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# Error bounds of Boole's formula for diverse function classes: Implications for Riemann–Liouville fractional integrals and numerical analysis

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**Abstract.** In numerical analysis, Boole's formula is a key tool for approximating definite integrals. Accurate approximation of these integrals is vital in numerical methods for solving differential equations, especially in the finite volume method, where high-quality integral approximations lead to improved results. This paper provides rigorous proof of integral inequalities for first-time differentiable convex functions within the context of fractional calculus. We begin by establishing an integral equality that involves fractional integrals, subsequently deriving Boole's formula-type inequalities for differentiable convex functions. This study examines important functional classes, including convex functions, Lipschitzian functions, bounded functions, and functions of bounded variation. Furthermore, we demonstrate the efficiency of derived inequalities through graphical representations, illustrating their application to specific functions and emphasizing their precision in approximating definite integrals.

## 1. Introduction and Preliminaries

Research studies on error bounds and their relationship with numerical integration are critically important in the mathematical literature, significantly contributing to advancing knowledge and techniques within the disciplines. Numerical integration, a method for calculating the area under a curve, can be described as quadrature for single-variable functions and cubature for functions with multiple variables. Since the time of the ancient Greeks, when they started using polygons to approximate the area of a circle, people have been struggling with challenges with integration. The invention of calculus in the 17th century was a game-changer, giving rise to fundamental rules for integration, which provided the framework for modern numerical methods. As the discipline of numerical analysis has advanced, various techniques have been created to improve the accuracy of integration. Among these, a key method of numerical integration is

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the Newton–Cotes formulas, referred to as quadrature formulas. These techniques approximate a function using values obtained at uniformly spaced intervals, employing polynomials of varying degrees. Among these methods, the trapezoidal rule stands out as one of the most elementary approaches to numerical integration, and it can be described as

$$\int_{\aleph}^{\Lambda} \mathsf{F}(\varpi)\, d\varpi \approx \frac{\Lambda - \aleph}{2} \left( \mathsf{F}(\aleph) + \mathsf{F}(\Lambda) \right). \tag{1}$$

Simpson's rule uses a quadratic polynomial to capture the better function's behaviour. It is worth noting that Thomas Simpson (1710–1761) created essential numerical integration techniques known as Simpson's law. The formula for Simpson 1/3 is stated as:

$$\int_{\aleph}^{\Lambda} \mathsf{F}(\varpi)\, d\varpi \approx \frac{(\Lambda - \aleph)}{6} \left[ \mathsf{F}(\aleph) + 4\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) + \mathsf{F}(\Lambda) \right]. \tag{2}$$

Simpson's second rule is another effective technique that employs a cubic polynomial for enhanced approximation, especially when more data points are accessible. It requires more accurate function evaluations but keeps the same order of error as Simpson's 1/3 rule. The formula for Simpson's 3/8 rule outlined as:

$$\int_{\aleph}^{\Lambda} \mathsf{F}(\varpi)\, d\varpi \approx \frac{(\Lambda - \aleph)}{8} \left[ \mathsf{F}(\aleph) + 3\mathsf{F}\left(\frac{2\aleph + \Lambda}{3}\right) + 3\mathsf{F}\left(\frac{\aleph + 2\Lambda}{3}\right) + \mathsf{F}(\Lambda) \right]. \tag{3}$$

To attain greater precision, we can utilize the five-point Boole's rule, which incorporates five data points within the interval. This principle is named after mathematician George Boole, recognized for his significant contributions to mathematical analysis and logic. The formulation of Boole's rule is defined as follows:

$$\int_{\aleph}^{\Lambda} \mathsf{F}(\varpi)\, d\varpi \approx \frac{2(\Lambda - \aleph)}{45} \left[ 7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph + \Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph + 3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda) \right]. \tag{4}$$

These approaches provide the foundation of numerical integration, offering dependable ways for estimating specified integrals in many applications. These methods enable precise approximations, facilitating decision-making and improving comprehension of complicated systems, providing them essential tools in both theoretical and applied mathematics. To know more about one can visit [8, 11] for numerical integration and its applications.

It is generally recognized that inequality is one of the most important study tools in mathematics. Fractional inequalities, especially those related to Jensen, Hermite–Hadamard (H.H.), Simpson, Milne, Euler Maclaurin and Boole's, are an important and broad area of mathematical analysis [1, 21]. Each of these inequalities offers valuable insights into the relationships established by fractional calculus, thereby contributing to a nuanced understanding of functions and their integral properties. The following Newton–Cotes quadrature, frequently employed in numerical integration, incorporates a three-point Simpson's-type inequality as well as Boole's inequality.

The Simpson 1/3 formula is described as follows:

**Theorem 1.1.** *Presume* $\mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$ *be a function that is four times differentiable and continuous function on* $(\aleph, \Lambda)$, *and assume* $\left\|\mathsf{F}^{(4)}\right\|_{\infty} = \sup_{\varpi \in (\aleph, \Lambda)} \left|\mathsf{F}^{(4)}(\varpi)\right| < \infty$. *Then, the subsequent inequality can be declared as follows:*

$$\left| \frac{1}{6} \left[ \mathsf{F}(\aleph) + 4\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) + \mathsf{F}(\Lambda) \right] - \frac{1}{\Lambda - \aleph} \int_{\aleph}^{\Lambda} \mathsf{F}(\varpi) d\varpi \right| \leq \frac{1}{2880} \left\|\mathsf{F}^{(4)}\right\|_{\infty} (\Lambda - \aleph)^4 .$$

The Simpson 3/8 rule is a recognized closed-type quadrature rule, which is expressed as follows in accordance with the Simpson 3/8 inequality:

**Theorem 1.2.** *Presume* $\mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$ *is a function continuously differentiable upto the forth order on* $(\aleph, \Lambda)$ *, and assume* $\left\|\mathsf{F}^{(4)}\right\|_\infty = \sup_{\varpi\in(\aleph,\Lambda)} \left|\mathsf{F}^{(4)}(\varpi)\right| < \infty$. *Then, the subsequent inequality can be expressed as follows:*

$$\left|\frac{1}{8}\left[\mathsf{F}(\aleph) + 3\mathsf{F}\left(\frac{2\aleph + \Lambda}{2}\right) + 3\mathsf{F}\left(\frac{\aleph + 2\Lambda}{2}\right) + \mathsf{F}(\Lambda)\right] - \frac{1}{\Lambda - \aleph}\int_\aleph^\Lambda \mathsf{F}(\varpi)d\varpi\right| \le \frac{1}{6480}\left\|\mathsf{F}^{(4)}\right\|_\infty (\Lambda - \aleph)^4 .$$

The Simpson 2/45 rule is a widely recognized closed-type quadrature rule, formulated according to its associated inequality as follows:

**Theorem 1.3.** *Presume* $\mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$, *is a six times continuously differentiable mapping on* $(\aleph, \Lambda)$ *and* $\|\mathsf{F}^{(6)}\| := \sup_{\varpi\in(\aleph,\Lambda)} \left|\mathsf{F}^{(6)}(\varpi)\right| < \infty$. *Then the subsequent inequality can be described as follows:*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph + \Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph + 3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] - \frac{1}{\Lambda - \aleph}\int_\aleph^\Lambda \mathsf{F}(\varpi)d\varpi\right|$$
$$\le \frac{1}{1935360}\left\|\mathsf{F}^{(6)}\right\|_\infty (\Lambda - \aleph)^6 .$$

In the realm of mathematical analysis, fractional calculus is recognized as a significant extension of traditional calculus, focusing on derivatives and integrals of arbitrary real or complex order. This specialized field of applied mathematics specifically addresses fractional-order derivatives and integration. For a brief introduction to this field, refer to [13, 14]. Although the concept discussed in [10] originated in 1695, most significant advancements have occurred in the past century. Over the last thirty years, fractional calculus has been utilized in numerous areas, including signal processing, physics, biosciences, engineering, and finance [15, 23, 25]. Characterized by its adaptability, fractional calculus facilitates the modelling of memory and hereditary traits in various materials and processes. This unique capability has been extensively leveraged in fields such as control theory, signal processing, and biophysics, resulting in more accurate and comprehensive models.
The Riemann–Liouville (R.L.) integral is a basic type of fractional integration that extends the idea of integration to fractional orders and offers a framework for identifying the accumulation of quantities in an advanced manner. The R.L. integral of order $\alpha$ is defined as

$$\mathcal{J}^\alpha_{\aleph+}\mathsf{F}(\varpi) = \frac{1}{\Gamma(\alpha)}\int_\aleph^\varpi (\varpi - \zeta)^{\alpha-1}\mathsf{F}(\zeta)d\zeta, \quad \varpi > \aleph$$

and

$$\mathcal{J}^\alpha_{\Lambda-}\mathsf{F}(\varpi) = \frac{1}{\Gamma(\alpha)}\int_\varpi^\Lambda (\zeta - \varpi)^{\alpha-1}\mathsf{F}(\zeta)d\zeta, \quad \varpi < \Lambda$$

respectively. Here, $\Gamma(\alpha)$ is the well-known Gamma function and $\mathcal{J}^0_{\aleph+}\mathsf{F}(\varpi) = \mathcal{J}^0_{\Lambda-}\mathsf{F}(\varpi) = \mathsf{F}(\varpi)$.
The study of fractional calculus is a mathematical variation on classical calculus that deals with derivatives and integrals of any arbitrary real or complex order. It is frequently employed in control theory, signal processing, and biology. It addresses mathematical issues related to anomalous diffusion and wave propagation in complicated environments. Fractional Calculus focusses extensively on R.L. integrals to analyse complex systems. It is associated with the theory of inequalities such as H.H., Simpson, Newton, and Milne-type inequality.

