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Qualitative analysis of a neutral fractional stochastic differential equation with G-Lévy noise

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Abstract. This paper investigates the existence and uniqueness of solutions to Caputo-type neutral fractional stochastic differential equations driven by multiplicative and fractional noises within the framework of the G-Lévy process, where the Hurst index satisfies $H \in \left(\frac{1}{2}, 1\right)$. The analysis employs Cauchy's inequality and Gronwall's inequality as essential mathematical tools to obtain rigorous estimates and establish the well-posedness of the system. To validate the theoretical findings, a detailed comparison is carried out between the exact solution and its approximation obtained via the Picard iterative method, with particular emphasis on evaluating the associated error bounds. Furthermore, an exponential estimation for the solutions is derived, providing deeper insight into their long-term behavior. Finally, two carefully designed illustrative examples are presented to demonstrate the applicability and effectiveness of the proposed theoretical framework.

1. Introduction

Over the past decades, fractional differential equations (FDEs) have become an essential mathematical framework for capturing memory and hereditary properties in diverse scientific processes. Their broad applicability spans disciplines such as physics, chemistry, engineering, and medicine [24, 29]. Researchers have successfully employed FDEs in various contexts, including the fractional modeling of influenza [14], the tuberculosis model [13], and the development of numerical algorithms for solving real-world fractional models [2, 20]. For more study, we recommend some other resources such as [28, 30, 35].

The study of neutral differential equations (NDEs) has attracted considerable attention due to their diverse applications in finance, population dynamics, and control theory. Foundational contributions were made by Hale and Lunel [8], who developed the basic theoretical framework for deterministic NDEs. Later, Liu [16] extended this research by addressing optimal control problems involving neutral differential systems. In recent years, several studies have continued to explore various aspects of neutral stochastic differential equations (NSDEs) under different conditions and modeling frameworks [1, 3, 12, 17, 18].

Recently, growing interest has been directed toward the theory of nonlinear expectation because of its significance in dealing with uncertainty modeling, risk evaluation, and superhedging in financial systems.

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In particular, substantial progress has been made in extending the sublinear expectation (**SLE**) framework to account for volatility ambiguity, leading to the development of the G-Brownian motion ($\mathcal{GBM}$) theory. Peng [22] developed G-Brownian motion as a way to include the unidentified volatility in financial models. Uncertainty issues affecting an undominated family of probability measures are intimately related to its theory. Other linkages have been found in the analysis of equations whose solutions depend on the entire trajectory of a process, typically represented by backward stochastic differential equations, in addition to the subject of financial mathematics. Thus, coupled G-expectation and $\mathcal{GBM}$ are attractive mathematical concepts. Accordingly, $\mathcal{GBM}$ together with its associated G-expectation are considered fascinating structures within mathematical theory. The studies by Soner et al. [32], Wang et al. [38], Geo [6] and others are recommended for further information. The behavior of the solution including its existence, uniqueness, stability, moment estimations, and its varied and continuous dependence on initial conditions has been thoroughly investigated in [15, 23]. L*é*vy process-based stochastic differential equations (**SDE***s*) are used in many different domains, such as biology [10] to simulate disease propagation, physics [39] to depict distinct phenomena, genetics [5] to examine animal movement, and finance [4] to predict market dynamics. $\mathcal{GBM}$ is insufficient for depicting the financial world when it comes to handling volatility uncertainty in financial models. The continuous route trait, which is common to both $\mathcal{GBM}$ and ordinary Brownian motion, frequently makes them unsuitable for precise modeling. Consequently, it makes sense that Hu and Peng [9] created the jump-based approach, which they named the G-L*é*vy process (GLP). **SLE** was later expressed as an upper-expectation by Ren [26], who introduced a novel method. The author of [19] investigated the integration theory for the GLP with finite activity, determined the It*ô* formula for the generic G-It*ô* L*é*vy process, and developed the integral based on the jump measure associated with the pure jump GLP. The author in [36] has graciously established the existence and derived exponential estimates for solutions of **SDE***s* governed by GLP. In [6], the author derived the Burkholder Davis Gundy (**BDG**) inequality in the context of G-stochastic calculus related to $\mathcal{GBM}$. The work of the author [31] marks a major breakthrough in the study of **SDE***s* by being the first to investigate their quasi-sure exponential stability under the framework of the GLP. Yuan et al. [41] investigated discrete-time feedback stabilization for neutral stochastic functional differential equations driven by a GLP. Gueye et al. [7] investigated backward **SDE***s* driven by GLP with double reflexions. Wang et al. [37] examined the existence of solutions to **SDE***s* driven by a GLP with discontinuous coefficients. Ullah et al. [33] discuss solutions to stochastic functional differential equations governed by the GLP and their exponential estimates.

Suppose $\mathfrak{R}_0^d = \mathfrak{R}^d \setminus \{0\}$ where, $\mathfrak{R}^d$ represent the d-dimensional $(d-D)$ Euclidean space. We define $\mathbb{B}C((-\infty,0];\mathbb{R}^d)$ as the set of functions $\aleph$ that are continuous and bounded on $(-\infty,0]$, with values in $\mathfrak{R}^d$, equipped with the norm

$$||\aleph|| = \sup_{\theta\in(-\infty,0]} |\aleph(\theta)|.$$

Let $(\mathbb{S},\mathbb{F},\mathbb{P})$ be a complete probability space (CPs) and $\mathbb{F}_\mu = \sigma\{\mathcal{B}(v) : 0 \le v \le \mu\}$ denote the natural filtration on a CPs. It is assumed that the filtration $\{\mathbb{F}_\mu : \mu \ge 0\}$ satisfies the standard conditions. Furthermore, let the following functions be defined: $\vartheta_1 : [0,\chi] \times \mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^d$, $\vartheta_2 : [0,\chi] \times \mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^{d\times m}$, $\Lambda_1 : [0,\chi] \times \mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^{d\times m}$, $\Lambda_2 : [0,\chi] \times \mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^{d\times m}$, $\Pi : [0,\chi] \times \mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^{d\times m}$, $F : [0,\chi]\times\mathbb{B}C((-\infty,0];\mathfrak{R}^d) \to \mathfrak{R}^{d\times m}$, where each of these functions is assumed to be Borel measurable.

In [27] Ren et al. studied stochastic functional differential equations with infinite delay driven by $\mathcal{GBM}$ of the form:

$$d\beta(\mu) = \vartheta_1(\mu,\beta_\mu)\,d\mu + \vartheta_2(\mu,\beta_\mu)d\langle\mathscr{B},\mathscr{B}\rangle(\mu) + \Lambda_1(\mu,\beta_\mu)d\mathscr{B}(\mu), \tag{1}$$

where $\mu \in [0,\chi]$, the initial condition $\beta(0) \in \mathfrak{R}^d$ is given, $\{\langle\mathscr{B},\mathscr{B}\rangle(\mu), \mu \ge 0\}$ denotes the quadratic variation process of the $\mathcal{GBM}$ $\{\mathscr{B}(\mu), \mu \ge 0\}$, and ϑ_1, ϑ_2, and Λ_1 are given functions satisfying $\vartheta_1(\cdot,\beta)$, $\vartheta_2(\cdot,\beta)$, $\Lambda_1(\cdot,\beta) \in M_G^2([0,\chi];\mathfrak{R}^d)$ for all $\beta \in \mathfrak{R}^d$.

In [34], Ullah et al. investigated the Carath*é*odory approximation scheme in the context of **SDE***s* driven by a GLP of the form:

$$d\beta(\mu) = \vartheta_1(\mu, \beta(\mu))d\mu + \vartheta_2(\mu, \beta(\mu))d\langle \mathscr{B}, \mathscr{B}\rangle(\mu) + \Lambda_1(\mu, \beta(\mu))d\mathscr{B}(\mu) + \int_{\Re_0^d} F(\mu, \beta(\mu^-), q)L(d\mu, dq). \tag{2}$$

For $0 \leq \mu \leq \chi < \infty$, consider the process with initial condition $\beta(\mu_0) = \beta_0$ such that $\mathbb{E}[|\beta_0|^2] < \infty$. Note that $\beta(\mu^-)$ denotes the left-hand limit of $\beta(\mu)$. Here, $\vartheta_1(\cdot, \beta)$, $\vartheta_2(\cdot, \beta)$, $\Lambda_1(\cdot, \beta) \in M_G^2([0, \chi]; \Re^n)$ and $F(\cdot, \beta, \cdot) \in \mathbb{H}_G^2([0, \chi] \times \Re_0^d; \Re^n)$.

