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Some modifications of Szász–type operators including U-Charlier-Poisson polynomials

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Abstract. In this paper, we investigate a generalized Szász–type positive linear operator recently introduced in the literature and analyze its fundamental properties through the study of moments. In particular, we construct several important modifications, namely the Stancu, Kantorovich and Stancu–Kantorovich variants, and derive explicit expressions for their first- and second-order moments. These moment identities provide the basis for establishing various approximation properties of the operators. More specifically, they are instrumental in proving Korovkin-type theorems, estimating rates of convergence via the modulus of continuity and Peetre's K-functional, and examining approximation behavior in Lipschitz spaces. Furthermore, we employ these identities to obtain Voronovskaya-type asymptotic results. The findings presented here contribute to a deeper understanding of the approximation capabilities of generalized Szász–type operators and their modifications, thereby enriching the theory of positive linear operators.

1. Introduction

Positive linear operators have long been regarded as one of the most classical and effective tools in approximation theory. Since the pioneering result of Weierstrass and the construction of Bernstein polynomials [2], numerous modifications and generalizations have been proposed, among which the operators of Kantorovich [12, 13, 15], Stancu [10, 17], Durrmeyer [4, 8], and Szász [19] have played a fundamental role in shaping the field. In recent years, the introduction of generalized forms with additional parameters has attracted increasing attention [7, 9], as these formulations offer greater flexibility and allow for more refined approximation processes. Motivated by this line of research, the present work is devoted to the study of a generalized Szász–type operator recently introduced in [1, 18, 22]. We first establish its moment identities and then extend the construction to the Stancu, Kantorovich and Stancu–Kantorovich modifications. Building upon these results, we further derive approximation properties including Korovkin-type theorems, estimates of the rate of convergence with respect to the modulus of continuity and Peetre's K-functional, approximation in Lipschitz spaces, as well as Voronovskaya-type asymptotic formulas.

The convergence behavior of the Szász operators, as first introduced in [19], is characterized by:

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$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\Big(\frac{k}{n}\Big), \tag{1}$$

where $f \in C[0,\infty)$, $n \in N$, and $x \in [0,\infty)$.

Generalizations of Szász operators employing polynomials, particularly those defined via generating functions, have attracted significant interest in recent years due to their enhanced flexibility and applications in approximation theory [5]. Among these, a notable generalization can be obtained through the Appell polynomials introduced in [11]:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\Big(\frac{k}{n}\Big), \tag{2}$$

where $p_k(x) \geq 0$ for $x \in [0,\infty)$ and $g(1) \neq 0$.

Subsequently, Varma et al. employed Sheffer polynomials, which are defined by the following generating function:

$$R(z)S(xz) = \sum_{k=0}^{\infty} p_k(x) \frac{z^k}{k!}, \tag{3}$$

where R and S are analytic functions. Using these functions, the operators incorporating the Brenke-type polynomials are then defined as follows [21]:

$$L_n(f;x) = \frac{1}{R(1)S(nx)} \sum_{k=0}^{\infty} p_k(nx) f\Big(\frac{k}{n}\Big), \tag{4}$$

where $x \geq 0$ and $n \in N$.

On the other hand, Özmen and Erkuş [16] considered a special case of (3), employing the classical Charlier–Poisson polynomials, and defined the corresponding operators as follows:

$$e^z\Big(1 - \frac{z}{a}\Big)^u = \sum_{k=0}^{\infty} C_k(a,u) \frac{z^k}{k!}, a \neq 0. \tag{5}$$

Subsequently, Varma and Taşdelen [20] proposed linear positive operators based on the Charlier–Poisson polynomials, as given in (4):

$$L_n(f;x,a) = \frac{1}{e}\Big(1 - \frac{1}{a}\Big)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k(a, -(a-1)nx)}{k!} f\Big(\frac{k}{n}\Big), \tag{6}$$

where $a > 1$ and $x \geq 0$.

It is now possible to express the Charlier–Poisson polynomials [5], as defined in (5), in the following form:

$$e^{-\alpha w}(1+w)^x = \sum_{n=0}^{\infty} (-\alpha)^n C_n(x,\alpha) \frac{w^n}{n!}, \tag{7}$$

with $\alpha \neq 0$. Note that by setting $\alpha = a$, $w = -\frac{z}{a}$, and $x = u$ in (7), one recovers (5).

Subsequently, Clemente et al. [6] generalized this operator by introducing a new family of parametric U–Charlier–Poisson-type polynomials. This operator, denoted by $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$, is defined as follows:

$$J_n(f;x) = \frac{1}{\beta e + A_j(\lambda,\alpha)}\Big(1 - \frac{1}{a}\Big)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx; a, \beta, \lambda)}{k!(-a)^k} f\Big(\frac{k}{n}\Big), \tag{8}$$

where $f \in C[0,\infty)$, $n, J \in N$, $\beta, \lambda \in R$, $\alpha \neq 0$, $\beta e \neq -A_j(\lambda,\alpha)$, $x \in [0,\infty)$, $a > 1$,

$$A_j(\lambda,\alpha) = \lambda^n (-\alpha)^J \prod_{m=2}^{2+J} \frac{2^m - 1}{2^m - 2},$$

$$\sum_{n=0}^{\infty} \frac{G_n^{[2+J]}(x;\alpha,\beta,\lambda)}{(-a)^n} \frac{z^n}{n!} = (\beta e^z + Aj(\lambda,\alpha)) \left(1 - \frac{z}{a}\right)^x.$$

Note that by setting $\beta = 1$, $\lambda = 0$, and $z = x$ in (8), one recovers the classical Charlier–Poisson polynomials given in (6). Consequently, the generating function of $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ encompasses, as special cases, the generating function of the Charlier–Poisson polynomials, i.e., $C_n(x,\alpha) = G_n^{[2+J]}(x;\alpha,1,0)$.

By employing generating function identities [6], the first-order moments of the operator can be obtained as follows:

$$J_n(1;x) = 1, \tag{9}$$

$$J_n(t;x) = x + \frac{\beta e}{n\left(\beta e + A_j(\lambda,\alpha)\right)}, \tag{10}$$

$$J_n\left(t^2;x\right) = x^2 + x\left(\frac{1}{n(a-1)} + \frac{2\beta e}{n\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{1}{n}\right) + \frac{2\beta e}{n^2\left(\beta e + A_j(\lambda,\alpha)\right)}. \tag{11}$$

In this paper, we generalize the operator in (8) to the Kantorovich, Stancu and Stancu–Kantorovich types. To begin with, we introduce the Stancu-type generalization of the operator in (8) as follows:

$$J_n^*(f;x) = \frac{1}{\beta e + A_j(\lambda,\alpha)}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k} f\left(\frac{k+\zeta}{n+\mu}\right),$$

for $f \in C[0,\infty)$, $n, J \in N$, $\beta, \lambda \in R$, $\alpha \neq 0$, $\beta e \neq -A_j(\lambda,\alpha)$, $x \in [0,\infty)$, $a > 1$, and $\zeta, \mu \geq 0$.