In *2013*, Sarikaya et al [18] have reported H.H. type inequalities for fractional integrals. Their findings are outlined as:

**Theorem 1.4 (See [18]).** *Assume* $\mathsf{F} : [\aleph, \Lambda] \subset \mathbb{R}^+ \to \mathbb{R}$ *be a positive function and* $\mathsf{F} \in L_1[\aleph, \Lambda]$ *with* $\alpha > 0$. *If* $\mathsf{F}$ *is a convex function on* $[\aleph, \Lambda]$, *then the subsequent inequalities for fractional integrals holds:*

$$\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) \le \frac{\Gamma(\alpha + 1)}{2(\Lambda - \aleph)^\alpha}\left[\mathcal{J}^\alpha_{\aleph^+}\mathsf{F}(\Lambda) + \mathcal{J}^\alpha_{\Lambda^-}\mathsf{F}(\aleph)\right] \le \frac{\mathsf{F}(\aleph) + \mathsf{F}(\Lambda)}{2}. \tag{5}$$

In the same work, they acquired certain trapezoidal-type inequalities. On the other hand, Sarikaya and Yildirim [19] proposed a novel formulation of the H.H. inequality for R.L. fractional integrals.

**Theorem 1.5 (See [19]).** *Assume* $\mathsf{F} : [\aleph, \Lambda] \subset \mathbb{R}^+ \to \mathbb{R}$ *is a positive function* $\mathsf{F} \in L_1[\aleph, \Lambda]$ *with* $\alpha > 0$. *If* $\mathsf{F}$ *is a convex function on* $[\aleph, \Lambda]$, *then the subsequent inequalities for fractional integrals holds:*

$$\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\aleph) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\Lambda)\right] \leq \frac{\mathsf{F}(\aleph)+\mathsf{F}(\Lambda)}{2}. \tag{6}$$

In the same study, they also found some midpoint-type inequalities. For the Riemann integrals and R.L. fractional integrals, Mohammed and Brevik [22] have examined H.H. type inequalities and obtained several new inequalities. Kara et al [9] have investigated the lower and upper bounds for parameterized-type inequalities by employing the R.L. fractional integral operators. For $(\alpha, m)$-convex functions, Qi et al [6] have derived H.H. type inequalities by leveraging generalized fractional integral. Budak et al [7] have explored fractional variant of Milne-type inequalities with differentiable convex functions. Furthermore, they investigate several function classes like bounded, Lipschitz, and functions of bounded variation. In [5], authors investigated Euler-Maclaurin-type inequalities for various function classes by involving R.L. fractional integrals. For further study of inequalities involving fractional integrals, consult [4, 12, 17, 20] and references therein.

Inspired by current research, we investigate an integral of Boole's formula type by leveraging R.L. fractional integrals relevant to first-time differentiable convex functions. Boole's formula-type inequalities enable effective approximations for sixth-degree polynomials. This study presents numerical illustration, computational analyses, and visual representations to illustrate the significance and accuracy of the newly derived inequalities. These findings represent a substantial progression in numerical integration, providing both theoretical insights and practical implications to enhance the reliability and efficiency of integration techniques.

The structure of the paper is delineated as follows: Section 2, we discuss principal findings regarding Boole's formula-type inequalities for functions with convex derivatives, utilizing R.L. fractional integrals. Section 3 presents applications to numerical integration, including examples and graphical representations that substantiate and support newly formulated results. Lastly, Section 4 discusses concluding remarks on this study and highlights opportunities for future research.

## 2. Main Results

In this section, we provide some innovative fractional Boole's formula-type inequalities for convex functions that are differentiable at only one time. Let us first establish a new integral equality; then, using this equality, we will develop Boole's formula-type inequalities for the R.L. fractional integral for differentiable convex function.

**Lemma 2.1.** *If* $\mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$ *be a function that is absolutely continuous on* $(\aleph, \Lambda)$ *such that* $\mathsf{F}' \in L_1[\aleph, \Lambda]$. *Then, the following equality is valid*

$$\begin{aligned} &\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] \\ &\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right] \\ &= \frac{\Lambda-\aleph}{4}\int_0^1 \mathcal{K}(\xi,\alpha)\left[\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi, \end{aligned}$$

*where*

$$\mathcal{K}(\xi,\alpha) = \begin{cases} \xi^{\alpha} - \frac{7}{45}, & 0 \leq \xi < \frac{1}{2}, \\ \xi^{\alpha} - \frac{13}{15}, & \frac{1}{2} \leq \xi < 1. \end{cases}$$

*Proof.* By recognizing the basic rules of integration, it is sufficient to state that

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^{\frac{1}{2}} \left(\xi^\alpha - \frac{7}{45}\right)\left[\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi \\
&= \frac{2}{\Lambda-\aleph}\left(\xi^\alpha - \frac{7}{45}\right)\left[\mathsf{F}\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) + \mathsf{F}\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]\Bigg|_0^{\frac{1}{2}} \\
&\quad - \frac{2\alpha}{\Lambda-\aleph}\int_0^{\frac{1}{2}} \xi^{\alpha-1}\left[\mathsf{F}\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) + \mathsf{F}\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi \\
&= \frac{2}{\Lambda-\aleph}\left[\left(\frac{1}{2^\alpha} - \frac{7}{45}\right)\left(\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + \mathsf{F}\left(\frac{3\Lambda+\aleph}{4}\right)\right) + \frac{7}{45}\left(\mathsf{F}(\aleph) + \mathsf{F}(\Lambda)\right)\right] \\
&\quad - \frac{2\alpha}{\Lambda-\aleph}\int_0^{\frac{1}{2}} \xi^{\alpha-1}\left[\mathsf{F}\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) + \mathsf{F}\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi. \qquad (7)
\end{aligned}$$

In the same way, we find

$$\begin{aligned}
\mathcal{I}_2 &= \frac{2}{\Lambda-\aleph}\left[\frac{4}{15}\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) - \left(\frac{1}{2^\alpha} - \frac{13}{15}\right)\left(\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + \mathsf{F}\left(\frac{3\Lambda+\aleph}{4}\right)\right)\right] \\
&\quad - \frac{2\alpha}{\Lambda-\aleph}\int_{\frac{1}{2}}^1 \xi^{\alpha-1}\left[\mathsf{F}\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) + \mathsf{F}\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi. \qquad (8)
\end{aligned}$$

Consequently, we arrive at the subsequent equality by merging (7) and (8)

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 &= \frac{2}{45(\Lambda-\aleph)}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] \\
&\quad - \frac{2\alpha}{\Lambda-\aleph}\left[\int_0^1 \xi^{\alpha-1}\left[\mathsf{F}\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) + \mathsf{F}\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi\right]. \qquad (9)
\end{aligned}$$

When we make the substitutions $\varpi = \frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph$ and $\varpi = \frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda$ for $\xi \in [0,1]$ then the equality (9) can be expressed as

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 &= \frac{2}{45(\Lambda-\aleph)}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] \\
&\quad - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha+1}}\left[\mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^+}\mathsf{F}(\Lambda) + \mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^-}\mathsf{F}(\aleph)\right]. \qquad (10)
\end{aligned}$$

Ultimately, we achieve required equality by multiplying each side of (10) by $\frac{\Lambda-\aleph}{4}$, which completes the proof of Lemma 2.1. □

**Corollary 2.2.** *Choosing $\alpha = 1$ in Lemma 2.1, then the following equality holds*

$$\begin{aligned}
&\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] - \frac{1}{(\Lambda-\aleph)}\int_\aleph^\Lambda \mathsf{F}(\varpi)d\varpi \\
&= \frac{\Lambda-\aleph}{4}\int_0^1 \mathcal{K}(\xi,\alpha)\left[\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right]d\xi,
\end{aligned}$$

*where*

$$\mathcal{K}(\xi) = \begin{cases} \xi - \frac{7}{45}, & 0 \le \xi < \frac{1}{2}, \\ \xi - \frac{13}{15}, & \frac{1}{2} \le \xi < 1. \end{cases}$$

*2.1. Boole's-type Inequalities Through Fractional Integral for Convex Functions*

This section will establish the error bounds for Weddle's rule utilizing a newly formulated identity, alongside principles of convexity, absolute properties, power means, and Hölder's inequality for integrals.

