Filomat 40:3 (2026), 917–930
https://doi.org/10.2298/FIL2603917C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hybrid Chlodowsky-Jain-Appell operators

Merve Çil[a,*], Mehmet Ali Özarslan[a]

[a] *Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, Türkiye*

Abstract. In this paper, by using Jain-Appell operators and classical Bernstein-Chlodowsky operators, we introduce Hybrid Chlodowsky-Jain-Appell operators. We investigate approximation properties of these new operators by using modulus of continuity, partial moduli of continuities, weighted modulus of continuity and bivariate Lipschitz class functionals. We introduce new operators of GBS type by using new defined Hybrid Chlodowsky-Jain-Appell operators. Furthermore, we investigate the approximation properties for new GBS version of the operators by using mixed modulus of continuity and bivariate Lipschitz class functional.

1. Introduction

The goal of approximation theory is to approximate functions by simpler functions such as polynomials. Weierstrass' theorem was proved by the operators that developed in 1912 which are called as Bernstein operators [3]. This theorem shows that every continuous function on a closed set can be uniformly approximated with the help of sequence of polynomials. Many linear positive operators have been investigated with the Weierstrass' theorem to study approximation properties in many function spaces. These operators includes Szasz, Baskakov, Lupas, Meyer-König and Zeller, Bleimann-Butzer-Hahn, and many others. Interest in the exploration of different linear positive operators has grown significantly over the past few years([1], [7], [8], [16]).

Paul Appell, in [2], introduced the polynomial sequence $P_n(\xi)$, which are referred as Appell polynomials satisfying the following monomiality property,

$$DP_n(\xi) = nP_{n-1}(\xi), \quad D \equiv \frac{d}{d\xi}. \tag{1}$$

The Appell polynomials can also be defined using the generating function

$$A(v)e^{v\xi} = \sum_{k=0}^{\infty} p_k(\xi)v^k,$$

2020 *Mathematics Subject Classification*. Primary 41A10; Secondary 41A25.

Keywords. Jakimovski-Leviatan operators, Chlodowsky Szasz-Mirakyan operators, Jain-Appell operators, Modulus of continuity, Lipschitz class.

Received: 09 August 2025; Revised: 28 September 2025; Accepted: 02 December 2025

Communicated by Dragan S. Djordjević

* Corresponding author: Merve Çil

Email addresses: merve.cil@emu.edu.tr (Merve Çil), mehmetali.ozarslan@emu.edu.tr (Mehmet Ali Özarslan)

where $A(v) = \sum_{n=0}^{\infty} a_n v^n$, with $A(1) \neq 1$, represents an analytic function within the disk $|v| < r$ where $r > 1$. Jakamovski and Leviatan [12] introduced a set of operators denoted as $P_n(h;\xi)$, using Appell polynomials. The definition of these operators is as follows:

$$P_n(h;\xi) = \frac{e^{-n\xi}}{A(1)} \sum_{k=0}^{\infty} p_k(n\xi) h\left(\frac{k}{n}\right), \quad \xi \geq 0, \tag{2}$$

Here, $P_n(h;\xi)$ involves the exponential term $e^{-n\xi}$ and the Appell polynomials $p_k(n\xi)$.
A famous member of $P_n(h;\xi)$, the Szasz-Mirakjan operators were used for approximating continuous functions on the unbounded interval $[0,\infty)$. These operators are defined as:

$$S_n(h;\xi) := e^{-n\xi} \sum_{k=0}^{\infty} \frac{(n\xi)^k}{k!} h\left(\frac{k}{n}\right),$$

where $n \in \mathbb{N}$, $\xi \in [0,\infty)$, and h is a sufficiently smooth function that ensures the convergence of the series. The function h is a member of a particular subset of the space $C[0,\infty)$ such that the above series is convergent. In the last few years, there have been a number of modifications or generalizations of the Szasz-Mirakjan operators. We mention two of these generalization which are Jain-Pethe and Jain-Appell operators. These two operators are important in the construction of Hybrid Chlodowsky-Jain-Appell operators.
The Jain-Pethe operators are defined [14] as follows:

$$S_n^{(\alpha)}(h;\xi) = \frac{1}{(1+n\alpha)^{(\xi/\alpha)}} \sum_{k=0}^{\infty} h\left(\frac{k}{n}\right)\left(\frac{n}{1+n\alpha}\right)^k \frac{\xi^{(k,-\alpha)}}{k!}, \tag{3}$$

where $\xi^{(k,-\alpha)} = \xi(\xi+\alpha)(\xi+2\alpha)\cdots(\xi+(k-1)\alpha), (k \in \mathbb{N} := 1,2,\cdots)$ and $\xi^{(0,-\alpha)} = 1$. These operators can be described as the gamma transform of the Szasz-Mirakjan operators.
The Jain-Appell operators, were defined by [16]

$$C_n^{(\alpha)}(h;\xi) = \frac{1}{(1+n\alpha)^{(\xi/\alpha)} A(1)} \sum_{k=0}^{\infty} h\left(\frac{k}{n}\right) p_k^{\sim(\alpha)}(\xi;n) \tag{4}$$

where

$$p_j^{\sim(\alpha)}(\xi;n) = \sum_{i=0}^{j} \frac{a_i}{(k-i)!}\left(\frac{n}{1+n\alpha}\right)^{k-i} \xi^{(k-i,-\alpha)}$$

and these operators are the gamma transform of the Jakimovski-Leviatan operators. Jain-Appell operators include the Jain-Pethe operators as well as several interesting new operators such as the Appell-Baskakov and Appell-Lupaş operators. It can be shown that the Appell polynomials can be applied to derive the classical operators of Baskakov and Lupaş in the sense of Jakimovski-Leviatan.
The classical Bernstein-Chodowsky polynomials, denoted as $B_n^c(h;\xi)$, have the following form:

$$B_n^c(h;\xi) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\xi}{a_n}\right)^k \left(1 - \frac{\xi}{a_n}\right)^{n-k} h\left(a_n \frac{k}{n}\right). \tag{5}$$

Here, $0 \leq \xi \leq a_n$ and a_n is a sequence of positive numbers satisfying the conditions $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{n} = 0$. Chlodowsky introduced the classical Bernstein-Chodowsky polynomials in 1932, as an extension of Bernstein polynomials on an unbounded intervals.
Most of the problems in Applied Sciences are investigated in the higher dimensions and approximating to a functions in higher dimensions is very important in analysing these problems. Especially the problems studied in the unbounded domains needs a bivariate operators that are defined on them. In the present

paper, combining the classical Bernstein-Chodowsky operators (5) and the Jain-Appel operators (4), originally defined for univariate functions, we can extend these operators to the bivariate case. Therefore, we introduce the Hybrid Chlodowsky-Jain-Appell operators by

$$C_{n,m}^{(\alpha)}(h;\xi,\eta)=\frac{1}{(1+m\alpha)^{(\eta/\alpha)}A(1)}\sum_{k=0}^{n}\sum_{j=0}^{\infty}\binom{n}{k}\left(\frac{\xi}{a_n}\right)^{k}\left(1-\frac{\xi}{a_n}\right)^{n-k}\times h\left(a_n\frac{k}{n},\frac{j}{m}\right)p_j^{\sim(\alpha)}(\eta;m)$$

for all $n,m\in\mathbb{N}$, $h\in C([0,\infty)\times[0,\infty))$ such that the above series is convergent. The operator in (8) can be expressed as the tensorial product of ${}_{\xi}B_n^c$ and ${}_{\eta}C_m^{(\alpha)}$, that is , $C_{n,m}^{(\alpha)}={}_{\xi}B_n^c\circ{}_{\eta}C_m^{(\alpha)}$ where

$${}_{\xi}B_n^c(h;\xi)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\xi}{a_n}\right)^{k}\left(1-\frac{\xi}{a_n}\right)^{n-k}h\left(a_n\frac{k}{n}\right)$$

and

$${}_{\eta}C_m^{(\alpha)}(h;\eta)=\frac{1}{(1+m\alpha)^{(\eta/\alpha)}A(1)}\sum_{j=0}^{\infty}h\left(\frac{j}{m}\right)p_j^{\sim(\alpha)}(\eta;m).$$