Next, we define the Kantorovich-type generalization of the operators in (8) as follows:

$$J_n^{**}(f;x) = \frac{n}{\beta e + A_j(\lambda,\alpha)}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt,$$

where $f \in C[0,\infty)$, $n, J \in N$, $\beta, \lambda \in R$, $\alpha \neq 0$, $\beta e \neq -A_j(\lambda,\alpha)$, $x \in [0,\infty)$, $a > 1$.

Furthermore, we introduce the Stancu–Kantorovich-type extension of the operators in (8) as follows:

$$J_n^{***}(f;x) = \frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k} \int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}} f(t)\,dt,$$

for $f \in C[0,\infty)$, $n, J \in N$, $\beta, \lambda \in R$, $\alpha \neq 0$, $\beta e \neq -A_j(\lambda,\alpha)$, $x \in [0,\infty)$, $a > 1$, and $\zeta, \mu \geq 0$.

2. Approximation Properties

We have derived the moments of the generalized Szász–type operator along with its Stancu, Kantorovich and Stancu–Kantorovich modifications. These moment identities form the basis for establishing Korovkin-type theorems, estimating rates of convergence, and obtaining Voronovskaya-type results.

Lemma 2.1. *For $n \in N$ and $x \geq 0$, the operators satisfy the equalities given in* (9)–(11), *which are obtained with the aid of the following identities* [6]*:*

$$\sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx; a, \beta, \lambda)}{k!(-a)^k} = \left(\beta e + A_j(\lambda, \alpha)\right)\left(1 - \frac{1}{a}\right)^{-(a-1)nx}, \tag{12}$$

$$\sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx; a, \beta, \lambda)}{k!(-a)^k} = \left[\beta e + nx\left(\beta e + A_j(\lambda, \alpha)\right)\right]\left(1 - \frac{1}{a}\right)^{-(a-1)nx}, \tag{13}$$

$$\sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx; a, \beta, \lambda)}{k!(-a)^k} = \left[n^2\left(x^2 + x\phi\right)\left(\beta e + A_j(\lambda, \alpha)\right) + 2\beta e\right]\left(1 - \frac{1}{a}\right)^{-(a-1)nx}, \tag{14}$$

where

$$\Phi = \frac{1}{n(a-1)} + \frac{2\beta e}{n\left(\beta e + A_j(\lambda, \alpha)\right)} + \frac{1}{n}.$$

The foregoing equalities imply the following result:

$$J_n(1; x) = 1,$$

$$J_n(t; x) = x + \frac{\beta e}{n\left(\beta e + A_j(\lambda, \alpha)\right)},$$

$$J_n\left(t^2; x\right) = x^2 + x\left(\frac{1}{n(a-1)} + \frac{2\beta e}{n\left(\beta e + A_j(\lambda, \alpha)\right)} + \frac{1}{n}\right) + \frac{2\beta e}{n^2\left(\beta e + A_j(\lambda, \alpha)\right)}.$$

Lemma 2.2. *For $n \in N$, $x \geq 0$, $\beta e \neq -A_j(\lambda, \alpha)$, $a > 1$, and $\zeta, \mu \geq 0$, the operators can be expressed in the following form:*

$$J_n^*(1; x) = 1, \tag{15}$$

$$J_n^*(t; x) = \frac{n}{n+\mu}x + \frac{\zeta}{n+\mu} + \frac{\beta e}{(n+\mu)\left(\beta e + A_j(\lambda, \alpha)\right)}, \tag{16}$$

$$J_n^*\left(t^2; x\right) = \frac{n^2}{(n+\mu)^2}x^2 + \frac{n}{(n+\mu)^2}\left(\frac{a}{a-1} + 2\zeta + \frac{2\beta e}{\beta e + A_j(\lambda, \alpha)}\right)x + \frac{2\beta e(\zeta+1)}{(n+\mu)^2\left(\beta e + A_j(\lambda, \alpha)\right)} + \frac{\zeta^2}{(n+\mu)^2}. \tag{17}$$

Lemma 2.3. *For $n \in N$, $x \geq 0$, $a > 1$, and $\beta e \neq -A_j(\lambda, \alpha)$, the operators are given as follows:*

$$J_n^{**}(1; x) = 1, \tag{18}$$

$$J_n^{**}(t; x) = x + \frac{\beta e}{n\left(\beta e + A_j(\lambda, \alpha)\right)} + \frac{1}{2n}, \tag{19}$$

$$J_n^{**}\left(t^2; x\right) = x^2 + \frac{1}{n}\left(\frac{2a-1}{a-1} + \frac{2\beta e}{\beta e + A_j(\lambda, \alpha)}\right)x + \frac{3\beta e}{n^2\left(\beta e + A_j(\lambda, \alpha)\right)} + \frac{1}{3n^2}. \tag{20}$$

Lemma 2.4. *For $n \in N$, $x \geq 0$, $\beta e \neq -A_j(\lambda, \alpha)$, $a > 1$,and $\zeta, \mu \geq 0$, the operators are characterized by the following expressions:*

$$J_n^{***}(1; x) = 1, \tag{21}$$

$$J_n^{***}(t;x) = \frac{n}{n+\mu}x + \frac{\beta e}{(n+\mu)\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{2\zeta+1}{2(n+\mu)}, \tag{22}$$

$$\begin{aligned} J_n^{***}\left(t^2;x\right) = {} & \frac{n^2}{(n+\mu)^2}x^2 + \frac{n}{(n+\mu)^2}\left(\frac{2a-1}{a-1} + 2\zeta + \frac{2\beta e}{\beta e + A_j(\lambda,\alpha)}\right)x \\ & + \frac{(2\zeta+3)\beta e}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta^2+\zeta+\frac{1}{3}}{(n+\mu)^2}. \end{aligned} \tag{23}$$

Theorem 2.5. *Let $S := \left\{f : [0,\infty) \to \mathbb{R} : \left|f(x)\right| \le Me^{Ax}\right\}$, where $A \in \mathbb{R}$. If $f \in C[0,\infty) \cap S$, then*

$$\lim_{n\to\infty} J_n(f;x) = f,$$

$$\lim_{n\to\infty} J_n^{*}(f;x) = f,$$

$$\lim_{n\to\infty} J_n^{**}(f;x) = f,$$

$$\lim_{n\to\infty} J_n^{***}(f;x) = f.$$

It follows that the above operators converge uniformly on each compact subset of $[0,\infty)$ [14].

3. Rate of Convergence

Let $F \in C[0,\infty)$. For $\delta > 0$, the modulus of continuity of F, denoted by $\omega(F;\delta)$ is defined as

$$\omega(F;\delta) := \sup_{x,y\in[0,\infty)} \left|F(x) - F(y)\right|,$$

where $\left|x-y\right| < \delta$.