Inspired by [27, 34], we investigate the following Caputo-type neutral fractional stochastic differential equations driven by the GLP

$$\begin{aligned} {}^cD^{\jmath}\left[\beta(\mu) - \delta I^{\jmath}\Pi(\mu, \beta_\mu)\right] &= \vartheta_1(\mu, \beta_\mu) + \vartheta_2(\mu, \beta_\mu)d\langle \mathscr{B}, \mathscr{B}\rangle(\mu) + \Lambda_1(\mu, \beta_\mu)d\mathscr{B}(\mu) + \Lambda_2(\mu, \beta_\mu)d\mathscr{B}^H(\mu) \\ &\quad + \int_{\Re_0^d} F(\mu, \beta_{\mu-}, q)\mathcal{L}(d\mu, dq), \end{aligned} \tag{3}$$

where ${}^cD^{\jmath}$ is the Caputo fractional derivative of order $\jmath$ $(0 < \jmath < 1)$ and $I^{\jmath}$ is the Riemann-Liouville integral, $\delta \in \Re$, on $\mu \in [0, \chi]$, with the initial value $\sigma(0)$ belonging $\Re^d$, $\beta_\mu = \{\beta(\mu + \theta), -\infty < \theta \leq 0\}$. Additionally, $\beta_{\mu-}$ denotes the left-hand limit of β_μ, and $\mathscr{B}(\mu)$ represents a $d - D$ $\mathcal{GBM}$ and $\mathscr{B}^H(\mu)$ represent fractional Brownian motion ($\mathcal{FBM}$). The functions $\vartheta_1(\cdot, \beta)$, $\vartheta_2(\cdot, \beta)$, $\Lambda_1(\cdot, \beta)$, $\Lambda_2(\cdot, \beta)$ and $\Pi(\cdot, \beta)$ belong to the space $M_G^2((-\infty, \chi]; \Re^d)$, while $F(\cdot, \beta, \cdot)$ belongs to the space $\mathbb{H}_G^2((-\infty, \chi] \times \Re_0^d; \Re^d)$ for each $\chi \in \Re^d$. Please refer to [36]. The initial condition for equation (3) is specified as

$$\beta_0 = \sigma = \{\sigma(\theta) : -\infty < \theta \leq 0\}, \tag{4}$$

is $\mathbb{F}_0$-measurable, a random variable with values in $\mathbb{B}C((-\infty, 0]; \Re^d)$, such that $\sigma \in M_G^2((-\infty, \chi]; \Re^d)$.

The characteristics of $\mathcal{FBM}$ depend on the Hurst exponent H. When $H = \frac{1}{2}$, the $\mathcal{FBM}$ behaves like a standard Brownian motion. If $H > \frac{1}{2}$, the process exhibits positive correlation in its increments, indicating long-range dependence. In contrast, when $H < \frac{1}{2}$, the increments are negatively correlated, reflecting short-range dependence or anti-persistence.

The main contributions of this paper are summarized as follows:

- Investigated Caputo-type neutral fractional **SDE***s* driven by the GLP.
- Established existence and uniqueness results for the proposed model.
- Derived an exponential estimate for the obtained solutions.
- Compared exact and Picard approximate solutions with error analysis.
- Presented an illustrative example to verify the theoretical findings.

The structure of the article is as follows: Section 2 outlines the fundamental definitions and lemmas related to fractional calculus and the G-framework. In Section 3, we investigate the existence and uniqueness of solutions to neutral fractional **SDE***s* driven by the GLP, along with an analysis of their boundedness. This section also includes an error estimate between the exact and approximate solutions. Section 4 is devoted to establishing an exponential estimate for the solutions of fractional **SDE***s* influenced by the GLP. In Section 5, two examples are given to show how the proposed results can be applied in practice.

2. Preliminaries

Important notations and initial findings inside the G-framework that will form the basis of the next discussion are presented in this section. Consider the space $\mathbb{S}_\chi = C([0, \chi]; \Re^d)$, comprising continuous mappings from $[0, \chi]$ into $\Re^d$. For any $\chi > 0$, we define the space $L_{ip}(\mathbb{S}_\chi)$ as follows:

$$L_{ip}(\mathbb{S}_\chi) = \left\{\varphi(\mathscr{B}(\mu_1), \mathscr{B}(\mu_2), \ldots, \mathscr{B}\mu_d)) \,\middle|\, d \geq 1,\ \mu_1, \ldots, \mu_d \in [0, \chi],\ \varphi \in C_{b.L_{ip}}(\Re^{d\times m})\right\},$$

where $C_{b.L_{ip}}(\Re^{d\times m})$ denotes the set of all bounded Lipschitz continuous functions on $\Re^{d\times m}$. A functional $\mathbb{E}$ acting on $L_{ip}(\mathbb{S}_\chi)$ is referred to as a **SLE** if it satisfies the following properties for all $\beta_1, \beta_2 \in L_{ip}(\mathbb{S}_\chi)$:

(a) *Monotonicity:* $\mathbf{E}[\beta_1] \geq \mathbb{E}[\beta_2]$ if $\beta_1 \geq \beta_2$.

(b) *Constant preserving:* $\mathbf{E}[a] = a$ for any constant $a \in \Re$.

(c) *Sub-additivity:* $\mathbf{E}[\beta_1 + \beta_2] \leq \mathbf{E}[\beta_1] + \mathbf{E}[\beta_2]$.

(d) *Positive homogeneity:* $\mathbf{E}[a\beta_1] = a\mathbf{E}[\beta_1]$ for all $a \geq 0$.

For any $\mu \leq \chi$, we have the inclusion $L_{ip}(\mathbb{S}_\mu) \subseteq L_{ip}(\mathbb{S}_\chi)$, and we define $L_{ip}(\mathbb{S}) = \bigcup_{n=1}^{\infty} L_{ip}(\mathbb{S}_n)$. For each $p \geq 1$, the space $L_G^p(\mathbb{S})$ is defined as the completion of $L_{ip}(\mathbb{S})$ under the norm

$$\|\mathbb{X}\|_p := \left(\hat{\mathbb{E}}[|\mathbb{X}|^p]\right)^{1/p},$$

which endows it with a Banach space structure. Furthermore, for $0 \leq \mu \leq \chi < \infty$, the following inclusions hold:

$$L_G^p(\mathbb{S}_\chi) \subseteq L_G^p(\mathbb{S}_\chi) \subseteq L_G^p(\mathbb{S}).$$

The triplet $(\mathbb{S}, L_{ip}(\mathbb{S}_\chi), \mathbb{E})$ is referred to as a **SLE** space. Given $p \geq 1$, a partition of the interval $[0,\chi]$ is defined as a finite set $\mathbb{A}_\chi^{\mathcal{N}} = \{0 = \mu_0 < \mu_1 < \cdots < \mu_{\mathcal{N}} = \chi\}$. For each $p \geq 1$, the space of simple processes $M_G^{p,0}([0,\chi])$ consists of processes of the form

$$\lambda_\mu(q) = \sum_{i=0}^{\mathcal{N}-1} \Phi_{\mu_i}(q) I_{[\mu_i,\mu_{i+1}]}(\mu),$$

where $\Phi_{\mu_i}(q) \in L_G^p(S_{\mu_i})$ and $\{0 = \mu_0 < \mu_1 < \cdots < \mu_{\mathcal{N}} = \chi\}$ is a partition of the interval $[0,\chi]$. The space $M_G^{p,0}([0,\chi])$ is then completed w.r.t the following norm:

$$\|\lambda\|_p = \left(\int_0^\chi \hat{\mathbb{E}}[|\lambda(\ell)|^p]\, d\ell\right)^{1/p}.$$

This completed space is denoted by $M_G^p(0,\chi)$.

Definition 2.1. *[21] Suppose $\lambda_\mu \in M_G^p(0,\chi)$ for some $p \geq 1$. The corresponding Itô integral in the G-framework is defined as:*

$$\int_0^\chi \lambda(\ell)\, d\mathscr{B}(\ell) = \sum_{j=0}^{\mathcal{N}-1} \Phi_j \left(\mathscr{B}(\mu_{j+1}) - \mathscr{B}(\mu_j)\right).$$

Definition 2.2. *[21] A process $\{\langle\mathscr{B}\rangle(\mu)\}_{\mu\geq 0}$, where $\langle\mathscr{B}\rangle(0) = 0$, is called the G-quadratic variation process, which is defined by:*

$$\langle\mathscr{B}\rangle(\mu) = \lim_{\mathcal{N}\to\infty} \sum_{j=0}^{\mathcal{N}-1} \left(\mathscr{B}(\mu_{j+1}^{\mathcal{N}}) - \mathscr{B}(\mu_j^{\mathcal{N}})\right)^2 = \mathscr{B}(\mu)^2 - 2\int_0^\mu \mathscr{B}(\ell)\, d\mathscr{B}(\ell).$$

Define $\triangle_{0,\chi}$ as a function from $M_G^{0,1}(0,\chi)$ to $L_G^2(F_\chi)$, given by:

$$\triangle_{0,\chi}(\lambda) = \int_0^\chi \lambda(\ell)\, d\langle\mathscr{B}\rangle(\ell) = \sum_{j=0}^{\mathcal{N}-1} \Phi_j \left(\langle\mathscr{B}\rangle_{(\mu_{j+1})} - \mathscr{B}(\mu_j)\right).$$

The above operator $\triangle_{0,\chi}$ admits an extension to the space $M_G^1(0,\chi)$, where for each $\lambda \in M_G^1(0,\chi)$, it is defined as:

$$\int_0^\chi \lambda(\ell)\, d\langle\mathscr{B}\rangle(\ell) = \triangle_{0,\chi}(\lambda).$$

Assume that $\mathcal{U}$ is a weakly compact set associated with the **SLE**. The corresponding capacity $\widehat{c}$ is defined by:

$$\hat{c}(\mathbb{J}) = \sup_{P\in \mathcal{U}} P(\mathbb{J}), \quad \mathbb{J} \in F_{\chi}.$$

The set $\mathbb{J}$ is classified as polar if $\hat{c}(\mathbb{J}) = 0$. Additionally, if a characteristic persists beyond a polar set, it is true q.s.

Lemma 2.3. *[21] Let $y \in L_G^p$ such that $\mathbb{E}[|y|^p]$ is finite, with $p \geq 0$. Then,*

$$\hat{c}(|y| > \eta) \leq \frac{\mathbb{E}[|y|^p]}{\eta},$$

for every $\eta > 0$.