Letting $A_{a_n}:\{(\xi,\eta):0\le\xi\le a_n,\eta\ge0\}$ also we define the operators $C_{n,m}^{*(\alpha)}$ as

$$C_{n,m}^{*(\alpha)}(h;\xi,\eta)=\begin{cases}C_{n,m}^{(\alpha)}(h;\xi,\eta), & (\xi,\eta)\in A_{a_n}\\ h(\xi,\eta), & (\xi,\eta)\in[0,\infty)\times[0,\infty)\setminus A_{a_n}.\end{cases}\tag{6}$$

Recently, a similar approach in constructing two variable operators, as in (8), was considered in [20].

In the construction, we used Appell polynomials which are a general family including many important members such as Bernoulli, Euler, Genocchi and Hermite polynomials. Therefore, rather than introducing and investigating each operator one by one, we prefer to introduce the general family and focus on the main approximation results.

The paper organised in order to examine the approximation properties of the bivariate operators mentioned in (8). For these purposes, we use the Lipschitz class functionals and the moduli of continuity. In particular, we obtain the order of convergence of these operators in weighted spaces ,we take into account the modulus of continuity introduced in [11]. We also introduce the GBS version of the operators in (8) and utilize the mixed modulus of smoothness to study the approximation properties of the GBS operators. With these methods, we study approximations of these operators and these methods are used to examine how well the operators approximate functions and whether they improve and converge in specific function spaces, especially when weighted conditions are involved.

2. Moments of the Hybrid Chlodowsky-Jain-Appell operators

To investigate the approximation properties of new defined Hybrid Chlodowsky-Jain-Appell operators, we consider a basic test functions defined as $w_{i,j}=\mathrm{t}^{i}\mathfrak{s}^{j}$, where $0\le i\le4$ and $0\le j\le4$.

Lemma 2.1. *The following moments, hold true*

$$
\begin{aligned}
C_{n,m}^{(\alpha)}(w_{0,0};\xi,\eta) =&1,\\
C_{n,m}^{(\alpha)}(w_{0,0};\xi,\eta) =&1,\\
C_{n,m}^{(\alpha)}(w_{1,0};\xi,\eta) =&\xi,\\
C_{n,m}^{(\alpha)}(w_{0,1};\xi,\eta) =&\eta+\frac{1}{m}\frac{A'(1)}{A(1)},\\
C_{n,m}^{(\alpha)}(w_{2,0};\xi,\eta) =&\xi^2+\frac{\xi}{n}(a_n-\xi),\\
C_{n,m}^{(\alpha)}(w_{0,2};\xi,\eta) =&\eta^2+\alpha\eta+\eta\frac{1}{m}\left(\frac{A(1)+2A'(1)}{A(1)}\right)+\frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right)\\
C_{n,m}^{(\alpha)}(w_{3,0};\xi,\eta) =&\xi^3+\frac{\xi(a_n-\xi)}{n^2}(\xi(3n-2)+a_n),\\
C_{n,m}^{(\alpha)}(w_{0,3};\xi,\eta) =&\eta^3+3\eta^2\alpha+2\eta\alpha^2+(\eta^2+\eta\alpha)\frac{1}{m}\left(\frac{4A(1)+3A'(1)}{A(1)}\right)+\eta\frac{1}{m^2}\left(\frac{A(1)+8A'(1)+3A''(1)}{A(1)}\right)\\
&+\frac{1}{m^3}\left(\frac{A'(1)+4A''(1)+A'''(1)}{A(1)}\right),\\
C_{n,m}^{(\alpha)}(w_{4,0};\xi,\eta) =&\xi^4+\frac{\xi^3(a_n-\xi)}{n^3}(6n^2-5n+2)+\frac{\xi(a_n-\xi)^2}{n^3}(2\xi(3n-2)+a_n)+\frac{\xi^2a_n(a_n-\xi)(n-1)}{n^3},\\
C_{n,m}^{(\alpha)}(w_{0,4};\xi,\eta) =&\eta^4+6\eta^3\alpha+11\eta^2\alpha^2+6\eta\alpha^3+(\eta^3+3\eta^2\alpha+2\eta\alpha^2)\frac{1}{m}\left(\frac{10A(1)+4A'(1)}{A(1)}\right)\\
&+(\eta^2+\eta\alpha)\frac{1}{m^2}\left(\frac{14A(1)+30A'(1)+6A''(1)}{A(1)}\right)\\
&+\eta\frac{1}{m^3}\left(\frac{A(1)+26A'(1)+30A''(1)+4A'''(1)}{A(1)}\right)\\
&+\frac{1}{m^4}\left(\frac{A'(1)+14A''(1)+10A'''(1)+A^{(4)}(1)}{A(1)}\right).
\end{aligned}
$$

Proof. We can easily show the above results using the fact that the operators can be written in the form of the tensorial product $C_{n,m}^{(\alpha)}={}_{\xi}B_n^c\circ{}_{\eta}C_m^{(\alpha)}$ ans using the moments of each univariate operators ${}_{\xi}B_n^c$ and ${}_{\eta}C_m^{(\alpha)}$. □

Lemma 2.2. *As a result of Lemma 2.1, the following central moments holds true:*

$$
\begin{aligned}
C_{n,m}^{(\alpha)}((w_{1,0}-\xi)^2;\xi,\eta) =&\frac{\xi}{n}\left(a_n-\xi\right),\\
C_{n,m}^{(\alpha)}((w_{0,1}-\eta)^2;\xi,\eta) =&\alpha\eta+\eta\frac{1}{m}+\frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right),\\
C_{n,m}^{(\alpha)}((w_{1,0}-\xi)^4;\xi,\eta) =&\left(\frac{3}{n^2}-\frac{6}{n^3}\right)\xi^4-\frac{6a_n(n-2)}{n^3}\xi^3+a_n^2\left(\frac{3}{n^2}-\frac{7}{n^3}\right)\xi^2+\frac{a_n^3}{n^3}\xi\\
C_{n,m}^{(\alpha)}((w_{0,1}-\eta)^4;\xi,\eta) =&\left(3\alpha^2+14\alpha\frac{1}{m}+\frac{1}{m^2}\left(\frac{10A(1)+4A'(1)}{A(1)}\right)\right)\eta^2+\left(6\alpha^3+\alpha^2\frac{1}{m}\left(\frac{20A(1)+8A'(1)}{A(1)}\right)\right.\\
&\left.+\alpha\frac{1}{m^2}\left(\frac{14A(1)+30A'(1)+6A''(1)}{A(1)}\right)+\frac{1}{m^3}\left(\frac{A(1)+22A'(1)+14A''(1)}{A(1)}\right)\right)\eta\\
&+\frac{1}{m^4}\left(\frac{A'(1)+14A''(1)+10A'''(1)+A^{(4)}(1)}{A(1)}\right).
\end{aligned}
$$