**Theorem 2.3.** *If all the assumptions in Lemma 2.1 are accomplished and* $|\mathsf{F}'|$ *is convex on* $[\aleph, \Lambda]$*, then one can prove the following inequality*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq\frac{\Lambda-\aleph}{4}\left(\Delta_1(\alpha)+\Delta_2(\alpha)\right)\left[|\mathsf{F}'(\aleph)|+|\mathsf{F}'(\Lambda)|\right],\end{aligned}\tag{11}$$

*where*

$$\begin{aligned}\Delta_1(\alpha)&=\int_0^{\frac{1}{2}}\left|\xi^{\alpha}-\frac{7}{45}\right|d\xi\\&=\begin{cases}\frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1}+\left(\frac{7}{45}\right)^{\frac{1}{\alpha}+1}\left(\frac{2\alpha}{\alpha+1}\right)-\frac{7}{90}, & 0<\alpha<\frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex]\frac{7}{90}-\frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1}, & \alpha\geq\frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\end{cases}\end{aligned}$$

*and*

$$\begin{aligned}\Delta_2(\alpha)&=\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|d\xi\\&=\begin{cases}\frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)}-\frac{13}{30}, & 0<\alpha\leq\frac{\ln\left(\frac{13}{15}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex]\frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{1+\frac{1}{\alpha}}+\frac{2^{\alpha+1}+1}{2^{\alpha+1}(\alpha+1)}-\frac{13}{10}, & \alpha>\frac{\ln\left(\frac{13}{15}\right)}{\ln\left(\frac{1}{2}\right)}.\end{cases}\end{aligned}$$

*Proof.* By referencing Lemma 2.1 and leveraging the convexity of $|\mathsf{F}'|$, we acquire

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq\frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left|\xi^{\alpha}-\frac{7}{45}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\right.\\&\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\right]\\&\leq\frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left|\xi^{\alpha}-\frac{7}{45}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|+\frac{2-\xi}{2}|\mathsf{F}(\aleph)|+\frac{\xi}{2}|\mathsf{F}'(\aleph)|+\frac{2-\xi}{2}|\mathsf{F}(\Lambda)|\right)d\xi\right.\\&\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|+\frac{2-\xi}{2}|\mathsf{F}(\aleph)|+\frac{\xi}{2}|\mathsf{F}'(\aleph)|+\frac{2-\xi}{2}|\mathsf{F}(\Lambda)|\right)d\xi\right]\end{aligned}\tag{12}$$

$$= \frac{\Lambda - \aleph}{4}\left[\int_0^{\frac{1}{2}}\left|\xi^\alpha - \frac{7}{45}\right|d\xi + \int_{\frac{1}{2}}^1\left|\xi^\alpha - \frac{13}{15}\right|d\xi\right][|\mathsf{F}'(\aleph)| + |\mathsf{F}'(\Lambda)|].$$

We have reached the conclusion of the proof of Theorem 2.3. □

**Remark 2.4.** *By choosing $\alpha = 1$ in Theorem 2.3, then we have*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\rho)\right] - \frac{1}{\rho-\aleph}\int_\aleph^\rho \mathsf{F}(\varpi)d\varpi\right|$$
$$\leq \frac{239(\Lambda-\aleph)}{6480}[|\mathsf{F}'(\aleph)| + |\mathsf{F}'(\Lambda)|],$$

*which is reported by Shehzadi et al in [3, Theorem 3].*

**Theorem 2.5.** *If all the assumptions in Lemma 2.1 are accomplished and $|\mathsf{F}'|^q$, $q > 1$ is convex on $[\aleph, \Lambda]$, then the subsequent inequality is valid*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\rho)\right]\right. \tag{13}$$
$$\left. -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^\alpha}\left[\mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^+}\mathsf{F}(\Lambda) + \mathcal{J}^\alpha_{\left(\frac{\aleph+\rho}{2}\right)^-}\mathsf{F}(\aleph)\right]\right|$$
$$\leq \frac{\Lambda-\aleph}{4}\left[\gamma_1(\alpha,p)\left\{\left(\frac{|\mathsf{F}'(\Lambda)|^q + 7|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}} + \left(\frac{|\mathsf{F}'(\aleph)|^q + 7|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right.$$
$$\left. +\gamma_2(\alpha,p)\left\{\left(\frac{3|\mathsf{F}'(\Lambda)|^q + 5|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}} + \left(\frac{3|\mathsf{F}'(\aleph)|^q + 5|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right],$$

*where $p^{-1} + q^{-1} = 1$,*

$$\gamma_1(\alpha,p) = \left(\int_0^{\frac{1}{2}}\left|\xi^\alpha - \frac{7}{45}\right|^p d\xi\right)^{\frac{1}{p}},$$

*and*

$$\gamma_2(\alpha,p) = \left(\int_{\frac{1}{2}}^1\left|\xi^\alpha - \frac{13}{15}\right|^p d\xi\right)^{\frac{1}{p}}.$$

*Proof.* By involving Hölder's inequality (12), it becomes

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.$$
$$\left. -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^\alpha}\left[\mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^+}\mathsf{F}(\Lambda) + \mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^-}\mathsf{F}(\aleph)\right]\right|$$
$$\leq \frac{\Lambda-\aleph}{4}\left[\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha - \frac{7}{45}\right|^p d\xi\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.\right.$$
$$\left.\left. +\left(\int_0^{\frac{1}{2}}\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|^{p}d\xi\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{1}{2}}^{1}\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right\}\right].$$

By exploiting the convexity $|\mathsf{F}'|^q$, we conclude that

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|$$
$$\leq\frac{\Lambda-\aleph}{4}\left[\gamma_1(\alpha,p)\left\{\left(\int_0^{\frac{1}{2}}\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)\right|^{q}d\xi\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right\}\right.$$
$$\left.+\gamma_2(\alpha,p)\left\{\left(\int_{\frac{1}{2}}^{1}\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)\right|^{q}d\xi\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right\}\right]$$
$$\leq\frac{\Lambda-\aleph}{4}\left[\gamma_1(\alpha,p)\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\aleph)|^q\right)d\xi\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^{\frac{1}{2}}\left(\frac{\xi}{2}|\mathsf{F}'(\aleph)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\Lambda)|^q\right)d\xi\right)^{\frac{1}{q}}\right\}+\gamma_2(\alpha,p)\left\{\left(\int_{\frac{1}{2}}^{1}\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\aleph)|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left(\frac{\xi}{2}|\mathsf{F}'(\aleph)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\Lambda)|^q\right)d\xi\right)^{\frac{1}{q}}\right\}\right]$$
$$=\frac{\Lambda-\aleph}{4}\left[\gamma_1(\alpha,p)\left\{\left(\frac{|\mathsf{F}'(\Lambda)|^q+7|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathsf{F}'(\aleph)|^q+7|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right.$$
$$\left.+\gamma_2(\alpha,p)\left\{\left(\frac{3|\mathsf{F}'(\Lambda)|^q+5|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3|\mathsf{F}'(\aleph)|^q+5|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right].$$

We have successfully concluded the proof for Theorem 2.5. □

**Corollary 2.6.** *Choosing $\alpha = 1$ in Theorem 2.5, then we derive proceeding inequality*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]-\frac{1}{\rho-\aleph}\int_{\aleph}^{\rho}\mathsf{F}(\varpi)d\varpi\right|$$
$$\leq\frac{\Lambda-\aleph}{4}\left[\left(\frac{7^{p+1}+\left(\frac{31}{2}\right)^{p+1}}{45^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathsf{F}'(\Lambda)|^q+7|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathsf{F}'(\aleph)|^q+7|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right.$$
$$\left.+\left(\frac{2^{p+1}+\left(\frac{11}{2}\right)^{p+1}}{15^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|\mathsf{F}'(\Lambda)|^q+5|\mathsf{F}'(\aleph)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3|\mathsf{F}'(\aleph)|^q+5|\mathsf{F}'(\Lambda)|^q}{16}\right)^{\frac{1}{q}}\right\}\right].$$

