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On the strong Slater condition of linear systems with an evenly convex constraint set

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Abstract. This paper deals with the stability of the intersection of a given evenly convex set X with the solution set of a given linear system σ whose coefficients can be arbitrarily perturbed. More precisely, we focus on the set of strong Slater points of σ in X, by analyzing firstly the case when X is a closed convex set, and then the more general case when X is evenly convex. We shall establish dual characterizations for the aforementioned set of strong Slater points by following well-known characterizations of the solution set of a linear system. Finally, we apply our results to analyze the consistency of systems with both strict and weak inequalities defined by lower semicontinuous convex functions.

1. Introduction

Stability in optimization is a fundamental concept that analyzes whether certain elements related to a problem change when small perturbations are introduced into the model's data. This analysis is crucial in both theoretical and practical contexts, since in many real-world applications (such as economics, engineering, or artificial intelligence) the data are often subject to uncertainty or noise.

In the seventies, Robinson [19] developed the first systematic studies on stability in linear systems, analyzing how small perturbations in the data affected the solution set. Since then, numerous authors have studied stability conditions for the feasible set [3, 4, 12, 13], the optimal value and the optimal set [5], the uniqueness of optimal solution [16] or even the boundary of the feasible set [10] of an optimization problem.

In general, the study of stability of a linear system allows us to determine if the system remains consistent (or inconsistent) under sufficiently small perturbations (due to either computing or measurement errors) of the data. However, in most cases it is necessary to consider that some of the constraints of such a system cannot be perturbed (e.g., the sign constraints). In such a case, it is useful to consider an exact constraint set X which could represent either the solution set of the system formed by all the exact constraints or a discrete subset of $\mathbb{R}^n$ (as $\mathbb{Z}^n$ or $\{0,1\}^n$ in integer or Boolean programming).

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In this paper we shall consider a non-empty evenly convex set [7] $X \subset \mathbb{R}^n$ which can be expressed as

$$X := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\},$$

and a linear inequality system,

$$\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in L\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, W, S and L are arbitrary pairwise disjoint index sets (possibly infinite) such that $W \cup S \neq \emptyset$, $L \neq \emptyset$ and the coefficients are given by two functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$, being $T := W \cup S \cup L$ and, $a_t := a(t)$ and $b_t := b(t)$ for all $t \in T$. We denote by F and F^X the solution set of σ in $\mathbb{R}^n$ and X, respectively, that is,

$$F := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in L\}$$

and $F^X = F \cap X$. We say that σ is consistent (with respect to X) if $F^X \neq \emptyset$.

In a more general context, with X an arbitrary non-empty set and σ possibly containing equality constraints, [1] and [2] analyze the effect on F^X of small changes in the coefficients of σ and generalize different stability concepts studied in [11] and [14] for the classical setting, where $X = \mathbb{R}^n$ and σ contains only inequality constraints.

If $X \subset \mathbb{R}^n$ is an arbitrary non-empty exact constraint set and the system σ in (1) is consistent (i.e., $F^X := F \cap X \neq \emptyset$), it is said that σ is *stably consistent* when it remains consistent under sufficiently small perturbations of the data (see [1] and [11] for details about how the size of perturbations is measured).

In the classical context, where $X = \mathbb{R}^n$, [11, Theorem 3.1] establishes six characterizations of the stably consistent systems, including the so-called *strong Slater condition*. It is said that σ in (1) satisfies the strong Slater condition if there exist $\overline{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $\langle a_t, \overline{x}\rangle + \varepsilon \leq b_t$ for all $t \in L$ and, in such a case, $\overline{x}$ is called a *strong Slater point* of σ. We shall denote by F_{SS} the set of all the strong Slater points of σ. Properties and geometry of the set F_{SS} have been recently studied in [22], where it is also obtained a dual characterization for the non-emptiness of this set (i.e., for the fulfillment of the strong Slater condition) in terms of the data of σ, following similar characterizations provided for the solution set of a system.

In this paper, we analyze the stability of F^X for the particular case in which X is an evenly convex set (i.e., the intersection of a family of open halfspaces). More precisely, we focus on the set of strong Slater points of σ in X, by analyzing firstly the case when X is a closed convex set, and then the more general case when X is evenly convex.

The paper is organized as follows. Section 2 contains the necessary notation and some basic results on convex as well as evenly convex sets to be used later. In addition, several results about the relationship between stable consistency and the strong Slater condition, obtained in previous works, are gathered. In Section 3, we analyze the strong Slater condition for linear systems with a closed convex constraint set, obtaining dual characterizations, for both the solution set and the set of strong Slater points, in terms of the data of σ and X. Moreover, we analyze conditions under which the set of strong Slater points of σ in X is contained in a closed/open halfspace. Section 4 is devoted to extend the results in Section 3 to the case in which the constraint set is evenly convex. Finally, Section 5 presents an application to systems with both strict and weak inequalities defined by lower semicontinuous convex functions.

2. Preliminaries

We begin this section by introducing the notation and basic definitions used throughout the paper. As usual in convex analysis (see, e.g., [17, 20]), for a non-empty set $A \subset \mathbb{R}^n$ we denote by $\operatorname{conv} A$, $\operatorname{cone} A$, $\operatorname{aff} A$ and $\dim A$ the convex hull of A, the convex cone generated by A and the origin, the affine hull of A and the dimension of $\operatorname{aff} A$, respectively. We consider $\operatorname{cone} \emptyset := \{0_n\}$ where 0_n is the zero vector in $\mathbb{R}^n$. By $\mathbb{R}_+$ and $\mathbb{R}_{++}$ we denote the sets of non-negative and positive real numbers, respectively, being $\mathbb{R}_+ A := \{\lambda x : \lambda \geq 0,\ x \in A\}$ and $\mathbb{R}_{++} A := \{\lambda x : \lambda > 0,\ x \in A\}$ cones in $\mathbb{R}^n$ with $0_n \in \mathbb{R}_+ A$. The smallest convex cone containing $A \cup \{0_n\}$ is $\operatorname{cone} A = \mathbb{R}_+ \operatorname{conv} A$. For an index set T, the space of generalized finite sequences, $\mathbb{R}^{(T)}$, is the linear space of those functions $\lambda : T \to \mathbb{R}$ whose support, $\operatorname{supp}\lambda := \{t \in T : \lambda_t \neq 0\}$, is finite. The convex cone of the non-negative generalized finite sequences is denoted by $\mathbb{R}_+^{(T)}$. From the topological side, given $A \subset \mathbb{R}^n$, we denote by $\operatorname{cl} A$, $\operatorname{int} A$ and $\operatorname{rint} A$, the closure, the interior and the relative interior of A, respectively. The Minkowski sum for sets is denoted in the usual way.

A set $A \subset \mathbb{R}^n$ is said to be *evenly convex* (see [7]) if it is the intersection of some family, possibly empty, of open halfspaces. The *evenly convex hull* of a set $A \subset \mathbb{R}^n$, denoted by $\operatorname{eco} A$, is the smallest evenly convex set which contains A, or equivalently, it is the intersection of all the open halfspaces containing A. From the definition, given $\overline{x} \in \mathbb{R}^n$, $\overline{x} \notin \operatorname{eco} A$ if and only if there exists $z \in \mathbb{R}^n$ such that $\langle z, x - \overline{x}\rangle < 0$ for all $x \in A$. In particular,

$$0_n \notin \operatorname{eco} A \quad \text{if and only if} \quad \{\langle x, z\rangle < 0, x \in A\} \text{ is consistent.} \tag{2}$$

Further properties of the evenly convex hull operator are collected in [6, Chapter 1].

The next proposition recovers some results about the convex hull and the evenly convex hull of the sum of certain sets. More precisely, statements (*i*), (*ii*) and (*iii*) can be found in [23, Theorem 4.12], [15, Lemma 2.1] and [15, Proposition 2.2], respectively.

Proposition 2.1. *Let A, B be non-empty sets in $\mathbb{R}^n$. Then the following statements hold:*

(*i*) $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$.

(*ii*) $\operatorname{conv}(A + \mathbb{R}_+B) = \operatorname{conv} A + \operatorname{cone} B$.

(*iii*) $\operatorname{eco}(A + \mathbb{R}_+B) = \operatorname{eco}(\operatorname{conv} A + \operatorname{cone} B)$.

Given two subsets $C, D \subset \mathbb{R}^n$, C is said to be *openly separated from* D [18] if there exists an open halfspace $H = \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$ such that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$. Analogously, C is said to be *closedly separated from* D if there exists a closed halfspace $H = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$ such that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$. Thus, C is openly separated from D if and only if D is closedly separated from C. Moreover, it is easy to prove that, if D is a cone containing the origin, then C is openly separated from D if and only if there exists $a \in \mathbb{R}^n$ such that $\langle a, x\rangle < \langle a, y\rangle$ for all $x \in C, y \in D$ (equivalently, $\langle a, x\rangle < 0 \leq \langle a, y\rangle$ for all $x \in C, y \in D$).