Proof. The proof follows from Lemma 2.1 and the linearity of these operators. □

Corollary 2.3. *Using the given conditions on (a_n) and using Lemma 2.1 and Lemma 2.2, we can derive the following estimates:*

$$C_{n,m}^{(\alpha)}((w_{1,0}-\xi)^2;\xi,\eta) = \mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi) \text{ as } n\to\infty$$
$$C_{n,m}^{(\alpha)}((w_{0,1}-\eta)^2;\xi,\eta) \leq \Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)$$
$$C_{n,m}^{(\alpha)}((w_{1,0}-\xi)^4;\xi,\eta) = \mathbb{O}\left(\frac{a_n}{n}\right)(\xi^4+\xi^3+\xi^2+\xi) \text{ as } n\to\infty$$
$$C_{n,m}^{(\alpha)}((w_{0,1}-\eta)^4;\xi,\eta) \leq \Upsilon(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta^2+\eta+1)$$

where $\alpha=\alpha_m\to 0$ with $\alpha_m=\mathbb{O}\left(\frac{1}{m}\right)$, $\Lambda(A)$ and $\Upsilon(A)$ represent specific constants which is depending on the function A.

3. Approximation results

Let $\theta(\xi,\eta)=1+\xi^2+\eta^2$ be weighted function defined by

$$B_\theta(\mathbb{R}_+^2)=\{h: h\in |h(\xi,\eta)|\leq M_h\theta(\xi,\eta) \text{ for } M_h>0\}.$$

For $\mathbb{R}_+^2=\left\{(\xi,\eta)\in\mathbb{R}_+^2 : \xi,\eta\in[0,\infty)\right\}$ we can also assume following classes of functions:

$$C^{(r)}(\mathbb{R}_+^2)=\left\{h\in C(\mathbb{R}_+^2)\,\middle|\, h \text{ is } r\text{-times continuously differentiable}\right\}$$
$$C_\theta(\mathbb{R}_+^2)=\left\{h: h\in B_\theta(\mathbb{R}_+^2)\cap C(\mathbb{R}_+^2)\right\}$$
$$C_\theta^k(\mathbb{R}_+^2)=\left\{h: h\in C_\theta(\mathbb{R}_+^2) \text{ such that } \lim_{\xi,\eta\to\infty}\frac{h(\xi,\eta)}{\theta(\xi,\eta)}=k_h<\infty\right\}$$
$$C_\theta^0(\mathbb{R}_+^2)=\left\{h: h\in C_\theta(\mathbb{R}_+^2) \text{ such that } \lim_{\xi,\eta\to\infty}\frac{h(\xi,\eta)}{\theta(\xi,\eta)}=0\right\}$$

The norm on B_θ is defined as $\| h \|_\theta = \sup_{\xi,\eta\in\mathbb{R}_+^2}\frac{h(\xi,\eta)}{\theta(\xi,\eta)}$.

Lemma 3.1 ([9],[10]). *Let $\theta(\xi,\eta)=1+\xi^2+\eta^2$ be a weight function defined on $\mathbb{R}_+^2$. Then, any positive linear operator $\{J_{n,m}\}_{n,m\geq 1}$ acting from C_θ to B_θ satisfies the following property:*

$$\| J_{n,m}(\theta;\xi,\eta) \|_\theta \leq C$$

where $C>0$ is a positive real constant.

Theorem 3.2 ([9],[10]). *For any positive linear operator $\{J_{n,m}\}_{n,m\geq 1}$ acting $C_\theta^k(\mathbb{R}_+^2)$ to $B_\theta(\mathbb{R}_+^2)$ and satisfying*

$$\lim_{n,m\to\infty} \| J_{n,m}(1;\xi,\eta)-1 \|_\theta = 0$$
$$\lim_{n,m\to\infty} \| J_{n,m}(\mathfrak{t};\xi,\eta)-\xi \|_\theta = 0$$
$$\lim_{n,m\to\infty} \| J_{n,m}(\mathfrak{s};\xi,\eta)-\eta \|_\theta = 0$$
$$\lim_{n,m\to\infty} \| J_{n,m}(\mathfrak{t}^2+\mathfrak{s}^2;\xi,\eta)-(\xi^2+\eta^2) \|_\theta = 0$$

we have

$$\lim_{n,m\to\infty} \| J_{n,m}(h)-h \|_\theta = 0.$$

the rest of the paper, we assume that $\alpha = \alpha_n$, where $\alpha_n \to 0$ as $n \to \infty$.

Theorem 3.3. *For all $h \in C^k_\theta(\mathbb{R}^2_+)$, the operators $\{C^{*(\alpha)}_{n,m}\}_{n,m>1}$ satisfy*

$$\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(h) - h \|_\theta = 0. \tag{7}$$

Proof.

$$\begin{aligned}
&\| C^{*(\alpha_n)}_{n,m}(\theta;\xi,\eta) \|_\theta \\
=\;& \sup_{\xi,\eta\in\mathbb{R}^2_+} \frac{C^{*(\alpha_n)}_{n,m}(1+\mathrm{t}^2+\mathfrak{s}^2;\xi,\eta)}{1+\xi^2+\eta^2} \\
\le\;& \sup_{\xi,\eta\in R^2_+/A_{a_n}} \frac{C^{*(\alpha_n)}_{n,m}(1+\mathrm{t}^2+\mathfrak{s}^2;\xi,\eta)}{1+\xi^2+\eta^2} + \sup_{\xi,\eta\in A_{a_n}} \frac{C^{*(\alpha_n)}_{n,m}(1+\mathrm{t}^2+\mathfrak{s}^2;\xi,\eta)}{1+\xi^2+\eta^2} \\
\le\;& 1 + \sup_{\xi,\eta\in A_{a_n}} \left[\frac{1}{1+\xi^2+\eta^2}\left(C^{*(\alpha_n)}_{n,m}(\mathrm{t}^2;\xi,\eta) + C^{*(\alpha_n)}_{n,m}(\mathfrak{s}^2;\xi,\eta)\right)\right] \\
=\;& 1 + \sup_{\xi,\eta\in A_{a_n}} \left[\frac{1}{1+\xi^2+\eta^2}\left(\xi^2 + \frac{\xi}{n}(a_n-\xi) + \eta^2 + \alpha_n\eta + \eta\frac{1}{m}\left(\frac{A(1)+2A'(1)}{A(1)}\right) + \frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right)\right)\right] \\
\le\;& 1 + \max_{\substack{0\le\xi\le a_n\\ \eta\ge 0}} \left(\xi^2 + \frac{\xi}{n}(a_n-\xi)\right) + \max_{\substack{0\le\xi\le a_n\\ \eta\ge 0}} \left(\eta^2 + \alpha_n\eta + \eta\frac{1}{m}\left(\frac{A(1)+2A'(1)}{A(1)}\right) + \frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right)\right) \\
\le\;& 1 + M \le N
\end{aligned}$$

as $n,m \to \infty$, N is positive number, and thus

$$\| C^{*(\alpha_n)}_{n,m}(\theta;\xi,\eta) \|_\theta \le N.$$

Since $\theta(\xi,\eta) = 1+\xi^2+\eta^2$ and $\| C^{*(\alpha_n)}_{n,m}(\theta;\xi,\eta) \|_\theta \le N$ we get $C_\theta(\mathbb{R}^2_+) \to B_\theta(\mathbb{R}^2_+)$