Moreover, it is well known that,

$$\left|F(x) - F(y)\right| \le \omega(F;\delta)\left(\frac{\left|x-y\right|}{\delta} + 1\right). \tag{24}$$

In addition, if f is uniformly continuous, it satisfies

$$\left|F(x) - F(y)\right| \le \omega(F;\delta). \tag{25}$$

Define the set S as

$$S := \left\{F : [0,\infty) \to \mathbb{R} : |F(x)| \le Me^{Ax},\ A \in \mathbb{R},\ M \in \mathbb{R}^+\right\}.$$

The next propositions collect several important properties of the operators discussed above.

Theorem 3.1. *Let $f \in \widetilde{C}[0,\infty) \cap S$. The operators J_n^{*} obey the inequality:*

$$\left|J_n^{*}(f;x) - f(x)\right| \le 2\omega\left(f;\Upsilon_n^{*}(x;\alpha,\beta,\lambda)\right),$$

with

$$\Upsilon_n^{*}(x;\alpha,\beta,\lambda) = \frac{1}{n+\mu}\left[\mu^2x^2 + \left(\frac{na}{a-1} - 2\zeta\mu - \frac{2\beta e\mu}{\beta e + A_j(\lambda,\alpha)}\right)x + \frac{2\beta e(\zeta+1)}{\beta e + A_j(\lambda,\alpha)} + \zeta^2\right]^{1/2}$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous function on $[0,\infty)$.

Proof. By virtue of the definition of the new operators J_n^* and Lemma 2.2, we obtain

$$\begin{aligned}\left|J_n^*(f;x)-f(x)\right| &= \left|\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(f\left(\frac{k+\zeta}{n+\mu}\right)-f(x)\right)\right| \\ &\leq \frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|f\left(\frac{k+\zeta}{n+\mu}\right)-f(x)\right|.\end{aligned}$$

This way of (24) follows:

$$\begin{aligned}\left|J_n^*(f;x)-f(x)\right| &\leq \frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(1+\frac{1}{\delta}\left|\frac{k+\zeta}{n+\mu}-x\right|\right)\omega(f;\delta) \\ &\leq \left\{1+\frac{1}{\delta}\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|\frac{k+\zeta}{n+\mu}-x\right|\right\}\omega(f;\delta).\end{aligned}$$

Alternatively, by applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|\frac{k+\zeta}{n+\mu}-x\right| \\ &\leq \left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{1/2} \\ &\times\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(\frac{k+\zeta}{n+\mu}-x\right)^2\right]^{1/2} \\ &=\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(\frac{k+\zeta}{n+\mu}-x\right)^2\right]^{1/2} \\ &=\sqrt{J_n^*\left((t-x)^2;x\right)} \\ &=\frac{1}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{2\beta e(\zeta+1)}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]^{1/2} \\ &=\sqrt{\Upsilon_n^*(x;\alpha,\beta,\lambda)}.\end{aligned}$$

Consequently, the above expression yields

$$\left|J_n^*(f;x)-f(x)\right|\leq\left\{1+\frac{1}{\delta}\sqrt{\Upsilon_n^*(x;\alpha,\beta,\lambda)}\right\}\omega(f;\delta).$$

Setting $\delta=\sqrt{\Upsilon_n^*(x;\alpha,\beta,\lambda)}$, eads to the desired conclusion, which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f\in\widetilde{C}[0,\infty)\cap S$. It follows that the operators J_n^{**} satisfy the inequality given below:*

$$\left|J_n^{**}(f;x)-f(x)\right|\leq 2\omega(f;\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)),$$

with

$$\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)=\frac{1}{n}\left[\frac{na}{a-1}x+\frac{3\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{3}\right]^{1/2}$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous function on $[0,\infty)$.

Proof. Combining the definition of the new operators J_n^{**} with Lemma 2.3, we obtain

$$\left|J_n^{**}(f;x)-f(x)\right| = \left|\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(f(t)-f(x))dt\right|$$
$$\leq \frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left|f(t)-f(x)\right|dt.$$

This way of (24) follows:

$$\left|J_n^{**}(f;x)-f(x)\right| \leq \frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(1+\frac{1}{\delta}|t-x|\right)\omega(f;\delta)\,dt$$
$$\leq \left\{1+\frac{1}{\delta}\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\,dt\right\}\omega(f;\delta).$$

By the Cauchy–Schwarz inequality, it follows that

$$\sum_{k=0}^{\infty}\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\,dt$$
$$\leq\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{1/2}$$
$$\times\left[\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^2dt\right]^{1/2}$$
$$=\left[\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^2dt\right]^{1/2}$$
$$=\sqrt{J_n^{**}\left((t-x)^2;x\right)}$$
$$=\frac{1}{n}\left[\frac{na}{a-1}x+\frac{3\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{3}\right]^{1/2}$$
$$=\sqrt{\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)}.$$

It follows from the above expression that

$$\left|J_n^{**}(f;x)-f(x)\right|\leq\left\{1+\frac{1}{\delta}\sqrt{\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)}\right\}\omega(f;\delta).$$

By taking $\delta=\sqrt{\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)}$, we obtain the desired result, thereby completing the proof of Theorem 3.2. □

Theorem 3.3. *Let* $f \in \widetilde{C}[0,\infty) \cap S$. *The operators* J_n^{***} *then satisfy the following inequality:*

$$\left|J_n^{***}(f;x) - f(x)\right| \leq 2\omega(f;\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)),$$

with

$$\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda) = \frac{1}{n+\mu}\left[\mu^2x^2 + \left(\frac{na}{a-1} - (2\zeta+1)\mu - \frac{2\beta e\mu}{\beta e + A_j(\lambda,\alpha)}\right)x + \frac{(2\zeta+3)\beta e}{\beta e + A_j(\lambda,\alpha)} + \zeta^2 + \zeta + \frac{1}{3}\right]^{1/2}$$

where $\widetilde{C}[0,\infty)$ *is the space of uniformly continuous function on* $[0,\infty)$.