Lemma 2.4. *Let $\Upsilon \in M_G^p(0,\chi)$ with $p \geq 2$. Then, the following inequality holds:*

$$\mathbb{E}\left[\sup_{0\leq t\leq \chi}\left|\int_0^t \Upsilon(\ell)\, d\mathscr{B}(\ell)\right|^p\right] \leq \zeta\mathbb{E}\left[\int_0^t |\Upsilon(\ell)|^2\, d\ell\right]^{p/2},$$

where $0 < \zeta = m_2\chi^{\frac{p}{2}-1} < \infty$, and m_2 is a positive value that depends on p.

Lemma 2.5. *Let $\Upsilon \in M_G^p(0,\chi)$ with $p \geq 1$. Then, the following inequality holds:*

$$\mathbb{E}\left[\sup_{0\leq t\leq \chi}\left|\int_0^t \Upsilon(\ell)\, d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\right|^p\right] \leq \gamma\mathbb{E}\left[\int_0^t |\Upsilon(\ell)|^2\, d\ell\right]^{p/2},$$

where $0 < \gamma = m_1\chi^{p-1} < \infty$, and m_1 is a positive value that depends on p.

Definition 2.6. *[11] The fractional integral of order $\jmath$ from 0 to y of the function ϑ_1 is expressed as*

$$I_{0,y}^{\jmath}\vartheta_1(y) = \frac{1}{\Gamma(\jmath)}\int_0^y (y-\zeta)^{\jmath-1}\vartheta_1(\zeta)\, d\zeta, \quad for, \ \ y > 0,\ \jmath > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.7. *[1] The Caputo derivative of fractional order $\jmath$ for the function $\vartheta_1$is defined as*

$${}^{C}D_{0,y}^{\jmath}\vartheta_1(y) = \frac{1}{\Gamma(n-\jmath)}\int_0^y (y-\zeta)^{n-\jmath-1}\vartheta_1^{(n)}(\zeta)\, d\zeta, \quad where\, n = \lfloor \jmath \rfloor + 1.$$

Definition 2.8. *[40] Let $u(\mu)$ be a continuous function on $[0,\chi]$, and suppose there exists a continuous non-decreasing function $\varphi(\mu)$ such that*

$$u(\mu) \leq \varphi(\mu) + \int_0^{\mu} \jmath(\ell)u(\ell)\, d\ell, \quad \mu \in [0,\chi],$$

where $\jmath(\mu)$ is a given continuous function. Then, the solution $u(\mu)$ satisfies the following inequality:

$$u(\mu) \leq \varphi(\mu)\exp\left(\int_0^{\mu} \jmath(\ell)\, d\ell\right), \quad \mu \in [0,\chi].$$

Lemma 2.9. *[25] Let $\varphi : I \to L_2^0$ be a function that satisfies*

$$\int_0^{\chi} \|\varphi(\mu)\|_{L_2^0}\, d\mu < \infty.$$

Then, the following inequality holds:

$$\mathbb{E}\left\|\int_0^{\tau} \varphi(\mu)\, d\mathscr{B}^H(\mu)\right\|^2 \leq 2H\,\mu^{2H-1}\int_0^{\tau} \mathbb{E}\|\varphi(\mu)\|^2_{L_2^0}\, d\mu.$$

In this manuscript we put $m_3 = 2H\,\mu^{2H-1}$, where m_3 is positive value.

Definition 2.10. *[36] The process $\{Y(\mu), \mu \geq 0\}$, constructed on a* **SLE** *space $(\mathbb{S}, L_{ip}(\mathbb{S}_\chi), \mathbb{E})$, is termed a GLP provided that it fulfills the following five fundamental characteristics.*

1. *$Y(\mu)$=0.*
2. *For any $\ell, \mu \geq 0$, the increment $Y(\mu+\ell) - Y(\ell)$ is independent of the collection $\{Y(\mu_1), Y(\mu_2), \dots, Y(\mu_m)\}$ for every $m \in \mathbb{N}$ and for all partitions satisfying $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m \leq \mu$.*
3. *The distribution of the increment $Y(\mu+\ell) - Y(\ell)$ is invariant with respect to the starting time ℓ, and thus depends only on the length of the increment μ, for all $\ell, \mu \geq 0$.*

Furthermore, the process $\{Y(\mu)\}_{\mu\geq 0}$ is classified as a GLP if the following additional conditions hold:

4. *There exists a Lévy process in $2-$ dimension, denoted by $\{(Y^c(\mu), Y^d(\mu))\}_{\mu\geq 0}$, such that for every $\mu \geq 0$, the decomposition*

$$Y(\mu) = Y^c(\mu) + Y^d(\mu)$$

holds, where $Y^c(\mu)$ and $Y^d(\mu)$ represent the continuous and jump components, respectively.

5. *The processes $Y^c(\mu)$ and $Y^d(\mu)$ satisfy the following properties:*

$$\lim_{\mu\to 0^+} \frac{\mathbb{E}[|Y^c(\mu)|^3]}{\mu} = 0, \quad \textit{and} \quad \mathbb{E}[|Y^d(\mu)|] < C\mu, \quad \forall\, \mu \geq 0,$$

where C is a constant that depends on the properties of the process $Y(\mu)$.

We denote by $\mathbb{H}_G^{\phi}([0,\chi] \times \mathfrak{R}_0^d)$ the space consisting of all basic (elementary) random fields on $[0,\mu] \times \mathfrak{R}_0^d \times S$, which can be represented in the following form:

$$F(h,q)(\omega) = \sum_{i=1}^{n-1}\sum_{j=1}^{m} \Phi_{i,j}\mathbf{1}_{(\mu_i,\mu_{i+1}]}(h)\,\psi_j(q),$$

consider $n, m \in \mathbb{N}$ with $0 \leq \mu_1 < \mu_2 < \cdots < \mu_n \leq \chi$, and let $\{\psi_j\}_{j=1}^m \subset C_{b,lip}(\mathfrak{R}^d)$ be a collection of non-overlapping functions such that $\aleph_j(0) = 0$. The coefficients $\Phi_{i,j}$ are expressed as $\Phi_{i,j} = \varphi_{i,j}(\beta_{\mu_1}, \dots, \beta_{\mu_i} - \beta_{\mu_{i-1}})$, where $\varphi_{i,j} \in C_{b,lip}(\mathfrak{R}^{d\times i})$. This space is equipped with the norm given by

$$\|F\|_{\mathbb{H}_G^p([0,\chi]\times\mathfrak{R}_0^d)} = \mathbb{E}\left[\int_0^{\chi} \sup_{v\in\nu} \int_{\mathfrak{R}_0^d} |F(\ell,q)|^p v(dq)\, d\ell\right]^{\frac{1}{p}}, \quad p = 1, 2.$$

Definition 2.11. *Consider the Itô integral of* $F \in \mathbb{H}_G^{\phi}([0,\chi] \times \Re_0^d)$ *with respect to the discontinuity measure* $\mathcal{L}$, *which is defined as follows:*

$$\int_0^{\mu} \int_{\Re_0^d} F(\ell, q)\mathcal{L}(d\ell, dq) = \sum_{v<\ell\leq\mu} F\left(\ell, \Delta\beta(\ell)\right), \quad \textit{quasi surely.}$$

Suppose $\mathbb{H}_G^p([0,\chi] \times \Re_0^d)$ *denote the topological closure of* $\mathbb{H}_G^{\phi}([0,\chi] \times \Re_0^d)$ *with respect to the norm* $\|F\|_{\mathbb{H}_G^p([0,\chi]\times\Re_0^d)}$, *for* $p = 1, 2$. *The Itô integral can be extended to this space, and for* $p = 1, 2$, *the extended integral takes values in* $L_G^p(S_\chi)$. *The following* **BDG**-*type inequality applies to these integrals.*

Lemma 2.12. *Let* $F(\ell, q) \in \mathbb{H}_G^2([0,\chi] \times \Re_0^d)$. *Then, there exists a cadlag modification* $\hat{\beta}(\mu)$ *of the process* $\beta(\mu) = \int_0^{\mu} \int_{\Re_0^d} F(\ell, q)\mathcal{L}(d\ell, dq)$ *such that* $\forall\ \mu \in [0,\chi]$ *and* $p \geq 2$, *the following condition is satisfied:*

$$\mathbb{E}\left[\sup_{0\leq\ell\leq\mu} |\hat{\beta}(\mu)|^2\right] \leq m_4 \mathbb{E}\left[\int_0^{\mu} \int_{\Re_0^d} F^2(\ell, q)v(dq)d\ell\right], \quad m_4 > 0.$$

In this manuscript, we put $m_5 = \delta^2$ and $Q = \mu^{2\jmath-1}$.

3. Uniqueness and Existence of Solutions for Caputo-Type Neutral Fractional SDEs Governed by GLP

Definition 3.1. *A càdlàg process* $\beta(\mu)$, *adapted to the filtration* F_μ *and belonging to the space* $M_G^2((-\infty,\chi]; \mathbb{R}^d)$, *is considered to satisfy equation (3) along with the initial condition(4) if it satisfies*

$$\begin{aligned}\beta(\mu) = \sigma(0) &+ \frac{1}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \vartheta_1(\ell, \beta_\ell)d\ell + \frac{1}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \vartheta_2(\ell, \beta_\ell)d\langle\mathscr{B}, \mathscr{B}\rangle(\ell) \\ &+ \frac{1}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \Lambda_1(\ell, \beta_\ell)d\mathscr{B}(\ell) + \frac{1}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \Lambda_2(\ell, \beta_\ell)d\mathscr{B}^H(\ell) \\ &+ \frac{1}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \int_{\Re_0^d} F(\ell, \beta_{\ell^-}, q)\mathcal{L}(d\ell, dq) \\ &+ \frac{\delta}{\Gamma(\jmath)} \int_0^{\mu} (\mu-\ell)^{\jmath-1} \Pi(\ell, \beta_\ell)d\ell. \end{aligned} \tag{5}$$

Let $\beta(\mu)$ and $y(\mu)$ be two solutions of equation (3). The solution $\beta(\mu)$ is considered unique if it coincides with $y(\mu)$ for all μ, that is

$$\mathbb{E}\left[|\beta(\mu) - y(\mu)|^2\right] = 0,$$

holds q.s.