$$\begin{aligned}
\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(1;\xi,\eta) - 1 \|_\theta &= \lim_{n,m\to\infty}\left(\sup_{\xi,\eta\in\mathbb{R}^2_+} \frac{1-1}{1+\xi^2+\eta^2}\right) = 0, \\
\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(\mathrm{t};\xi,\eta) - \xi \|_\theta &= \lim_{n,m\to\infty}\left(\sup_{\xi,\eta\in\mathbb{R}^2_+} \frac{\xi-\xi}{1+\xi^2+\eta^2}\right) = 0, \\
\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(\mathfrak{s};\xi,\eta) - \eta \|_\theta &= \lim_{n,m\to\infty}\left(\sup_{\xi,\eta\in\mathbb{R}^2_+} \frac{\eta + \frac{1}{m}\frac{A'(1)}{A(1)} - \eta}{1+\xi^2+\eta^2}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(\mathrm{t}^2+\mathfrak{s}^2;\xi,\eta) - (\xi^2+\eta^2) \|_\theta \\
=\;& \lim_{n,m\to\infty}\left(\sup_{\xi,\eta\in\mathbb{R}^2_+}\left[\frac{1}{1+\xi^2+\eta^2}\left(C^{*(\alpha_n)}_{n,m}(\mathrm{t}^2;\xi,\eta) + C^{*(\alpha_n)}_{n,m}(\mathfrak{s}^2;\xi,\eta) - \xi^2 - \eta^2\right)\right]\right. \\
=\;& \lim_{n,m\to\infty}\left(\sup_{\xi,\eta\in\mathbb{R}^2_+}\left[\frac{\xi^2 + \frac{\xi}{n}(a_n-\xi) - \xi^2}{1+\xi^2+\eta^2}\right]\right) + \sup_{\xi,\eta\in\mathbb{R}^2_+}\left[\frac{\eta^2 + \alpha_n\eta + \eta\frac{1}{m}\left(\frac{A(1)+2A'(1)}{A(1)}\right) + \frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right) - \eta^2}{1+\xi^2+\eta^2}\right]\Bigg) \\
=\;& 0.
\end{aligned}$$

Therefore, $\lim_{n,m\to\infty} \| C^{*(\alpha_n)}_{n,m}(\mathrm{t}^2+\mathfrak{s}^2;\xi,\eta) - (\xi^2+\eta^2) \|_\theta = 0$, which completes the proof. □

4. Main Results

The aim of this section is to find the degree of approximation provided by the operators in (8) within the continuous functions space defined on a compact set denoted as $I_{uv} := [0,u] \times [0,v]$ which is the subset of $[0,\infty) \times [0,\infty)$. The goal is to understand the convergence behaviour and the accuracy of approximation. The full modulus of continuity is defined by

$$\omega(h;\varepsilon) = \sup\{\mid h(k,\ell) - h(\xi,\eta) \mid : (k,\ell),(\xi,\eta) \in I_{uv} \text{ and } \sqrt{(k-\xi)^2 + (\ell-\eta)^2} \leq \varepsilon\}.$$

The partial moduli of continuities for variables ξ and η are defined as

$$\omega_1(h;\varepsilon) = sup\{|h(\xi_1,\eta) - h(\xi_2,\eta)| : \eta \in [0,v] \text{ and } \mid \xi_1 - \xi_2 \mid \leq \varepsilon\}$$

and

$$\omega_2(h;\varepsilon) = sup\{|h(\xi,\eta_1) - h(\xi,\eta_2)| : \xi \in [0,u] \text{ and } \mid \eta_1 - \eta_2 \mid \leq \varepsilon\}.$$

It can be easily seen that partial moduli of continuities possess the same properties of the usual modulus of continuity.

Theorem 4.1. *We have the inequality for any $(\xi,\eta) \in I_{uv}$ and $h \in C(I_{uv})$ as follows:*

$$\mid C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) \mid \leq 2\omega(h;\varrho_{n,m}(\xi,\eta)) \tag{8}$$

where $\varrho_{n,m}(\xi,\eta) = \left(\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi) + \Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)\right)^{\frac{1}{2}}$.

Proof. As a result of the definition of complete modulus of continuity, we have

$$\begin{aligned}
&\mid C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) \mid \\
&\leq C_{n,m}^{(\alpha_n)}(\mid h(\mathfrak{t},\mathfrak{s}) - h(\xi,\eta) \mid;\xi,\eta) \\
&\leq C_{n,m}^{(\alpha_n)}\left(\omega\left(h;\sqrt{(\mathfrak{t}-\xi)^2+(\mathfrak{s}-\eta)^2}\right);\xi,\eta\right) \\
&\leq \omega(h;\varrho_{n,m})\left\{1 + \frac{1}{\varrho_{n,m}}C_{n,m}^{(\alpha_n)}\left(\sqrt{(\mathfrak{t}-\xi)^2+(\mathfrak{s}-\eta)^2};\xi,\eta\right)\right\}.
\end{aligned}$$

Using Corollary 2.3 and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\mid C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) \mid \\
&\leq \omega(h;\varrho_{n,m}(\xi,\eta))\left\{1 + \frac{1}{\varrho_{n,m}(\xi,\eta)}\left\{C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2 + (w_{0,1}-\eta)^2;\xi,\eta)\right\}^{\frac{1}{2}}\right\} \\
&\leq \omega(h;\varrho_{n,m}(\xi,\eta))\left\{1 + \frac{1}{\varrho_{n,m}(\xi,\eta)}\left\{C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta) + C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)\right\}^{\frac{1}{2}}\right\} \\
&\leq \omega(h;\varrho_{n,m}(\xi,\eta))\left\{1 + \frac{1}{\varrho_{n,m}(\xi,\eta)}\left\{\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi) + \Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)\right\}^{\frac{1}{2}}\right\}.
\end{aligned}$$

This inequality provides the desired result. □

Theorem 4.2. *For any function h in the continuous functions space on the compact set I_{uv} and for all $(\xi,\eta) \in I_{uv}$, the following estimate holds true:*

$$\mid C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) \mid \leq 2(\omega_1(h;\varepsilon_n(\xi)) + \omega_2(h;\nu_m(\eta)))$$

where

$$\varepsilon_n(\xi)^2 = \frac{\xi}{n}(a_n - \xi) \quad \text{and} \quad \nu_m(\eta)^2 = \alpha_n\eta + \eta\frac{1}{m} + \frac{1}{m^2}\left(\frac{A'(1) + A''(1)}{A(1)}\right).$$

Proof. With the Cauchy-Schwarz inequality and the definition of partial moduli of continuity along with the moments given in Lemma 2.2 , we can establish the following inequality:

$$\begin{aligned}
&| C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) | \\
&\leq C_{n,m}^{\alpha_n}(| h(\mathfrak{t},\mathfrak{s}) - h(\xi,\eta) |;\xi,\eta) \\
&\leq C_{n,m}^{(\alpha_n)}(| h(\mathfrak{t},\mathfrak{s}) - h(\xi,\mathfrak{s}) |;\xi,\eta) + C_{n,m}^{(\alpha_n)}(| h(\xi,\mathfrak{s}) - h(\xi,\eta) |;\xi,\eta) \\
&\leq C_{n,m}^{(\alpha_n)}(\omega_1(h;| \mathfrak{t}-\xi |);\xi,\eta) + C_{n,m}^{(\alpha_n)}(\omega_2(h;| \mathfrak{s}-\eta |);\xi,\eta) \\
&\leq \omega_1(h;\varepsilon_n(\xi))\left[1+\frac{1}{\varepsilon_n(\xi)}C_{n,m}^{(\alpha_n)}(| \mathfrak{t}-\xi |;\xi,\eta)\right] + \omega_2(h;\nu_m(\eta))\left[1+\frac{1}{\nu_m(\eta)}C_{n,m}^{(\alpha_n)}(| \mathfrak{s}-\eta |;\xi,\eta)\right] \\
&\leq \omega_1(h;\varepsilon_n(\xi))\left[1+\frac{1}{\varepsilon_n(\xi)}\left(C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)\right)^{\frac{1}{2}}\right] + \omega_2(h;\nu_m(\eta))\left[1+\frac{1}{\nu_m(\eta)}\left(C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)\right)^{\frac{1}{2}}\right]
\end{aligned}$$

choosing $\varepsilon_n(\xi)^2 = \frac{\xi}{n}(a_n - \xi)$ and $\nu_m(\eta)^2 = \alpha_n\eta + \eta\frac{1}{m} + \frac{1}{m^2}\left(\frac{A'(1)+A''(1)}{A(1)}\right)$, we obtain the required result.
□

