{k=0}^{n} e^{\frac{1}{s}\binom{\alpha+k-1}{k}\left(1-\frac{1}{s}\right)^{n-k}}. \tag{35}$$

Proof. Applying the MLT to the multiplicative differential equation (21), and since $\mathcal{L}_m\{1\} = 1$, we have $\mathcal{L}_m\{(y^{**})^x\} \oplus \mathcal{L}_m\{(y^*)^{1+\alpha-x}\} \oplus \mathcal{L}_m\{y^n\} = 1$, where $y = \tilde{L}_n^{(\alpha)}$. Applying the properties from Proposition 2.8 along with (32), we get

$$\mathcal{L}_m\{(y^{**})^x\} = \frac{1}{\frac{d^*}{ds}\left(\mathcal{L}_m\{y^{**}\}\right)} = \frac{1}{\frac{d^*}{ds}\left(\frac{1}{y(0)^s y^*(0)}(\mathcal{L}_m\{y\})^{s^2}\right)} = \frac{e^{\binom{n+\alpha}{n}}}{\left(\frac{d^*}{ds}[\mathcal{L}_m\{y\}]\right)^{s^2}(\mathcal{L}_m\{y\})^{2s}},$$

and

$$\begin{aligned}\mathcal{L}_m\{(y^*)^{1+\alpha-x}\} &= \left(\frac{(\mathcal{L}_m\{y\})^s}{y(0)}\right)^{1+\alpha}\left(\frac{d^*}{ds}[\mathcal{L}_m\{y^*\}]\right)\\ &= \left(\frac{(\mathcal{L}_m\{y\})^s}{e^{\binom{n+\alpha}{n}}}\right)^{1+\alpha}\left(\frac{d^*}{ds}\left[\frac{(\mathcal{L}_m\{y\})^s}{y(0)}\right]\right) = \frac{(\mathcal{L}_m\{y\})^{s(1+\alpha)+1}}{e^{(1+\alpha)\binom{n+\alpha}{n}}}\left(\frac{d^*}{ds}[\mathcal{L}_m\{y\}]\right)^s.\end{aligned}$$

By substituting and rearranging, we obtain the first-order multiplicative differential equation

$$\left[\frac{d^*}{ds}[\mathcal{L}_m\{y\}]\right](\mathcal{L}_m\{y\})^{\frac{n+s(\alpha-1)+1}{s(1-s)}} = e^{\frac{\alpha}{s(1-s)}\binom{n+\alpha}{n}}. \tag{36}$$

Setting $\alpha = 0$ and applying the method of separation of variables to this first-order multiplicative differential equation (see [14]) yields (34). Moreover, applying the MLT to (33) and using (34), we obtain (35). □

Remark 4.5. *By applying the methods presented in [14], we can derive the multiplicative solution of the multiplicative differential equation* (36)*. In such case, we obtain*

$$\mathcal{L}_m\{\tilde{L}_n^{(\alpha)}(x)\}(s) = \exp\left(\binom{n+\alpha}{n}\frac{\alpha}{s^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{(s-1)^k}{n+\alpha-k}\right),$$

which provides an alternative expression of (35)*.*

The above proposition provides a method for computing the MLT of functions that can be expanded as a series of MLPs. In classical calculus, this result was established by Tricomi in [38, 39] and has been widely used in various fields. This result is commonly known as Tricomi's formula.

Theorem 4.6. *The analytic function $F_m(s)$ is regular at infinity and can be represented as a multiplicative series of the form*

$$F_m(s) = \prod_{n=0}^{\infty}\left[e^{f_n} \odot \left(\prod_{k=0}^{n} e^{\frac{1}{s}\binom{\alpha+k-1}{k}\left(1-\frac{1}{s}\right)^{n-k}}\right)\right], \tag{37}$$

if and only if $F_m(s)$ is the MLT of $f(x)$ a positive function on $[0,\infty)$. In this case, $f(x)$ can be represented by the multiplicative Laguerre series

$$f(x) = \prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n^{(\alpha)}(x) = \prod_{n=0}^{\infty} e^{f_n L_n^{(\alpha)}(x)} \tag{38}$$

which is absolutely and uniformly convergent for $x > 0$.

In particular, if $\alpha = 0$, under the above conditions, we have

$$F_m(s) = \prod_{n=0}^{\infty} e^{f_n} \odot e^{\frac{1}{s}\left(1-\frac{1}{s}\right)^n} \Leftrightarrow f(x) = \prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n(x). \tag{39}$$

Moreover, the coefficient f_n is said to be the n-th multiplicative Laguerre-Fourier coefficient and it is given by

$$f_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^{\infty} \ln(f(x)) L_n^{(\alpha)}(x)\omega(x)dx. \tag{40}$$

Proof. Using the properties of the MLT (Proposition 2.8) along with (35), we obtain

$$F_m(s) = \mathcal{L}_m\left\{\prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n^{(\alpha)}(x)\right\} = \prod_{n=0}^{\infty}\left(\mathcal{L}_m\{\tilde{L}_n^{(\alpha)}(x)\}\right)^{f_n} = \prod_{n=0}^{\infty}\left(\prod_{k=0}^{n} e^{\frac{1}{s}\binom{\alpha+k-1}{k}\left(1-\frac{1}{s}\right)^{n-k}}\right)^{f_n}.$$

If $F_m(s)$ is analytic at infinity and (37) holds, then from (9) and the above equation, it follows that $F_m(s)$ is the MLT of the positive function (38). In this case, since

$$f(x) = \prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n(x) = \exp\left(\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(x)\right)$$

and the classical series $\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(x)$ is absolutely and uniformly convergent for $x > 0$ (see [29, Proposition 1]), it follows that the representation in (38) also converges absolutely and uniformly for $x > 0$. The converse is similar.

Moreover, if $\ln(f(x))$ has the following expansion $\ln(f(x)) = \sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(x)$, then by using the orthogonality property of the classical Laguerre polynomials and (30), we obtain (40). □

Notice that, from equation (39) along with properties 2 and 6 of Proposition 2.8 for MLT, we obtain

$$\mathcal{L}_m\left\{\prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n(2bx)\right\}(s) = \prod_{n=0}^{\infty} e^{f_n} \odot e^{\frac{1}{s}\left(1-\frac{2b}{s}\right)^n}. \tag{41}$$

This result is equivalent to

$$\mathcal{L}_m^{-1}\left\{\prod_{n=0}^{\infty} e^{f_n} \odot e^{\frac{1}{s}\left(1-\frac{2b}{s}\right)^n}\right\} = \prod_{n=0}^{\infty} e^{f_n} \odot \tilde{L}_n(2bx). \tag{42}$$

In what follows, we consider the mean absolute error, defined as $\mathbf{E}_c^d(N,f,g) := \int_c^d |f(x)-g(x)|dx$, where $c < d$, and the functions f, g are absolutely integrable functions on the interval $[c,d]$. $\mathbf{E}_c^d(N,f,g)$ quantifies the average difference of two functions in $[c,d]$. That is, if g is an approximation of f, then $\mathbf{E}_c^d(N,f,g)$ describes how good such approximation is on the specified domain.