In this article, we establish the necessary conditions for linear growth and Lipschitz continuity, as follows:

(H_1) For every $\beta \in \mathbb{B}C((-\infty, 0]; \Re^d)$, a positive constant ξ_1 exists such that

$$\begin{aligned}&|\vartheta_1(\mu,\beta)|^2 \vee |\vartheta_2(\mu,\beta)|^2 \vee |\Pi(\mu,\beta)|^2 \vee |\Lambda_1(\mu,\beta)|^2 \vee \int_{\Re_0^d} |F(\mu,\beta,q)|^2 v(dq) \\ &\quad \vee |\Lambda_2(\mu,\beta)|^2 \leq \xi_1(1+|\beta|^2).\end{aligned}$$

(H_2) $\forall\ \beta, y \in \mathbb{B}C((-\infty, 0]; \Re^d)$, a positive constant ξ_2 exists such that

$$\begin{aligned}&|\vartheta_1(\mu,y) - \vartheta_1(\mu,\beta)|^2 \vee |\vartheta_2(\mu,y) - \vartheta_2(\mu,\beta)|^2 \vee |\Pi(\mu,y) - \Pi(\mu,\beta)|^2 \vee |\Lambda_1(\mu,y) - \Lambda_1(\mu,\beta)|^2 \\ &\quad \vee \int_{\Re^d} |F(\mu,y,q) - F(\mu,\beta,q)|^2 v(dq) \vee |\Lambda_2(\mu,y) - \Lambda_2(\mu,\beta)|^2 \leq \xi_2|y-\beta|^2.\end{aligned}$$

In the subsequent lemma, we demonstrate that every solution $\beta(\mu)$ to equation (3) exhibits boundedness, indicating that $\beta(\mu)$ belongs to the space $M_G^2((-\infty,\chi];\mathfrak{R}^d)$.

Lemma 3.2. *Consider a function $\beta(\mu)$ that satisfies equation (3) along with the initial condition specified in (4). Assuming the expectation $\mathbb{E}|\beta|^2$ is finite and that the growth condition (H_1) is met. Then*

$$\mathbb{E}\left[\sup_{-\infty\leq\ell\leq\mu}|\beta(\ell)|^2\right]\leq\mathbb{E}\|\sigma\|^2+7\left[\left(1+\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)(\Gamma(\jmath))^2}\right)\mathbb{E}\|\sigma\|^2+\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)(\Gamma(\jmath))^2}\right]e^{\frac{7\xi_1\mathcal{M}Q\chi}{(1-2\jmath)(\Gamma(\jmath))^2}},$$

where $\mathcal{M}=\chi+m_1\chi+m_2+m_3+m_4+m_5$ where m_i are positive constants.

Proof. We now turn our attention to equation (5) and make use of the associated inequality $\left|\sum_{j=1}^7 b_j\right|^2\leq 7\sum_{j=1}^7|b_j|^2$, to derive

$$\begin{aligned}|\beta(\mu)|^2\leq{}&7|\sigma(0)|^2+7\left|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta_\ell)d\ell\right|^2+7\left|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta_\ell)d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\right|^2\\&+7\left|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta_\ell)d\mathscr{B}(\ell)\right|^2+7\left|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta_\ell)d\mathscr{B}^H(\ell)\right|^2\\&+7\left|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\int_{\mathbb{R}_0^d}F(\ell,\beta_{\ell^-},q)\mathcal{L}(d\ell,dq)\right|^2\\&+7\left|\frac{\delta}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta_\ell)d\ell\right|^2.\end{aligned}\tag{6}$$

Based on the Cauchy inequality, Lemmas 2.4, 2.5, 2.9, 2.12 and G-expectation, we obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta(\ell)|^2\Big]\leq{}&7\mathbb{E}|\sigma(0)|^2+\frac{7Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\left|\vartheta_1(\ell,\beta_\ell)\right|^2d\ell\\&+\frac{7m_1Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\left|\vartheta_2(\ell,\beta_\ell)\right|^2d\ell\\&+\frac{7m_2Q}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\left|\Lambda_1(\ell,\beta_\ell)\right|^2d\ell\\&+\frac{7m_3Q}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\left|\Lambda_2(\ell,\beta_\ell)\right|^2d\ell\\&+\frac{7m_4Q}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\int_{\mathfrak{R}_0^d}\left|F(\ell,\beta_{\ell^-},q)\right|^2v(dq)d\ell\\&+\frac{7m_5Q}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\int_0^\mu\left|\Pi(\ell,\beta_\ell)\right|^2d\ell.\end{aligned}$$

By utilizing hypothesis (H_1), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta(\ell)|^2\Big]\leq{}&7\|\sigma\|^2+\frac{7\xi_1Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}(1+|\beta_\ell|^2)d\ell\\&+\frac{7\xi_1Qm_1\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}(1+|\beta_\ell|^2)d\ell\\&+\frac{7\xi_1Qm_2}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}(1+|\beta_\ell|^2)d\ell\end{aligned}$$

$$
\begin{aligned}
&+\frac{7\xi_1 Q m_3}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}(1+|\beta_\ell|^2)d\ell\\
&+\frac{7\xi_1 Q m_4}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}(1+|\beta_\ell|^2)d\ell\\
&+\frac{7\xi_1 Q m_5}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}(1+|\beta_\ell|^2)d\ell\\
\leq& 7\|\sigma\|^2+\frac{7\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\chi\\
&+\frac{7\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\chi\int_0^{\mu}\mathbb{E}|\beta_\ell|^2 d\ell\\
\leq& 7\mathbb{E}\|\sigma\|^2+\frac{7\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\chi\\
&+\frac{7\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\int_0^{\mu}\Big[\mathbb{E}\|\sigma\|^2+\mathbb{E}\Big(\sup_{0\leq u\leq\ell}|\beta(u)|^2\Big)\Big]d\ell\\
\leq& 7\mathbb{E}\|\sigma\|^2+\frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2}+\frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2\\
&+\frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\leq u\leq\ell}|\beta(u)|^2\Big]d\ell,
\end{aligned}
$$

where $\mathcal{M}=\chi+m_1\chi+m_2+m_3+m_4+m_5$. By applying Gronwall's inequality, we obtain the following result.

$$
\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta(\ell)|^2\Big]\leq 7\left[(1+\frac{\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2+\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{\frac{7\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}}.
$$

Observing that

$$
\mathbb{E}\Big[\sup_{-\infty<\ell\leq\mu}|\beta(\ell)|^2\Big]\leq\mathbb{E}\|\sigma\|^2+\sup_{0\leq\ell\leq\mu}\mathbb{E}|\beta(\ell)|^2,
$$

this implies

$$
\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta(\ell)|^2\Big]\leq\mathbb{E}\|\sigma\|^2+7\left[(1+\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2+\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{\frac{7\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}}.
$$

For $\mu\in[0,\chi]$, let $\beta^0(\mu)=\sigma(0)$ and $\beta^0(0)=\sigma$. For every natural number n, we define: $\beta^n(0)=\sigma$ and define the Picard iteration process as:

$$
\begin{aligned}
\beta^n(\mu)=&\,\zeta(0)+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta_\ell^{n-1})d\ell+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta_\ell^{n-1})d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta_\ell^{n-1})d\mathscr{B}(\ell)+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta_\ell^{n-1})d\mathscr{B}^H(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\int_{\aleph_0^d}F(\ell,\beta_{\ell^-}^{n-1},q)\mathcal{L}(d\ell,dq)\\
&+\frac{\delta}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta_\ell^{n-1})d\ell.
\end{aligned}\tag{7}
$$

Next, we focus on proving the existence and uniqueness of the solution, as well as deriving an error bound for the difference between the exact solution $\beta(\mu)$ and the Picard approximations $\beta^n(\mu)$, where $n=\mathbb{N}$. □

Theorem 3.3. *Assume that conditions* (H_1) *and* (H_2) *are satisfied, and that* $\mathbb{E}\|\sigma\|^2 < \infty$. *Under these conditions, the solution to equation (3) is unique and càdlàg, with* $\beta(\mu) \in M_G^2((-\infty,\chi];\mathfrak{R}^d)$. *In addition,* $\forall\ n \geq 1$, *the Picard approximations* $\beta^n(\mu)$ *and the exact solution* $\beta(\mu)$ *satisfy the following inequality:*

$$\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)-\beta(\ell)|^2\Big] \leq \mathcal{Z}\,(\Psi\mu)^n\,\frac{1}{n!}e^{\Psi\mu},$$

where,

$$\mathcal{Z} = \frac{6\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5)(1+\mathbb{E}\|\sigma\|^2)\chi$$
$$\Psi = \frac{6\xi_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5)$$

and m_i *are positive constants.*

Proof. Let $\{\beta^n\}_{n\geq1}$ be the sequence generated by the Picard iteration as defined in equation (7). It is evident that the initial function $\beta^0(\mu)$ belongs to the space $M_G^2\big((-\infty,\chi];\mathfrak{R}^d\big)$. By applying the inequality $\big|\sum_{j=1}^7 b_j\big|^2 \leq 7\sum_{j=1}^7 |b_j|^2$, together with Lemmas 2.4, 2.5, and 2.12, as well as the Cauchy-Schwarz inequality and the assumption (H_1), we can establish the following result.