To evaluate the degree of approximation provided by the hybrid operators defined in equation (8), we utilize the concept of the Lipschitz class. In the bivariate case, we defined the Lipschitz class functions $Lip_M(\gamma_1,\gamma_2)$ with $0 < \gamma_1 \leq 1$ and $0 < \gamma_2 \leq 1$ as follows:

$$| h(k,\ell) - h(\xi,\eta) | \leq M \,| k-\xi |^{\gamma_1} | \ell-\eta |^{\gamma_2}.$$

where h belongs to the continuous function space.

Theorem 4.3. *Let $h \in Lip_M(\gamma_1,\gamma_2)$. We have*

$$| C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) | \leq M\varepsilon_n^{\gamma_1}(\xi)\nu_m^{\gamma_2}(\eta),$$

where $\varepsilon_n(\xi)$ and $\nu_m(\eta)$ are the same as Theorem 4.2.

Proof. Since $h \in Lip_M(\gamma_1,\gamma_2)$, we may write

$$\begin{aligned}
| C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) | &\leq C_{n,m}^{(\alpha_n)}(| h(\mathfrak{t},\mathfrak{s}) - h(\xi,\eta) |;\xi,\eta) \\
&\leq C_{n,m}^{(\alpha_n)}(M\,| \mathfrak{t}-\xi |^{\gamma_1} | \mathfrak{s}-\eta |^{\gamma_2};\xi,\eta) \\
&\leq M\ {}_{\xi}B_n^c(| \mathfrak{t}-\xi |^{\gamma_1};\xi,\eta)\ {}_{\eta}C_m^{(\alpha_n)}(| \mathfrak{s}-\eta |^{\gamma_2};\xi,\eta)
\end{aligned}$$

Applying the Hölder's inequality with $(p_1,q_1) = \left(\frac{2}{\gamma_1},\frac{2}{2-\gamma_1}\right)$ and $(p_2,q_2) = \left(\frac{2}{\gamma_2},\frac{2}{2-\gamma_2}\right)$, we have

$$\begin{aligned}
&| C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta) | \\
&\leq M\ {}_{\xi}B_n^c((w_{1,0}-\xi)^2;\xi,\eta)^{\gamma_1/2}\ {}_{\xi}B_n^c(w_{0,0};\xi,\eta)^{(2-\gamma_1)/2} \cdot {}_{\eta}C_m^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)^{\gamma_2/2}\ {}_{\eta}C_m^{(\alpha_n)}(w_{0,0};\xi,\eta)^{(2-\gamma_2)/2} \\
&= M\varepsilon_n^{\gamma_1}(\xi)\nu_m^{\gamma_2}(\eta)
\end{aligned}$$

This proves the theorem. □

To determine the approximation order of the bivariate operators in (8) within a weight space, we introduce the weighted modulus of continuity, which is defined as follows:

$$\omega_\theta(h;\varepsilon_1,\varepsilon_2) = \sup_{\xi,\eta\in[0,\infty)}\ \sup_{|h_1|\leq\varepsilon_1,|h_2|\leq\varepsilon_2} \frac{| h(\xi+h_1,\eta+h_2) - f(\xi,\eta) |}{\theta(\xi,\eta)\theta(h_1,h_2)}. \tag{9}$$

Theorem 4.4. *If h belongs to the class $C_\theta^k(\mathbb{R}_+^2)$, then, for values of n and m that are sufficiently large, the following estimate is satisfied:*

$$\sup_{\xi,\eta\in[0,\infty)} \frac{|\, C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta)\,|}{\theta(\xi,\eta)^3} \leq C\omega_\theta(h;\varepsilon_n',\nu_m'),$$

where $\varepsilon_n' = \left(\frac{a_n}{n}\right)^{1/2}$, $\nu_m' = \left(\frac{\sigma(A)}{m}\right)^{1/2}$, $\sigma(A) = \max\{\Lambda(A),\Upsilon(A)\}$ and C is a constant depending on n, m.

Proof. We may write

$$|\, h(\mathfrak{t},\mathfrak{s}) - h(\xi,\eta)\,| \leq 8(1+\xi^2+\eta^2)\omega_\theta(h;\varepsilon_n',\nu_m')\left(1+\frac{|\,\mathfrak{t}-\xi\,|}{\varepsilon_n'}\right)\left(1+\frac{|\,\mathfrak{s}-\eta\,|}{\nu_m}\right)(1+(\mathfrak{t}-\xi)^2)(1+(\mathfrak{s}-\eta)^2).$$

Thus,

$$\begin{aligned}
&|\, C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta)\,| \\
&\leq 8(1+\xi^2+\eta^2)\omega_\theta(h;\varepsilon_n',\nu_m') \\
&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\xi}{a_n}\right)^k\left(1-\frac{\xi}{a_n}\right)^{n-k}\left(1+\frac{1}{\varepsilon_n'}\left|k\frac{a_n}{n}-\xi\right|\right)\left(1+\left(k\frac{a_n}{n}-\xi\right)^2\right)\sum_{j=0}^{\infty}\frac{1}{(1+m\alpha_n)^{(\eta/\alpha_n)}A(1)}p_j^{\sim(\alpha_n)}(\eta;m) \\
&\times\left(1+\frac{1}{\nu_m'}\left|\frac{j}{m}-\eta\right|\right)\left(1+\left(\frac{j}{m}-\eta\right)^2\right).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we can derive

$$\begin{aligned}
&|\, C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta)\,| \\
&\leq 8(1+\xi^2+\eta^2)\omega_\theta(h;\varepsilon_n',\nu_m') \\
&\times\Bigg[1+C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)+\frac{1}{\varepsilon_n'}\sqrt{C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)} \\
&+\frac{1}{\varepsilon_n'}\sqrt{C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)C_{n,m}^{(\alpha_n)}((w_{1,0}-\xi)^4;\xi,\eta)}\Bigg] \\
&\times\Bigg[1+C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)+\frac{1}{\nu_m'}\sqrt{C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)} \\
&+\frac{1}{\nu_m'}\sqrt{C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^4;\xi,\eta)}\Bigg].
\end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned}
&|\, C_{n,m}^{(\alpha_n)}(h;\xi,\eta) - h(\xi,\eta)\,| \\
&\leq 8(1+\xi^2+\eta^2)\omega_\theta(h;\varepsilon_n',\nu_m') \\
&\times\Bigg[1+\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi)+\frac{1}{\varepsilon_n}\sqrt{\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi)} \\
&+\frac{1}{\varepsilon_n'}\sqrt{\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi)+\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^4+\xi^3+\xi^2+\xi)}\Bigg] \\
&\times\Bigg[1+\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)+\frac{1}{\nu_m'}\sqrt{\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)} \\
&+\frac{1}{\nu_m'}\sqrt{\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)+\Upsilon(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta^2+\eta+1)}\Bigg].
\end{aligned}$$

Taking $\varepsilon'_n = \left(\frac{a_n}{n}\right)^{1/2}$, $\nu'_m = \left(\frac{\sigma(A)}{m}\right)^{1/2}$ with $\sigma(A) = \max\{\Lambda(A), \Upsilon(A)\}$, we reach the desired solution. $\square$

5. GBS operator constructed by the Hybrid Chlodowsky-Jain-Appell operators and their approximation results

In this section, we give the definition of GBS Hybrid Chlodowsky-Jain-Appell operators and we give approximation results for these new defined operators.