Example 4.7. *Consider the positive function $f(x) = e^{-ax}$ with $a \in \mathbb{R}$. By applying* (40) *and using the orthogonality of the classical Laguerre polynomials, we obtain the coefficients $f_0 = -a(\alpha+1)$, $f_1 = a$, and $f_n = 0$ for $n \geq 2$. Thus, from* (38)*, the multiplicative series can be expressed as a finite product as follows:*

$$e^{-ax} = \prod_{n=0}^{1} e^{f_n L_n^{(\alpha)}(x)} = e^{-a(\alpha+1)L_0^{(\alpha)}(x)+aL_1^{(\alpha)}(x)}, \tag{43}$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = -x+\alpha+1$. Since $f(x)$ has exponential form , the approximation process using multiplicative series is more efficient than employing the classical Laguerre series. For instance, $f(x)$ can be approximated in a classical way by using partial sums of Laguerre expansions as follows:

$$e^{-ax} \approx \sum_{n=0}^{N} \frac{a^n}{(1+a)^{1+\alpha+n}} L_n^{(\alpha)}(x) = \Sigma(x, N, \alpha, a), \quad N \geq 0, \quad \Re\mathfrak{e}(\alpha) > \frac{1}{2}. \tag{44}$$

Figures 1 and 2 illustrate that, regardless of changes in the values of a and N, a considerable error can be observed. Moreover, the error increases significantly as we move away from the origin. In contrast, approximating f using a multiplicative series leads to a finite product representation, as shown in (43)*, providing a more accurate and stable approximation.*

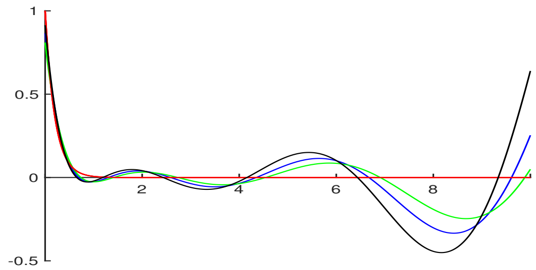

Figure 1: Plots $f(x) = e^{-5x}$ (red), $\Sigma(x, 10, 0, 5)$ (blue), $\Sigma(x, 10, \frac{1}{4}, 5)$ (green), and $\Sigma(x, 10, -\frac{1}{4}, 5)$ (black) on $[0, 10]$.

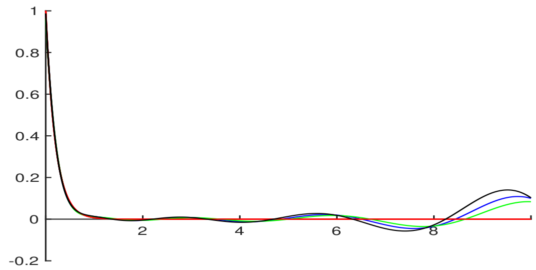

Figure 2: Plots $f(x) = e^{-5x}$ (red), $\Sigma(x, 20, 0, 5)$ (blue), $\Sigma(x, 20, \frac{1}{4}, 5)$ (green), and $\Sigma(x, 20, -\frac{1}{4}, 5)$ (black) on $[0, 10]$.

Moreover, if we consider the mean absolute error between $f(x) = e^{-5x}$ and the partial sums of Laguerre expansions (44) *with $a = 5$, as can be seen in Table 2, for $\alpha = 0, 2, 4, 8$ there is a reduction of the mean absolute error in* $[0, 10]$ *when N increases.*

N	$\mathbf{E}_0^{10}(N, f, \Sigma)_{\alpha=0}$	$\mathbf{E}_0^{10}(N, f, \Sigma)_{\alpha=2}$	$\mathbf{E}_0^{10}(N, f, \Sigma)_{\alpha=4}$	$\mathbf{E}_0^{10}(N, f, \Sigma)_{\alpha=8}$
0	1.680591	0.234532	0.200815	0.200042
1	4.502656	0.268524	0.200903	0.200042
2	5.669825	0.248570	0.200222	0.200038
5	2.161020	0.232325	0.193532	0.199951
10	0.964084	0.145893	0.164753	0.198674
15	0.492376	0.084480	0.123346	0.193412
20	0.218291	0.042975	0.083946	0.181358
25	0.090215	0.021205	0.053044	0.161875
30	0.036968	0.010303	0.031658	0.136895

Table 2: Mean absolute errors for $f(x) = e^{-ax}$ with $a = 5$ and $\alpha = 0,\ 2,\ 4,\ 8$ on $[0, 10]$.

On the contrary, the mean absolute error between $f(x) = e^{-5x}$ and the partial products of MLPs for $N = 0$ is

$$\mathbf{E}_0^{10}\left(0, e^{-5x}, \prod_{n=0}^{0} e^{f_n} \odot \tilde{L}_n(x)\right) = 0.25121,$$

$$\mathbf{E}_0^{10}\left(0, e^{-5x}, \prod_{n=0}^{0} e^{f_n} \odot \tilde{L}_n^{(2)}(x)\right) = 0.20000,$$

$$\mathbf{E}_0^{10}\left(0, e^{-5x}, \prod_{n=0}^{0} e^{f_n} \odot \tilde{L}_n^{(4)}(x)\right) = 0.20000,$$

and for $N \geq 1$ (for any $\alpha > -1$) we have

$$\mathbf{E}_0^{10}\left(0, e^{-5x}, \prod_{n=0}^{N} e^{f_n} \odot \tilde{L}_n^{(\alpha)}(x)\right) = 0.$$

That is, the mean absolute error is zero since we have an exact formula. As result, for this function in particular the approximation using partial products of MLPs is more efficient than partial sums of Laguerre polynomials.