**Theorem 2.7.** *If all the assumptions in Lemma 2.1 are accomplished and* $|\mathsf{F}'|^q$, $q \geq 1$ *is convex on* $[\aleph, \Lambda]$, *then the subsequent inequality is valid*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\
&\leq \frac{\Lambda-\aleph}{4}\left[(\Delta_1(\alpha))^{1-\frac{1}{q}}\left\{(\Delta_3(\alpha)|\mathsf{F}'(\Lambda)|^q + \Delta_5(\alpha)|\mathsf{F}'(\aleph)|^q)^{\frac{1}{q}}\right.\right.\\
&\left.+(\Delta_3(\alpha)|\mathsf{F}'(\aleph)|^q + \Delta_5(\alpha)|\mathsf{F}'(\Lambda)|^q)^{\frac{1}{q}}\right\} + (\Delta_2(\alpha))^{1-\frac{1}{q}}\\
&\left.\times\left\{(\Delta_4(\alpha)|\mathsf{F}'(\Lambda)|^q + \Delta_6(\alpha)|\mathsf{F}'(\aleph)|^q)^{\frac{1}{q}} + (\Delta_4(\alpha)|\mathsf{F}'(\aleph)|^q + \Delta_6(\alpha)|\mathsf{F}'(\Lambda)|^q)^{\frac{1}{q}}\right\}\right],
\end{aligned}$$

*where*

$$\begin{aligned}
\Delta_3(\alpha) &= \int_0^{\frac{1}{2}}\left|\xi^{\alpha} - \frac{7}{45}\right|\frac{\xi}{2}d\xi\\
&= \begin{cases} \left(\frac{7}{45}\right)^{1+\frac{2}{\alpha}}\left(\frac{\alpha}{2(\alpha+2)}\right) + \left(\frac{1}{\alpha+2}\right)\left(\frac{1}{2}\right)^{\alpha+3} - \frac{7}{720}, & 0 < \alpha < \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] \frac{7}{720} - \frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+3}, & \alpha \geq \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\end{cases}\\
\Delta_4(\alpha) &= \int_{\frac{1}{2}}^{1}\left|\xi^{\alpha} - \frac{13}{15}\right|\frac{\xi}{2}d\xi\\
&= \begin{cases} \frac{2^{\alpha+2}-1}{2^{\alpha+3}(\alpha+2)} - \frac{13}{80}, & 0 < \alpha \leq \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] \frac{\alpha}{2(\alpha+2)}\left(\frac{13}{15}\right)^{1+\frac{2}{\alpha}} + \frac{2^{\alpha+2}+1}{2^{\alpha+3}(\alpha+2)} - \left(\frac{13}{48}\right), & \alpha > \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\end{cases}\\
\Delta_5(\alpha) &= \int_0^{\frac{1}{2}}\left|\xi^{\alpha} - \frac{7}{45}\right|\frac{2-\xi}{2}d\xi\\
&= \begin{cases} \frac{2\alpha}{\alpha+1}\left(\frac{7}{45}\right)^{1+\frac{1}{\alpha}} - \frac{\alpha}{2(\alpha+2)}\left(\frac{7}{45}\right)^{1+\frac{2}{\alpha}} + \left(\frac{1}{\alpha+1}\right)\left(\frac{1}{2}\right)^{\alpha+1} & \\ -\left(\frac{1}{\alpha+2}\right)\left(\frac{1}{2}\right)^{\alpha+3} - \frac{49}{720}, & 0 < \alpha < \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] -\frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1} - \frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+3} + \frac{49}{720}, & \alpha \geq \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\end{cases}
\end{aligned}$$

*and*

$$\begin{aligned}
\Delta_6(\alpha) &= \int_{\frac{1}{2}}^{1}\left|\xi^{\alpha} - \frac{13}{15}\right|\frac{2-\xi}{2}d\xi\\
&= \begin{cases} \frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)} - \frac{2^{\alpha+1}-1}{2^{\alpha+3}(\alpha+2)} + \frac{13}{48}, & 0 < \alpha \leq \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] \frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{1+\frac{1}{\alpha}} - \frac{\alpha}{2(\alpha+2)}\left(\frac{13}{15}\right)^{1+\frac{2}{\alpha}} & \\ +\frac{2^{\alpha+1}+1}{2^{\alpha+1}(\alpha+1)} - \frac{2^{\alpha+1}-1}{2^{\alpha+3}(\alpha+2)} - \frac{247}{240}, & \alpha > \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)}.\end{cases}
\end{aligned}$$

*Proof.* Based on Lemma 2.1 and by utilizing the power mean inequality after considering the modulus, we achieve

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq\frac{\Lambda-\aleph}{4}\left[\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|d\xi\right)^{1-\frac1q}\left\{\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)\right|^q d\xi\right)^{\frac1q}\right.\right.\\&\left.+\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|^q d\xi\right)^{\frac1q}\right\}+\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|d\xi\right)^{1-\frac1q}\\&\left.\times\left\{\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)\right|^q d\xi\right)^{\frac1q}+\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|^q d\xi\right)^{\frac1q}\right\}\right].\end{aligned}$$

As $|\mathsf{F}'|^q$ is convex on the interval $[\aleph,\Lambda]$, we acquire

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq\frac{\Lambda-\aleph}{4}\left[\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|d\xi\right)^{1-\frac1q}\left\{\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\aleph)|^q\right)d\xi\right)^{\frac1q}\right.\right.\\&\left.+\left(\int_0^{\frac12}\left|\xi^{\alpha}-\frac{7}{45}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\aleph)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\Lambda)|^q\right)d\xi\right)^{\frac1q}\right\}+\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|d\xi\right)^{1-\frac1q}\\&\times\left\{\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\Lambda)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\aleph)|^q\right)d\xi\right)^{\frac1q}\right.\\&\left.\left.+\left(\int_{\frac12}^1\left|\xi^{\alpha}-\frac{13}{15}\right|\left(\frac{\xi}{2}|\mathsf{F}'(\aleph)|^q+\frac{2-\xi}{2}|\mathsf{F}'(\Lambda)|^q\right)d\xi\right)^{\frac1q}\right\}\right].\end{aligned}$$

The proof of Theorem 2.7 has been finalized. □

**Corollary 2.8.** *Choosing $\alpha=1$ in Theorem 2.7, then we attain proceeding inequality*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]-\frac{1}{\rho-\aleph}\int_{\aleph}^{\rho}\mathsf{F}(\varpi)d\varpi\right|\\&\leq\frac{\Lambda-\aleph}{1440}\left[\left(\frac{1157}{45}\right)\left\{\left(\frac{25672|\mathsf{F}'(\Lambda)|^q+130523|\mathsf{F}'(\aleph)|^q}{156195}\right)^{\frac1q}+\left(\frac{25672|\mathsf{F}'(\aleph)|^q+130523|\mathsf{F}'(\Lambda)|^q}{156195}\right)^{\frac1q}\right\}\right.\\&\left.+\left(\frac{137}{5}\right)\left\{\left(\frac{2038|\mathsf{F}'(\Lambda)|^q+4127|\mathsf{F}'(\aleph)|^q}{6165}\right)^{\frac1q}+\left(\frac{2038|\mathsf{F}'(\aleph)|^q+4127|\mathsf{F}'(\Lambda)|^q}{6165}\right)^{\frac1q}\right\}\right].\end{aligned}$$

*2.2. Fractional Boole's type Inequalities for Bounded and Lipschitzian Functions*

This segment focuses on presenting fractional Boole's-type inequalities specifically for bounded and Lipschitzian functions.