Proof. Using the definition of J_n^{***} and Lemma 2.4, one finds that

$$\begin{aligned}\left|J_n^{***}(f;x) - f(x)\right| &= \left|\frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(f(t)-f(x))\,dt\right| \\ &\leq \frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}\left|f(t)-f(x)\right|dt.\end{aligned}$$

By using (24), it follows

$$\begin{aligned}\left|J_n^{***}(f;x) - f(x)\right| &\leq \frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}\left(1+\frac{1}{\delta}|t-x|\right)\omega(f;\delta)\,dt \\ &\leq \left\{1 + \frac{1}{\delta}\frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}|t-x|\,dt\right\}\omega(f;\delta).\end{aligned}$$

On the other hand, the Cauchy–Schwarz inequality implies that

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}|t-x|\,dt \\ &\leq \left[\frac{1}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{1/2} \\ &\times\left[\frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(t-x)^2dt\right]^{1/2} \\ &= \left[\frac{n+\mu}{\beta e + A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(t-x)^2dt\right]^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{J_n^{***}\left((t-x)^2;x\right)} \\
&= \frac{1}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-(2\zeta+1)\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{(2\zeta+3)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2+\zeta+\frac{1}{3}\right]^{1/2} \\
&= \sqrt{\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)}.
\end{aligned}
$$

Thus, from the above expression, we obtain

$$
\left|J_n^{***}(f;x)-f(x)\right| \le \left\{1+\frac{1}{\delta}\sqrt{\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)}\right\}\omega(f;\delta).
$$

Letting $\delta = \sqrt{\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)}$, we arrive at the asserted result, thereby completing the proof of Theorem 3.3. $\square$

We proceed by introducing the Lipschitz class of order θ, denoted $Lip_M(\theta)$ $(0<\theta\le 1,\ M>0)$, which is defined as:

$$
Lip_M(\theta) = \{h \in C_B[0,\infty) : |h(t)-h(x)| \le M|t-x|^{\theta},\ t,x\in[0,\infty)\}. \tag{26}
$$

Based on this expression, we obtain the following result, providing an explicit bound for the approximation error of the operators applied to the function $h \in Lip_M(\theta)$.

Theorem 3.4. *Let $f \in Lip_M(\theta)$, then for $x \ge 0$*

$$
\left|J_n^{*}(f;x)-f(x)\right| \le M\left(\Upsilon_n^{*}(x;\alpha,\beta,\lambda)\right)^{\theta}
$$

where

$$
\Upsilon_n^{*}(x;\alpha,\beta,\lambda) = \frac{1}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{2\beta e(\zeta+1)}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]^{1/2}.
$$

Proof. Applying the operator J_n^*, we obtain

$$
\begin{aligned}
\left|J_n^{*}(f;x)-f(x)\right| &= \left|\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(f\left(\frac{k+\zeta}{n+\mu}\right)-f(x)\right)\right| \\
&\le \frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|f\left(\frac{k+\zeta}{n+\mu}\right)-f(x)\right|.
\end{aligned}
$$

By using (26), it follows

$$
\left|J_n^{*}(f;x)-f(x)\right| \le \frac{M}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|\frac{k+\zeta}{n+\mu}-x\right|^{\theta}.
$$

Alternatively, by applying the Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{M}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left|\frac{k+\zeta}{n+\mu}-x\right|^{\theta} \\
&\le M\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{(2-\theta)/2}
\end{aligned}
$$

$$\times\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(\frac{k+\zeta}{n+\mu}-x\right)^2\right]^{\theta/2}$$
$$=M\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\left(\frac{k+\zeta}{n+\mu}-x\right)^2\right]^{\theta/2}$$
$$=M\left(J_n^*\left((t-x)^2;x\right)\right)^{\theta/2}$$
$$=\frac{M}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{2\beta e(\zeta+1)}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]^{1/2}$$
$$=M\left(\Upsilon_n^*(x;\alpha,\beta,\lambda)\right)^{\theta}.$$

This completes the proof of Theorem 3.4. □

Theorem 3.5. *Let $f\in Lip_M(\theta)$. It follows that the operators J_n^{**} satisfy the inequality given below:*

$$\left|J_n^{**}(f;x)-f(x)\right|\leq M\left(\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)\right)^{\theta}$$

with

$$\Upsilon_n^{\Diamond}(x;\alpha,\beta,\lambda)=\frac{1}{n}\left[\frac{na}{a-1}x+\frac{3\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{3}\right]^{1/2}.$$

Proof. Combining the definition of the new operators J_n^{**} with Lemma 2.3, we obtain

$$\left|J_n^{**}(f;x)-f(x)\right|=\left|\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(f(t)-f(x))\,dt\right|$$
$$\leq\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left|f(t)-f(x)\right|dt.$$

By using (26), it follows

$$\left|J_n^{**}(f;x)-f(x)\right|\leq\frac{Mn}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|^{\theta}\,dt.$$

By the Hölder inequality, it follows that

$$\sum_{k=0}^{\infty}\frac{Mn}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|^{\theta}\,dt$$
$$\leq M\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{(2-\theta)/2}$$
$$\times\left[\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^2dt\right]^{\theta/2}$$

$$
\begin{aligned}
&=M\left[\frac{n}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^2dt\right]^{\theta/2}\\
&=M\left(J_n^{**}\left((t-x)^2;x\right)\right)^{\theta/2}\\
&=\frac{M}{n}\left[\frac{na}{a-1}x+\frac{3\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{3}\right]^{1/2}\\
&=M\left(\Upsilon_n^{\diamond}(x;\alpha,\beta,\lambda)\right)^{\theta}.
\end{aligned}
$$

This completes the proof of Theorem 3.5. □

Theorem 3.6. *Let $f\in Lip_M(\theta)$. The operators J_n^{***} satisfy the following bound:*

$$
\left|J_n^{***}(f;x)-f(x)\right|\leq M\left(\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)\right)^{\theta}
$$

with

$$
\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)=\frac{1}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-(2\zeta+1)\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{(2\zeta+3)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2+\zeta+\frac{1}{3}\right]^{1/2}.
$$

Proof. By the definition of J_n^{***} and Lemma 2.4, the following result is established:

$$
\begin{aligned}
\left|J_n^{***}(f;x)-f(x)\right|&=\left|\frac{n+\mu}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(f(t)-f(x))\,dt\right|\\
&\leq\frac{n+\mu}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}\left|f(t)-f(x)\right|dt.
\end{aligned}
$$

This way of (26) follows:

$$
\left|J_n^{***}(f;x)-f(x)\right|\leq\frac{M(n+\mu)}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}|t-x|^{\theta}\,dt.
$$

By the Hölder inequality, it follows that

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{M(n+\mu)}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}|t-x|^{\theta}\,dt\\
&\leq M\left[\frac{1}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\right]^{(2-\theta)/2}\\
&\times\left[\frac{n+\mu}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(t-x)^2dt\right]^{\theta/2}
\end{aligned}
$$

$$=M\left[\frac{n+\mu}{\beta e+A_j(\lambda,\alpha)}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!(-a)^k}\int_{\frac{k+\zeta}{n+\mu}}^{\frac{k+\zeta+1}{n+\mu}}(t-x)^2dt\right]^{\theta/2}$$
$$=M\left(J_n^{***}\left((t-x)^2;x\right)\right)^{\theta/2}$$
$$=\frac{M}{n+\mu}\left[\mu^2x^2+\left(\frac{na}{a-1}-(2\zeta+1)\mu-\frac{2\beta e\mu}{\beta e+A_j(\lambda,\alpha)}\right)x+\frac{(2\zeta+3)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2+\zeta+\frac{1}{3}\right]^{1/2}$$
$$=M\left(\Upsilon_n^{\blacklozenge}(x;\alpha,\beta,\lambda)\right)^{\theta}.$$

This completes the proof of Theorem 3.6. □

Based on [3], the Peetre's K-functional for the space of continuous functions can be expressed as

$$K(\Lambda;\delta)=\inf_{\Theta\in C^2[0,b]}\left\{\|\Lambda-\Theta\|_{C[0,b]}+\delta\|\Theta\|_{C^2[0,b]}\right\}.$$

We proceed to establish the rate of convergence through Peetre's K-functional.