On the other hand, according to Theorem 4.6, the MLT of $f(x)$ is given by (37)*:*

$$F_m(s) = \prod_{n=0}^{1}\left[e^{f_n} \odot \left(\prod_{k=0}^{n} e^{\frac{1}{s}\binom{\alpha+k-1}{k}\left(1-\frac{1}{s}\right)^{n-k}}\right)\right] = e^{-\frac{a}{s}(\alpha+1)+\frac{a}{s}(1-\frac{1}{s})+\frac{a}{s}\alpha} = e^{-\frac{a}{s^2}}.$$

This is precisely the MLT of $f(x)$, which, using the Definition 2.6, is given by $\mathcal{L}_m\left\{e^{-ax}\right\} = e^{-\frac{a}{s^2}}$.

In general, if we consider a function of the form $f(x) = e^{P(x)}$, where $P(x)$ is a real polynomial of degree N, then

$$e^{P(x)} = \prod_{n=0}^{N} e^{f_n} \odot \tilde{L}_n^{(\alpha)}(x)$$

where f_n is as in (40) with $\ln(f(x)) = P(x)$. In contrast, to approximate $f(x)$ with the the classical series of Laguerre polynomials requires an infinite number of terms. In conclusion, MC and multiplicative series of MLPs can be a more efficient tool to approximate exponential functions.

On the other hand, from a numerical computation point of view, the number of operations to compute both the partial sums of Laguerre polynomials and the partial products of MLPs are the same, although it is more expensive to compute products and exponentials than sums and differences. That is, computing multiplicative series can be, in general, more expensive. However, as illustrated in the example and the previous comments, the increased computational cost can be compensated when dealing with exponential growth functions.

5. Numerical inversion of the multiplicative Laplace transform

In this section, we apply the results previously derived to introduce two numerical methods to approximate the inverse MLT. In both methods, we use MLPs with $\alpha = 0$. All plots and tables were generated using Matlab R2024a.

5.1. Tricomi's method for the numerical inversion multiplicative Laplace transform

The following proposition introduces the first method to approximate the inverse MLT, based on Tricomi's formula. Henceforth, this method will be referred to as the multiplicative Tricomi's method.

Proposition 5.1. *Let $F_m(s) = e^{\phi(s)}$ be an analytic function at $s = 2b$ with $b > 0$. Define*

$$\Phi_T(z) := F_m\left(\frac{2b}{1-z}\right) \odot e^{\frac{2b}{1-z}}, \tag{45}$$

and let $f(x)$ be a smooth, positive function with bounded exponential growth. If $f(x)$ admits the following multiplicative Laguerre series representation:

$$f(x) = \prod_{n=0}^{\infty} \left(e^{f_n} \odot \tilde{L}_n(2bx)\right), \; b > 0, \; x \geq 0, \tag{46}$$

then $f(x)$ is the inverse MLT of $F_m(s)$ if and only if

$$e^{f_n} = \left(\Phi_T^{*(n)}(0)\right)^{\frac{1}{n!}}, \;\; n = 0, 1, 2, \ldots. \tag{47}$$

Furthermore, the series for $f(x)$ is absolutely and uniformly convergent for all $x > 0$.

Proof. By taking $s = \frac{2b}{1-z} \Leftrightarrow z = 1 - \frac{2b}{s}$, if $F_m(s)$ is analytic at $s = 2b$, then $\Phi_T(z)$ is analytic at $z = 0$. In this case, there exists $R_0 > 0$ such that

$$\Phi_T(z) = \exp\left(\sum_{n=0}^{\infty} a_n z^n\right) = \prod_{n=0}^{\infty} e^{a_n} \odot e^{z^n}, \;\; a_n \in \mathbb{C}, \;\; |z| < R_0. \tag{48}$$

From (48) and (45), we get

$$\prod_{n=0}^{\infty} e^{a_n} \odot e^{z^n} = F_m\left(\frac{2b}{1-z}\right) \odot e^{\frac{2b}{1-z}}. \tag{49}$$

As a consequence, using (5), the coefficients of the Taylor multiplicative series (48) are given by

$$e^{a_n} = \left(\Phi_T^{*(n)}(0)\right)^{\frac{1}{n!}}, \;\; n = 0, 1, 2, \ldots. \tag{50}$$

Assume that $f(x)$ admits a multiplicative Laguerre series representation as in (46). By substituting $s = \frac{2b}{1-z}$ in (49) and rearranging, we obtain

$$F_m(s) = \prod_{n=0}^{\infty} e^{a_n} \odot e^{\frac{1}{s}\left(1-\frac{2b}{s}\right)^n}. \tag{51}$$

From (42), it follows that

$$\mathcal{L}_m^{-1}\{F_m(s)\} = \prod_{n=0}^{\infty} \left(e^{a_n} \odot \tilde{L}_n(2bx)\right). \tag{52}$$

By comparing (46) with (52) and using (50), we conclude that $f(x)$ is the inverse MLT of $F_m(s)$ if and only if e^{f_n} is as in (46) for $n = 0, 1, 2, \ldots$. Furthermore, since (51) is analytic at infinity from Theorem (4.6), the positive function $f(x)$ is absolutely and uniformly convergent for $x > 0$. □

The positive constant b in (46) is often chosen to improve the convergence of the multiplicative Laguerre series [27]. Furthermore, in classical calculus, some methods have been developed to numerically determine the value of the coefficient f_n of (2) (with $\sigma = b$) in terms of the associated Laplace transform. These methods include algebraic manipulation, fast Fourier transform techniques, Cauchy's integral theorem, Weeks' algorithm, and its subsequent modification, see [11, 22, 27] among others. However, in this manuscript, we do not consider the problem of computing the numerical values of the multiplicative coefficients e^{f_n}. Instead, we directly use e^{f_n} as given in (47).

In this case, the function $f(x)$, which represents the inverse MLT of $F_m(s)$, is approximated in (46) by the truncated multiplicative series

$$\tilde{f}_T(x,b,N) = \prod_{n=0}^{N} \left(\Phi_T^{*(n)}(0)\right)^{\frac{1}{n!}} \odot \tilde{L}_n(2bx), \; N = 0,1,\ldots, \; x \geq 0. \tag{53}$$

An algorithm is now presented to approximate the inverse MLT of a specified function based on Tricomi's method.

Algorithm 1 Approximation of the inverse MLT via Tricomi's method.