Proposition 2.2. *Let $C, D \subset \mathbb{R}^n$. Then, $0_n \notin \operatorname{eco}(C + \mathbb{R}_+D)$ if and only if $-C$ is openly separated from $\mathbb{R}_+D$.*

Proof. As $\mathbb{R}_+D$ is a cone containing the origin, $-C$ is openly separated from $\mathbb{R}_+D$ if and only if there exists $a \in \mathbb{R}^n$ such that $\langle a, -x\rangle < \langle a, y\rangle$ for all $x \in C, y \in \mathbb{R}_+D$. Equivalently, we have that $\langle -a, x + y\rangle < 0$, for all $x \in C, y \in \mathbb{R}_+D$ and, by (2), this condition characterizes $0_n \notin \operatorname{eco}(C + \mathbb{R}_+D)$. □

As a consequence of Propositions 2.1 and 2.2, one has that $0_n \notin \operatorname{eco}(C + \mathbb{R}_+D)$ if and only if $-\operatorname{conv} C$ is openly separated from $\operatorname{cone} D$. A further straightforward consequence of Proposition 2.2 is the following result.

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^{n+1}$. Then,*

$$0_{n+1} \notin \operatorname{eco}\Big((0_n, 1) + \mathbb{R}_+\Omega\Big) \text{ if and only if } (0_n, -1) \notin \operatorname{cl}\operatorname{cone}\Omega.$$

Proposition 2.4. *Let A, B, C be non-empty sets in $\mathbb{R}^n$. Then,*

$$0_n \notin \operatorname{eco}\Big((A + \mathbb{R}_+C) \cup B\Big) \text{ if and only if } 0_n \notin \operatorname{eco}\Big((A \cup B) + \mathbb{R}_+C\Big).$$

Proof. ($\Leftarrow$) Clearly, $A + \mathbb{R}_+C \subset (A \cup B) + \mathbb{R}_+C$ and $B \subset B + \mathbb{R}_+C \subset (A \cup B) + \mathbb{R}_+C$. Hence, $(A + \mathbb{R}_+C) \cup B \subset (A \cup B) + \mathbb{R}_+C$ and so, $\operatorname{eco}\Big((A + \mathbb{R}_+C) \cup B\Big) \subset \operatorname{eco}\Big((A \cup B) + \mathbb{R}_+C\Big)$. This shows that $0_n \notin \operatorname{eco}\Big((A \cup B) + \mathbb{R}_+C\Big)$ implies $0_n \notin \operatorname{eco}\Big((A + \mathbb{R}_+C) \cup B\Big)$.

($\Rightarrow$) Assume that $0_n \notin \operatorname{eco}\Big((A + \mathbb{R}_+C) \cup B\Big)$. By (2), there exists $z \in \mathbb{R}^n$ such that $\langle z, x\rangle < 0$ for all $x \in (A + \mathbb{R}_+C) \cup B$. Particularly, one has $\langle z, b\rangle < 0$ for all $b \in B$, and $\langle z, a + \delta c\rangle < 0$ for all $a \in A$, $c \in C$ and $\delta \geq 0$. It easily follows that $\langle z, c\rangle \leq 0$ for all $c \in C$. Then, $\langle z, b + \delta c\rangle < 0$ for all $b \in B$, $c \in C$ and $\delta \geq 0$, and so $\langle z, x\rangle < 0$ for all $x \in (A \cup B) + \mathbb{R}_+C$. This means, again in virtue of (2), that $0_n \notin \operatorname{eco}\Big((A \cup B) + \mathbb{R}_+C\Big)$. □

Now we recall two polar operators in the literature. For a proper set $A \subset \mathbb{R}^n$, we consider

$$A^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 0,\ \forall x \in A\},$$
$$A^e := \{y \in \mathbb{R}^n : \langle x, y\rangle < 0,\ \forall x \in A\},$$

assuming that $\mathbb{R}^n$ is the polar in the first sense of $\{0_n\}$ (and in the second sense of $\emptyset$), and conversely. The set A° is a closed convex cone containing the origin, while A^e is an evenly convex cone omitting the origin. Furthermore, one has $A^{\circ\circ} = \operatorname{cl}\operatorname{cone} A$, and so $A = A^{\circ\circ}$ if and only if A is a closed convex cone. In the same way, one has $A = A^{ee}$ if and only if A is an evenly convex cone omitting the origin.

We shall denote by F and F_{SS}, the solution set and the set of strong Slater points of σ in (1), respectively, that is,

$$F := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in L\},$$
$$F_{SS} := \{x \in \mathbb{R}^n : \exists\, \varepsilon > 0, \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in L\}.$$

Obviously, F is a closed convex set. Concerning the geometry of F_{SS}, it has been recently studied in [22] joint with further characterizations and properties of the set of strong Slater points. Moreover, for a consistent system σ as in (1), [22, Proposition 3.5] establishes the equivalence between the strong Slater condition and a geometrical condition depending on the coefficients of σ, recovering the following result which can be found in [11, Theorem 3.1] and [1, Lemma 1].

Proposition 2.5. *Given a consistent system σ as in* (1)*, the following statements are equivalent to each other:*

(i) *σ is stably consistent.*

(ii) *σ satisfies the strong Slater condition.*

(iii) $0_{n+1} \notin \operatorname{cl}\operatorname{conv}\{(a_t, b_t), t \in L\}$.

Given a non-empty set $X \subset \mathbb{R}^n$, the system σ in (1) is said to satisfy the *strong Slater condition with respect to X* if there exist $\overline{x} \in X$ (called *strong Slater point*) and $\overline{\varepsilon} > 0$ such that $\langle a_t, \overline{x}\rangle + \overline{\varepsilon} \leq b_t$ for all $t \in L$. We shall denote by F_{SS}^X the set of strong Slater points with respect to X, having that $F_{SS}^X = F_{SS} \cap X$. Analogously, $F_{SS}^{\operatorname{cl} X} = F_{SS} \cap \operatorname{cl} X$. From this definition, one has the following set containments:

$$\begin{array}{ccccc} F^X & \subset & F^{\operatorname{cl} X} & \subset & F \\ \cup & & \cup & & \cup \\ F_{SS}^X & \subset & F_{SS}^{\operatorname{cl} X} & \subset & F_{SS} \end{array} \tag{3}$$

In this context, in [1] the authors introduce a new geometrical condition, the so-called *G-consistency*, in order to extend condition (iii) in Proposition 2.5 to the general case. It is said that σ is G-consistent if the condition

$$0_{n+1} \notin \operatorname{cl}\Big(K(X) + \operatorname{conv}\{(a_t, b_t), t \in L\}\Big)$$

holds, where $K(X) := \Big\{(\omega, \gamma) \in \mathbb{R}^{n+1} : \langle \omega, x\rangle \leq \gamma, \forall x \in X\Big\}$ is a closed convex cone (the so-called *weak dual cone of X*, when X is a closed convex set [8]). The following result, obtained from [1], establishes the relationships between the stable consistency, the G-consistency and the strong Slater condition in this general setting.

Proposition 2.6. *If $F^X \neq \emptyset$, then the following statements hold:*

(i) *If σ is stably consistent, then σ is G-consistent.*

(ii) *If σ is G-consistent and X is convex, then $F_{SS}^{\operatorname{cl} X} \neq \emptyset$.*

(iii) *If $F_{SS}^X \neq \emptyset$, then σ is stably consistent.*

In the particular case that X is a closed convex set, it can be expressed as the solution set of a certain linear system. For this case, the following characterization is obtained as a consequence of Proposition 2.6.

Corollary 2.7. *Let* $X := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W\}$. *If* $F^X \neq \emptyset$, *then the following statements are equivalent to each other:*

(*i*) σ *is stably consistent.*

(*ii*) $0_{n+1} \notin \operatorname{cl}(\operatorname{cone}\{(a_t, b_t), t \in W\} + \operatorname{conv}\{(a_t, b_t), t \in L\})$.

(*iii*) $F_{SS}^X \neq \emptyset$.