A function defined on the rectangle $\mathbb{A} = ([u, v] \times [d, e])$ is considered B-continuous if, for every $(\xi, \eta) \in \mathbb{A}$, the function satisfies the following property:

$$\lim_{(a,b)\to(\xi,\eta)} \Delta_{(a,b)} h(\xi, \eta) = 0,$$

where $\Delta_{(a,b)} h(\xi, \eta) = h(\xi, \eta) - h(\xi, b) - h(a, \eta) + h(a, b)$.

We define $\mathbb{C}_b(\mathbb{A})$ as the space of all B-continuous functions on the rectangle $\mathbb{A}$. Similarly, $\mathbb{B}(\mathbb{A})$ represents the all bounded functions space defined on $\mathbb{A}$, and $\mathbb{C}(\mathbb{A})$ denotes the all continuous functions space on $\mathbb{A}$, endowed with the sup-norm $\| \cdot \|_\infty$. It is known that $\mathbb{C}(\mathbb{A}) \subset \mathbb{C}_b(\mathbb{A})$ ([4]).

The mixed modulus of smoothness of $h \in \mathbb{C}_b(\mathbb{A})$ is defined as

$$\omega_{mixed}(h; \varepsilon_1, \varepsilon_2) := \sup\{| \Delta_{(\xi+h_1, \eta+h_2)} h(\xi, \eta) |\}, \tag{10}$$

where the supremum is obtained by considering all possible (ξ, η) pairs in the set $\mathbb{A}$. Let $(h_1, h_2) \in [0, \infty) \times [0, \infty)$, such that $(\xi + h_1, \eta + h_2) \in \mathbb{A}$. $0 <| h_1 |\leq \varepsilon_1$, $0 <| h_2 |\leq \varepsilon_2$, and where $\Delta_{(a,b)} h(\xi, \eta)$ is defined as above. Marchaud defined the mixed modulus of continuity with upper bounds and the total modulus of continuity [15].

A function with real values, defined on the set $\mathbb{A}$, is uniformly B-continuous if and only if the following condition is satisfied:

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \omega_{mixed}(h; \varepsilon_1, \varepsilon_2) = 0. \tag{11}$$

Furthermore, for any non-negative values of λ_1 and λ_2, the following inequality is valid:

$$\omega_{mixed}(h; \lambda_1\varepsilon_1, \lambda_2\varepsilon_2) \leq (1+]\lambda_1[)(1+]\lambda_2[)\omega_{mixed}(h; \varepsilon_1, \varepsilon_2), \tag{12}$$

where the notation $]\lambda[$ represents the floor function applied to the value λ, which means it denotes the greatest integer that is less than or equal to λ. A function $h : \mathbb{A} \to \mathbb{R}$ is called as Bögel differentiable [5], of $(\xi, \eta) \in \mathbb{A}$, if

$$\lim_{(a,b)\to(\xi,\eta)} \frac{\Delta_{(a,b)} h(\xi, \eta)}{(a - \xi)(b - \eta)} = \mathbb{D}_B h(\xi, \eta) < \infty.$$

Here, the term $\mathbb{D}_\mathbb{B}$ denotes the B-derivative of the function f, and the space of all B-differentiable functions is represented as $\mathbb{D}_b(\mathbb{A})$.

In this section, we introduce a specialized version of the operators (8).

For any function h belonging to the space $\mathbb{C}_b(\mathbb{A})$, the GBS operators corresponding to $C_{n,m}^{(\alpha_n)}(h; \xi, \eta)$ operators are defined as follows:

$$\begin{aligned} K_{n,m}^{(\alpha_n)}(h; \xi, \eta) = & \sum_{k=0}^{n} \sum_{j=0}^{\infty} \binom{n}{k} \left(\frac{\xi}{a_n}\right)^k \left(1 - \frac{\xi}{a_n}\right)^{n-k} \frac{1}{(1 + m\alpha_n)^{(\eta/\alpha_n)} A(1)} \\ & \times p_j^{\sim(\alpha_n)}(\eta; m) \left[h\left(k\frac{a_n}{n}, \eta\right) + h\left(\xi, \frac{j}{m}\right) - h\left(k\frac{a_n}{n}, \frac{j}{m}\right) \right]. \end{aligned} \tag{13}$$

Let $I_{cd};:=[0,c]\times[0,d]\subset\mathbb{A}_{a_n}$.
The important difference between Hybrid Chlodowsky-Jain-Appell operators and the GBS kind of these operators is Hybrid Chlodowsky-Jain-Appell operators requires less input then the GBS kind. The values of the function at the points $\frac{ka_n}{n}$ and $\frac{j}{m}$ is enough to get the approximation results for Hybrid Chlodowsky-Jain-Appell operators. On the other hand, for GBS Hybrid Chlodowsky-Jain-Appell operators we need to know the values at the points $\frac{ka_n}{n}$, $\frac{j}{m}$ and the lines passing through these points. From our observations, we can say that the GBS Hybrid Chlodowsky-Jain-Appell operators provides better approximation with more input.

Theorem 5.1. *For every function h belonging to the set $\mathbb{C}_b(I_{cd})$ and for all (ξ,η) in I_{cd}, with (13) the following estimates holds true*

$$\mid K_{n,m}^{(\alpha_n)}(h;\xi,\eta)-h(\xi,\eta)\mid\leq 4\omega_{mixed}(h;\phi_n,\psi_m),$$

where $\phi_n=\left(\frac{a_n}{n}(c^2+c)\right)^{1/2}$, $\psi_m=\left(\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(d+1)\right)^{1/2}$ and $\Lambda(A)$ is a constant.