Require: $N \geq 0$, $b > 0$, $\{L_n(x)\}_{n=0}^{N}$ the classical Laguerre polynomials, and $F_m(s)$ a analytic function at $s = 2b$.
1: **Initialize:** $\tilde{f}_T(x) \leftarrow 1$
2: **Initialize:** $\Phi_T(z) \leftarrow \left(F_m\left(\frac{2b}{1-z}\right)\right)^{\frac{2b}{1-z}}$
3: **for** $n = 0$ to N **do**
4: $\psi(n) \leftarrow \exp\left((\ln \Phi_T(z))^{(n)}(0)\right)$
5: $\tilde{f}_T(x) \leftarrow \tilde{f}_T(x) \cdot (\psi(n))^{\frac{1}{n!}L_n(2bx)}$
6: **end for**
7: **return** $\tilde{f}_T(x,b,N) \leftarrow \tilde{f}_T(x)$ ▷ Truncated multiplicative series (53).

Now, we illustrate the above results with a numerical example.

Example 5.2. *Consider the function* $F_m(s) = 2^{\frac{s}{s^2+1}}$, *which is analytic at* $s = 2b$ *for any* $b > 0$. *From Example 3.6, the positive function* $y = f(x) = 2^{\cos x} = \mathcal{L}_m^{-1}\{2^{\frac{s}{s^2+1}}\}$ *is a solution of the classical differential equation* (19). *In this case, we have* $\Phi_T(z) = 2^{\frac{(2b)^2}{(2b)^2+(1-z)^2}}$, *so the corresponding approximation is given by*

$$2^{\cos x} \approx \tilde{f}_T(x,b,N) = \prod_{n=0}^{N} \left(2^{\frac{1}{n!}\left(\frac{(2b)^2}{(2b)^2+(1-z)^2}\right)^{(n)}\Big|_{z=0}} \right) \odot \tilde{L}_n(2bx), \; N = 0,1,\ldots, \; x \geq 0.$$

Figures 3 and 4 show that a more accurate approximation of f *is obtained when N increases, for the cases when* $b = 1$ *and* $b = 2$. *Moreover, for* $b = 1$ *the approximation is significantly more accurate than for* $b = 2$ *when comparing multiplicative polynomials of the same degree. Moreover, it can be inferred from Table 3 that the cases* $b = 1$ *and* $b = 4$ *have a faster convergence for larger N, since their mean absolute error is similar and it is lower than the other cases.*

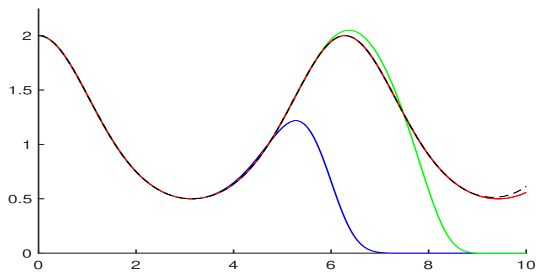

Figure 3: Plots $f(x) = 2^{\cos x}$ (red), $\tilde{f}_T(x,1,6)$ (blue), $\tilde{f}_T(x,1,9)$ (green), and $\tilde{f}_T(x,1,13)$ (black) on $[0,10]$.

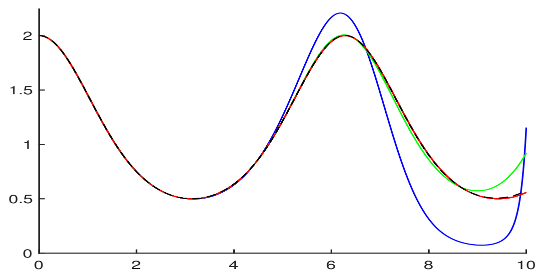

Figure 4: Plots $f(x) = 2^{\cos x}$ (red), $\tilde{f}_T(x,2,6)$ (blue), $\tilde{f}_T(x,2,9)$ (green), and $\tilde{f}_T(x,2,13)$ (black) on $[0,10]$.

N	$\mathbf{E}_0^{10}\left(N,f,\tilde{f}_T\right)_{b=\frac{1}{2}}$	$\mathbf{E}_0^{10}\left(N,f,\tilde{f}_T\right)_{b=1}$	$\mathbf{E}_0^{10}\left(N,f,\tilde{f}_T\right)_{b=2}$	$\mathbf{E}_0^{10}\left(N,f,\tilde{f}_T\right)_{b=4}$
0	5.572586	8.420373	7.173369	8.912670
1	6.223941	6.060505	6.500677	5.148688
2	13.957701	7.539675	6.966991	7.766821
5	4.327238	5.153757	4.589747	5.811868
10	0.421964	0.245563	0.127360	3.399798
15	0.098554	0.000655	0.001319	2.066255
20	0.014414	7.275091×10^{-7}	3.086705×10^{-5}	9.054827×10^{-5}
25	0.001243	7.933560×10^{-10}	7.115391×10^{-7}	3.361390×10^{-9}
30	3.201083×10^{-4}	6.281056×10^{-13}	1.279487×10^{-8}	1.221779×10^{-13}

Table 3: Mean absolute errors for $f(x)=2^{\cos x}$ with $b=\frac{1}{2},1,2,4$ on $[0,10]$.

5.2. Weeks' method for numerical inversion of the multiplicative Laplace transform

A generalization of the previous approach is the multiplicative Weeks' method. In classical calculus, this method has been widely studied; see [9, 16, 21, 40] among others.

Let σ_0 denote the abscissa of convergence of the MLT $F_m(s)=e^{\phi(s)}$. That is, all the singularities of $F_m(s)$ are located in the half-plane $\Re\mathfrak{e}(s)<\sigma_0$. Moreover, assume that $F_m(s)$ is analytic at infinity and define

$$\Phi_W(z):=F_m\left(\sigma+b\frac{z+1}{1-z}\right)\odot e^{\frac{2b}{1-z}},\ b>0,\ \sigma>\sigma_0. \tag{54}$$

Notice that if $b=\sigma$, then the expressions (45) and (54) coincide. The condition $b>0$ ensures that the multiplicative weighted Laguerre polynomials $\tilde{L}_n(2bx)\odot e^{e^{-bx}}$ exhibit well-behaved asymptotic properties for large x. In particular, they satisfy the bound $|\tilde{L}_n(2bx)\odot e^{e^{-bx}}|<e$, since $|L_n(2bx)e^{-bx}|<1$, see [22].