Proof. We just need to show that the G-consistency of σ is equivalent to statement (*ii*). Since $X = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W\}$, the nonhomogeneous Farkas Lemma for linear semi-infinite systems (see, e.g., [14, Corollary 3.1.2]) establishes that

$$K(X) = \operatorname{cl}\operatorname{cone}\{(a_t, b_t), t \in W; (0_n, 1)\}.$$

Then, since $\operatorname{cone}\{(a_t, b_t), t \in W\} \subset K(X)$, the G-consistency of σ implies condition (*ii*). Now, we prove the converse implication. Indeed, if

$$0_{n+1} \in \operatorname{cl}\Big(K(X) + \operatorname{conv}\{(a_t, b_t), t \in L\}\Big) = \operatorname{cl}\Big(\operatorname{cone}\{(a_t, b_t), t \in W; (0_n, 1)\} + \operatorname{conv}\{(a_t, b_t), t \in L\}\Big),$$

then there exist sequences $\{\mu^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(W)}$, $\{\gamma^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(L)}$ with $\sum_{t\in L}\lambda_t^k = 1$ for every $k \in \mathbb{N}$, such that $0_{n+1} = \lim\limits_{k\to\infty} \sum_{t\in W}\mu_t^k(a_t, b_t) + \sum_{t\in L}\lambda_t^k(a_t, b_t) + \gamma^k(0_n, 1)$. Taking an arbitrary $\overline{x} \in F^X$, one gets

$$0 = \langle 0_{n+1}, (\overline{x}, -1)\rangle = \lim_{k\to\infty} \sum_{t\in W}\mu_t^k(\langle a_t, \overline{x}\rangle - b_t) + \sum_{t\in L}\lambda_t^k(\langle a_t, \overline{x}\rangle - b_t) - \gamma^k.$$

Therefore, $\lim\limits_{k\to\infty}\sum_{t\in W}\mu_t^k(\langle a_t, \overline{x}\rangle - b_t) = 0$, $\lim\limits_{k\to\infty}\sum_{t\in L}\lambda_t^k(\langle a_t, \overline{x}\rangle - b_t) = 0$ and $\lim\limits_{k\to\infty}\gamma^k = 0$. Hence,

$$0_{n+1} = \lim_{k\to\infty}\sum_{t\in W}\mu_t^k(a_t, b_t) + \sum_{t\in L}\lambda_t^k(a_t, b_t)$$

and so,

$$0_{n+1} \in \operatorname{cl}(\operatorname{cone}\{(a_t, b_t), t \in W\} + \operatorname{conv}\{(a_t, b_t), t \in L\}),$$

which concludes the proof. □

Observe that we can recover Proposition 2.5 from the above result when $X = \mathbb{R}^n$ (taking $(a_t, b_t) = 0_{n+1}$ for all $t \in W$).

3. Strong Slater condition with a closed convex constraint set

In this section we are interested in the analysis of the strong Slater condition of the linear system $\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in L\}$ introduced in (1) with a closed convex constraint set of the form

$$X := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W\}. \tag{4}$$

In other words, we are interested on the set

$$F_{SS}^X = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W;\ \exists\,\varepsilon > 0, \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in L\}.$$

Our objective is to obtain the counterpart for F_{SS}^X of the well-known characterizations of F^X, the solution set of σ with respect to X.

We firstly observe that F^X is nothing else that the solution set of the linear system $\{\langle a_t, x\rangle \le b_t,\ t \in T\}$, where $T = L \cup W$. Throughout the paper, in order to shorten the notation, we associate to this linear system the sets

$$C_T := \{(a_t, b_t), t \in T\} \text{ and}$$
$$D := C_T \cup \{(0_n, 1)\},$$

which allow to define two prominent cones in linear semi-infinite programming (see, e.g., [14]) associated to that system, say the *second order moment cone*, defined by $N := \operatorname{cone} C_T$, and the *characteristic cone*, defined by $K := \operatorname{cone} D$, having that $K = N + \mathbb{R}_+\{(0_n, 1)\}$. Applying the classical polarity operator to D, we get

$$D^\circ = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + b_t x_{n+1} \le 0, t \in T;\ x_{n+1} \le 0\}.$$

Next result summarizes several characterizations of the set F^X of solutions of σ with respect to the closed convex set X in (4).

Theorem 3.1 (Characterizations of F^X). *For $\overline{x} \in \mathbb{R}^n$, one has*

$\overline{x} \in F^X$ *if and only if* $(\overline{x}, -1) \in D^\circ$.

Furthermore, the following statements are equivalent:

(i) *σ is consistent with respect to X (i.e., $F^X \neq \emptyset$).*

(ii) *There exists $(x, x_{n+1}) \in D^\circ$ such that $x_{n+1} \neq 0$.*

(iii) $(0_n, -1) \notin \operatorname{cl} K$.

(iv) $(0_n, -1) \notin \operatorname{cl} N$.

(v) $0_{n+1} \notin \operatorname{eco}\Big((0_n, 1) + \mathbb{R}_+ C_T\Big)$.

Proof. The first statement and the equivalence between (i), (ii) and (iii) follow from [22, Theorem 3.1] applied to the system $\{\langle a_t, x\rangle \le b_t,\ t \in T\}$. The equivalence between (iii) and (iv) follows from [14, Lemma 4.1], whereas the equivalence between (iv) and (v) follows from Corollary 2.3. □

Now, we associate to F^X_{SS} the sets

$$\mathcal{C} := C^0_W \cup C^{-1}_L \cup \{(0_n, 1, -1)\} \text{ and}$$
$$\mathcal{D} := \mathcal{C} \cup \{(0_n, 0, 1)\},$$

where $C^0_W := C_W \times \{0\} = \{(a_t, b_t, 0), t \in W\}$ and $C^{-1}_L := C_L \times \{-1\} = \{(a_t, b_t, -1), t \in L\}$. In this case, one has

$$\begin{aligned}\mathcal{D}^\circ = \{(x, x_{n+1}, x_{n+2}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : &\langle a_t, x\rangle + b_t x_{n+1} \le 0,\ t \in W;\\ &\langle a_t, x\rangle + b_t x_{n+1} - x_{n+2} \le 0,\ t \in L;\\ &x_{n+1} - x_{n+2} \le 0;\ x_{n+2} \le 0\}.\end{aligned}$$

Inspired by [21, 22] and proceeding similarly as in former lines, we will consider the following cones associated to F^X_{SS}, say $\mathcal{N} := \operatorname{cone} \mathcal{C}$ and $\mathcal{K} := \operatorname{cone} \mathcal{D}$, having that $\mathcal{K} = \mathcal{N} + \mathbb{R}_+\{(0_n, 0, 1)\}$. Now we are in a position to establish corresponding characterizations of the set F^X_{SS} of the strong Slater points of σ with respect to the closed convex set X in (4).

Theorem 3.2 (Characterizations of F^X_{SS}). *For $\overline{x} \in \mathbb{R}^n$, one has*

$$\overline{x} \in F^X_{SS} \text{ if and only if } (\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}^\circ \text{ for some } \overline{\varepsilon} \in]0, 1]. \tag{5}$$

Furthermore, the following statements are equivalent:

(*i*) *σ satisfies the strong Slater condition with respect to X (i.e., $F_{SS}^X \neq \emptyset$).*

(*ii*) *There exists $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$ such that $x_{n+2} \neq 0$.*

(*iii*) $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{K}$.

(*iv*) $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{N}$.

(*v*) $0_{n+2} \notin \operatorname{eco}\left((0_n, 0, 1) + \mathbb{R}_+ C\right)$.

Proof. The first statement in (5) easily follows from the definition of F_{SS}^X and the expression of $\mathcal{D}^\circ$.

[(*i*) ⇔ (*ii*)] It is a straightforward consequence of (5).

[(*ii*) ⇔ (*iii*)] Assume that $x_{n+2} = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$. In this case, since

$$\langle (0_n, 0, -1), (x, x_{n+1}, x_{n+2}) \rangle = 0 \leq 0,$$

for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$, then one has that $(0_n, 0, -1) \in \operatorname{cl} \mathcal{K} = \mathcal{D}^{\circ\circ}$. Conversely, if $(0_n, 0, -1) \in \operatorname{cl} \mathcal{K} = \mathcal{D}^{\circ\circ}$, then

$$-x_{n+2} = \langle (0_n, 0, -1), (x, x_{n+1}, x_{n+2}) \rangle \leq 0$$

for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$. Since $x_{n+2} \leq 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$, one concludes that $x_{n+2} = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$.

Finally, the equivalence between (*iii*) and (*iv*) follows from [14, Lemma 4.1], whereas the equivalence between (*iv*) and (*v*) follows from Corollary 2.3. □

Observe that whenever $X = \{x \in \mathbb{R}^n : \langle 0_n, x \rangle \leq 0\} = \mathbb{R}^n$, Theorem 3.2 reduces to [22, Theorem 3.3]. Furthermore, we are not explicitly assuming in Theorem 3.2 the non-emptiness of F^X to characterize the existence of strong Slater points within X (the non-emptiness of F_{SS}^X), although it is a condition which is implicit as we can see in the following result.