**Theorem 2.9.** *If all conditions of Lemma 2.1 are accomplished. If there exist $m, M \in \mathbb{R}$ such that $m \le \mathsf{F}'(\xi) \le M$ for $\xi \in [\aleph, \Lambda]$, then we have the following Boole's type inequality for R.L. fractional integrals:*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\le\frac{\Lambda-\aleph}{4}[\Delta_1(\alpha)+\Delta_2(\alpha)](M-m).\end{aligned}\tag{14}$$

*Here, $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ are described as in Theorem 2.3.*

*Proof.* With the assistance of Lemma 2.1, we establish

$$\begin{aligned}&\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\\&\le\frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left(\xi^{\alpha}-\frac{7}{45}\right)\left(\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\frac{m+M}{2}\right)d\xi\right.\\&+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha}-\frac{13}{15}\right)\left(\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\frac{m+M}{2}\right)d\xi\\&+\int_0^{\frac{1}{2}}\left(\xi^{\alpha}-\frac{7}{45}\right)\left(\frac{m+M}{2}-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right)d\xi\\&\left.+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha}-\frac{13}{15}\right)\left(\frac{m+M}{2}-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right)d\xi\right].\end{aligned}\tag{15}$$

Through the use of modulus properties in (15), we achieve

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\&\le\frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left|\xi^{\alpha}-\frac{7}{45}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\frac{m+M}{2}\right|d\xi\right.\\&+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\frac{m+M}{2}\right|d\xi\\&+\int_0^{\frac{1}{2}}\left|\xi^{\alpha}-\frac{7}{45}\right|\left|\frac{m+M}{2}-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\\&\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{13}{15}\right|\left|\frac{m+M}{2}-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\right].\end{aligned}$$

From the statements $m \leq \mathsf{F}'(\xi) \leq M$ for $\xi \in [\aleph, \Lambda]$, we get

$$\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \frac{m+M}{2}\right| \leq \frac{M-m}{2}, \tag{16}$$

and

$$\left|\frac{m+M}{2} - \mathsf{F}'(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda)\right| \leq \frac{M-m}{2}. \tag{17}$$

Through the application of inequalities (16) and (17), we reach

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{(\frac{\aleph+\Lambda}{2})^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{(\frac{\aleph+\Lambda}{2})^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq \frac{(\Lambda-\aleph)(M-m)}{4}\left[\int_0^{\frac{1}{2}}\left(\xi^{\alpha} - \frac{7}{45}\right)d\xi + \int_{\frac{1}{2}}^{1}\left(\xi^{\alpha} - \frac{13}{15}\right)d\xi\right]\\&= \frac{\Lambda-\aleph}{4}[\Delta_1(\alpha) + \Delta_2(\alpha)](M-m).\end{aligned}$$

Thus, proof has been finalized. □

**Remark 2.10.** *Choosing $\alpha = 1$ in Theorem 2.9, then we attain the proceeding inequality*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] - \frac{1}{\Lambda-\aleph}\int_{\aleph}^{\Lambda}\mathsf{F}(\varpi)d\varpi\right|\\&\leq \frac{239(\Lambda-\aleph)}{6480}(M-m),\end{aligned}$$

*which is presented by Shehzadi et al in [3, Theorem 6].*

**Corollary 2.11.** *Under conditions of Theorem 2.9, if there exist $M \in \mathbb{R}^+$ such that $|\mathsf{F}'(\xi)| \leq M$ for all $\xi \in [\aleph, \Lambda]$, then we attain*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{(\frac{\aleph+\Lambda}{2})^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{(\frac{\aleph+\Lambda}{2})^{-}}\mathsf{F}(\aleph)\right]\right|\\&\leq \frac{\Lambda-\aleph}{2}[\Delta_1(\alpha) + \Delta_2(\alpha)]M.\end{aligned}$$

**Remark 2.12.** *Assume $\alpha = 1$ in Corollary 2.11, then we attain following inequality*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] - \frac{1}{\Lambda-\aleph}\int_{\aleph}^{\Lambda}\mathsf{F}(\varpi)d\varpi\right|\\&\leq \frac{239(\Lambda-\aleph)}{3240}M,\end{aligned}$$

*which is established by Shehzadi et al in [3, Corollary 1].*

**Theorem 2.13.** *If all assumptions of Lemma 2.1 are accomplished. If* $\mathsf{F}'$ *is a L-Lipschitzian on* $[\aleph, \Lambda]$, *then we have the subsequent inequality*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.$$
$$\left. -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|$$
$$\leq \frac{L(\Lambda-\aleph)^{2}}{4}\left[\Delta_7(\alpha) + \Delta_8(\alpha)\right],$$

*where*

$$\Delta_7(\alpha) = \int_0^{\frac{1}{2}}\left|\xi^{\alpha} - \frac{7}{45}\right|(1-\xi)d\xi$$
$$= \begin{cases} \frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1} + \frac{2\alpha}{\alpha+1}\left(\frac{7}{45}\right)^{\frac{1}{\alpha}+1} - \frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2} - \frac{\alpha}{\alpha+2}\left(\frac{7}{45}\right)^{1+\frac{2}{\alpha}} - \frac{7}{120}, & 0 < \alpha < \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)}, \\ -\frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1} + \frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2} - \frac{7}{120}, & \alpha \geq \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)}, \end{cases}$$

*and*

$$\Delta_8(\alpha) = \int_{\frac{1}{2}}^{1}\left|\xi^{\alpha} - \frac{13}{15}\right|(1-\xi)d\xi$$
$$= \begin{cases} \frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)} - \frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)} - \frac{13}{120}, & 0 < \alpha \leq \frac{\ln\left(\frac{13}{15}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\left(\frac{13}{15}\right)^{1+\frac{2}{\alpha}} + \frac{2^{\alpha+1}+1}{2^{\alpha+1}(\alpha+1)} - \frac{2^{\alpha+2}+1}{2^{\alpha+2}(\alpha+2)} - \frac{91}{120}, & \alpha > \frac{\ln\left(\frac{13}{15}\right)}{\ln\left(\frac{1}{2}\right)}. \end{cases}$$

*Proof.* By leveraging Lemma 2.1, we have

$$\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]$$
$$- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]$$
$$= \frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left(\xi^{\alpha} - \frac{7}{45}\right)\left(\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right)d\xi\right.$$
$$\left. + \int_{\frac{1}{2}}^{1}\left(\xi^{\alpha} - \frac{13}{15}\right)\left(\mathsf{F}'\left(\frac{\xi}{2}\Lambda + \frac{2-\xi}{2}\aleph\right) - \mathsf{F}'\left(\frac{\xi}{2}\aleph + \frac{2-\xi}{2}\Lambda\right)\right)d\xi\right].$$

Since $\mathsf{F}'$ is $L$-Lipschitzian function, we observe

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]\right.$$
$$\left. -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|$$

$$\leq \frac{\Lambda-\aleph}{4}\left[\int_0^{\frac{1}{2}}\left|\xi^\alpha-\frac{7}{45}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{13}{15}\right|\left|\mathsf{F}'\left(\frac{\xi}{2}\Lambda+\frac{2-\xi}{2}\aleph\right)-\mathsf{F}'\left(\frac{\xi}{2}\aleph+\frac{2-\xi}{2}\Lambda\right)\right|d\xi\right]$$
$$\leq \frac{\Lambda-\aleph}{4}\left[\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha-\frac{7}{45}\right|d\xi+\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{13}{15}\right|d\xi\right)(\Lambda-\aleph)(1-\xi)L\right]$$
$$=\frac{L(\Lambda-\aleph)^2}{4}\left[\Delta_7(\alpha)+\Delta_8(\alpha)\right].$$

Hence, the proof has been concluded. □

**Remark 2.14.** *Assume $\alpha = 1$ in Theorem 2.13, then we acquire the proceeding inequality*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]-\frac{1}{\Lambda-\aleph}\int_\aleph^\Lambda \mathsf{F}(\varpi)d\varpi\right|$$
$$\leq \frac{80627(\Lambda-\aleph)^2}{4374000}L,$$

*which is obtained by Shehzadi et al in [3, Theorem 7].*

### 2.3. Boole's Type Inequality in the Fractional Context for Functions of Bounded Variation

In this part, we provide a proof for a Boole's rule-type inequality applicable to functions of bounded variation.