The Weeks' method assumes that a smooth, positive function $f(x)$, with bounded exponential growth and given by the inverse MLT of $F_m(s)$, admits the following multiplicative series representation in terms of MLPs:

$$f(x)=\prod_{n=0}^{\infty}\left(e^{f_n}\odot\tilde{L}_n(2bx)\odot e^{e^{(\sigma-b)x}}\right),\ x\geq 0. \tag{55}$$

Using the multiplicative first shifting property from Proposition 2.8 and (41), we obtain the MLT of (55):

$$\begin{aligned}F_m(s)&=\mathcal{L}_m\left\{\prod_{n=0}^{\infty}\left(e^{f_n}\odot\tilde{L}_n(2bx)\odot e^{e^{(\sigma-b)x}}\right)\right\}(s)\\&=\mathcal{L}_m\left\{\prod_{n=0}^{\infty}\left(e^{f_n}\odot\tilde{L}_n(2bx)\right)\right\}(s)\Big|_{s\to s-(\sigma-b)}=\prod_{n=0}^{\infty}e^{f_n\frac{1}{s-\sigma+b}\left(1-\frac{2b}{s-\sigma+b}\right)^n}.\end{aligned}$$

By substituting $s=\sigma+b\frac{z+1}{1-z}\Leftrightarrow z=1-\frac{2b}{s-\sigma+b}$, we obtain

$$F_m\left(\sigma+b\frac{z+1}{1-z}\right)\odot e^{\frac{2b}{1-z}}=\prod_{n=0}^{\infty}e^{f_nz^n}=\Phi_W(z). \tag{56}$$

Hence, the coefficients e^{f_n} in the multiplicative expansion (55) are also the coefficients of the Taylor multiplicative series (56). The radius of convergence of (56) is strictly greater than one, due to the choice of $F_m(s)$ being analytic at infinity and $\sigma>\sigma_0$ (see [22]). Moreover, the multiplicative series converges uniformly within this radius (see [22]). As a consequence, the multiplicative expansion (55) also converges uniformly. Therefore, from (56) we conclude

$$e^{f_n}=\left(\Phi_W^{*(n)}(0)\right)^{\frac{1}{n!}},\ n=0,1,2,\ldots. \tag{57}$$

Therefore, the function $f(x)$, which represents the inverse MLT of $F_m(s)$, is approximated by the truncated multiplicative series

$$\tilde{f_W}(x,b,\sigma,N) = \prod_{n=0}^{N} \left(\Phi_W^{*(n)}(0)\right)^{\frac{1}{n!}} \odot \tilde{L}_n(2bx) \odot e^{e^{(\sigma-b)x}},\ N = 0,1,\ldots,\ x \geq 0. \tag{58}$$

The algorithm below provides a procedure for approximating the inverse MLT of a specified function based on Weeks' method.

Algorithm 2 Approximation of the inverse MLT via Weeks' method.

Require: $N \geq 0$, $b > 0$, $\{L_n(x)\}_{n=0}^{N}$ the classical Laguerre polynomials, $F_m(s) = e^{\phi(s)}$ an analytic function at infinity, and $\sigma > \sigma_0$.

1: **Initialize:** $\tilde{f_W}(x) \leftarrow 1$
2: **Initialize:** $\Phi_W(z) \leftarrow \left(F_m\left(\sigma + b\frac{z+1}{1-z}\right)\right)^{\frac{2b}{1-z}}$
3: **for** $n = 0$ to N **do**
4: $\quad \psi(n) \leftarrow \exp\left((\ln \Phi_W(z))^{(n)}(0)\right)$
5: $\quad \tilde{f_W}(x) \leftarrow \tilde{f_W}(x) \cdot (\psi(n))^{\frac{1}{n!}L_n(2bx)e^{(\sigma-b)x}}$
6: **end for**
7: **return** $\tilde{f_W}(x,b,N) \leftarrow \tilde{f_W}(x)$ ▷ Truncated multiplicative series (58).

On the other hand, the choice of appropriate values for the parameters σ_0, σ y b can be quite complicated and even chaotic. The multiplicative series (55) is highly sensitive to these parameters, and inappropriate choices can result in a slower or less accurate convergence. Nevertheless, in [13] the authors provide useful guidelines for the choice of these parameters. Although those recommendations were originally developed for the classical Weeks' method, this is not a limitation since the multiplicative Weeks' method is structurally similar, the main difference lies in the use of an exponential basis.

- **Choice of the parameter** σ_0. It is necessary to know or determine the numerical value of the abscissa of convergence σ_0 of the MLT $F_m(s)$. This can be defined either as the limit of the set of values of $\mathfrak{Re}(s)$ for which this integral converges, or, alternatively, as the maximum of the real parts of the singular points s_j of $F_m(s)$.
- **Choice of the parameter** σ. Suppose $T > 0$, and consider the interval $0 \leq x \leq T$. Two cases arise:
  1. For large values of T, it is necessary to choose $\sigma - \sigma_0 > 0$ small. In this case, the multiplicative series (2) converges slowly and thus the multiplicative expansion (55) also converges slowly. As a result, more terms are needed to achieve an accurate approximation.
  2. For small values of T, it is possible to take a larger $\sigma - \sigma_0 > 0$. This leads to faster convergence of the multiplicative expansion (55) and a more efficient approximation with fewer terms.
- **Choice of the parameter** b. Once σ has been fixed, a suitable choice for b is any value that satisfies $b \geq \sigma - \sigma_0 > 0$. This condition guarantees the convergence of the multiplicative expansion (55) (see [13]).

The following example illustrates the algorithm and the choice of the parameters.

Example 5.3. *We consider the MLT* $F_m(s) = e^{\frac{1}{s+a}}$ *($s > -a$) of an unknown function* $f(x)$*. In this case, using equations* (56) *and* (57)*, we obtain*

$$\Phi_W(z) = \exp\left(\frac{2b}{b+\sigma+a+(b-a-\sigma)z}\right), \text{ and } f_n = \frac{2b(\sigma-b+a)^n}{(\sigma+b+a)^{n+1}},\ n = 0,1,\ldots.$$

As a consequence, from (55), *we have*

$$f(x) = \prod_{n=0}^{\infty} \exp\left(\frac{2b(\sigma - b + a)^n}{(\sigma + b + a)^{n+1}} L_n(2bx) e^{(\sigma - b)x} \right), \; x \geq 0.$$

In fact, the function $F_m(x) = e^{\frac{1}{s+a}}$ *is the MLT of* $f(x) = e^{e^{-ax}}$.