$$\mathbb{E}\Big[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)|^2\Big] \leq 7\mathbb{E}\|\sigma\|^2 + \frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2} + \frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu \mathbb{E}\Big[\sup_{0\leq u\leq\ell}|\beta^{n-1}(u)|^2\Big]d\ell,$$

where $\mathcal{M} = \chi + m_1\chi + m_2 + m_3 + m_4 + m_5$. Observing that

$$\max_{1\leq n\leq j}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^{n-1}(\ell)|^2\right] \leq \max\left\{\mathbb{E}\|\sigma\|^2, \max_{1\leq n\leq j}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)|^2\right]\right\} \leq \mathbb{E}\|\sigma\|^2 + \max_{1\leq n\leq j}E\left[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)|^2\right],$$

so we have

$$\max_{1\leq n\leq j}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)|^2\right] \leq 7\mathbb{E}\|\sigma\|^2 + \frac{7\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2} + \frac{14\xi_1 Q\mathcal{M}\chi}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu \max_{1\leq n\leq j}\mathbb{E}\Big[\sup_{0\leq u\leq\ell}|\beta^n(u)|^2\Big]d\ell.$$

Utilizing Grownwall inequality

$$\max_{1\leq n\leq j}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)|^2\right] \leq 7\left[\Big(1 + \frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\Big)\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{\frac{7\xi_1\mathcal{M}Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}}.$$

Since j can be any value, setting $\mu = \chi$ gives

$$\mathbb{E}\sup_{0\leq\ell\leq\chi}|\beta^n(\ell)|^2 \leq 7\left[(1 + \frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{\frac{7\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}}. \tag{8}$$

Based on the sequence $\{\beta^n(\mu);\mu\geq 0\}$ given in equation (7), we obtain

$$\begin{aligned}\beta^1(\mu)-\beta^0(\mu)=&\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta^0_\ell)d\ell+\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta^0_\ell)d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\\&+\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta^0_\ell)d\mathscr{B}(\ell)+\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta^0_\ell)d\mathscr{B}^H(\ell)\\&+\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\int_{\Re^d_0}F(\ell,\beta^0_{\ell^-},q)\mathcal{L}(d\ell,dq)\\&+\frac{\delta}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta^0_\ell)d\ell.\end{aligned}$$

Based on (H_1), the Cauchy inequality, Lemmas 2.4, 2.5, 2.9, 2.12 and G-expectation, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq\ell\leq\chi}|\beta^1(\ell)-\beta^0(\ell)|^2\right]\leq&\frac{\xi_1Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\&+\frac{\xi_1m_1Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\&+\frac{\xi_1m_2Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\&+\frac{\xi_1m_3Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\&+\frac{\xi_1m_4Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\&+\frac{\xi_1m_5Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\\leq&\frac{\xi_1Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\int_0^\mu(1+\mathbb{E}\|\sigma\|^2)d\ell\\\leq&\mathcal{Z}.\end{aligned}$$

Let $\mathcal{Z}=\frac{6\xi_1Q}{(1-2\jmath)\Gamma(\jmath)^2}\left[(\chi+m_1\chi+m_2+m_3+m_4+m_5)(1+\mathbb{E}\|\sigma\|^2)\chi\right]$. Then, using similar presumptions and justifications, (H_2) follows

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^2(\ell)-\beta^1(\ell)|^2\right]\leq&\frac{6\xi_2Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\\&\times\int_0^\mu\mathbb{E}|\beta^1(\ell)-\beta^0(\ell)|^2\,d\ell\\\leq&\frac{6\xi_2Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\\&\times\int_0^\mu\mathbb{E}\sup_{0\leq v\leq\ell}|\beta^1(v)-\beta^0(v)|^2\,d\ell\\\leq&\frac{6\xi_2Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\mathcal{Z}\mu.\end{aligned}$$

Likewise, we get

$$\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^3(\ell)-\beta^2(\ell)|^2\right]\leq\mathcal{Z}\left[\frac{6\xi_2Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\right]\frac{\mu^2}{n!}.$$

Consequently, ∀ $n \geq 0$, we assert that

$$\mathbb{E}\left[\sup_{0\leq\ell\leq\mu} |\beta^{n+1}(\ell) - \beta^{n}(\ell)|^2\right] \leq \mathcal{Z}\frac{\Psi^n \mu^n}{n!}, \tag{9}$$

where

$$\mathcal{Z} = \frac{6\xi_1 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5)(1 + \mathbb{E}\|\sigma\|^2)\chi$$
$$\Psi = \frac{6\xi_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5).$$

Using the mathematical induction, we confirm that (9) is true for any $n \geq 0$. At n = 0, it has been established. Assume (9) is true for some $n \geq 0$. Using the same reasoning as before, we arrive at

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq\ell\leq\mu} |\beta^{n+2}(\ell) - \beta^{n+1}(\ell)|^2\right] &\leq \frac{6\xi_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5)\\ &\quad\times \int_0^{\mu} \mathbb{E}|\beta_s^{n+1} - \beta_\ell^{n})|^2\, d\ell\\ &\leq \Psi \int_0^{\mu} \mathbb{E}\Big[\sup_{0\leq v\leq\ell} |\beta^{n+1}(v) - \beta^{n}(v)|^2\Big] d\ell\\ &\leq \Psi \int_0^{\mu} \mathcal{Z}\frac{\Psi^n\mu^n}{n!} d\ell\\ &\leq \mathcal{Z}\frac{\Psi^{n+1}\mu^{n+1}}{(n+1)!}.\end{aligned} \tag{10}$$

This indicates that for $n+1$, (9) is true. Consequently, induction (9) is true for every $n \geq 0$. Lemma 2.3 grants us:

$$\begin{aligned}\hat{c}\left\{\sup_{0\leq\ell\leq\chi} |\beta^{n+1}(\ell) - \beta^{n}(\ell)|^2 > \frac{1}{2^n}\right\} &\leq 2^n \mathbb{E}\left[\sup_{0\leq\ell\leq\chi} |\beta^{n+1}(\ell) - \beta^{n}(\ell)|^2\right]\\ &\leq \mathcal{Z}\,[2\Psi\mu]^n \,\frac{1}{n!}.\end{aligned}$$

For nearly every ω, a positive integer $n_0 = n_0(\omega)$ exists since $\sum_{n=0}^{\infty} \frac{k[2\Psi\mu]^n}{n!} < \infty$, according to the Borel-Cantelli lemma,

$$\sup_{0\leq\mu\leq\chi} |\beta^{n+1}(\mu) - \beta^{n}(\mu)|^2 \leq \frac{1}{2^n}, \quad \text{as} \quad n \geq n_0. \tag{11}$$

It indicates that, quasi surely, the partial sums

$$\beta^0(\mu) + \sum_{j=0}^{n-1}\left[\beta^{j+1}(\mu) - \beta^{j}(\mu)\right] = \beta^n(\mu),$$

the sequence exhibits uniform convergence on $\mu \in (-\infty, \chi]$, with $\beta(\mu)$ representing the limit. Therefore, $\beta^n(\mu)$ converges uniformly to $\beta(\mu)$ on the interval $\mu \in (-\infty, \chi]$. This guarantees that $\beta(\mu)$ is F_μ-adapted and cȁdlȁg. Moreover, from equation (9), it is clear that the sequence $\{\beta^n(\mu) : n \geq 1\}$ forms a Cauchy sequence in L_G^2. Consequently, $\beta^n(\mu)$ converges to $\beta(\mu)$ in L_G^2, which implies that

$$\mathbb{E}\left[|\beta^n(\mu) - \beta(\mu)|^2\right] \to 0, \quad \text{as} \quad n \to \infty.$$

By letting $n \to \infty$ in equation (8), we obtain

$$\mathbb{E}\Big[\sup_{0\le\ell\le\chi}|\beta(\ell)|^2\Big] \le 7\left[(1+\frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}}. \tag{12}$$