Theorem 3.7. *Let $f\in C[0,\infty)$, then for $x\geq 0$*

$$\left|J_n^*(f;x)-f(x)\right|\leq 2K(f;\chi_n^*(\alpha,\beta,\lambda;b))$$

where

$$\chi_n^*(\alpha,\beta,\lambda;b)=\frac{1}{4(n+\mu)^2}\left[\mu^2b^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta\mu e}{\beta e+A_j(\lambda,\alpha)}\right)b+\frac{2(\zeta+1)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]$$
$$+\frac{1}{2(n+\mu)}\left(\mu b+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right).$$

Proof. Let $g\in C^2[0,\infty)$. Expanding the function in a Taylor series at the point x, we get

$$g(t)=g(x)+(t-x)g'(x)+(t-x)^2\frac{g''(x)}{2}.$$

By using the operator J_n^* on both sides of the preceding equation, it follows that

$$J_n^*(g;x)=g(x)+g'(x)J_n^*(t-x;x)+\frac{g''(x)}{2}J_n^*\left((t-x)^2;x\right).$$

From the triangle inequality,

$$\left|J_n^*(g;x)-g(x)\right|\leq\left|g'(x)\right|\left|J_n^*(t-x;x)\right|+\frac{\left|g''(x)\right|}{2}\left|J_n^*\left((t-x)^2;x\right)\right|.$$

By maximizing both sides of the above inequality over $[0,b]$ and utilizing the first and second moments of the operator J_n^*, we see that

$$\left\|J_n^*(g;x)-g(x)\right\|_{C[0,b]}\leq\frac{1}{n+\mu}\left(\mu b+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)\left\|g'(x)\right\|_{C[0,b]}$$
$$+\frac{1}{2(n+\mu)^2}\left[\mu^2b^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta\mu e}{\beta e+A_j(\lambda,\alpha)}\right)b+\frac{2(\zeta+1)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]\left\|g''(x)\right\|_{C[0,b]}.$$

Using the property of $C^2[0,b)$, we get

$$\left\|J_n^*(g;x)-g(x)\right\|_{C[0,b]}\le\left\{\frac{1}{n+\mu}\left(\mu b+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)\right.$$
$$\left.+\frac{1}{2(n+\mu)^2}\left[\mu^2b^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta\mu e}{\beta e+A_j(\lambda,\alpha)}\right)b+\frac{2(\zeta+1)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]\right\}\left\|g(x)\right\|_{C^2[0,b]}. \quad (27)$$

Using the linearity and positivity of the operator J_n^*, we obtain

$$\begin{aligned}\left|J_n^*(f;x)-f(x)\right|&=\left|J_n^*(f;x)-J_n^*(g;x)+J_n^*(g;x)-g(x)+g(x)-f(x)\right|\\&\le\left|J_n^*(f;x)-J_n^*(g;x)\right|+\left|J_n^*(g;x)-g(x)\right|+\left|f(x)-g(x)\right|.\end{aligned}$$

By considering the supremum over $[0,b]$ on both sides of the preceding inequality, it follows that

$$\left\|J_n^*(f;x)-f(x)\right\|_{C[0,b]}\le2\left\|f(x)-g(x)\right\|_{C[0,b]}+\left\|J_n^*(g;x)-g(x)\right\|_{C[0,b]}.$$

Here, we apply inequality (27),

$$\left\|J_n^*(f;x)-f(x)\right\|_{C[0,b]}\le2\left\{\left\|f(x)-g(x)\right\|_{C[0,b]}+\left[\frac{1}{2(n+\mu)}\left(\mu b+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)\right.\right.$$
$$\left.\left.+\frac{1}{4(n+\mu)^2}\left[\mu^2b^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta\mu e}{\beta e+A_j(\lambda,\alpha)}\right)b+\frac{2(\zeta+1)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]\left\|g(x)\right\|_{C^2[0,b]}\right\}.$$

Applying the infimum over $g\in C^2[0,b]$ to both sides of the above inequality, we then choose

$$\begin{aligned}\chi_n^*(\alpha,\beta,\lambda;b)=&\frac{1}{4(n+\mu)^2}\left[\mu^2b^2+\left(\frac{na}{a-1}-2\zeta\mu-\frac{2\beta\mu e}{\beta e+A_j(\lambda,\alpha)}\right)b+\frac{2(\zeta+1)\beta e}{\beta e+A_j(\lambda,\alpha)}+\zeta^2\right]\\&+\frac{1}{2(n+\mu)}\left(\mu b+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right),\end{aligned}$$

this yields the desired conclusion. □

Theorem 3.8. *Let $f\in C[0,\infty)$, then for $x\ge0$*

$$\left|J_n^{**}(f;x)-f(x)\right|\le2K\left(f;\chi_n^{\diamond}(\alpha,\beta,\lambda;b)\right)$$

where

$$\chi_n^{\diamond}(\alpha,\beta,\lambda;b)=\frac{3n+2}{12n^2}+\frac{a}{a-1}\frac{b}{2n}+\frac{(n+3)\beta e}{2n^2\left(\beta e+A_j(\lambda,\alpha)\right)}.$$

Proof. Let $g\in C^2[0,\infty)$. Using the Taylor expansion at x, we have

$$g(t)=g(x)+(t-x)g'(x)+(t-x)^2\frac{g''(x)}{2}.$$

Operating with J_n^{**} on both sides of the above equation, we have

$$J_n^{**}(g;x)=g(x)+g'(x)J_n^{**}(t-x;x)+\frac{g''(x)}{2}J_n^{**}\left((t-x)^2;x\right).$$

From the triangle inequality, we obtain

$$\left|J_n^{**}(g;x)-g(x)\right|\le\left|g'(x)\right|\left|J_n^{**}(t-x;x)\right|+\frac{\left|g''(x)\right|}{2}\left|J_n^{**}\left((t-x)^2;x\right)\right|.$$

Taking the maximum on $[0,b]$ and employing the first and second moments of J_n^{**}, one finds that

$$\begin{aligned}\left\|J_n^{**}(g;x)-g(x)\right\|_{C[0,b]} \leq & \left(\frac{\beta e}{n\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{1}{2n}\right)\left\|g'(x)\right\|_{C[0,b]} \\ & +\left(\frac{a}{a-1}\frac{b}{n}+\frac{3\beta e}{n^2\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{1}{3n^2}\right)\left\|g''(x)\right\|_{C[0,b]}.\end{aligned}$$