In this case we have $\sigma_0 = -a$, *so taking into account the previous comments about the choice of the parameters, setting* $\sigma = a = 2$ *and given that* $\sigma - \sigma_0 = 4$, *then we can choose* $b \geq 4$. *In Table 4 we present the mean absolute errors for values close to* $\sigma - \sigma_0 = 4$.

N	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=4.1}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=4.2}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=4.5}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=4.8}$
0	0.024786	0.047624	0.106691	0.154830
1	7.641490×10^{-4}	0.002828	0.014284	0.030274
2	2.524369×10^{-5}	1.788606×10^{-4}	0.002006	0.006138
5	1.084664×10^{-9}	5.392163×10^{-8}	6.593886×10^{-6}	6.015012×10^{-5}
10	1.933722×10^{-16}	7.745445×10^{-14}	5.319749×10^{-10}	3.061855×10^{-8}
15	1.222468×10^{-16}	1.155787×10^{-16}	3.117515×10^{-14}	1.360649×10^{-11}
20	1.174733×10^{-16}	1.143644×10^{-16}	1.225808×10^{-16}	2.380223×10^{-15}
25	1.174733×10^{-16}	1.143644×10^{-16}	1.219146×10^{-16}	1.563350×10^{-16}
30	1.174733×10^{-16}	1.143644×10^{-16}	1.219146×10^{-16}	1.563350×10^{-16}

Table 4: Mean absolute errors for $f(x) = e^{e^{-2x}}$ with $\sigma = 2$, and $b = 4.1,\ 4.2,\ 4.5,\ 4.8$ on $[0, 10]$.

On the other hand, Table 5 shows the mean absolute error for values afar from $\sigma - \sigma_0 = 4$.

N	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=5}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=6}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=8}$	$\mathbf{E}_0^{10}\left(N, f, \tilde{f}_W\right)_{b=10}$
0	0.182387	0.284185	0.397328	0.459370
1	0.042159	0.104216	0.209269	0.285552
2	0.010044	0.038580	0.109589	0.175549
5	1.585969×10^{-4}	0.002201	0.016658	0.041684
10	1.801854×10^{-7}	2.159754×10^{-5}	0.000826	0.004234
15	1.917256×10^{-10}	2.177889×10^{-7}	4.291099×10^{-5}	4.534032×10^{-4}
20	9.938997×10^{-14}	2.037308×10^{-9}	2.247647×10^{-6}	4.918743×10^{-5}
25	1.555868×10^{-16}	1.052363×10^{-11}	1.172801×10^{-7}	5.357011×10^{-6}
30	1.555868×10^{-16}	7.979381×10^{-15}	5.642207×10^{-9}	5.837429×10^{-7}

Table 5: Mean absolute errors for $f(x) = e^{e^{-2x}}$ with $\sigma = 2$, and $b = 5,\ 6,\ 8,\ 10$ on $[0, 10]$.

For this example, it can be observed that for different values close to $b = 4$ *the mean absolute error decreases as N increases. However, when b moves away from* $\sigma - \sigma_0 = 4$, *the mean absolute error increases with respect to the case when b is closer to* 4.

5.3. *Application to the solution of a nonlinear classical second-order differential Equation*

Here, we use the previous results to analytically and numerically solve a nonlinear second-order classical differential equation that would be very difficult to solve with classical calculus techniques. This example highlights the importance of studying both the analytic and numerical properties of multiplicative differential equations.

Consider the nonlinear second-order classical differential equation with initial conditions

$$\begin{cases} y'' + \left(2 - \frac{y'}{y}\right) y' + 3y \ln y = 0, \\ y(0) = y'(0) = 1. \end{cases} \tag{59}$$

The presence of nonlinear terms makes it difficult to solve it by using conventional methods. However, according to Theorem 3.5, the solution of this nonlinear equation coincides with the solution of the linear multiplicative differential equation:

$$\begin{cases} y^{**} \oplus (y^*)^2 \oplus (y)^3 = 1, \\ y(0) = 1, \; y^*(0) = e. \end{cases} \tag{60}$$

By applying properties of the MLT and denoting $Y = \mathcal{L}_m\{y\} = F_m(s)$, we obtain

$$\left(\frac{Y^{s^2}}{y(0)^s y^*(0)}\right) \oplus \left(\frac{Y^s}{y(0)}\right)^2 \oplus Y^3 = 1.$$

It follows that $F_m(s) = e^{\frac{1}{s^2+2s+3}}$. Using the inverse MLT, we obtain the solution of (59) and (60)

$$y(x) = e^{\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}x\right)e^{-x}}, \; x \geq 0.$$

In this case, the expressions (45) and (54) take the following form:

$$\Phi_T(z) = \exp\left(\frac{2b(1-z)}{(2b)^2 + (1-z)[4b + 3(1-z)]}\right),$$
$$\Phi_W(z) = \exp\left(\frac{2b(1-z)}{[\sigma(1-z) + b(1+z)]^2 + (1-z)[(2\sigma+3)(1-z) + 2b(1+z)]}\right).$$

Since $\sigma_0 = -1$, setting $\sigma = -0.5$ we can choose $b \geq \sigma - \sigma_0 = 0.5$. Figures 5 and 6 illustrate the approximated solution using multiplicative Tricomi's method and multiplicative Weeks' method, respectively, with $b = 1$ and $N = 10,\ 15$, respectively. At first sight, it appears that multiplicative Weeks' method produces a more accurate and faster approximation compared to the multiplicative Tricomi's method.

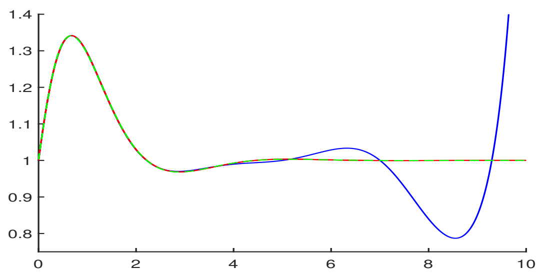

Figure 5: Plots $y(x) = e^{\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}x\right)e^{-x}}$ (red line), $\tilde{f}_T(x, 1, 10)$ (blue line), and $\tilde{f}_W(x, 1, -0.5, 10)$ (green line) on $[0, 10]$.