Proposition 3.3. *Let X be a closed convex set as in* (4). *The following statements are equivalent:*

(*i*) *$F_{SS}^X \neq \emptyset$.*

(*ii*) *$F^X \neq \emptyset$ and $0_{n+1} \notin \operatorname{cl} \operatorname{conv}(C_L + \mathbb{R}_+ C_W)$.*

Proof. [(*i*) ⇒ (*ii*)] If $F_{SS}^X \neq \emptyset$, then $F^X \neq \emptyset$ by (3). Now assume, on the contrary, that $0_{n+1} \in \operatorname{cl} \operatorname{conv}(C_L + \mathbb{R}_+ C_W) = \operatorname{cl}(\operatorname{conv} C_L + \operatorname{cone} C_W)$. Then, there exist sequences $\{\mu^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(W)}$, $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(L)}$ with $\sum_{t\in L} \lambda_t^k = 1$ for every $k \in \mathbb{N}$, such that $0_{n+1} = \lim_{k\to\infty} \sum_{t\in W} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t)$. Hence, one can write

$$(0_n, 0, -1) = \lim_{k\to\infty} \sum_{t\in W} \mu_t^k (a_t, b_t, 0) + \sum_{t\in L} \lambda_t^k (a_t, b_t, -1),$$

and so $(0_n, 0, -1) \in \operatorname{cl} \mathcal{N}$. This implies by Theorem 3.2 that $F_{SS}^X = \emptyset$ and so, a contradiction.

[(*ii*) ⇒ (*i*)] If $F_{SS}^X = \emptyset$, then $(0_n, 0, -1) \in \operatorname{cl} \mathcal{N}$ by Theorem 3.2. Hence, there exist sequences $\{\mu^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(W)}$, $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(L)}$ and $\{\delta^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$(0_n, 0, -1) = \lim_{k\to\infty} \sum_{t\in W} \mu_t^k (a_t, b_t, 0) + \sum_{t\in L} \lambda_t^k (a_t, b_t, -1) + \delta^k (0_n, 1, -1). \tag{6}$$

If $\{\delta^k\}$ is unbounded, then we may assume $\lim_{k\to\infty} \delta^k = +\infty$. Hence, from (6), one has

$$0_{n+2} = \lim_{k\to\infty} \sum_{t\in W} (\delta^k)^{-1} \mu_t^k (a_t, b_t, 0) + \sum_{t\in L} (\delta^k)^{-1} \lambda_t^k (a_t, b_t, -1) + (0_n, 1, -1),$$

which implies

$$(0_n, -1) = \lim_{k\to\infty} \sum_{t\in W} (\delta^k)^{-1} \mu_t^k (a_t, b_t) + \sum_{t\in L} (\delta^k)^{-1} \lambda_t^k (a_t, b_t) \in \operatorname{cl} N$$

and so, by Theorem 3.1, $F^X = \emptyset$. Assume now that $\{\delta^k\}$ is bounded. Then, it contains a convergent subsequence and, for brevity, we write $\lim_{k\to\infty} \delta^k = \delta \geq 0$. From (6), we have

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in W} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t) + \delta^k (0_n, 1), \tag{7}$$

$$1 = \lim_{k\to\infty} \sum_{t\in L} \lambda_t^k + \delta^k. \tag{8}$$

Since $F^X \neq \emptyset$, for any $\overline{x} \in F^X$ one has

$$0 = \langle 0_{n+1}, (\overline{x}, -1) \rangle = \lim_{k\to\infty} \sum_{t\in W} \mu_t^k (\langle a_t, \overline{x} \rangle - b_t) + \sum_{t\in L} \lambda_t^k (\langle a_t, \overline{x} \rangle - b_t) - \delta^k.$$

Therefore, $\lim\limits_{k\to\infty} \sum_{t\in W} \mu_t^k (\langle a_t, \overline{x} \rangle - b_t) = 0$, $\lim\limits_{k\to\infty} \sum_{t\in L} \lambda_t^k (\langle a_t, \overline{x} \rangle - b_t) = 0$ and $\delta = \lim\limits_{k\to\infty} \delta^k = 0$. Taking this into account in (7) and (8), we get that $0_{n+1} = \lim\limits_{k\to\infty} \sum_{t\in W} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t)$ and $1 = \lim\limits_{k\to\infty} \sum_{t\in L} \lambda_t^k$. Since $\gamma^k := \sum_{t\in L} \lambda_t^k > 0$ for k large enough, then

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in W} (\gamma^k)^{-1} \mu_t^k (a_t, b_t) + \sum_{t\in L} (\gamma^k)^{-1} \lambda_t^k (a_t, b_t)$$

with $\sum_{t\in L} (\gamma^k)^{-1} \lambda_t^k = 1$, which shows that $0_{n+1} \in \operatorname{cl}(\operatorname{cone} C_W + \operatorname{conv} C_L)$. □

Observe that, by applying Proposition 2.1(*ii*) to condition (*ii*) in Proposition 3.3, we recover the equivalence between conditions (*ii*) and (*iii*) in Corollary 2.7.

As a consequence of Proposition 3.3 and Theorem 3.1, one has that $F_{SS}^X \neq \emptyset$ if and only if $(0_n, -1) \notin \operatorname{cl} N$ and $0_{n+1} \notin \operatorname{cl} \operatorname{conv}(C_L + \mathbb{R}_+ C_W)$, that is, in order to guarantee the fulfillment of the strong Slater condition with respect to a closed convex set, one has to check two conditions in the space $\mathbb{R}^{n+1}$. Theorem 3.2 shows that these two conditions are indeed equivalent to a unique condition in the space $\mathbb{R}^{n+2}$.

A weak or strict inequality is said to be a *consequence* (or a *consequent relation*) of the set F_{SS}^X provided that F_{SS}^X is contained in the (closed or open) halfspace defined by such an inequality.

Proposition 3.4 (Consequent weak relations of F_{SS}^X). *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^X is non-empty. Then, the following statements are equivalent:*

(*i*) *$\langle a, x \rangle \leq b$ is a consequence of F_{SS}^X.*

(*ii*) *$(a, b, 0) \in \operatorname{cl} \mathcal{K}$.*

(*iii*) *$(a, b) \in \operatorname{cl} K$.*

(*iv*) *$\langle a, x \rangle \leq b$ is a consequence of F^X.*

Proof. [(*i*) ⇒ (*ii*)] Assume that $\langle a, x \rangle \leq b$ is a consequence of F_{SS}^X, that is, $F_{SS}^X \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$. We shall prove that $(a, b, 0) \in \mathcal{D}^{\circ\circ} = \operatorname{cl} \mathcal{K}$. For that purpose, given $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$, we distinguish two cases:

- Case 1: $x_{n+1} \leq x_{n+2} < 0$. In this case, we see that $\left(\frac{x}{-x_{n+1}}, -1, \frac{x_{n+2}}{-x_{n+1}}\right) \in \mathcal{D}^\circ$ where $0 < \frac{x_{n+2}}{x_{n+1}} \leq 1$. According to Theorem 3.2, $\overline{x} := \frac{-1}{x_{n+1}} x \in F_{SS}^X$ and so, by assumption, $\langle a, \overline{x} \rangle \leq b$, that is, $\langle a, x \rangle + b x_{n+1} \leq 0$.

- Case 2: $x_{n+1} \leq x_{n+2} = 0$. For $\overline{x} \in F_{SS}^X$, let $\overline{\varepsilon} \in]0,1]$ be such that $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}^\circ$. Then,

$$(x^\lambda, x_{n+1}^\lambda, x_{n+2}^\lambda) := (1-\lambda)(x, x_{n+1}, 0) + \lambda(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}^\circ$$

  for all $\lambda \in]0,1[$. Since $x_{n+1}^\lambda \leq x_{n+2}^\lambda < 0$, the vector $(x^\lambda, x_{n+1}^\lambda, x_{n+2}^\lambda)$ corresponds to the Case 1. Thus,

$$(1-\lambda)(\langle a, x\rangle + bx_{n+1}) + \lambda(\langle a, \overline{x}\rangle - b) = \langle a, x^\lambda\rangle + bx_{n+1}^\lambda \leq 0$$

  for all $\lambda \in]0,1[$. By taking limits when $\lambda \to 0$, one has $\langle a, x\rangle + bx_{n+1} \leq 0$.

Since $\langle a, x\rangle + bx_{n+1} \leq 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}^\circ$, then $(a, b, 0) \in \mathcal{D}^{\circ\circ} = \operatorname{cl} \mathcal{K}$.

[$(ii) \Rightarrow (iii)$] It is a straightforward consequence of the definitions of the cones $\mathcal{K}$ and K.

[$(iii) \Rightarrow (iv)$] It follows from [14, Corollary 3.1.2].

[$(iv) \Rightarrow (i)$] It easily follows since $F_{SS}^X \subset F^X$. □

We observe from the above result that the set of all the weak inequalities which are consequence of F_{SS}^X, coincides with the weak dual cone of its closure, F^X. To see that $\operatorname{cl} F_{SS}^X = F^X$ whenever F_{SS}^X is non-empty, we firstly observe that

$$\operatorname{cl} F_{SS}^X = \operatorname{cl}(F_{SS} \cap X) \subset \operatorname{cl} F_{SS} \cap \operatorname{cl} X = F \cap X = F^X$$

since X is closed and $\operatorname{cl} F_{SS} = F$ (see [22, Lemma 2.1(i)]). Now, given $\overline{x} \in F^X$ and $\widetilde{x} \in F_{SS}^X$, it follows that $\overline{x} = \lim_{\lambda \downarrow 0} x^\lambda$ where $x^\lambda := \lambda\widetilde{x} + (1-\lambda)\overline{x} \in F_{SS}^X$ for all $\lambda \in]0,1[$. Thus, $\overline{x} \in \operatorname{cl} F_{SS}^X$.