Using the property of $C^2[0,b)$, we get

$$\left\|J_n^{**}(g;x)-g(x)\right\|_{C[0,b]} \leq \left[\frac{3n+2}{6n^2}+\frac{a}{a-1}\frac{b}{n}+\frac{(n+3)\beta e}{n^2\left(\beta e+A_j(\lambda,\alpha)\right)}\right]\left\|g(x)\right\|_{C^2[0,b]}. \tag{28}$$

Using the linearity and positivity of the operator J_n^{**}, we obtain

$$\begin{aligned}\left|J_n^{**}(f;x)-f(x)\right| &= \left|J_n^{**}(f;x)-J_n^{**}(g;x)+J_n^{**}(g;x)-g(x)+g(x)-f(x)\right| \\ &\leq \left|J_n^{**}(f;x)-J_n^{**}(g;x)\right|+\left|J_n^{**}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|.\end{aligned}$$

Maximizing both sides of the above inequality over $[0,b]$ yields

$$\left\|J_n^{**}(f;x)-f(x)\right\|_{C[0,b]} \leq 2\left\|f(x)-g(x)\right\|_{C[0,b]}+\left\|J_n^{**}(g;x)-g(x)\right\|_{C[0,b]}.$$

In this step, we make use of inequality (28),

$$\left\|J_n^{**}(f;x)-f(x)\right\|_{C[0,b]} \leq 2\left\{\left\|f(x)-g(x)\right\|_{C[0,b]}+\frac{3n+2}{12n^2}+\frac{a}{a-1}\frac{b}{2n}+\frac{(n+3)\beta e}{2n^2\left(\beta e+A_j(\lambda,\alpha)\right)}\right\}\left\|g(x)\right\|_{C^2[0,b]}.$$

By considering the infimum over $g\in C^2[0,b]$ on both sides of the preceding inequality, we select

$$\chi_n^{\diamond}(\alpha,\beta,\lambda,;b)=\frac{3n+2}{12n^2}+\frac{a}{a-1}\frac{b}{2n}+\frac{(n+3)\beta e}{2n^2\left(\beta e+A_j(\lambda,\alpha)\right)}.$$

Hence, the result is established. □

Theorem 3.9. *Let $f\in C[0,\infty)$, then for $x\geq 0$*

$$\left|J_n^{***}(f;x)-f(x)\right| \leq 2K\left(f;\chi_n^{\blacklozenge}(\alpha,\beta,\lambda;b)\right)$$

where

$$\begin{aligned}\chi_n^{\blacklozenge}(\alpha,\beta,\lambda,;b)=&\frac{\mu^2}{2(n+\mu)^2}b^2+\left(\frac{a}{a-1}\frac{n}{(n+\mu)^2}-\frac{\mu(2\zeta+1-n-\mu)}{(n+\mu)^2}-\frac{2\beta e\mu}{(n+\mu)^2(\beta e+A_j(\lambda,\alpha))}\right)\frac{b}{2} \\ &+\frac{(2\zeta+n+\mu+3)\beta e}{2(n+\mu)^2\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{\zeta^2+\zeta+\frac{1}{3}}{2(n+\mu)^2}+\frac{2\zeta+1}{4(n+\mu)}.\end{aligned}$$

Proof. Let $g\in C^2[0,\infty)$. Applying the Taylor expansion at the point x, we obtain

$$g(t)=g(x)+(t-x)g'(x)+(t-x)^2\frac{g''(x)}{2}.$$

Applying J_n^{***} to both sides of the above equation yields

$$J_n^{***}(g;x) = g(x) + g'(x) J_n^{***}(t-x;x) + \frac{g''(x)}{2} J_n^{***}\left((t-x)^2;x\right).$$

From the triangle inequality,

$$\left|J_n^{***}(g;x) - g(x)\right| \le \left|g'(x)\right| \left|J_n^{***}(t-x;x)\right| + \frac{\left|g''(x)\right|}{2} \left|J_n^{***}\left((t-x)^2;x\right)\right|.$$

Considering the maximum over $[0,b]$ on both sides of the preceding inequality and applying the first and second moments of J_n^{***}, it follows that

$$\begin{aligned}\left\|J_n^{***}(g;x) - g(x)\right\|_{C[0,b]} \le & \left(\frac{\mu b}{n+\mu} + \frac{\beta e}{(n+\mu)\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta}{n+\mu} + \frac{1}{2(n+\mu)}\right)\left\|g'(x)\right\|_{C[0,b]} \\ & + \left[\frac{\mu^2}{(n+\mu)^2} b^2 + \left(\frac{a}{a-1}\frac{n}{(n+\mu)^2} - \frac{(2\zeta+1)\mu}{(n+\mu)^2} - \frac{2\beta e \mu}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)}\right) b \right. \\ & \left. + \frac{(2\zeta+3)\beta e}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta^2+\zeta+\frac{1}{3}}{(n+\mu)^2}\right]\left\|g''(x)\right\|_{C[0,b]}.\end{aligned}$$

Using the property of $C^2[0,b)$, we get

$$\begin{aligned}\left\|J_n^{***}(g;x) - g(x)\right\|_{C[0,b]} \le & \left[\frac{\mu b}{n+\mu} + \frac{\beta e}{(n+\mu)\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta}{n+\mu} + \frac{1}{2(n+\mu)} + \frac{\mu^2}{(n+\mu)^2} b^2 \right. \\ & + \left(\frac{a}{a-1}\frac{n}{(n+\mu)^2} - \frac{(2\zeta+1)\mu}{(n+\mu)^2} - \frac{2\beta e \mu}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)}\right) b \\ & \left. + \frac{(2\zeta+3)\beta e}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta^2+\zeta+\frac{1}{3}}{(n+\mu)^2}\right]\left\|g(x)\right\|_{C^2[0,b]}. \end{aligned} \tag{29}$$

Using the linearity and positivity of the operator J_n^{***}, we obtain

$$\begin{aligned}\left|J_n^{***}(f;x) - f(x)\right| &= \left|J_n^{***}(f;x) - J_n^{***}(g;x) + J_n^{***}(g;x) - g(x) + g(x) - f(x)\right| \\ &\le \left|J_n^{***}(f;x) - J_n^{***}(g;x)\right| + \left|J_n^{***}(g;x) - g(x)\right| + \left|f(x) - g(x)\right|.\end{aligned}$$

Taking the maximum of both sides of the inequality above on $[0,b]$ leads to

$$\left\|J_n^{***}(f;x) - f(x)\right\|_{C[0,b]} \le 2\left\|f(x) - g(x)\right\|_{C[0,b]} + \left\|J_n^{***}(g;x) - g(x)\right\|_{C[0,b]}.$$