After that, we must confirm that $\beta(\mu)$ meets equation (3). With assumption (H_2) in mind and applying the same reasoning as before, we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Big[\vartheta_1(\ell,\beta_\ell^n)-\vartheta_1(\ell,\beta_\ell)\Big]d\ell\Big|^2\Big]\\
&+\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Big[\vartheta_2(\ell,\beta_\ell^n)-\vartheta_2(\ell,\beta_\ell)\Big]d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\Big|^2\Big]\\
&+\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Big[\Lambda_1(\ell,\beta_\ell^n)-\Lambda_1(\ell,\beta_\ell)\Big]d\mathscr{B}(\ell)\Big|^2\Big]\\
&+\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Big[\Lambda_2(\ell,\beta_\ell^n)-\Lambda_2(\ell,\beta_\ell)\Big]d\mathscr{B}^H(\ell)\Big|^2\Big]\\
&+\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{1}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\int_{\mathfrak{R}_0^d}\Big[F(\ell,\beta_{\ell^-}^n,q)-F(\ell,\beta_{\ell^-},q)\Big]\mathcal{L}(d\ell,dq)\Big|^2\Big]\\
&+\mathbb{E}\Big[\sup_{0\le\ell\le\chi}\Big|\frac{\delta}{\Gamma(\jmath)}\int_0^\mu(\mu-\ell)^{\jmath-1}\Big[\Pi(\ell,\beta_\ell^n)-\Pi(\ell,\beta_\ell)\Big]d\ell\Big|^2\Big]\\
&\le\frac{Q\mu}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[|\vartheta_1(\ell,y_\ell)-\vartheta_1(\ell,\beta_\ell)|^2\Big]d\ell\\
&\quad+\frac{m_1Q\mu}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[|\vartheta_2(\ell,y_\ell)-\vartheta_2(\ell,\beta_\ell)|^2\Big]d\ell\\
&\quad+\frac{m_2Q}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[|\Lambda_1(\ell,y_\ell)-\Lambda_1(\ell,\beta_\ell)|^2\Big]d\ell\\
&\quad+\frac{m_3Q}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[|\Lambda_2(\ell,y_\ell)-\Lambda_2(\ell,\beta_\ell)|^2\Big]d\ell\\
&\quad+\frac{m_4Q}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\int_{\mathbb{R}_0^d}\mathbb{E}\Big[|F(\ell,y_\ell,q)-F(\ell,\beta_\ell,q)|^2\Big]v(dq)d\ell\\
&\quad+\frac{m_5Q}{(1-2\jmath)\,\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[|\Pi(\ell,y_\ell)-\Pi(\ell,\beta_\ell)|^2\Big]d\ell\\
&\le\frac{\xi_2Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[\sup_{0\le v\le\ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
&\quad+\frac{\xi_2m_1Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[\sup_{0\le v\le\ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
&\quad+\frac{\xi_2m_2Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[\sup_{0\le v\le\ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
&\quad+\frac{\xi_2m_3Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\mathbb{E}\Big[\sup_{0\le v\le\ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
&\quad+\frac{\xi_2m_4Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^\mu\int_{\mathbb{R}_0^d}\mathbb{E}\Big[\sup_{0\le v\le\ell}|\beta^n(v)-\beta(v)|^2\Big]v(dq)d\ell
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi_2 m_5 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le v\le \ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
\le\ &\frac{\xi_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi+m_1\chi+m_2+m_3+m_4+m_5)\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le v\le \ell}|\beta^n(v)-\beta(v)|^2\Big]d\ell\\
&\to 0\quad \text{as}\quad n\to\infty. \qquad (13)
\end{aligned}
$$

Alternatively, we say

$$
\begin{aligned}
\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta_\ell^n) &\to \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta_\ell)\quad \text{in } L_G^2,\\
\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta_\ell^n)\,d\langle\mathscr{B},\mathscr{B}\rangle(\ell) &\to \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta_\ell)\,d\langle\mathscr{B},\mathscr{B}\rangle(\ell),\quad \text{in } L_G^2,\\
\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta_\ell^n)\,d\mathscr{B}^H(\ell) &\to \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta_\ell)\,d\mathscr{B}^H(\ell),\quad \text{in } L_G^2,\\
\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta_\ell^n)\,d\mathscr{B}(\ell) &\to \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta_\ell)\,d\mathscr{B}(\ell),\quad \text{in } L_G^2,\\
\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\int_{\Re_0^d}F(\ell,\beta_{\ell-}^n,q)\,\mathcal{L}(d\ell,dq) &\to \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\int_{\Re_0^d}F(\ell,\beta_{\ell-},q)\,\mathcal{L}(d\ell,dq)\,,\text{in } L_G^2,\\
\frac{\delta}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta_\ell^n) &\to \frac{\delta}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta_\ell)\quad \text{in } L_G^2.
\end{aligned}
$$

With the limits $n\to\infty$ in (7), we have for $\mu\in[0,\chi]$,

$$
\begin{aligned}
\lim_{n\to\infty}\beta^n(\mu) = \sigma(0) &+ \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\lim_{n\to\infty}\vartheta_1(\ell,\beta_\ell^{n-1})d\ell + \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\lim_{n\to\infty}\vartheta_2(\ell,\beta_\ell^{n-1})d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\lim_{n\to\infty}\Lambda_1(\ell,\beta_\ell^{n-1})d\mathscr{B}(\ell) + \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\lim_{n\to\infty}\Lambda_2(\ell,\beta_\ell^{n-1})d\mathscr{B}^H(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\int_{\Re_0^d}\lim_{n\to\infty}F(\ell,\beta_{\ell^-}^{n-1},q)\mathcal{L}(d\ell,dq)\\
&+\frac{\delta}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\lim_{n\to\infty}\Pi(\ell,\beta_\ell^{n-1})d\ell,
\end{aligned}
$$

this leads to

$$
\begin{aligned}
\beta(\mu) = \sigma(0) &+ \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_1(\ell,\beta_\ell)d\ell + \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\vartheta_2(\ell,\beta_\ell)d\langle\mathscr{B},\mathscr{B}\rangle(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_1(\ell,\beta_\ell)d\mathscr{B}(\ell) + \frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Lambda_2(\ell,\beta_\ell)d\mathscr{B}^H(\ell)\\
&+\frac{1}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\int_{\Re_0^d}F(\ell,\beta_{\ell^-},q)\mathcal{L}(d\ell,dq)\\
&+\frac{\delta}{\Gamma(\jmath)}\int_0^{\mu}(\mu-\ell)^{\jmath-1}\Pi(\ell,\beta_\ell)d\ell, \qquad (14)
\end{aligned}
$$

μ falls between 0 and χ. This demonstrates that $\beta(\mu)$ is the answer to (3). Let us consider the case where equation (3) yields two distinct solutions, $\beta(\mu)$ and $y(\mu)$, in order to demonstrate unicity/uniqueness. Based on related reasoning, we arrive at

$$
\mathbb{E}\left[\sup_{0\le\ell\le\mu}|y(\ell)-\beta(\ell)|^2\right] \le \frac{6Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\vartheta_1(\ell,y_\ell)-\vartheta_1(\ell,\beta_\ell)|^2 d\ell
$$

$$+ \frac{6m_1 Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\vartheta_2(\ell, y_\ell) - \vartheta_2(\ell, \beta_\ell)|^2 d\ell$$
$$+ \frac{6m_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\Lambda_1(\ell, y_\ell) - \Lambda_1(\ell, \beta_\ell)|^2 d\ell$$
$$+ \frac{6m_3 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\Lambda_2(\ell, y_\ell) - \Lambda_2(\ell, \beta_\ell)|^2 d\ell$$
$$+ \frac{6m_4 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\int_{\Re_0^d}\mathbb{E}|F(\ell, y_\ell, q) - F(\ell, \beta_\ell, q)|^2 v(dq)d\ell$$
$$+ \frac{6m_5 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\Pi(\ell, y_\ell) - \Pi(\ell, \beta_\ell)|^2 d\ell.$$

Due to assumption (H_2), we can obtain

$$\mathbb{E}\left[\sup_{0\le\ell\le\mu}|y(\ell) - \beta(\ell)|^2\right] \le \frac{6\xi_2 Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell$$
$$+ \frac{6\xi_2 m_1 Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell$$
$$+ \frac{6\xi_2 m_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell$$
$$+ \frac{6\xi_2 m_3 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell$$
$$+ \frac{6\xi_2 m_4 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\int_{\Re_0^d}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell$$
$$+ \frac{6\xi_2 m_5 Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell.$$

Where $\mu \in [0, \chi]$, so

$$\mathbb{E}\left[\sup_{0\le\ell\le\mu}|y(\ell) - \beta(\ell)|^2\right] \le \frac{6\xi_2 Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5)\int_0^{\mu}\mathbb{E}\Big[\sup_{0\le u\le\ell}|y(u) - \beta(u))|^2\Big]d\ell.$$

It is evident that

$$\sup_{-\infty<u\le\chi}|y(u)|^2 \le |\sigma|^2 + \sup_{0<u\le\chi}|y(u)|^2,$$

and by leveraging the Grownwall inequality with identical initial conditions, the conclusion can be derived,

$$\mathbb{E}\Big[\sup_{-\infty<\ell\le\mu}|y(\ell) - \beta(\ell)|^2\Big] = 0, \tag{15}$$

this shows that $\beta(\mu) = y(\mu)$ holds quasi-surely for all $\mu \in (-\infty, \chi]$. To conclude, the error estimation needs to be verified. Using equations (5) and (7), along with the same reasoning as before, we obtain the following results

$$\mathbb{E}\left[\sup_{0\le\ell\le\mu}|\beta^n(\ell) - \beta(\ell)|^2\right] \le \frac{6Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\vartheta_1(\ell, \beta^n(\ell)) - \vartheta_1(\ell, \beta_\ell)|^2 d\ell$$
$$+ \frac{6m_1 Q\mu}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\mathbb{E}|\vartheta_2(\ell, \beta^n(\ell)) - \vartheta_2(\ell, \beta_\ell)|^2 d\ell$$

$$
\begin{aligned}
&+ \frac{6m_2Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu} \mathbb{E}|\Lambda_1(\ell,\beta^n(\ell)) - \Lambda_1(\ell,\beta_\ell)|^2 d\ell \\
&+ \frac{6m_3Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu} \mathbb{E}|\Lambda_2(\ell,\beta^n(\ell)) - \Lambda_2(\ell,\beta_\ell)|^2 d\ell \\
&+ \frac{6m_4Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu}\int_{\Re_0^d} \mathbb{E}|F(\ell,\beta^n(\ell),q) - F(\ell,\beta_\ell,q)|^2 v(dq) d\ell \\
&+ \frac{6m_5Q}{(1-2\jmath)\Gamma(\jmath)^2}\int_0^{\mu} \mathbb{E}|\Pi(\ell,\beta^n(\ell)) - \Pi(\ell,\beta_\ell)|^2 d\ell \\
\leq\ & \frac{6\xi_2Q}{(1-2\jmath)\Gamma(\jmath)^2}(\chi + m_1\chi + m_2 + m_3 + m_4 + m_5) \\
&\times \int_0^{\mu} \mathbb{E}\Big[\sup_{0\leq v\leq \ell} |\beta^n(v) - \beta(v))|^2\Big] d\ell \\
\leq\ & \Psi \int_0^{\mu} \mathbb{E}\Big[\sup_{0\leq v\leq \ell} |\beta^n(v) - \beta^{n-1}(v)|^2\Big] d\ell \\
&+ \Psi \int_0^{\mu} \mathbb{E}\Big[\sup_{0\leq v\leq \ell} |\beta^{n-1}(v) - \beta(v)|^2\Big] d\ell.
\end{aligned}
$$