Proposition 3.5 (Consequent strict relations of F_{SS}^X). *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^X is non-empty. Then, $\langle a, x\rangle < b$ is a consequence of F_{SS}^X if and only if*

$$(0_n, 0, -1) \in \operatorname{cl} \operatorname{cone}\left(\mathcal{D} \cup \{(-a, -b, 0)\}\right).$$

Proof. Since F_{SS}^X is non-empty, then the set containment $F_{SS}^X \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$ is equivalent to

$$\{x \in \mathbb{R}^n : \langle -a, x\rangle \leq -b;\ \langle a_t, x\rangle \leq b_t,\ t \in W;\ \exists\, \varepsilon > 0, \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in L\} = \emptyset,$$

and this is equivalent, by Theorem 3.2, to $(0_n, 0, -1) \in \operatorname{cl} \operatorname{cone}(\mathcal{D} \cup \{(-a, -b, 0)\})$. □

Proposition 3.6. *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^X is non-empty. If $(a, b, c) \in \operatorname{cl} \mathcal{K}$ for some $c < 0$, then $\langle a, x\rangle < b$ is a consequence of F_{SS}^X.*

Proof. Assume that $(a, b, c) \in \operatorname{cl} \mathcal{K}$ for some $c < 0$ and let $\overline{x} \in F_{SS}^X$. Since $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}^\circ$ for some $\overline{\varepsilon} \in]0,1]$ by Theorem 3.2, and $\operatorname{cl} \mathcal{K} = \mathcal{D}^{\circ\circ}$, then

$$\langle a, \overline{x}\rangle - b < \langle a, \overline{x}\rangle - b - c\overline{\varepsilon} = \langle (a, b, c), (\overline{x}, -1, -\overline{\varepsilon})\rangle \leq 0.$$

Hence, $\langle a, \overline{x}\rangle < b$, and so $F_{SS}^X \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$. □

As a consequence of the previous results, by considering the case $X = \mathbb{R}^n$ we recover [22, Propositions 3.6 and 3.7].

Corollary 3.7 (Consequent relations of F_{SS}). *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS} is non-empty. Then,*

(*i*) *$\langle a, x\rangle \leq b$ is a consequence of F_{SS} if and only if*

$$(a, b, 0) \in \operatorname{cl} \operatorname{cone}\{(a_t, b_t, -1), t \in L; (0_n, 1, -1); (0_n, 0, 1)\}.$$

(*ii*) *$\langle a, x\rangle < b$ is a consequence of F_{SS} if and only if*

$$(0_n, 0, -1) \in \operatorname{cl} \operatorname{cone}\{(-a, -b, 0); (a_t, b_t, -1), t \in L; (0_n, 1, -1); (0_n, 0, 1)\}.$$

4. Strong Slater condition with an evenly convex constraint set

In this section we are interested in the analysis of the strong Slater condition of the linear system $\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in L\}$ introduced in (1) with an evenly convex constraint set of the form

$$X := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W; \langle a_t, x\rangle < b_t,\ t \in S\}, \tag{9}$$

whose closure, according to [9, Proposition 1.1], is

$$\operatorname{cl} X = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W \cup S\}. \tag{10}$$

In other words, we are interested on the set

$$F_{SS}^X = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in W; \langle a_t, x\rangle < b_t,\ t \in S;\ \exists\, \varepsilon > 0, \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in L\}.$$

Proceeding as in Section 3, our objective is to obtain the counterpart for F_{SS}^X of the well-known characterizations of F^X, the solution set of σ with respect to X.

Let $T := L \cup W \cup S$. We firstly observe that F^X is nothing else that the solution set of the linear system $\{\langle a_t, x\rangle \leq b_t,\ t \in L \cup W; \langle a_t, x\rangle < b_t,\ t \in S\}$, whose consistency is equivalent to the consistency of the system

$$\{\langle a_s + \alpha a_v, x\rangle < b_s + \alpha b_v, (s, v, \alpha) \in S \times (L \cup W) \times \mathbb{R}_+\}. \tag{11}$$

We associate to this system the set $H := (C_S + \mathbb{R}_+ C_{L\cup W}) \cup \{(0_n, 1)\}$. Applying the e-polarity operator to H, we get

$$H^e = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + b_t x_{n+1} \leq 0,\ t \in L \cup W; \\ \langle a_s, x\rangle + b_s x_{n+1} < 0,\ s \in S;\ x_{n+1} < 0\}.$$

Theorem 4.1 (Characterizations of F^X). *For $\overline{x} \in \mathbb{R}^n$, one has*

$\overline{x} \in F^X$ *if and only if* $(\overline{x}, -1) \in H^e$.

Furthermore, the following statements are equivalent:

(*i*) *σ is consistent with respect to X (i.e., $F^X \neq \emptyset$).*

(*ii*) $H^e \neq \emptyset$.

(*iii*) $0_{n+1} \notin \operatorname{eco} H$.

(*iv*) $0_{n+1} \notin \operatorname{eco}(C_S + \mathbb{R}_+ C_{L\cup W})$ *and* $(0_n, -1) \notin \operatorname{cl}\operatorname{cone} C_T$.

(*v*) $0_{n+1} \notin \operatorname{eco}\Big((C_S \cup \{(0_n, 1)\}) + \mathbb{R}_+ C_{L\cup W}\Big)$.

Proof. The first statement and the equivalence between (*i*), (*ii*) and (*iii*) follow from [22, Theorem 3.2] applied to the system in (11). The equivalence between (*iii*) and (*iv*) follows from [15, Lemma 3.1], whereas the equivalence between (*iii*) and (*v*) follows from Proposition 2.4. □

Remark 4.2. Observe that statement (*iii*) above can be equivalently written as $0_{n+1} \notin \operatorname{eco} \mathbb{R}_{++} H$ as a consequence of (2). The set $\operatorname{eco} \mathbb{R}_{++} H$ is precisely the so-called *strict dual cone* (see [8]) of the solution set of the system in (11), provided that it is non-empty.

Now, we shall consider the set F_S of Slater points of σ in (1), that is,

$$F_S := \{x \in \mathbb{R}^n : \langle a_t, x\rangle < b_t,\ t \in L\}.$$

Obviously, F_S is an evenly convex set such that $F_{SS} \subset F_S \subset F$. Further geometric properties and dual characterizations for this set have been recently established in [22]. It is also natural to consider the set $F_S^X := F_S \cap X$. The following result, which is a straightforward consequence of Theorem 4.1, characterizes the consistency of F_S^X, which is equivalent to the consistency of the system

$$\{\langle a_v + \alpha a_w, x\rangle < b_v + \alpha b_w, (v, w, \alpha) \in (S \cup L) \times W \times \mathbb{R}_+\},$$

in terms of the set $\widetilde{H} := (C_{S\cup L} + \mathbb{R}_+ C_W) \cup \{(0_n, 1)\}$ defined from the coefficients of the system. In this case, we observe that

$$\begin{aligned}\widetilde{H}^e = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : &\langle a_t, x\rangle + b_t x_{n+1} \le 0,\ t \in W;\\ &\langle a_s, x\rangle + b_s x_{n+1} < 0,\ s \in S \cup L;\ x_{n+1} < 0\}.\end{aligned}$$

Corollary 4.3 (Characterizations of F_S^X). *For $\overline{x} \in \mathbb{R}^n$, one has*

$$\overline{x} \in F_S^X \text{ if and only if } (\overline{x}, -1) \in \widetilde{H}^e.$$

Furthermore, the following statements are equivalent:

(i) *σ satisfies the Slater condition with respect to X (i.e., $F_S^X \neq \emptyset$).*

(ii) $\widetilde{H}^e \neq \emptyset$.

(iii) $0_{n+1} \notin \operatorname{eco} \widetilde{H}$.

(iv) $0_{n+1} \notin \operatorname{eco}(C_{S\cup L} + \mathbb{R}_+ C_W)$ *and* $(0_n, -1) \notin \operatorname{cl}\operatorname{cone} C_T$.

(v) $0_{n+1} \notin \operatorname{eco}\left((C_{S\cup L} \cup \{(0_n, 1)\}) + \mathbb{R}_+ C_W\right)$.