Inequality (29) is employed here,

$$\begin{aligned}\left\|J_n^{***}(f;x) - f(x)\right\|_{C[0,b]} \le & 2\left\{\left\|f(x) - g(x)\right\|_{C[0,b]}\right. \\ & + \frac{\mu b}{2(n+\mu)} + \frac{\beta e}{2(n+\mu)\left(\beta e + A_j(\lambda,\alpha)\right)} + \frac{\zeta}{2(n+\mu)} + \frac{1}{4(n+\mu)} \\ & + \left[\frac{\mu^2}{2(n+\mu)^2} b^2 + \left(\frac{a}{a-1}\frac{n}{(n+\mu)^2} - \frac{(2\zeta+1)\mu}{(n+\mu)^2} - \frac{2\beta e \mu}{(n+\mu)^2\left(\beta e + A_j(\lambda,\alpha)\right)}\right)\frac{b}{2}\end{aligned}$$

$$+\frac{(2\zeta+3)\,\beta e}{2(n+\mu)^2\left(\beta e+A_j\,(\lambda,\alpha)\right)}+\frac{\zeta^2+\zeta+\frac{1}{3}}{2(n+\mu)^2}\Bigg]\Bigg\}\left\|g\,(x)\right\|_{C^2[0,b]}.$$

Taking the infimum over $g \in C^2\,[0,b]$ on both sides of the above inequality, we choose

$$\begin{aligned}\chi_n^{\blacklozenge}(\alpha,\beta,\lambda,;b)=&\frac{\mu^2}{2\,(n+\mu)^2}b^2+\left(\frac{a}{a-1}\frac{n}{(n+\mu)^2}-\frac{\mu(2\zeta+1-n-\mu)}{(n+\mu)^2}-\frac{2\beta e\mu}{(n+\mu)^2(\beta e+A_j\,(\lambda,\alpha))}\right)\frac{b}{2}\\&+\frac{(2\zeta+n+\mu+3)\,\beta e}{2(n+\mu)^2\left(\beta e+A_j\,(\lambda,\alpha)\right)}+\frac{\zeta^2+\zeta+\frac{1}{3}}{2(n+\mu)^2}+\frac{2\zeta+1}{4(n+\mu)}.\end{aligned}$$

Consequently, the desired result follows. □

Theorem 3.10. *Let f, f' and $f'' \in C[0,\infty)$. It can be shown that the following limit is valid:*

$$\lim_{n\to\infty} n\,[J_n^*\,(f;x)-f\,(x)]=\left(-\mu x+\zeta+\frac{\beta e}{\beta e+A_j\,(\lambda,\alpha)}\right)f'\,(x)+\frac{a}{a-1}\frac{xf''\,(x)}{2}.$$

Proof. Expanding the function in a Taylor series at the point x, we obtain

$$f\,(t)=f\,(x)+(t-x)\,f'\,(x)+(t-x)^2\,\frac{f''\,(x)}{2}+\psi\,(t,x)\,(t-x)^2$$

where

$$\psi\,(t,x)=\frac{f''\,(\upsilon)-f''\,(x)}{2}$$

here, υ is between x and t, and $\lim_{t\to x}\psi\,(t,x)=0$. Using the operator J_n^* on both sides of the above equation yields

$$n\,[J_n^*\,(f;x)-f\,(x)]=nf'\,(x)\,J_n^*\,(t-x;x)+n\frac{f''\,(x)}{2}J_n^*\left((t-x)^2\,;x\right)+nJ_n^*\left(\psi\,(t,x)\,(t-x)^2\,;x\right).$$

Taking limit when $n\to\infty$ leads to

$$\lim_{n\to\infty} n\,[J_n^*\,(f;x)-f\,(x)]=f'\,(x)\lim_{n\to\infty} nJ_n^*\,(t-x;x)+\frac{f''\,(x)}{2}\lim_{n\to\infty} nJ_n^*\left((t-x)^2\,;x\right)+\lim_{n\to\infty} nJ_n^*\left(\psi\,(t,x)\,(t-x)^2\,;x\right).$$

Since the moments are known, it follows that

$$\begin{aligned}\lim_{n\to\infty} nJ_n^*\,(t-x;x)&=\lim_{n\to\infty}\left(\frac{-\mu n}{n+\mu}x+\frac{\zeta n}{n+\mu}+\frac{\beta en}{(n+\mu)\left(\beta e+A_j\,(\lambda,\alpha)\right)}\right)\\&=-\mu x+\zeta+\frac{\beta e}{\beta e+A_j\,(\lambda,\alpha)},\\\lim_{n\to\infty} nJ_n^*\left((t-x)^2\,;x\right)&=\lim_{n\to\infty}\left[\frac{n\mu^2}{(n+\mu)^2}x^2+\left(\frac{n^2a}{(n+\mu)^2\,(a-1)}-\frac{2\zeta\mu n}{(n+\mu)^2}-\frac{2\beta\mu en}{(n+\mu)^2\left(\beta e+A_j\,(\lambda,\alpha)\right)}\right)x\right.\\&\left.\quad+\frac{\zeta^2n}{(n+\mu)^2}+\frac{2\beta e\,(\zeta+1)\,n}{(n+\mu)^2\left(\beta e+A_j\,(\lambda,\alpha)\right)}\right]\\&=\frac{a}{a-1}x.\end{aligned}$$

Applying the above limits, we reach that

$$\lim_{n\to\infty} n\left[J_n^*(f;x)-f(x)\right]=\left(-\mu x+\zeta+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)f'(x)+\frac{a}{a-1}\frac{xf''(x)}{2}+\lim_{n\to\infty} nJ_n^*\left(\psi(t,x)(t-x)^2;x\right).$$

It can be shown that the limit on the right-hand side of the above equation is zero. This follows readily from the Cauchy–Schwarz inequality:

$$nJ_n^*\left(\psi(t,x)(t-x)^2;x\right)\le\sqrt{J_n^*\left(\psi^2(t,x);x\right)}\sqrt{n^2J_n^*(t-x)^4;x)}.$$

Using the Korovkin theorem [14], it follows that

$$\lim_{n\to\infty} J_n^*\left(\psi^2(t,x);x\right)=\psi^2(x,x)=0,$$

noting that $\psi^2(x,x)=0$, $\psi(t,x)\in C[0,\infty)$, $\psi(t,x)$ is bounded as $t\to\infty$ and taking into account that

$$J_n^*(t-x)^4;x)=O\left(\frac{1}{n^2}\right),$$

we conclude that the proof is complete. □

Theorem 3.11. *Let f, f' and $f''\in C[0,\infty)$. The below limit is satisfied:*

$$\lim_{n\to\infty} n\left[J_n^{**}(f;x)-f(x)\right]=\left(\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{2}\right)f'(x)+\frac{a}{a-1}\frac{xf''(x)}{2}.$$