Considering (9), we get

$$
\mathbb{E}\left[\sup_{0\leq \ell\leq \mu} |\beta^n(\ell) - \beta(\ell)|^2\right] \leq \mathcal{Z}\frac{\Psi^n\mu^n}{n!} + \Psi\int_0^{\mu} \mathbb{E}\Big[\sup_{0\leq v\leq \ell} |\beta^{n-1}(v) - \beta(v)|^2\Big] d\ell.
$$

Utilizing Grownwall inequality

$$
\mathbb{E}\left[\sup_{0\leq \ell\leq \mu} |\beta^n(\ell) - \beta(\ell)|^2\right] \leq \mathcal{Z}\frac{\Psi^n\mu^n}{n!} e^{\Psi\mu}.
$$

This provides the error estimate between the exact solution $\beta(\mu)$ and the Picard approximate solutions $\beta^n(\mu)$, for $n \geq 0$, of problem (3). □

4. Estimating Exponential Behavior with GLP

Assuming there is a single solution $\beta(\mu)$ to problem (3) for $\mu \in [0,\infty)$, we can display the exponential estimates. Here is how we now calculate the exponential estimate for (3).

Theorem 4.1. *Assuming the validity of conditions* (H_1) *and* (H_2)*, the following inequality holds:*

$$
\lim_{n\to\infty} \sup \frac{1}{\mu} \log|\beta(\mu)| \leq \frac{7}{2}\Big(\frac{\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}\Big),
$$

where $\mathcal{M} = \chi + m_1\chi + m_2 + m_3 + m_4 + m_5$ *with* m_i *representing positive values.*

Proof. We know from claim (12) that

$$
\mathbb{E}\Big[\sup_{0\leq \ell\leq \chi} |\beta(\ell)|^2\Big] \leq 7\left[(1 + \frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right] e^{\frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}}. \tag{16}
$$

Considering (16), for every $\Im = 1, 2, \ldots,$ we obtain

$$
\mathbb{E}\Big[\sup_{\Im-1\leq \mu\leq \Im} |\beta(\ell)|^2\Big] \leq 7\left[(1 + \frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right] e^{\frac{\xi_1\Lambda Q\Im}{(1-2\jmath)\Gamma(\jmath)^2}}.
$$

Applying Lemma 2.3 to every $\varepsilon > 0$, we obtain

$$\hat{c}\left\{w : \sup_{\mathfrak{I}-1\leq\mu\leq\mathfrak{I}} |\beta(\mu)|^2 > e^{\left(\frac{7\xi_1 Q\Lambda}{(1-2\jmath)\Gamma(\jmath)^2}+\varepsilon\right)\mathfrak{I}}\right\} \leq \frac{\mathbb{E}\left[\sup_{\ell-1\leq\mu\leq\ell} |\beta(\mu)|^2\right]}{e^{\left(\frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}+\varepsilon\right)\mathfrak{I}}}$$
$$\leq 7\left[(1 + \frac{2\xi_1\mathcal{M}Q\chi)}{(1-2\jmath)\Gamma(\jmath)^2}\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{-\varepsilon\mathfrak{I}}.$$

It is evident that the series

$$\sum_{\mathfrak{I}=1}^{\infty} 7\left[\left(1 + \frac{2\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right)\mathbb{E}\|\sigma\|^2 + \frac{\xi_1\mathcal{M}Q\chi}{(1-2\jmath)\Gamma(\jmath)^2}\right]e^{-\varepsilon\mathfrak{I}}$$

converges. As a result, invoking the Borel-Cantelli lemma implies that for almost every outcome $\omega \in \Omega$, there exists a random index $\mathfrak{I}_0 = \mathfrak{I}_0(\omega)$ such that

$$\sup_{\mathfrak{I}-1\leq\mu\leq\mathfrak{I}} |\beta(\mu)|^2 \leq e^{\left(\frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}+\varepsilon\right)\mathfrak{I}}, \quad \text{as} \quad \mathfrak{I} \geq \mathfrak{I}_0.$$

This means that for $\mathfrak{I} > \mathfrak{I}_0$ and $\mathfrak{I} - 1 \leq \mu \leq \mathfrak{I}$, we obtain

$$|\beta(\mu)| \leq e^{\frac{1}{2}\left(\frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2}+\varepsilon\right)\mathfrak{I}}.$$

Therefore

$$\lim_{\mu\to\infty}\sup \frac{1}{\mu}\log|\beta(\mu)| \leq \frac{1}{2}\Big(\frac{7\xi_1 Q\mathcal{M}}{(1-2\jmath)\Gamma(\jmath)^2} + \varepsilon\Big).$$

Since ε is arbitrary, the intended phrase becomes the result. □

5. Simulation Analysis

To clarify the main findings, illustrations are presented.

Example 5.1. *Examine the scalar fractional stochastic differential equation governed by a GLP*

$${}^{c}D^{\jmath}\beta_\mu = \beta_\mu + \beta_\mu d\langle\mathscr{B}\rangle_\mu + \sin(\beta_\mu)d\mathscr{B}_\mu + \cos(\beta_\mu)d\mathscr{B}^H_\mu + \int_{q\geq1}\beta_\mu\mathcal{L}(d\mu, dq) + \beta_\mu. \tag{17}$$

Let $\beta(0) = \sigma(0)$, and assume that the G-Lévy measure is defined by $\nu(dq) = \frac{dq}{(1+|q|^2)}$. The stochastic differential equation in question (17) will have a unique solution if it satisfies the conditions (H_1) and (H_2).

For this scenario, we define the following functions:

$$\vartheta_1(\mu,\beta_\mu) = \beta_\mu, \quad \vartheta_2(\mu,\beta_\mu) = \beta_\mu, \quad \Lambda_1(\mu,\beta_\mu) = \sin(\beta_\mu), \quad \Lambda_2(\mu,\beta_\mu) = \cos(\beta_\mu),$$
$$F(\mu,\beta_\mu,q) = \beta_\mu, \quad \Pi(\mu,\beta_\mu) = \beta_\mu.$$

Next, we proceed to establish the growth condition for this model.

$$|\vartheta_1(\mu,\beta(\mu))|^2 + |\vartheta_2(\mu,\beta(\mu))|^2 + |\Lambda_1(\mu,\beta(\mu))|^2 + |\Pi(\mu,\beta(\mu))|^2 + \int_{c\geq1}|F(\mu,\beta(\mu),c)|^2\,\nu(dc) + |\Lambda_2(\mu,\beta(\mu))|^2$$

$$= |\beta_\mu|^2 + |\beta_\mu|^2 + |\sin(\beta_\mu)|^2 + |\beta_\mu|^2 + \int_{c\geq1}|\beta_\mu|^2\,\nu(dq) + |\cos(\beta_\mu)|^2$$

$$
\begin{aligned}
&\leq 5|\beta|^2 + \int_{c\geq 1} |\beta|^2 \frac{dq}{1+|q|^2} \\
&\leq \left(5+\frac{\pi}{2}\right) + \left(5+\frac{\pi}{2}\right)|\beta|^2 \\
&= \left(5+\frac{\pi}{2}\right)\left(1+|\beta|^2\right).
\end{aligned}
$$

To verify the Lipschitz property, we analyze the boundedness of the differences between the system components with respect to their arguments.

$$
\begin{aligned}
&|\vartheta_1(\mu,y)-\vartheta_1(\mu,\beta)|^2 + |\vartheta_2(\mu,y)-\vartheta_2(\mu,\beta)|^2 + |\Lambda_1(\mu,y)-\Lambda_1(\mu,\beta)|^2 + |\Pi(\mu,y)-\Pi(\mu,\beta)|^2 \\
&\quad + \int_{q\geq 1} |F(\mu,y,q)-F(\mu,\beta,q)|^2 \, \nu(dq) + |\Lambda_2(\mu,y)-\Lambda_2(\mu,\beta)|^2 \\
&\leq 5\,|\beta_\mu - y_\mu|^2 + \int_{q\geq 1} |\beta_\mu - y_\mu|^2 \, \nu(dq) \\
&= 5\,|\beta_\mu - y_\mu|^2 + \int_{q\geq 1} |\beta_\mu - y_\mu|^2 \frac{dq}{1+|q|^2} \\
&\leq \left(5+\frac{\pi}{2}\right)|\beta_\mu - y_\mu|^2.
\end{aligned}
$$

Consequently, the scalar fractional **SDE** (17) *guarantees a unique solution.*

The problem (17) *also satisfied*

$$
\mathbb{E}\left[\sup_{0\leq \ell\leq \mu} |\beta^n(\ell)-\beta(\ell)|^2\right] \leq \mathcal{Z}\frac{\Psi^n \mu^n}{n!} e^{\Psi\mu},
$$

the error estimate between the exact solution $\beta(\mu)$ *and the Picard approximate solutions* $\beta^n(\mu)$*, for* $n \geq 0$.