Next, proceeding as in the previous section, we associate to F_{SS}^X the sets

$$\begin{aligned}&\mathcal{C} := C_W^0 \cup C_L^{-1} \cup \{(0_n, 1, -1)\},\\ &\mathcal{H} := (C_S^0 + \mathbb{R}_+ \mathcal{C}) \cup \{(0_n, 0, 1)\},\end{aligned}$$

where $C_S^0 := C_S \times \{0\} = \{(a_t, b_t, 0), t \in S\}$, $C_W^0 := C_W \times \{0\} = \{(a_t, b_t, 0), t \in W\}$ and $C_L^{-1} := C_L \times \{-1\} = \{(a_t, b_t, -1), t \in L\}$. In this case, one has

$$\begin{aligned}\mathcal{H}^e = \{(x, x_{n+1}, x_{n+2}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : &\langle a_t, x\rangle + b_t x_{n+1} < 0,\ t \in S;\\ &\langle a_t, x\rangle + b_t x_{n+1} \le 0,\ t \in W;\\ &\langle a_t, x\rangle + b_t x_{n+1} - x_{n+2} \le 0,\ t \in L;\\ &x_{n+1} - x_{n+2} \le 0;\ x_{n+2} < 0\}.\end{aligned}$$

Theorem 4.4 (Characterizations of F_{SS}^X). *For $\overline{x} \in \mathbb{R}^n$, one has*

$$\overline{x} \in F_{SS}^X \text{ if and only if } (\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{H}^e \text{ for some } \overline{\varepsilon} \in]0, 1]. \tag{12}$$

Furthermore, the following statements are equivalent:

(i) *σ satisfies the strong Slater condition with respect to X (i.e., $F_{SS}^X \neq \emptyset$).*

(ii) $\mathcal{H}^e \neq \emptyset$.

(iii) $0_{n+2} \notin \operatorname{eco} \mathcal{H}$.

(iv) $0_{n+2} \notin \operatorname{eco}(C_S^0 + \mathbb{R}_+ \mathcal{C})$ *and* $(0_n, 0, -1) \notin \operatorname{cl}\operatorname{cone}(C_S^0 \cup \mathcal{C})$.

(*v*) $0_{n+2} \notin \operatorname{eco}\Big((C_S^0 \cup \{(0_n, 0, 1)\}) + \mathbb{R}_+ C\Big)$.

Proof. The equivalence in (12) easily follows from the definition of F_{SS}^X and the expression of $\mathcal{H}^e$.

[(*i*) ⇔ (*ii*)] It is a straightforward consequence of (12).

[(*ii*) ⇔ (*iii*)] Let $(x, x_{n+1}, x_{n+2}) \in \mathcal{H}^e$. If $0_{n+2} \in \operatorname{eco}\mathcal{H} \subset \mathcal{H}^{ee}$, then

$$\langle (0_n, 0, 0), (x, x_{n+1}, x_{n+2}) \rangle < 0,$$

meaning $0 < 0$ which is impossible. Thus, $0_{n+2} \notin \operatorname{eco}\mathcal{H}$. Conversely, if $0_{n+2} \notin \operatorname{eco}\mathcal{H}$, by (2), there exists (x, x_{n+1}, x_{n+2}) such that $\langle (x, x_{n+1}, x_{n+2}), (u, v, w) \rangle < 0$ for all $(u, v, w) \in \mathcal{H}$. Hence, $(x, x_{n+1}, x_{n+2}) \in \mathcal{H}^e$ and so $\mathcal{H}^e \neq \emptyset$.

[(*iii*) ⇔ (*iv*)] It follows from [15, Lemma 3.1].

[(*iii*) ⇔ (*v*)] It is a consequence of Proposition 2.4. □

According to Remark 4.2, statements (*i*) to (*v*) in the above Theorem are also equivalent to the condition $0_{n+2} \notin \operatorname{eco}\mathbb{R}_{++}\mathcal{H}$.

Proposition 4.5. *Let X be an evenly convex set as in* (9)*. The following statements are equivalent:*

(*i*) $F_{SS}^X \neq \emptyset$.

(*ii*) $F^X \neq \emptyset$ *and* $0_{n+1} \notin \operatorname{cl}\operatorname{conv}(C_L + \mathbb{R}_+ C_{W\cup S})$.

(*iii*) $F^X \neq \emptyset$ *and* $0_{n+1} \notin \operatorname{cl}(\operatorname{conv}(C_L \cup \{(0_n, 1)\}) + \operatorname{cone} C_{W\cup S})$.

Proof. [(*i*) ⇒ (*ii*)] If $F_{SS}^X \neq \emptyset$, then $F^X \neq \emptyset$ and $F_{SS}^{\operatorname{cl} X} \neq \emptyset$ by (3). As X is a non-empty evenly convex set defined as in (9), by [9, Proposition 1.1], we have

$$\operatorname{cl} X := \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t,\ t \in W \cup S\}, \tag{13}$$

and by applying Proposition 3.3, $F_{SS}^{\operatorname{cl} X} \neq \emptyset$ implies $0_{n+1} \notin \operatorname{cl}\operatorname{conv}(C_L + \mathbb{R}_+ C_{W\cup S})$.

[(*ii*) ⇒ (*iii*)] Assume that $F^X \neq \emptyset$ and $0_{n+1} \in \operatorname{cl}(\operatorname{conv}(C_L \cup \{(0_n, 1)\}) + \operatorname{cone} C_{W\cup S})$. Then, there exist sequences $\{\mu^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(W\cup S)}$, $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(L)}$ and $\{\delta^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $\sum_{t\in L} \lambda_t^k + \delta^k = 1$ for every $k \in \mathbb{N}$, such that $0_{n+1} = \lim\limits_{k\to\infty} \sum_{t\in W\cup S} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t) + \delta^k (0_n, 1)$. As $\{\delta^k\}$ is bounded, then it contains a convergent subsequence and, for brevity, we write $\lim_{k\to\infty} \delta^k = \delta \geq 0$. Then we have

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in W\cup S} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t) + \delta(0_n, 1), \tag{14}$$

$$1 = \lim_{k\to\infty} \sum_{t\in L} \lambda_t^k + \delta. \tag{15}$$

If $\delta = 0$, then (14) and (15) turn into $0_{n+1} = \lim\limits_{k\to\infty} \sum_{t\in W\cup S} \mu_t^k (a_t, b_t) + \sum_{t\in L} \lambda_t^k (a_t, b_t)$ and $1 = \lim\limits_{k\to\infty} \sum_{t\in L} \lambda_t^k$. Since $\gamma^k := \sum_{t\in L} \lambda_t^k > 0$ for k large enough, then

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in W\cup S} (\gamma^k)^{-1} \mu_t^k (a_t, b_t) + \sum_{t\in L} (\gamma^k)^{-1} \lambda_t^k (a_t, b_t)$$

with $\sum_{t\in L} (\gamma^k)^{-1} \lambda_t^k = 1$, which shows that $0_{n+1} \in \operatorname{cl}(\operatorname{cone} C_{W\cup S} + \operatorname{conv} C_L) = \operatorname{cl}\operatorname{conv}(C_L + \mathbb{R}_+ C_{W\cup S})$ and (*ii*) fails.

If $\delta > 0$, then (14) becomes

$$(0_n, -1) = \lim_{k\to\infty} \sum_{t\in W\cup S} \delta^{-1} \mu_t^k (a_t, b_t) + \sum_{t\in L} \delta^{-1} \lambda_t^k (a_t, b_t),$$

which implies that $(0_n, -1) \in \operatorname{cl}\operatorname{cone} C_T$, being $T = L \cup W \cup S$. By applying Theorem 3.1, we obtain that $F^{\operatorname{cl} X} = \emptyset$, which contradicts $F^X \neq \emptyset$ and (*ii*) fails again.

[(*iii*) $\Rightarrow$ (*i*)] On the one hand, if $F^X \neq \emptyset$, by Theorem 4.1, one has $0_{n+1} \notin \operatorname{eco} H = \operatorname{eco}((C_S + \mathbb{R}_+ C_{L\cup W}) \cup \{(0_n, 1)\})$ and so, by (2), there exists $(u, v) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle (u, v), (x, x_{n+1})\rangle < 0$ for all $(x, x_{n+1}) \in H$. In particular, $\langle (u, v), (0_n, 1)\rangle = v < 0$ and, by letting $\widetilde{u} := -\frac{u}{v}$, one has that

$$\langle (\widetilde{u}, -1), (a_s, b_s) + \lambda(a_t, b_t)\rangle < 0, \tag{16}$$

for all $s \in S$, $t \in L \cup W$ and $\lambda \geq 0$. From (16), we obtain $\langle (\widetilde{u}, -1), (a_s, b_s)\rangle < 0$ for all $s \in S$ by letting $\lambda = 0$, and $\langle (\widetilde{u}, -1), (a_t, b_t)\rangle \leq 0$ for all $t \in L \cup W$, by taking limits when $\lambda \to \infty$ after dividing by λ.