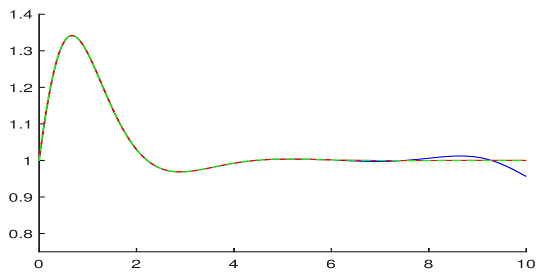

Figure 6: Plots $y(x) = e^{\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}x\right)e^{-x}}$ (red line), $\tilde{f}_T(x, 1, 15)$ (blue line), and $\tilde{f}_W(x, 1, -0.5, 15)$ (green line) on $[0, 10]$.

However, Table 6 shows that, for larger values of N, the mean absolute error generated by Tricomi's method is smaller than the mean absolute error generated by the Weeks' method.

N	Tricomi's method		Weeks' method	
	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_T\right)_{b=1}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_T\right)_{b=5}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_W\right)_{b=1}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_W\right)_{b=5}$
0	1.811982	0.975320	0.338518	0.447504
1	3.778614	19.540803	0.260349	0.374860
2	6.135666	8.285467	0.095946	0.269092
5	5.311679	7.302854	0.036310	0.073683
10	0.690709	5.351663	0.003220	0.013332
15	0.030597	3.271044	8.475483×10^{-4}	0.006264
20	9.457852×10^{-4}	1.372400	2.625599×10^{-4}	0.003090
25	3.107117×10^{-5}	0.063786	6.076312×10^{-5}	0.001340
30	6.311106×10^{-7}	1.939664×10^{-6}	1.113459×10^{-5}	5.174536×10^{-4}

Table 6: Mean absolute errors for $y(x)=e^{\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}x\right)e^{-x}}$ with $\sigma=-0.5$ and $b=1,\ 5$ on $[0,10]$.

Now, considering the parameters $\sigma=2$ and $b\geq\sigma-\sigma_0=4$, Table 7 shows the mean absolute errors obtained with both methods with $b=4.5$ and $b=5$. In this case, the multiplicative Weeks' method produces a smaller mean absolute errors when N increases, compared with the erros produced by Tricomi's method. Moreover, the convergence speed of the error is greater. As a result, the Weeks' method is more efficient for this selection of the parameters.

N	Tricomi's method		Weeks' method	
	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_T\right)_{b=4.5}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_T\right)_{b=5}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_W\right)_{b=4.5}$	$\mathbf{E}_0^{10}\left(N,y,\tilde{f}_W\right)_{b=5}$
0	1.026411	0.975320	0.415539	0.425183
1	17.344966	19.540803	0.193250	0.245082
2	8.234676	8.285467	0.102797	0.100969
5	7.191053	7.302854	0.023104	0.040632
10	5.181549	5.351663	0.008018	0.003443
15	2.893741	3.271044	2.450855×10^{-4}	0.001189
20	1.664429	1.372400	2.728235×10^{-5}	1.133764×10^{-4}
25	0.001637	0.063786	8.077798×10^{-9}	7.765518×10^{-4}
30	4.887999×10^{-8}	1.939664×10^{-6}	3.214810×10^{-11}	5.576192×10^{-10}

Table 7: Mean absolute errors for $y(x)=e^{\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}x\right)e^{-x}}$ with $\sigma=2$ and $b=4.5,\ 5$ on $[0,10]$.

Thus, the methods are sensitive to the choice of the parameters. As previously stated, several contributions have addressed the problem of properly choosing the parameters, but a general criteria has not been established (see [13]).

Finally, the classical Laplace transform nor the Weeks method is used to solve (59) because the Laplace transform of the product of functions and logarithms is quite complicated. As a consequence, it is not possible to obtain the Laplace transform of (59) explicitly and, as a consequence, its inverse cannot be approximated by using the Weeks classical method.

6. Conclusions

Multiplicative calculus can be used as an alternative tool to solve nonlinear classical differential equations whose solutions are not easily found using traditional methods of classical calculus, such as the ones presented in Examples 3.4, 3.6, and the equation (59). We present several examples that show that this approach presents numerical advantages compared with the classical approach, especially when dealing with some classes of functions such as exponential functions. In particular, Example (4.7) shows that the approximation of an exponential function using multiplicative series of MLPs can be more efficient than using classical series of Laguerre polynomials. Lastly, numerical techniques such as the multiplicative Tricomi's method and multiplicative Weeks' method can be useful tools in the process of approximating the inverse MLT, since it provides an analytic formula for the domain function, as in (53) and (58), respectively.

Acknowledgement

The work of the first and third authors was supported by the Universidad Pedagógica y Tecnológica de Colombia sede Tunja Code SGI 4022. The work of the second author was supported by México's SECIHTI Grant CBF2023-2024-625.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. Government Printing Office, Washington, D.C., 1964; reprinted by Dover, New York, 1972. Chapter 22.

[2] Z. Altay, E. Yılmaz, M. Uşen and F. Demirbağ, *Radius Model for Some Cells in Human Body on Multiplicative Calculus*, J. Math. Sci. Model. **7** (2024), no. 3, 112–20.

[3] A. E. Bashirov et al., *On modeling with multiplicative differential equations*, Appl. Math. J. Chinese Univ. Ser. B. **26** (2011), no. 4, 425–438.

[4] A. E. Bashirov, E. Mısırlı, and A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (2008), no. 1, 36–48.

[5] B. Bilgehan, *Efficient approximation for linear and non-linear signal representation*, IET Signal Proc. (2015), no. 9, 260—266.

[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Springer, Berlin, 2001.

[7] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach, New York-London-Paris, 1978.

[8] A. M. Cohen, *Numerical methods for Laplace transform inversion*, Numerical Methods and Algorithms, 5, Springer, New York, 2007.

[9] B. Davies, *Integral transforms and their applications*, 3rd ed., Texts in Applied Mathematics, 41, Springer, New York, 2002.

[10] E. Fuentes, L. E. Garza and F. Velázquez, *On multiplicative Jacobi polynomials and function approximation through multiplicative series*, Rend. Circ. Mat. Palermo , II. Ser **74** (2025), Paper No. 156.

[11] B. Gabutti and P. Lepora, *The numerical performance of Tricomi's formula for inverting the Laplace transform*, Numer. Math. **51** (1987), no. 3, 369–380.

[12] Y. H. Gangadharaiah and N. Sandeep, *Engineering applications of the Laplace transform*, CRC Press, Boca Raton, FL, 2023.

[13] B. Garbow, G. Giunta and A. Murli, *Software for an implementation of Weeks' method for the inverse Laplace transform*, ACM Trans. Math. Software **14** (1988), no. 2, 163-170.