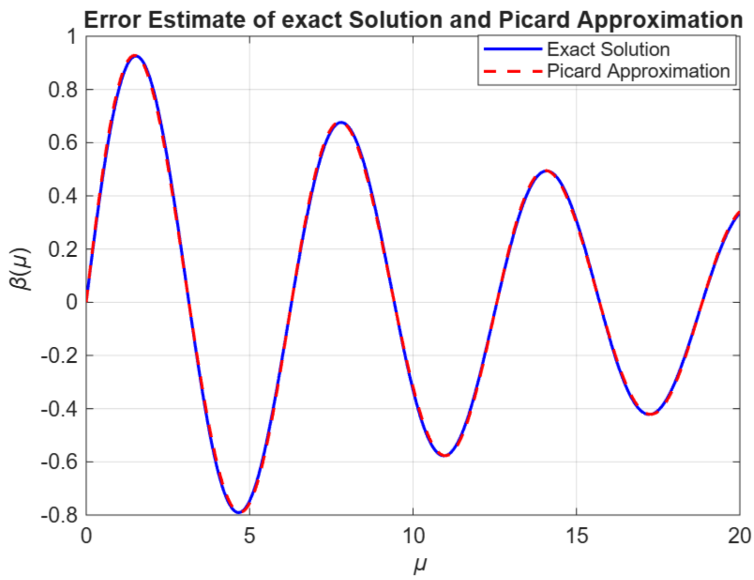

Figure 1: Simulation of Exact and Picard Solution

μ	Exact solution	Picard Approximation
0.0000	0.00000	0.00000
1.3378	0.91002	0.91843
2.6756	0.39306	0.37901
4.0134	-0.62630	-0.61018
5.3512	-0.61436	-0.62901
6.6890	0.28252	0.29291
8.0268	0.65946	0.65492
9.3645	0.03769	0.03618
10.702	-0.56060	-0.55412
12.040	-0.27511	-0.28462
13.378	0.37158	0.38181
14.716	0.40115	0.39238
16.054	-0.15179	-0.14607
17.391	-0.41648	-0.41835
18.729	-0.04711	-0.04898

Table 1: Table of Computed Values for the Exact and Picard Approximate Solutions

Example 5.2. *Consider the scalar fractional stochastic differential equation governed by a GLP*

$$^{c}D^{J}\beta_\mu = \frac{1}{2}\beta_\mu + \tanh(\beta_\mu)d\langle\mathscr{B}\rangle_\mu + \arctan(\beta_\mu)d\mathscr{B}_\mu + \frac{\beta_\mu}{1+\beta_\mu^2}d\mathscr{B}_\mu^H + \int_{q\geq 1}\frac{\beta_\mu}{1+\beta_\mu^2}\mathcal{L}(d\mu, dq) + \sin(\beta_\mu). \tag{18}$$

Let $\beta(0) = \sigma(0)$, and assume that the G-Lévy measure is given by

$$\nu(dq) = \frac{dq}{1+|q|^2}, \qquad q \geq 1.$$

For this model, we define the following functions:

$$\vartheta_1(\mu, \beta_\mu) = \frac{1}{2}\beta_\mu, \qquad \vartheta_2(\mu, \beta_\mu) = \tanh(\beta_\mu),$$

$$\Lambda_1(\mu, \beta_\mu) = \arctan(\beta_\mu), \qquad \Lambda_2(\mu, \beta_\mu) = \frac{\beta_\mu}{1+\beta_\mu^2},$$

$$F(\mu, \beta_\mu, q) = \frac{\beta_\mu}{1+\beta_\mu^2}, \qquad \Pi(\mu, \beta_\mu) = \sin(\beta_\mu).$$

Next, we proceed to establish the growth condition for this model.

$$|\vartheta_1(\mu, \beta(\mu))|^2 + |\vartheta_2(\mu, \beta(\mu))|^2 + |\Lambda_1(\mu, \beta(\mu))|^2 + |\Pi(\mu, \beta(\mu))|^2 + \int_{c\geq 1}|F(\mu, \beta(\mu), c)|^2\nu(dc) + |\Lambda_2(\mu, \beta(\mu))|^2$$

$$= \tfrac{1}{4}|\beta_\mu|^2 + |\tanh(\beta_\mu)|^2 + |\arctan(\beta_\mu)|^2 + |\sin(\beta_\mu)|^2 + \int_{c\geq 1}\left|\tfrac{\beta_\mu}{1+\beta_\mu^2}\right|^2\nu(dq) + \left|\tfrac{\beta_\mu}{1+\beta_\mu^2}\right|^2$$

$$= \left(\frac{5}{2} + \frac{\pi^2}{4} + \frac{\pi}{8}\right)\left(1 + |\beta_\mu|^2\right).$$

To verify the Lipschitz property, we analyze the boundedness of the differences between the system components with respect to their arguments.

$$|\vartheta_1(\mu, y) - \vartheta_1(\mu, \beta)|^2 + |\vartheta_2(\mu, y) - \vartheta_2(\mu, \beta)|^2 + |\Lambda_1(\mu, y) - \Lambda_1(\mu, \beta)|^2 + |\Pi(\mu, y) - \Pi(\mu, \beta)|^2$$

$$+\int_{q\geq 1}|F(\mu,y,q)-F(\mu,\beta,q)|^2\,\nu(dq)+|\Lambda_2(\mu,y)-\Lambda_2(\mu,\beta)|^2$$
$$\leq \Big(\frac{17}{4}+\frac{\pi}{2}\Big)|\beta_\mu-y_\mu|^2.$$

Consequently, the scalar fractional **SDE** (18) guarantees a unique solution.

The problem (18) also satisfied

$$\mathbb{E}\left[\sup_{0\leq\ell\leq\mu}|\beta^n(\ell)-\beta(\ell)|^2\right]\leq \mathcal{Z}\frac{\Psi^n\mu^n}{n!}e^{\Psi\mu},$$

the error estimate between the exact solution $\beta(\mu)$ and the Picard approximate solutions $\beta^n(\mu)$, for $n\geq 0$.

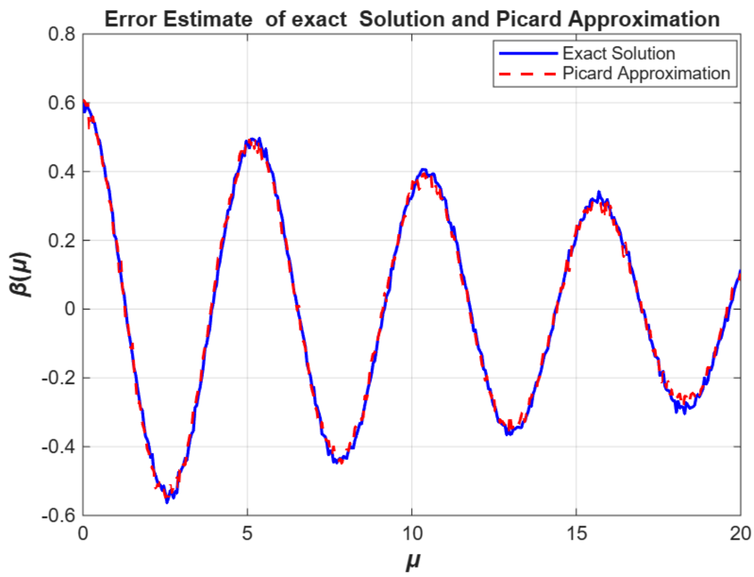

Figure 2: Simulation for Exact and Picard Solution.

μ	Exact solution	Picard Approximation
0.000	0.59876	0.60848
2.005	-0.39613	-0.41793
4.010	0.05483	0.07174
6.015	0.28146	0.25584
8.020	-0.43383	-0.40823
10.025	0.33137	0.37059
12.030	-0.11526	-0.11774
14.035	-0.13173	-0.11268
16.040	0.28437	0.29837
18.045	-0.30092	-0.26168

Table 2: Table of Computed Values for the Exact and Picard Approximate Solutions

6. Conclusion

This work establishes a comprehensive study of Caputo-type fractional stochastic differential equations driven by G-Lévy jumps, demonstrating the existence and uniqueness of solutions and providing a clear understanding of the system's behavior. Exponential estimates were derived to describe the long-term dynamics of the solutions, offering valuable insights into their evolution over time. Additionally, a detailed error analysis comparing exact solutions with numerical approximations obtained through the Picard iterative method highlights the accuracy and reliability of the results.

Directions for future work

Building on the results and methodology presented in this paper, several promising avenues can be explored to further advance the study of Caputo fractional stochastic differential equations driven by G-Lévy noise.

Extension to Alternative Fractional-Order Operators

Examine the applicability of the theoretical results to alternative fractional derivatives, including Caputo-Fabrizio, Prabhakar, and Hilfer operators. Conducting comparative analyses across these different fractional frameworks could offer deeper insights into the mathematical structures and physical interpretations associated with each type of fractional derivative.

Hybrid Systems with Random Switching

One may consider introducing random or Markovian switching mechanisms, resulting in hybrid stochastic fractional systems. These models are particularly relevant in the study of cyber-physical systems and offer a more complex and rich dynamical behavior for analysis.

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