**Theorem 2.15.** *If $\mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$ be a function of bounded variation on $[\aleph, \Lambda]$. Then, we have following inequality*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^\alpha}\left[\mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^+}\mathsf{F}(\Lambda)+\mathcal{J}^\alpha_{\left(\frac{\aleph+\Lambda}{2}\right)^-}\mathsf{F}(\aleph)\right]\right|$$
$$\leq \frac{1}{2}\max\left\{\frac{7}{45},\left|\left(\frac{1}{2}\right)^\alpha-\frac{7}{45}\right|,\left|\left(\frac{1}{2}\right)^\alpha-\frac{13}{15}\right|,\frac{2}{15}\right\}\bigvee_\aleph^\Lambda(\mathsf{F}),$$

*where $\bigvee_\aleph^\Lambda(\mathsf{F})$ represent the total variation of $\mathsf{F}$ on $[\aleph, \Lambda]$.*

*Proof.* Define mapping $\mathcal{K}(\varpi)$ by,

$$\mathcal{K}(\varpi)=\begin{cases}(\varpi-\aleph)^\alpha-\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^\alpha, & \aleph\leq\varpi<\frac{3\aleph+\Lambda}{4},\\ (\varpi-\aleph)^\alpha-\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^\alpha, & \frac{3\aleph+\Lambda}{4}\leq\varpi<\frac{\aleph+\Lambda}{2},\\ \frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^\alpha-(\Lambda-\varpi)^\alpha, & \frac{\aleph+\Lambda}{2}\leq\varpi<\frac{\aleph+3\Lambda}{4},\\ \frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^\alpha-(\Lambda-\varpi), & \frac{\aleph+3\Lambda}{4}\leq\varpi\leq\Lambda.\end{cases}$$

By performing integrating by parts, we have

$$\begin{aligned}
&\int_{\aleph}^{\Lambda} \mathcal{K}(\varpi) d\mathsf{F}(\varpi) \\
&= \int_{\aleph}^{\frac{3\aleph+\Lambda}{4}} \left((\varpi-\aleph)^{\alpha} - \frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right) d\mathsf{F}(\varpi) + \int_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}} \left((\varpi-\aleph)^{\alpha} - \frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right) d\mathsf{F}(\varpi) \\
&\quad + \int_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}} \left(\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - (\Lambda-\varpi)^{\alpha}\right) d\mathsf{F}(\varpi) + \int_{\frac{\aleph+3\Lambda}{4}}^{\Lambda} \left(\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - (\Lambda-\varpi)^{\alpha}\right) d\mathsf{F}(\varpi) \\
&= \left((\varpi-\aleph)^{\alpha} - \frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)\mathsf{F}(\varpi)\Big|_{\aleph}^{\frac{3\aleph+\Lambda}{4}} - \alpha\int_{\aleph}^{\frac{3\aleph+\Lambda}{4}} (\varpi-\aleph)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi \\
&\quad + \left((\varpi-\aleph)^{\alpha} - \frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)\mathsf{F}(\varpi)\Big|_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}} - \alpha\int_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}} (\varpi-\aleph)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi \\
&\quad + \left(\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - (\Lambda-\varpi)^{\alpha}\right)\mathsf{F}(\varpi)\Big|_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}} - \alpha\int_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}} (\Lambda-\varpi)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi \\
&\quad + \left(\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - (\Lambda-\varpi)^{\alpha}\right)\mathsf{F}(\varpi)\Big|_{\frac{\aleph+3\Lambda}{4}}^{\Lambda} - \alpha\int_{\frac{\aleph+3\Lambda}{4}}^{\Lambda} (\Lambda-\varpi)^{\alpha-1}\mathsf{F}(\varpi)d\varpi \\
&= \left(\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha} - \frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + \frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\mathsf{F}(\aleph) \\
&\quad - \alpha\int_{\aleph}^{\frac{3\aleph+\Lambda}{4}} (\varpi-\aleph)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi + \frac{2}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) \\
&\quad - \left(\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha} - \frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) - \alpha\int_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}} (\varpi-\aleph)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi \\
&\quad + \left(\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - \left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}\right)\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + \frac{2}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) \\
&\quad - \alpha\int_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}} (\Lambda-\varpi)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi + \frac{7}{45}\mathsf{F}(\Lambda) - \left(\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha} - \left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}\right)\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) \\
&\quad - \alpha\int_{\frac{\aleph+3\Lambda}{4}}^{\Lambda} (\Lambda-\varpi)^{\alpha-1}\,\mathsf{F}(\varpi)d\varpi \\
&= \left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\frac{1}{45}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] \\
&\quad - \alpha\int_{\aleph}^{\frac{\aleph+\Lambda}{2}} (\varpi-\aleph)^{\alpha-1}\mathsf{F}(\varpi)d\varpi - \alpha\int_{\frac{\aleph+\Lambda}{2}}^{\Lambda} (\Lambda-\varpi)^{\alpha-1}\mathsf{F}(\varpi)d\varpi \\
&= \frac{(\Lambda-\aleph)^{\alpha}}{2^{\alpha}}\frac{1}{45}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right] \\
&\quad - \Gamma(\alpha+1)\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right].
\end{aligned}$$

As a result, we can state

$$\frac{1}{90}\left[7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda)\right]$$

$$
\begin{aligned}
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\\
&=\frac{2^{\alpha-1}}{(\Lambda-\aleph)^{\alpha}}\int_{\aleph}^{\Lambda}\mathcal{K}(\varpi)d\mathsf{F}(\varpi).
\end{aligned}
$$

It is understood that if $g, \mathsf{F} : [\aleph, \Lambda] \to \mathbb{R}$ satisfy that $g$ is continuous on $[\aleph, \Lambda]$ and $\mathsf{F}$ possesses bounded variation on $[\aleph, \Lambda]$, then $\int\limits_{\aleph}^{\Lambda} g(\xi)d\mathsf{F}(\xi)$ exist and

$$
\left|\int\limits_{\aleph}^{\Lambda} g(\xi)d\mathsf{F}(\xi)\right| \leq \sup_{\xi\in[\aleph,\Lambda]}\left|g(\xi)\right|\bigvee_{\aleph}^{\Lambda}(\mathsf{F}). \tag{18}
$$

Alternatively, utilizing (18), we conclude

$$
\begin{aligned}
&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\
&\leq\frac{2^{\alpha-1}}{(\Lambda-\aleph)^{\alpha}}\left[\left|\int_{\aleph}^{\frac{3\aleph+\Lambda}{4}}\left((\varpi-\aleph)^{\alpha}-\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)d\mathsf{F}(\varpi)\right|+\left|\int_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}}\left((\varpi-\aleph)^{\alpha}-\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right)d\mathsf{F}(\varpi)\right|\right.\\
&\left.+\left|\int_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}}\left(\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-(\Lambda-\varpi)^{\alpha}\right)d\mathsf{F}(\varpi)\right|+\left|\int_{\frac{\aleph+3\Lambda}{4}}^{\Lambda}\left(\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-(\Lambda-\varpi)^{\alpha}\right)d\mathsf{F}(\varpi)\right|\right]\\
&\leq\frac{2^{\alpha-1}}{(\Lambda-\aleph)^{\alpha}}\left[\sup_{\varpi\in\left(\aleph,\frac{3\aleph+\Lambda}{4}\right)}\left|(\varpi-\aleph)^{\alpha}-\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right|\bigvee_{\aleph}^{\frac{3\aleph+\Lambda}{4}}(\mathsf{F})\right.\\
&+\sup_{\varpi\in\left(\frac{3\aleph+\Lambda}{4},\frac{\aleph+\Lambda}{2}\right)}\left|(\varpi-\aleph)^{\alpha}-\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right|\bigvee_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}}(\mathsf{F})\\
&+\sup_{\varpi\in\left(\frac{\aleph+\Lambda}{2},\frac{\aleph+3\Lambda}{4}\right)}\left|\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-(\Lambda-\varpi)^{\alpha}\right|\bigvee_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}}(\mathsf{F})\\
&\left.+\sup_{\varpi\in\left(\frac{\aleph+3\Lambda}{4},\Lambda\right)}\left|\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-(\Lambda-\varpi)^{\alpha}\right|\bigvee_{\frac{\aleph+3\Lambda}{4}}^{\Lambda}(\mathsf{F})\right]\\
&=\frac{2^{\alpha-1}}{(\Lambda-\aleph)^{\alpha}}\left[\max\left\{\left|\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}-\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right|,\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right\}\bigvee_{\aleph}^{\frac{3\aleph+\Lambda}{4}}(\mathsf{F})\right.\\
&+\max\left\{\frac{2}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha},\left|\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}-\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right|\right\}\bigvee_{\frac{3\aleph+\Lambda}{4}}^{\frac{\aleph+\Lambda}{2}}(\mathsf{F})\\
&+\max\left\{\left|\frac{13}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}\right|,\frac{2}{15}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}\right\}\bigvee_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}}(\mathsf{F})
\end{aligned}
$$

$$+\max\left\{\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha},\left|\frac{7}{45}\left(\frac{\Lambda-\aleph}{2}\right)^{\alpha}-\left(\frac{\Lambda-\aleph}{4}\right)^{\alpha}\right|\right\}\bigvee_{\frac{\aleph+3\Lambda}{4}}^{\Lambda}(\mathsf{F})\Bigg]$$