On the other hand, if $0_{n+1} \notin \operatorname{cl}(\operatorname{conv}(C_L \cup \{(0_n, 1)\}) + \operatorname{cone} C_{W\cup S})$, then there exist $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $\langle (\alpha, \beta), (x, x_{n+1})\rangle \leq \gamma < \langle (\alpha, \beta), (0_n, 0)\rangle = 0$, for all $(x, x_{n+1}) \in \operatorname{conv}(C_L \cup \{(0_n, 1)\}) + \operatorname{cone} C_{W\cup S}$. Then, by taking $(\widetilde{\alpha}, \widetilde{\beta}) := -\frac{1}{\gamma}(\alpha, \beta)$, we have $\langle (\widetilde{\alpha}, \widetilde{\beta}), (x, x_{n+1})\rangle \leq -1 < 0$, for all $(x, x_{n+1}) \in \operatorname{conv}(C_L \cup \{(0_n, 1)\}) + \operatorname{cone} C_{W\cup S}$. In particular, we have $\langle (\widetilde{\alpha}, \widetilde{\beta}), (0_n, 1)\rangle = \widetilde{\beta} < 0$ and

$$\langle (\widetilde{\alpha}, \widetilde{\beta}), (a_l, b_l) + \mu(a_t, b_t)\rangle < 0, \tag{17}$$

for all $l \in L$, $t \in W \cup S$ and $\mu \geq 0$. In the same way that in the previous case, we can obtain that $\langle (\widetilde{\alpha}, \widetilde{\beta}), (a_l, b_l)\rangle < 0$ for all $l \in L$, and $\langle (\widetilde{\alpha}, \widetilde{\beta}), (a_t, b_t)\rangle \leq 0$ for all $t \in W \cup S$.

Now, by consider $(a, b, c) := (\widetilde{u}, -1, 0) + (\widetilde{\alpha}, \widetilde{\beta}, -1)$, it is easy to prove that

$$\langle (a, b, c), (x, x_{n+1}, x_{n+2})\rangle < 0$$

for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{H} = (C_S^0 + \mathbb{R}_+ C) \cup \{(0_n, 0, 1)\}$, so that $0_{n+2} \notin \operatorname{eco} \mathcal{H}$ by (2), and $F_{SS}^X \neq \emptyset$ by Theorem 4.4. □

Whenever X is a closed convex set (i.e., if $S = \emptyset$), then Proposition 3.3 follows as a consequence of Proposition 4.5.

Next, we observe that if a linear system σ is consistent with respect to an evenly convex set X, then it satisfies the strong Slater condition with respect to X if and only if it does with respect to its closure $\operatorname{cl} X$, i.e., if and only if σ is stably consistent (see Corollary 2.7).

Corollary 4.6. *Let X be an evenly convex set as in* (9). *If $F^X \neq \emptyset$, then $F_{SS}^X \neq \emptyset$ if and only if $F_{SS}^{\operatorname{cl} X} \neq \emptyset$.*

Proof. If $F^X \neq \emptyset$, by Proposition 4.5, we have the equivalence between $F_{SS}^X \neq \emptyset$ and

$$0_{n+1} \notin \operatorname{cl}\operatorname{conv}(C_L + \mathbb{R}_+ C_{W\cup S}). \tag{18}$$

Moreover, by (3), $F^X \neq \emptyset$ implies $F^{\operatorname{cl} X} \neq \emptyset$ which together with (18) is equivalent to $F_{SS}^{\operatorname{cl} X} \neq \emptyset$ by (10) and Proposition 3.3. □

Although, under the assumption $F^X \neq \emptyset$, the strong Slater conditions with respect to X and with respect to $\operatorname{cl} X$ are equivalent, the sets F_{SS}^X and $F_{SS}^{\operatorname{cl} X}$ may not coincide, as the following example illustrates.

Example 4.7. Let $X := \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 < 0, -x_2 \leq 0\}$ and $\sigma = \{x_1 + x_2 \leq t,\ t \in]1, 2]\}$.

It is easy to see that $F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1\}$. Since $(0, 0) \in F_{SS}$ and $F_{SS} \neq \emptyset$ implies

$$\operatorname{int} F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 < 1\} \subset F_{SS} \subset F$$

(see [22, Lemma 2.1]), we need to check whether the points in the line $r = \{(\alpha, 1 - \alpha) : \alpha \in \mathbb{R}\}$ are strong Slater points or not. For each $\alpha \in \mathbb{R}$, the points $(\alpha, 1 - \alpha)$ are not strong Slater points since, for every $\varepsilon > 0$, one has $1 + \varepsilon > t$ for $t \in]1, \min\{1 + \varepsilon, 2\}[$. Consequently, $F_{SS} = \operatorname{int} F$.

In this case, $\operatorname{cl} X = \mathbb{R}_+^2$, $F^{\operatorname{cl} X} = \operatorname{conv}\{(0, 0), (1, 0), (0, 1)\}$, $F_{SS}^{\operatorname{cl} X} = F^{\operatorname{cl} X} \backslash (\operatorname{conv}\{(1, 0), (0, 1)\})$ and $F_{SS}^X = F_{SS}^{\operatorname{cl} X} \backslash (\operatorname{conv}\{(0, 0), (0, 1)\})$, so that $F_{SS}^{\operatorname{cl} X} \neq F_{SS}^X$ (see Figure 1).

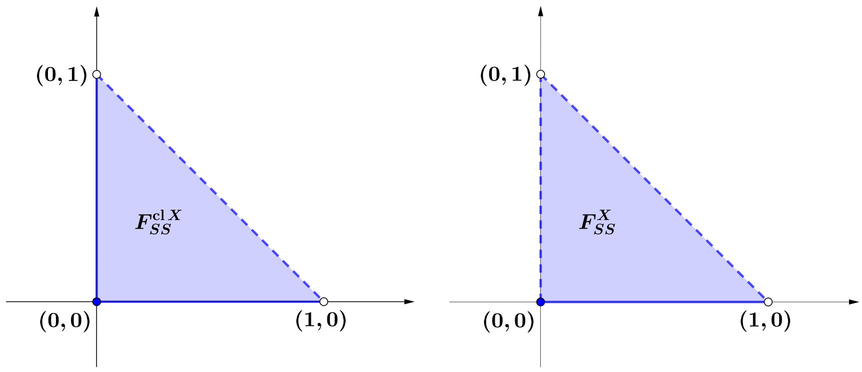

Figure 1: The sets $F_{SS}^{\mathrm{cl}\,X}$ and F_{SS}^{X} associated to σ in Example 4.7.

Now, we analyze the consequent relations of F_{SS}^{X} under the assumption of the even convexity of the constraint set X.

Proposition 4.8 (Consequent weak relations of F_{SS}^{X}). *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^{X} is non-empty. Then, $\langle a, x\rangle \leq b$ is a consequence of F_{SS}^{X} if and only if it is a consequence of $F_{SS}^{\mathrm{cl}\,X}$.*

The proof of this result is straightforward. We then refer to Proposicion 3.4 for further equivalent statements of consequent weak relations of F_{SS}^{X}.

Proposition 4.9 (Consequent strict relations of F_{SS}^{X}). *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^{X} is non-empty. Then, $\langle a, x\rangle < b$ is a consequence of F_{SS}^{X} if and only if*

$$0_{n+2} \in \operatorname{eco}\Big((C_S^0 \cup \{(0_n, 0, 1)\}) + \mathbb{R}_+(C \cup \{(-a, -b, 0)\})\Big).$$

Proof. Since F_{SS}^{X} is non-empty, then $\langle a, x\rangle < b$ is a consequence of F_{SS}^{X} (i.e., $F_{SS}^{X} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$) is equivalent to

$$\{x \in \mathbb{R}^n : \langle -a, x\rangle \leq -b;\ \langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S;\ \exists\, \varepsilon > 0, \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in L\} = \emptyset.$$

And this is equivalent, by Theorem 4.4, to $0_{n+2} \in \operatorname{eco}\Big((C_S^0 \cup \{(0_n, 0, 1)\}) + \mathbb{R}_+(C \cup \{(-a, -b, 0)\})\Big)$. □

Proposition 4.10. *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, and assume that F_{SS}^{X} is non-empty. If*

$$0_{n+1} \in \operatorname{eco}\Big((C_{S\cup L} + \mathbb{R}_+(C_W \cup \{(-a, -b)\})) \cup \{(0_n, 1)\}\Big), \tag{19}$$

then $\langle a, x\rangle < b$ is a consequence of F_{SS}^{X}.

Proof. Assume that (19) holds. Then, by Theorem 4.1, this means that the system

$$\{\langle a_t, x\rangle < b_t, t \in L; \langle a_t, x\rangle < b_t, t \in S; \langle a_t, x\rangle \leq b_t, t \in W; \langle a, x\rangle \geq b\}$$

is not consistent. This implies that if $\overline{x} \in F_S^X := F_S \cap X$ with $F_S := \{x \in \mathbb{R}^n : \langle a_t, x\rangle < b_t, t \in L\}$, then $\langle a, \overline{x}\rangle < b$. Since $\emptyset \neq F_{SS}^{X} \subset F_S^X$, one gets that $\langle a, x\rangle < b$ is a consequence of F_{SS}^{X}. □

5. Application to systems with convex inequalities

Now we apply the former results to provide necessary and sufficient conditions for the consistency of systems with both strict and weak inequalities of the form

$$\tau := \{f_t(x) \leq 0, t \in W; f_t(x) < 0, t \in S\}, \tag{20}$$

determined by proper lower semicontinuous convex functions $f_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. This kind of systems were analyzed in [8] in the context of set containments.