[14] S. G. Georgiev and K. Zennir, *Multiplicative differential equations—Vol. I*, Chapman & Hall/CRC, Boca Raton, FL, 2023.

[15] S. G. Georgiev, *Multiplicative differential geometry*, CRC Press, Boca Raton, FL, 2022.

[16] G. Giunta, G. Laccetti and M. Rizzardi, *More on the Weeks method for the numerical inversion of the Laplace transform*, Numer. Math. **54** (1988), no. 2, 193–200.

[17] S. Göktas, *A new type of Sturm-Liouville equation in the non-Newtonian calculus*, J. Funct. Spaces (2021), Art. ID 5203939, 8 pp.

[18] S. Göktas, *On the multiplicative Legendre equation*, Journal of Taibah University for Science **16** (2022), no. 1, 805—812.

[19] S. Göktaş, E. Yılmaz and A.C. Yar, *Some Spectral Properties of Multiplicative Hermite Equation*, Fundam. J. Math. Appl. **5** (March 2022), no. 1, 32-41.

[20] M. Grossman and R. Katz, *Non-Newtonian Calculus*, Lee Press, Rockport, Mass., 1972.

[21] S. U. Kamran, S. Khan and N. Mlaiki, *On the Approximation of Fractional-Order Differential Equations Using Laplace Transform and Weeks Method*, Symmetry **15** (2023), 1214.

[22] P. O. Kano, M. Brio and J. V. Moloney, *Application of Weeks method for the numerical inversion of the Laplace transform to the matrix exponential*, Commun. Math. Sci. **3** (2005), no. 3, 335–372.

[23] H. Y. Kosunalp, S. Bas and S. Kosunalp, *An Efficient Solution of Multiplicative Differential Equations through Laguerre Polynomials*, Symmetry **16** (2024), no. 6, 748.

[24] Laguerre, *Sur l'intégrale* $\int_x^\infty \frac{e^{-x}dx}{x}$, Bull. Soc. Math. France, **7** (1879), 72–81.

[25] M. A. Al-Gwaiz, *Sturm–Liouville Theory and Its Applications*, Springer–Verlag, London, 2008.

[26] G. G. Özbey et al., *Some properties of Bessel functions in multiplicative calculus*, Rend. Circ. Mat. Palermo (2) **74** (2025), no. 1, Paper No. 42, 16 pp.

[27] R. Piessens and M. Branders, *Numerical inversion of the Laplace transform using generalised Laguerre polynomials*, Proc. Inst. Electr. Engrs. **118** (1971), 1517–1522.

[28] A. K. Pogrebkov, *Multiplicative dynamical systems in terms of the induced dynamics*, Theor Math Phys. **204** (2020), 1201-1208.

[29] P. E. Ricci, D. Caratelli and F. Mainardi, *Tricomi's Method for the Laplace Transform and Orthogonal Polynomials*, Symmetry **13** (2021), no. 4, 589.

[30] M. Riza and H. Aktöre, *The Runge-Kutta method in geometric multiplicative calculus*, LMS J. Comput. Math. **18** (2015), no. 1, 539–554.

[31] J. Shen, T. Tang and L. L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Series in Computational Mathematics, 41, Springer Heidelberg, 2011.

[32] G. Singh and S. Bhalla, *Two step Newton's method with multiplicative calculus to solve the non-linear equations*, J. Comput. Anal. Appl. **31** (2023), no. 2, 171—179.

[33] G. Singh, S. Bhalla and R. Behl, *A Multiplicative Calculus Approach to Solve Applied Nonlinear Models*, Math. Comput. Appl. 2023, **28**, no. 2, 28.

[34] N. J. Sonin, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Math. Ann. **16** (1880), no. 1, 1–80.

[35] D. Stanley, *A Multiplicative Calculus, Problems*, resources, and issues. Res. Math. Educ. **9** (1999), no. 4, 310–326.

[36] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., New York, 1939.

[37] L. N. Trefethen, *Approximation theory and approximation practice*, extended edition, SIAM, Philadelphia, PA, [2020] ©2020.

[38] F. G. Tricomi, *Ancora sull'inversione della trasformazione di Laplace*, Rend. Lincei. **21** (1935), 420—426.

[39] F. G. Tricomi, *Trasformazione di Laplace e polinomi di Laguerre, I. Inversione della trasformazione; II. Alcune nuove formule sui polinomi di Laguerre*, Rend. Accad. Lincei **21** (1935), 232–239.

[40] J. A. C. Weideman, *Algorithms for parameter selection in the Weeks method for inverting the Laplace transform*, SIAM J. Sci. Comput. **21** (1999), no. 1, 111–128.

[41] N. Yalçın, *Multiplicative Chebyshev differential equations and multiplicative Chebyshev polynomials*, Thermal Sci. **26** (2022), Special issue 2, 785—79

[42] N. Yalçın, *The solutions of multiplicative Hermite differential equation and multiplicative Hermite polynomials*, Rend. Circ. Mat. Palermo (2) **70** (2021), no. 1, 9–21.

[43] N. Yalçın and E. Çelik, *Solution of multiplicative homogeneous linear differential equations with constant exponentials*, New Trends Math. Sci. **6** (2018), no. 2, 58–67

[44] N. Yalçın and E. Çelik, *The solution of multiplicative non-homogeneous linear differential equations*, J. Appl. Math. Comput. **2** (2018), no. 2, 27-36.

[45] N. Yalçın, E. Çelik and H. Gokdogan, *Multiplicative Laplace transform and its applications*, Optik. **127** (2016), no. 20, 9984–9995.

[46] N. Yalçın and M. Dedeturk, *Solutions of multiplicative linear differential equations via the multiplicative power series method*, Sigma, **41** (2023), no. 4, 837-847.

[47] N. Yalçın and M. Dedeturk, *Solutions of multiplicative ordinary differential equations via the multiplicative differential transform method*, AIMS Math. **6** (2021), no. 4, 3393–3409.

[48] M. Yalçın and H. Selvitopi, *Numerical methods for the multiplicative partial differential equations*, Open Math. **15** (2017), no. 1, 1344–1350.

[49] M. Ç. Yilmazer, S. Goktas, E. Yilmaz and M. Et, *Adams–Bashforth methods on bigeometric calculus*, AIMS Math. **10** (2025), no. 8, 18627–18640

[50] E. Yilmaz, *Multiplicative Bessel equation and its spectral properties*, Ric. Mat. **73** (2024), no. 3, 1289–1305.

.