$$\leq\frac{1}{2}\Bigg[\max\left\{\left|\left(\frac{1}{2}\right)^{\alpha}-\frac{7}{45}\right|,\frac{7}{45}\right\}\bigvee_{\aleph}^{\frac{3\aleph+\Lambda}{4}}(\mathsf{F})+\max\left\{\frac{2}{15},\left|\left(\frac{1}{2}\right)^{\alpha}-\frac{13}{15}\right|\right\}\bigvee_{\frac{3\aleph+\Lambda}{6}}^{\frac{\aleph+\Lambda}{2}}(\mathsf{F})$$

$$+\max\left\{\left|\frac{13}{15}-\left(\frac{1}{2}\right)^{\alpha}\right|,\frac{2}{15}\right\}\bigvee_{\frac{\aleph+\Lambda}{2}}^{\frac{\aleph+3\Lambda}{4}}(\mathsf{F})+\max\left\{\frac{7}{45},\left|\frac{7}{45}-\left(\frac{1}{2}\right)^{\alpha}\right|\right\}\bigvee_{\frac{\aleph+3\Lambda}{4}}^{\Lambda}(\mathsf{F})\Bigg]$$

$$\leq\frac{1}{2}\max\left\{\frac{7}{45},\left|\left(\frac{1}{2}\right)^{\alpha}-\frac{7}{45}\right|,\left|\left(\frac{1}{2}\right)^{\alpha}-\frac{13}{15}\right|,\frac{2}{15}\right\}\bigvee_{\aleph}^{\Lambda}(\mathsf{F}).$$

This concludes the proof of Theorem 2.15. □

**Remark 2.16.** *Assigning $\alpha = 1$ in Theorem 2.15, then we derive the following inequality*

$$\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]-\frac{1}{\Lambda-\aleph}\int_{\aleph}^{\Lambda}\mathsf{F}(\varpi)d\varpi\right|$$
$$\leq\frac{11}{60}\bigvee_{\aleph}^{\Lambda}(\mathsf{F}),$$

*which is demonstrated by Shehzadi et al in [3, Theorem 8].*

## 3. Computational Analysis

In this part, we present a subsetutional numerical study to validate the effectiveness of our newly obtained results. We established the practical applicability of the proposed inequalities through numerous computing experiments, particularly in approximating integrals of differentiable convex functions. The particular exapmles are further enhanced through the utilization of 2D plot models of the newly derived inequalities to analyze the numerical behavior of the graphical representations. All of these graphs are important for verifying the accuracy and importance of the theoretical results in practice.

**Example 3.1.** *Assume a function $\mathsf{F}:[\aleph,\Lambda]=[0,1]\to\mathbb{R}$ given by $\mathsf{F}(\xi)=\xi^6$ in Theorem 2.3 for all $\xi>0$, then we observe the left-hand side of* (11) *becomes*

$$\Bigg|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]$$
$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\Bigg|$$
$$=\left|\frac{55}{384}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{2}\right)^{+}}\mathsf{F}(1)+\mathcal{J}^{\alpha}_{\left(\frac{1}{2}\right)^{-}}\mathsf{F}(0)\right]\right|$$
$$=\left|\frac{55}{384}-\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha-1}\xi^{6}d\xi+\int_{0}^{\frac{1}{2}}\xi^{\alpha+5}d\xi\right]\right|$$
$$=\left|\frac{55}{384}-\frac{\alpha}{2^{1-\alpha}}\left[\frac{1}{2^{\alpha+6}(\alpha+6)}+\frac{(46080+\alpha\,(31848+\alpha\,(10834+\alpha\,(2325+\alpha\,(325+\alpha\,(27+\alpha))))))\,\Gamma(\alpha)}{2^{\alpha+6}\Gamma(\alpha+7)}\right]\right|.$$

*If we choose $\alpha = \frac{1}{2}$, then we attain*

$$\left|\frac{55}{384} - \frac{\alpha}{2^{1-\alpha}}\left[\frac{1}{2^{\alpha+6}(\alpha+6)} + \frac{(46080 + \alpha(31848 + \alpha(10834 + \alpha(2325 + \alpha(325 + \alpha(27+\alpha))))))\Gamma(\alpha)}{2^{\alpha+6}\Gamma(\alpha+7)}\right]\right| = 0.0979619. \tag{19}$$

*So, the right-hand side of* (11) *becomes*

$$\begin{cases} \frac{1}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1} + \left(\frac{7}{45}\right)^{\frac{1}{\alpha}+1}\left(\frac{2\alpha}{\alpha+1}\right) - \frac{7}{90}, & 0 < \alpha < \frac{\ln\left(\frac{7}{45}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{1+\frac{1}{\alpha}} + \frac{2^{\alpha+1}+1}{2^{\alpha+1}(\alpha+1)} - \frac{13}{10}, & \alpha > \frac{\ln\left(\frac{13}{15}\right)}{\ln\left(\frac{1}{2}\right)}. \end{cases}$$

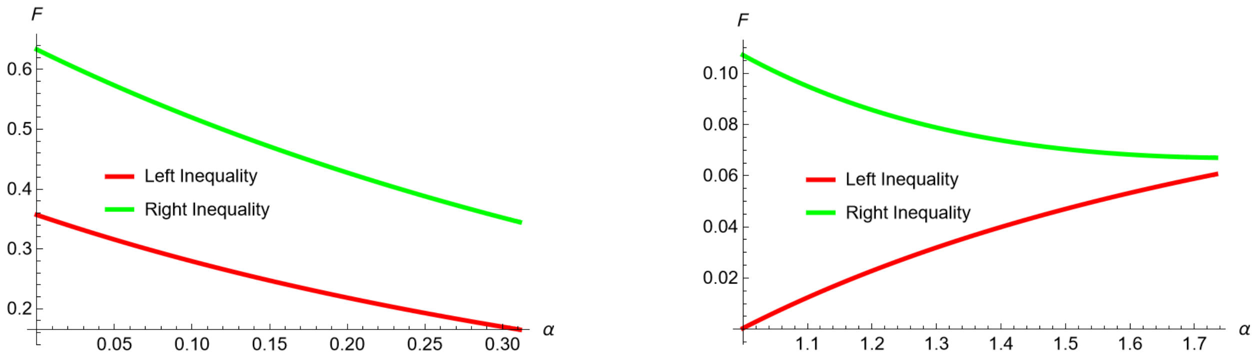
*If we choose $\alpha = \frac{1}{2}$ and specify the interval for numerical result, then we attain the right-hand side as:*

$$\frac{1}{4}\left(\frac{2\alpha}{\alpha+1}\left(\frac{7}{45}\right)^{\frac{1}{\alpha}+1} + \frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{\frac{1}{\alpha}+1} + \frac{1+2^{\alpha}}{2^{\alpha}(\alpha+1)} - \frac{62}{45}\right)[6(0)+6(1)] = 0.0924659. \tag{20}$$

*From* (19) *and* (20)*, it observe that left part is less than the right part of* (11)

$$0.0979619 < 0.0924659.$$

*This show that the inequality* (11) *is numerically valid.*




(a) 2D plot on the interval $0 < \alpha < \frac{\ln(\frac{7}{45})}{\ln(\frac{1}{2})}$ (b) 2D plot on the interval $\alpha > \frac{\ln(\frac{13}{15})}{\ln(\frac{1}{2})}$

Figure 1: Graph of both sides of (11) in Example 3.1, depending on $\alpha$, computed and plotted with Mathematica.