For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its effective domain is $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and its epigraph is $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^{n+1} : f(x) \leq r\}$. The Legendre-Fenchel conjugate of f is the function f^* defined, for every $x^* \in \mathbb{R}^n$, by $f^*(x^*) = \sup_{x \in \mathbb{R}^n}\{\langle x^*, x\rangle - f(x)\}$. If f is a proper lower semicontinuous convex function, then $f = f^{**}$.

We observe that the solution set of τ is not necessarily evenly convex (even when S is a singleton).

Example 5.1. Consider the functions $f_1(x) = x_1^2 + x_2^2 - 1$ defined on $\mathbb{R}^2$ and $f_2(x) = x_1 - \sqrt{x_2}$ defined on its effective domain $\operatorname{dom} f_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$. Both functions are proper lower semicontinuous convex (it is easy to see that their lower level sets are closed and convex), and so $f_i = f_i^{**}$ for $i = 1, 2$. We shall consider the system

$$\overline{\tau} = \{f_1(x) \leq 0, f_2(x) < 0\}.$$

Since $f_1^*(y) = \frac{\|y\|^2}{4} + 1$ and

$$f_2^*(y) = \begin{cases} \frac{-1}{4y_2}, & \text{if } y_1 = 1, y_2 < 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

one has that $X := \{x \in \mathbb{R}^2 : f_1(x) \leq 0\} = \{x \in \mathbb{R}^2 : y_1x_1 + y_2x_2 \leq \frac{\|y\|^2}{4} + 1, y \in \mathbb{R}^2\}$ and

$$\{f_2(x) < 0\} = \{\exists\, \varepsilon > 0, f_2(x) + \varepsilon \leq 0\} = \{\exists\, \varepsilon > 0, x_1 + y_2x_2 + \varepsilon \leq \frac{-1}{4y_2}, y_2 < 0\}.$$

Therefore, the solution set of $\overline{\tau}$ coincides with the set of strong Slater points of the linear system $\{x_1 - tx_2 \leq \frac{1}{4t}, t > 0\}$ with respect to the closed convex set X, say F_{SS}^X, which is not an evenly convex set. Observe that the open separation property from outside points fails at the origin, having that $(0, 0) \in (\operatorname{eco} F_{SS}^X) \backslash F_{SS}^X$ (see Figure 5.1).

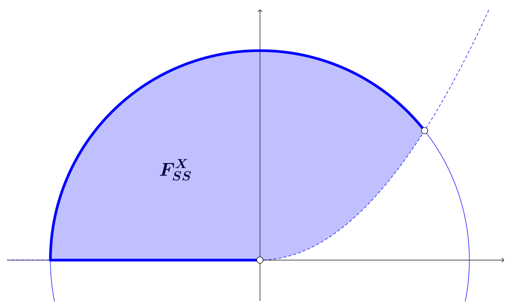

Figure 2: The solution set of $\overline{\tau}$.

Theorem 5.2 (Necessary and sufficient conditions for the consistency of τ). *Let $f_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lower semicontinuous convex functions for all $t \in W \cup S$. Consider the following statements:*

(*i*) $(0_n, 0, -1) \notin \operatorname{cl}\operatorname{cone}\left[\left(\bigcup_{t\in W} \operatorname{epi} f_t^* \times \{0\}\right) \cup \left(\bigcup_{t\in S} \operatorname{epi} f_t^* \times \{-1\}\right) \cup \{(0_n, 1, -1)\}\right]$;

(*ii*) $\tau = \{f_t(x) \leq 0, t \in W; f_t(x) < 0, t \in S\}$ *is consistent;*

(*iii*) $0_{n+1} \notin \operatorname{eco}\left[\{\{\operatorname{epi} f_t^*, t \in S\} + \mathbb{R}_+\{\operatorname{epi} f_t^*, t \in W\}\} \cup \{(0_n, 1)\}\right]$.

Then, one has (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*).

Proof. [(*i*) $\Rightarrow$ (*ii*)] According to Theorem 3.2(*iv*), (*i*) is equivalent to say that the system $\{\langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in S\}$ satisfies the strong Slater condition with respect to the closed convex set given by the solution set of the linear system $\{\langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in W\}$. Hence, there exist $\overline{x} \in \mathbb{R}^n$ and $\overline{\varepsilon} \in]0, 1]$ such that $\langle a, \overline{x}\rangle \leq f_t^*(a) + \delta$ for all $\delta \in \mathbb{R}_+, a \in \operatorname{dom} f_t^*, t \in W$, and

$$\langle a, \overline{x}\rangle + \overline{\varepsilon} \leq f_t^*(a) + \delta$$

for all $\delta \in \mathbb{R}_+, a \in \operatorname{dom} f_t^*, t \in S$. Thus, $f_t(\overline{x}) = f_t^{**}(\overline{x}) \leq 0$ for all $t \in W$ and $f_t(\overline{x}) < f_t(\overline{x}) + \overline{\varepsilon} = f_t^{**}(\overline{x}) + \overline{\varepsilon} \leq 0$ for all $t \in S$, which shows that τ is consistent.

[(*ii*) $\Rightarrow$ (*iii*)] If τ is consistent, then there exists $\overline{x} \in \mathbb{R}^n$ such that

$$\langle a, \overline{x}\rangle - f_t^*(a) \leq f_t^{**}(\overline{x}) = f_t(\overline{x}) < 0$$

for all $a \in \operatorname{dom} f_t^*, t \in S$, and

$$\langle a, \overline{x}\rangle - f_t^*(a) \leq f_t^{**}(\overline{x}) = f_t(\overline{x}) \leq 0$$

for all $a \in \operatorname{dom} f_t^*, t \in W$. Thus, the linear system $\{\langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in W\}$ is consistent with respect to the evenly convex set $X = \{x \in \mathbb{R}^n : \langle a, x\rangle < b, (a, b) \in \operatorname{epi} f_t^*, t \in S\}$ and so, by Theorem 4.1(*iii*), statement (*iii*) holds. □

As a straightforward consequence of this result we obtain [22, Theorem 5.2].

We observe that, in Theorem 5.2, the epigraphs of the functions f_t^*, $t \in W \cup S$, can be replaced by their corresponding graphs. Furthermore, one has that, if for every $t \in S$ there exists a compact set $C_t \subset \mathbb{R}^{n+1}$ such that $f_t(\cdot) = \max\{\langle a, \cdot\rangle - b : (a, b) \in C_t\}$ and for every $t \in W$ there exists a set $D_t \subset \mathbb{R}^{n+1}$ such that $f_t(\cdot) = \sup\{\langle a, \cdot\rangle - b : (a, b) \in D_t\}$, then the system τ is consistent if and only if

$$0_{n+1} \notin \operatorname{eco}\left[\{\{C_t, t \in S\} + \mathbb{R}_+\{D_t, t \in W\}\} \cup \{(0_n, 1)\}\right].$$

The proof of this fact follows easily from Theorem 4.1(*iii*), since, for every $t \in S$, $f_t(x) < 0$ if and only if $\langle a, x\rangle < b$ for all $(a, b) \in C_t$, and, for every $t \in W$, $f_t(x) \leq 0$ if and only if $\langle a, x\rangle \leq b$ for all $(a, b) \in D_t$.

We conclude by pointing out that, whenever the functions f_t, for all $t \in S$, in Theorem 5.2 are linear, then one has that:

- Statement (*ii*) is equivalent to the existence of Slater points of the linear inequality system $\sigma := \{\langle a_t, x\rangle \leq b_t, t \in S\}$ with respect to the closed convex set $X := \{x \in \mathbb{R}^n : f_t(x) \leq 0, t \in W\}$.
- Statement (*i*) is equivalent, by Theorem 3.2(*iii*), to the existence of strong Slater points of σ with respect to the set X rewritten as $\{x \in \mathbb{R}^n : \langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in W\}$.
- In this framework, statements (*ii*) and (*iii*) are equivalent in virtue of Theorem 4.1. However, (*i*) and (*ii*) are not equivalent in general.

To see the last statement, consider the linear system $\sigma = \{s_1 x \leq s_2, (s_1, s_2) \in S\}$ where $S = \{s \in \mathbb{R}^2 : s_1 \geq 0, s_2 \geq 0, s_1 + s_2 \neq 0\}$, and the constraint set $X = \{x \in \mathbb{R} : x \leq 1\}$. It is easy to check that $F^X = F =]-\infty, 0]$, $F_S^X = F_S =]-\infty, 0[$ (the set of Slater points of σ with respect to X) and $F_{SS}^X = F_{SS} = \emptyset$. Thus, condition (*ii*) in Theorem 5.2 holds, but condition (*i*) fails.

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