Proof. Using the definition of $\omega_{mixed}(h;\lambda_1\phi_n,\lambda_2\psi_m)$ and the elementary inequality, we have

$$\omega_{mixed}(h;\lambda_1\phi_n,\lambda_2\psi_m)\leq(1+]\lambda_1[)(1+]\lambda_2[)\times\omega_{mixed}(h;\phi_n,\psi_m),\quad \lambda_1,\lambda_2>0,$$

we get

$$\mid\Delta_{(\xi,\eta)}h(k,\ell)\mid\leq\omega_{mixed}(h;\mid k-\xi\mid,\mid\ell-\eta\mid)\leq\left(1+\frac{\mid k-\xi\mid}{\phi_n}\right)\left(1+\frac{\mid\ell-\eta\mid}{\psi_m}\right)\omega_{mixed}(h;\phi_n,\psi_m)$$

for every $(\xi,\eta),(k,\ell)\in I_{c,d}$ and for any $\phi_n,\psi_m>0$. Further, by the definition of $\Delta_{(\xi,\eta)}h(k,\ell)$, we write

$$h(\xi,\ell)+h(k,\eta)-h(k,\ell)=h(\xi,\eta)-\Delta_{(\xi,\eta)}h(k,\ell). \tag{14}$$

Applying the operator (8) to both sides of the equality (14), we get

$$K_{n,m}^{(\alpha_n)}(h;\xi,\eta)=h(\xi,\eta)C_{n,m}^{(\alpha_n)}(w_{0,0};\xi,\eta)-C_{(n,m)}^{(\alpha_n)}(\Delta_{(\xi,\eta)}h(k,\ell);\xi,\eta).$$

Since $C_{(n,m)}^{(\alpha_n)}(w_{0,0})=1$, applying the Cauchy-Schwarz inequality

$$\begin{aligned}&\mid K_{n,m}^{(\alpha_n)}(h;\xi,\eta)-h(\xi,\eta)\mid\\&\leq C_{(n,m)}^{(\alpha_n)}(\mid\Delta_{(\xi,\eta)}h(k,\ell)\mid;\xi,\eta)\\&\leq\Bigg(C_{(n,m)}^{(\alpha_n)}(w_{0,0};\xi,\eta)+\frac{1}{\phi_n}\sqrt{C_{(n,m)}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)}+\frac{1}{\psi_m}\sqrt{C_{(n,m)}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)}\\&+\frac{1}{\phi_n\psi_m}\sqrt{C_{(n,m)}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)}\sqrt{C_{(n,m)}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)}\Bigg)\times\omega_{mixed}(h;\phi_n,\psi_m).\end{aligned}$$

By Corollary 2.3 and for all $(\xi,\eta)\in I_{cd}$, we have

$$C_{(n,m)}^{(\alpha_n)}((w_{1,0}-\xi)^2;\xi,\eta)=\frac{\xi}{n}(a_n-\xi)\leq\frac{a_n}{n}(\xi^2+\xi)\leq\frac{a_n}{n}(c^2+c)$$

Similarly

$$C_{n,m}^{(\alpha_n)}((w_{0,1}-\eta)^2;\xi,\eta)\leq\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)\leq\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(d+1)$$

where $\Lambda(A)$ is a constant depending on A. Choosing $\phi_n=\left(\frac{a_n}{n}(c^2+c)\right)^{1/2}$ and $\psi_m=\left(\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(d+1)\right)^{1/2}$, the proof is completed. □

For $h \in Lip_M(\gamma_1, \gamma_2)$ and $(\xi, \eta) \in I_{cd}$, we have

$$| K_{n,m}^{(\alpha_n)}(h; \xi, \eta) - h(\xi, \eta) | \leq M \varepsilon_n(\xi)^{\gamma_1/2} \psi'_m(\eta)^{\gamma_2/2},$$

In the next theorem, with respect to the Lipschitz-class functional we derive the degree of approximation of $K_{n,m}^{(\alpha_n)}$ operators.
If h belongs to the space of all B-continuous functions $\mathbb{C}_b(I_{cd})$ and parameters $0 < \gamma_1 \leq 1$ and $0 < \gamma_2 \leq 1$, the Lipschitz class function $Lip_M(\gamma_1, \gamma_2)$ is defined as follows:

$$Lip_M(\gamma_1, \gamma_2) = \left\{ h \in C_b(I_{cd}) : | \Delta_{(\xi,\eta)} h(\mathfrak{t}, \mathfrak{s}) | \leq M \mid \mathfrak{t} - \xi |^{\gamma_1} | \mathfrak{s} - \eta |^{\gamma_2}, \quad \textit{for } (\mathfrak{t}, \mathfrak{s})(\xi, \eta) \in I_{cd} \right\}.$$

Theorem 5.2. *For $h \in Lip_M(\gamma_1, \gamma_2)$ and $(\xi, \eta) \in I_{cd}$, we have*

$$| U_{n,m}(h; \xi, \eta) - h(\xi, \eta) | \leq M \phi'_n(\xi)^{\gamma_1/2} \psi'_m(\eta)^{\gamma_2/2},$$

where $\phi'_n(\xi) = \| {}_\xi B_n^c((\mathfrak{t} - \xi)^2; \cdot) \|_\infty$, $\psi'_m(\eta) = \| {}_\eta C_m{}^{(\alpha_n)}((\mathfrak{s} - \eta)^2; \cdot) \|_\infty$ and M is a certain positive constant.

Proof. from the linearity of the operators, we have

$$\begin{aligned} K_{n,m}^{(\alpha_n)}(h; \cdot, \cdot) &= C_{n,m}^{(\alpha_n)}(h(\xi, \mathfrak{s}) + h(\mathfrak{t}, \eta) - h(\mathfrak{t}, \mathfrak{s}); \xi, \eta) \\ &= C_{n,m}^{(\alpha_n)}(h(\xi, \eta) - \Delta_{(\xi,\eta)} h(\mathfrak{t}, \mathfrak{s}); \xi, \eta) \\ &= h(\xi, \eta) C_{n,m}^{(\alpha_n)}(w_{0,0}; \xi, \eta) - C_{n,m}^{(\alpha_n)}(\Delta_{(\xi,\eta)} h(\mathfrak{t}, \mathfrak{s}); \xi, \eta) \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} | K_{n,m}^{(\alpha_n)}(h; \xi, \eta) - h(\xi, \eta) | &\leq C_{n,m}^{\alpha_n}(| \Delta_{(\xi,\eta)} h(\mathfrak{t}, \mathfrak{s}) |; \xi, \eta) \\ &\leq M C_{n,m}^{(\alpha_n)} (| \mathfrak{t} - \xi |^{\gamma_1} | \mathfrak{s} - \eta |^{\gamma_2}; \xi, \eta) \\ &= M C_{n,m}^{(\alpha_n)} (| \mathfrak{t} - \xi |^{\gamma_1}; \xi, \eta) C_{n,m}^{(\alpha_n)} (| \mathfrak{s} - \eta |^{\gamma_2}; \xi, \eta) . \end{aligned}$$

Now applying the Hölder's inequality with $(p_1, q_1) = \left(\frac{2}{\gamma_1}, \frac{2}{2-\gamma_1} \right)$ and $(p_2, q_2) = \left(\frac{2}{\gamma_2}, \frac{2}{2-\gamma_2} \right)$ we obtain

$$\begin{aligned} | K_{n,m}^{(\alpha_n)}(h; \xi, \eta) - h(\xi, \eta) | &\leq M {}_\xi B_n^c((\mathfrak{t} - \xi)^2; \xi)^{\gamma_1/2} {}_\xi B_n^c(w_0; \xi)^{(2-\gamma_1)/2} \\ &\quad \times {}_\eta C_m^{(\alpha_n)}((\mathfrak{s} - \eta)^2; \eta)^{\gamma_2/2} {}_\eta C_m^{(\alpha_n)}(w_0; \eta)^{(2-\gamma_2)/2}. \end{aligned}$$

Taking $\phi'_n(\xi) = \|_\xi \ B_n^c((\mathfrak{t} - \xi)^2; \cdot) \|_\infty$ and $\psi'_m(\eta) = \|_\eta \ C_m^{(\alpha_n)}((\mathfrak{s} - \eta)^2; \cdot) \|_\infty$, we get the desired solution. □