**Example 3.2.** *Consider a function $\mathsf{F} : [\aleph, \Lambda] = [0,1] \to \mathbb{R}$ given by $\mathsf{F}(\xi) = \xi^6$ in Theorem 2.5 for all $\xi > 0$, then we* *observe the left-hand side of* (13) *becomes*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7\mathsf{F}(\aleph)+32\mathsf{F}\left(\frac{3\aleph+\Lambda}{4}\right)+12\mathsf{F}\left(\frac{\aleph+\Lambda}{2}\right)+32\mathsf{F}\left(\frac{\aleph+3\Lambda}{4}\right)+7\mathsf{F}(\Lambda)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda-\aleph)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}}\mathsf{F}(\Lambda)+\mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}}\mathsf{F}(\aleph)\right]\right|\\
&=\left|\frac{55}{384}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{2}\right)^{+}}\mathsf{F}(1)+\mathcal{J}^{\alpha}_{\left(\frac{1}{2}\right)^{-}}\mathsf{F}(0)\right]\right|\\
&=\left|\frac{55}{384}-\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha-1}\xi^{6}d\xi+\int_{0}^{\frac{1}{2}}\xi^{\alpha+5}d\xi\right]\right|\\
&=\left|\frac{55}{384}-\frac{\alpha}{2^{1-\alpha}}\left[\frac{1}{2^{\alpha+6}(\alpha+6)}+\frac{(46080+\alpha(31848+\alpha(10834+\alpha(2325+\alpha(325+\alpha(27+\alpha))))))\Gamma(\alpha)}{2^{\alpha+6}\Gamma(\alpha+7)}\right]\right|.
\end{aligned}$$

*If we choose* $\alpha=\frac{1}{2}$, *then we attain*

$$\begin{aligned}
&\left|\frac{55}{384}-\frac{\alpha}{2^{1-\alpha}}\left[\frac{1}{2^{\alpha+6}(\alpha+6)}+\frac{(46080+\alpha(31848+\alpha(10834+\alpha(2325+\alpha(325+\alpha(27+\alpha))))))\Gamma(\alpha)}{2^{\alpha+6}\Gamma(\alpha+7)}\right]\right|\\
&=0.0979619.
\end{aligned}\tag{21}$$

*So, the right-hand side of* (13) *becomes*

$$\frac{1}{4}\left[\gamma_1(\alpha,2)\frac{3\left(1+\sqrt{7}\right)}{2}+\gamma_2(\alpha,2)\frac{3\left(\sqrt{3}+\sqrt{5}\right)}{2}\right],$$

*where*

$$\begin{cases}\gamma_1(\alpha,2)=\left(\frac{1}{2^{2\alpha+1}(2\alpha+1)}-\frac{14}{45(\alpha+1)2^{\alpha+1}}+\frac{49}{4050}\right)^{\frac{1}{2}},\\ \gamma_2(\alpha,2)=\left(\frac{2^{2\alpha+1}-1}{2^{2\alpha+1}(2\alpha+1)}-\frac{26(2^{\alpha+1}-1)}{15(\alpha+1)2^{\alpha+1}}+\frac{169}{450}\right)^{\frac{1}{2}}.\end{cases}$$

*If we choose* $\alpha=\frac{1}{2}$ *and specify the interval for numerical result, then we attain the right-hand side as:*

$$\begin{aligned}
&\frac{1}{4}\left[\left(\frac{1}{2^{2\alpha+1}(2\alpha+1)}-\frac{14}{45(\alpha+1)2^{\alpha+1}}+\frac{49}{4050}\right)^{\frac{1}{2}}\frac{3\left(1+\sqrt{7}\right)}{2}\right.\\
&\left.+\left(\frac{2^{2\alpha+1}-1}{2^{2\alpha+1}(2\alpha+1)}-\frac{26(2^{\alpha+1}-1)}{15(\alpha+1)2^{\alpha+1}}+\frac{169}{450}\right)^{\frac{1}{2}}\frac{3\left(\sqrt{3}+\sqrt{5}\right)}{2}\right]\\
&=0.43391.
\end{aligned}\tag{22}$$

*From* (21) *and* (22), *it observe that left part is less than the right part of* (13)

$$0.0979619<0.43391.$$

*This show that the inequality* (13) *is numerically valid.*

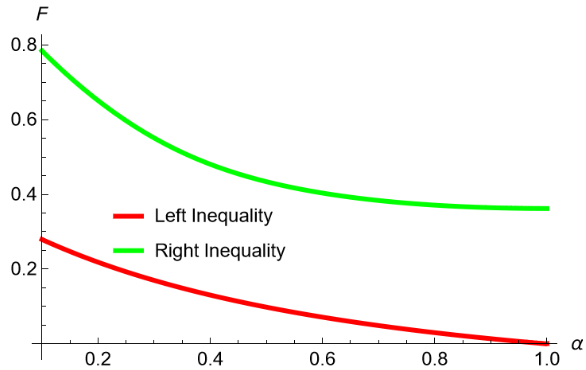



Figure 2: Graphical representation of inequalities of Theorem 2.5, and Example 3.2

**Example 3.3.** *Assume a function* $\mathsf{F} : [\aleph, \Lambda] = [0, 1] \to \mathbb{R}$ *with* $m = -1$ *and* $M = 1$ *given by* $\mathsf{F}(\xi) = \sin \xi$ *in Theorem 2.9, for* $\alpha = \frac{1}{2}$*, then we observe the left-hand side of* (14) *becomes*

$$\left| \frac{1}{90} \left[ 7\mathsf{F}(\aleph) + 32\mathsf{F}\left(\frac{3\aleph + \Lambda}{4}\right) + 12\mathsf{F}\left(\frac{\aleph + \Lambda}{2}\right) + 32\mathsf{F}\left(\frac{\aleph + 3\Lambda}{4}\right) + 7\mathsf{F}(\Lambda) \right] \right.$$
$$\left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Lambda - \aleph)^{\alpha}} \left[ \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{+}} \mathsf{F}(\Lambda) + \mathcal{J}^{\alpha}_{\left(\frac{\aleph+\Lambda}{2}\right)^{-}} \mathsf{F}(\aleph) \right] \right|$$
$$= 0.0117298, \tag{23}$$

*and the right-hand side of* (14) *for* $\alpha = \frac{1}{2}$ *becomes*

$$\frac{1}{4}\left( \frac{2\alpha}{\alpha+1}\left(\frac{7}{45}\right)^{\frac{1}{\alpha}+1} + \frac{2\alpha}{\alpha+1}\left(\frac{13}{15}\right)^{\frac{1}{\alpha}+1} + \frac{1+2^{\alpha}}{2^{\alpha}(\alpha+1)} - \frac{62}{45} \right)[M - m]$$
$$= 0.098389. \tag{24}$$

*From* (23) *and* (24)*, it observe that left part is less than the right part of* (14)

$$0.0117298 < 0.098389.$$

*This show that the inequality* (14) *is numerically valid.*

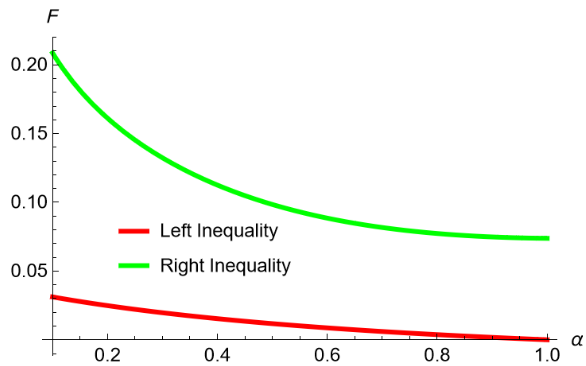



Figure 3: Graphical illustration of inequalities in Theorem 2.9 and Example 3.3

Table 1: Evaluation of Boole's rule accuracy for lower and higher degree polynomial

| Function | Exact Integral | Boole's Approximation | Absolute Error |
|---|---|---|---|
| $\xi^6$ | 0.142857 | 0.143229 | 0.000372 |
| $\xi^2$ | 0.3333 | 0.3333 | 0.0000 |

**Remark 3.4.** *For an algebraic function of degree five or lower, the left-hand side of all outcomes will perfectly match the exact integral when $\alpha = 1$, resulting in zero absolute error. On the other hand, when an algebraic polynomial of degree six or higher is selected, the left-hand side will be non-zero for $\alpha = 1$. This behavior demonstrates that Boole's formula is specifically formulated for polynomials of degree six or higher.*

## 4. Conclusion

The fundamental goal of this research is to present novel Boole's formula-type inequalities employing R.L. fractional integrals that apply to first-time differentiable convex functions. To attain this objective, we first established an integral equality linked to R.L. fractional integrals and then verified new Boole's formula inequalities for differentiable convex functions. The inequalities described in this work can help to determine the bounds of Boole's formula. Real-life applications based on the new findings are offered to increase the work's applicability. Furthermore, numerical examples and graphical analysis support the numerical validity of the conclusions. This study thoroughly investigates numerous classes of functions using unique approaches. This research extensively explored various classes of functions using specific methodologies. In approximation theory, such inequalities help in estimating errors when approximating functions using polynomials or other simpler functions. The conclusions given in this study are critical in the subject of integral inequalities, and they provide exciting prospects for new scholars to pursue additional extensions and the consequences of these results for various mathematical fields. This study has a significant impact on future research in generalized fractional integrals, convexity, $s$-convexity, and different types of convexity, as well as equations with higher-order derivatives.

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