Theorem 5.3. *If $h \in \mathbb{D}_b(I_{cd})$ and $\mathbb{D}_\mathbb{B} h \in B(I_{cd})$, then for each pair of $(\xi, \eta) \in I_{cd}$, the following estimates holds true;*

$$\begin{aligned} &| K_{n,m}^{(\alpha_n)}(h; \xi, \eta) - (\xi, \eta) | \\ &\leq C \left\{ 3 \| \mathbb{D}_\mathbb{B} h \|_\infty + 2\omega_{mixed}(h; \phi_n^*, \psi_m^*) \sqrt{\xi^2 + \xi} \sqrt{\eta + 1} \right\} \phi_n^* \psi_m^* \\ &+ \left\{ \omega_{mixed}(h; \phi_n^*, \psi_m^*) \left(\psi_m^* \sqrt{\xi^4 + \xi^3 + \xi^2 + \xi} \sqrt{\eta + 1} + \phi_n^* \sqrt{\eta^2 + \eta + 1} \sqrt{\xi^2 + \xi} \right) \right\}. \end{aligned}$$

for $\phi_n^* = \sqrt{\left(\frac{a_n}{n} \right)}$, $\psi_m^* = \sqrt{\sigma(A)}$, $\sigma(A) = \max\{\Lambda(A), \Upsilon(A)\}$ and C is a constant depending on n and m only.

Proof. By the hypothesis, we can write

$$\Delta_{(\xi,\eta)} h(k, \ell) = (k - \xi)(\ell - \eta) \mathbb{D}_\mathbb{B} h(\alpha_n, \beta), \quad \xi < \alpha_n < k; \ \eta < \beta < \ell.$$

Clearly,

$$\mathbb{D}_{\mathbb{B}}h(\alpha_n,\beta)=\Delta_{(\xi,\eta)}\mathbb{D}_{\mathbb{B}}h(\alpha_n,\beta)+\mathbb{D}_{\mathbb{B}}h(\alpha_n,\eta)+\mathbb{D}_{\mathbb{B}}h(\xi,\beta)-\mathbb{D}_{\mathbb{B}}h(\xi,\eta).$$

Since $\mathbb{D}_{\mathbb{B}}h\in B(I_{cd})$, we have from the above inequality that

$$\begin{aligned}
&\mid C_{n,m}^{(\alpha_n)}(\Delta_{(\xi,\eta)}h(k,\ell);\xi,\eta)\mid\\
&=\mid C_{n,m}^{(\alpha_n)}((k-\xi)(\ell-\eta)\mathbb{D}_{\mathbb{B}}h(\alpha_n,\beta);\xi,\eta)\mid\\
&\leq C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid\mid\ell-\eta\mid\mid\Delta_{(\xi,\eta)}\mathbb{D}_{\mathbb{B}}h(\alpha_n,\beta)\mid;\xi,\eta)\\
&+C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid\mid\ell-\eta\mid(\mid\mathbb{D}_{\mathbb{B}}h(\alpha_n,\eta)\mid+\mid\mathbb{D}_{\mathbb{B}}hf(\xi,\beta)\mid-\mid\mathbb{D}_{\mathbb{B}}h(\xi,\eta)\mid);\xi,\eta)\\
&\leq C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid\mid\ell-\eta\mid\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\mid\alpha_n-\xi\mid,\mid\beta-\eta\mid;\xi,\eta)+3\parallel\mathbb{D}_{\mathbb{B}}h\parallel_\infty C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid\mid\ell-\eta\mid;\xi,\eta).
\end{aligned}\tag{15}$$

Using the (12), we can write

$$\begin{aligned}
\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\mid\alpha_n-\xi\mid,\mid\beta-\eta\mid)&\leq\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\mid k-\xi\mid,\mid\ell-\eta\mid)\\
&\leq\left(1+\frac{1}{\phi_n^*}\mid k-\xi\mid\right)\left(1+\frac{1}{\psi_m^*}\mid\ell-\eta\mid\right)\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\phi_n^*,\psi_m^*).
\end{aligned}\tag{16}$$

Combining (15), (16) and using the Cauchy-Schwarz inequality, we obtain the following

$$\begin{aligned}
&\mid K_{n,m}^{(\alpha_n)}(h;\xi,\eta)-h(\xi,\eta)\mid=\mid C_{n,m}^{(\alpha_n)}\Delta_{(\xi,\eta)}h(k,\ell);\xi,\eta)\mid\\
&\leq3\parallel\mathbb{D}_{\mathbb{B}}h\parallel_\infty\sqrt{C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^2;\xi,\eta)}+\Big(C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid\mid\ell-\eta\mid;\xi,\eta\Big)\\
&+\frac{1}{\phi_n^*}C_{n,m}^{(\alpha_n)}((k-\xi)^2\mid\ell-\eta\mid;\xi,\eta)+\frac{1}{\psi_m^*}C_{n,m}^{(\alpha_n)}(\mid k-\xi\mid(\ell-\eta)^2;\xi,\eta)\\
&+\frac{1}{\phi_n^*\psi_m^*}C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^2;\xi,\eta)\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\phi_n^*\psi_m^*)\\
&\leq3\parallel\mathbb{D}_{\mathbb{B}}h\parallel_\infty\sqrt{C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^2;\xi,\eta)}+\left(\sqrt{C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^2;\xi,\eta}\right)\\
&+\frac{1}{\phi_n^*}\sqrt{C_{n,m}^{(\alpha_n)}((k-\xi)^4(\ell-\eta)^2;\xi,\eta)}+\frac{1}{\psi_m^*}\sqrt{C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^4;\xi,\eta)}\\
&+\frac{1}{\phi_n^*\psi_m^*}C_{n,m}^{(\alpha_n)}((k-\xi)^2(\ell-\eta)^2;\xi,\eta)\omega_{mixed}(\mathbb{D}_{\mathbb{B}}h;\phi_n^*\psi_m^*).
\end{aligned}$$

Since for $(\xi,\eta),(k,\ell)\in I_{cd}$ we have

$$C_{n,m}^{(\alpha_n)}((k-\xi)^{2i}(\ell-\eta)^{2j};\xi,\eta)={}_\xi B_n((k-\xi)^{2i};\xi)_\eta C_m{}^{(\alpha_n)}((\ell-\eta)^{2j};\eta)\tag{17}$$

for $i=\{1,2\}$ and $j=\{1,2\}$. From Lemma 2.3, we can write

$$\begin{aligned}
{}_\xi B_n((k-\xi)^2;\xi)&=\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^2+\xi)\\
{}_\xi B_n((k-\xi)^4;\xi)&=\mathbb{O}\left(\frac{a_n}{n}\right)(\xi^4+\xi^3+\xi^2+\xi)\\
{}_\eta C_m{}^{(\alpha_n)}((\ell-\eta)^2;\eta)&\leq\Lambda(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta+1)\\
{}_\eta C_m{}^{(\alpha_n)}((\ell-\eta)^4;\eta)&\leq\Upsilon(A)\mathbb{O}\left(\frac{1}{m}\right)(\eta^2+\eta+1)
\end{aligned}$$

combining (16) and (17), on choosing $\phi_n^*=\sqrt{\left(\frac{a_n}{n}\right)}$, $\psi_m^*=\sqrt{\sigma(A)}$ and $\sigma(A)=\max\{\Lambda(A),\Upsilon(A)\}$, we obtain the desired result. □

6. Concluding Remarks

Recently some approximation operators have been constructed based on certain special functions. Appell polynomials are one of them. Another one can be the Mittag-Leffler functions, see [18]. Research of this type can be extended to the new special functions considered for instance in the paper [6], [13], [19] and [17].

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