Proof. Using the Taylor expansion at x, we have

$$f(t)=f(x)+(t-x)f'(x)+(t-x)^2\frac{f''(x)}{2}+\psi(t,x)(t-x)^2$$

where

$$\psi(t,x)=\frac{f''(v)-f''(x)}{2}$$

with v between x and t, and $\lim_{t\to x}\psi(t,x)=0$, applying J_n^{**} to both sides gives

$$n\left[J_n^{**}(f;x)-f(x)\right]=nf'(x)J_n^{**}(t-x;x)+n\frac{f''(x)}{2}J_n^{**}\left((t-x)^2;x\right)+nJ_n^{**}\left(\psi(t,x)(t-x)^2;x\right).$$

Taking limit when $n\to\infty$ leads to

$$\lim_{n\to\infty} n\left[J_n^{**}(f;x)-f(x)\right]=f'(x)\lim_{n\to\infty} nJ_n^{**}(t-x;x)+\frac{f''(x)}{2}\lim_{n\to\infty} nJ_n^{**}\left((t-x)^2;x\right)+\lim_{n\to\infty} nJ_n^{**}\left(\psi(t,x)(t-x)^2;x\right).$$

Using the known values of the moments, we obtain

$$\begin{aligned}\lim_{n\to\infty} nJ_n^{**}(t-x;x)&=\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{2},\\ \lim_{n\to\infty} nJ_n^{**}\left((t-x)^2;x\right)&=\lim_{n\to\infty}\left(\frac{a}{a-1}x+\frac{3\beta e}{n\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{1}{3n}\right)\\ &=\frac{a}{a-1}x.\end{aligned}$$

From the above limits, we get

$$\lim_{n\to\infty} n\left[J_n^{**}(f;x)-f(x)\right]=\left(\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}+\frac{1}{2}\right)f'(x)+\frac{a}{a-1}\frac{xf''(x)}{2}+\lim_{n\to\infty} nJ_n^{**}\left(\psi(t,x)(t-x)^2;x\right).$$

The limit on the right-hand side of the preceding equation can be shown to vanish. Indeed, this is a direct consequence of the Cauchy–Schwarz inequality:

$$nJ_n^{**}\left(\psi(t,x)(t-x)^2;x\right)\leq \sqrt{J_n^{**}(\psi^2(t,x);x)}\sqrt{n^2J_n^{**}(t-x)^4;x)}.$$

Applying the Korovkin theorem [14], we have

$$\lim_{n\to\infty} J_n^{**}\left(\psi^2(t,x);x\right)=\psi^2(x,x)=0,$$

given that $\psi^2(x,x)=0$, $\psi(t,x)\in C[0,\infty)$ is bounded as $t\to\infty$ and

$$J_n^{**}(t-x)^4;x)=O\left(\frac{1}{n^2}\right),$$

the proof is complete. □

Theorem 3.12. *Let f, f' and $f''\in C[0,\infty)$. We have the following limit:*

$$\lim_{n\to\infty} n\left[J_n^{***}(f;x)-f(x)\right]=\left(-\mu x+\zeta+\frac{1}{2}+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)f'(x)+\frac{a}{a-1}\frac{xf''(x)}{2}.$$

Proof. Applying the Taylor expansion at the point x, we get

$$f(t)=f(x)+(t-x)f'(x)+(t-x)^2\frac{f''(x)}{2}+\psi(t,x)(t-x)^2$$

where

$$\psi(t,x)=\frac{f''(\upsilon)-f''(x)}{2}$$

where υ lies between x and t, with $\lim_{t\to x}\psi(t,x)=0$. By applying the operator J_n^{***} to both sides of the preceding equation, we obtain

$$n\left[J_n^{***}(f;x)-f(x)\right]=nf'(x)J_n^{***}(t-x;x)+n\frac{f''(x)}{2}J_n^{***}\left((t-x)^2;x\right)+nJ_n^{***}\left(\psi(t,x)(t-x)^2;x\right).$$

Taking limit when $n\to\infty$ leads to

$$\lim_{n\to\infty} n\left[J_n^{***}(f;x)-f(x)\right]=f'(x)\lim_{n\to\infty} nJ_n^{***}(t-x;x)+\frac{f''(x)}{2}\lim_{n\to\infty} nJ_n^{***}\left((t-x)^2;x\right)+\lim_{n\to\infty} nJ_n^{***}\left(\psi(t,x)(t-x)^2;x\right).$$

By employing the known moment values, we reach that

$$\begin{aligned}\lim_{n\to\infty} nJ_n^{***}(t-x;x)&=\lim_{n\to\infty} n\left(-\frac{\mu}{n+\mu}x+\frac{\beta e}{(n+\mu)\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{2\zeta+1}{2(n+\mu)}\right)\\&=-\mu x+\zeta+\frac{1}{2}+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)},\end{aligned}$$

$$\lim_{n\to\infty} nJ_n^{***}\left((t-x)^2;x\right)=\lim_{n\to\infty} n\left[\frac{\mu^2}{(n+\mu)^2}x^2+\left(\frac{na}{(n+\mu)^2(a-1)}-\frac{2\beta e\mu}{(n+\mu)^2\left(\beta e+A_j(\lambda,\alpha)\right)}-\frac{(2\zeta+1)\mu}{(n+\mu)^2}\right)x\right.$$

$$+\frac{(2\zeta+3)\beta e}{(n+\mu)^2\left(\beta e+A_j(\lambda,\alpha)\right)}+\frac{\zeta^2+\zeta+\frac{1}{3}}{(n+\mu)^2}\Bigg]$$
$$=\frac{a}{a-1}x.$$

Using the limits derived above, it follows that

$$\lim_{n\to\infty} n\left[J_n^{***}(f;x)-f(x)\right]=\left(-\mu x+\zeta+\frac{1}{2}+\frac{\beta e}{\beta e+A_j(\lambda,\alpha)}\right)f'(x)+\frac{a}{a-1}\frac{xf''(x)}{2}+\lim_{n\to\infty} nJ_n^{***}\left(\psi(t,x)(t-x)^2;x\right).$$

It is straightforward to verify that the limit on the right-hand side of the above equation is zero, using the Cauchy–Schwarz inequality then we have

$$nJ_n^{***}\left(\psi(t,x)(t-x)^2;x\right)\leq\sqrt{J_n^{***}(\psi^2(t,x);x)}\sqrt{n^2J_n^{***}(t-x)^4;x)}.$$

By virtue of the Korovkin theorem [14], the following result is established:

$$\lim_{n\to\infty} J_n^{***}\left(\psi^2(t,x);x\right)=\psi^2(x,x)=0,$$

since $\psi^2(x,x)=0$, $\psi(t,x)\in C[0,\infty)$, and $\psi(t,x)$ remains bounded as $t\to\infty$ together with the fact that

$$J_n^{***}((t-x)^4;x)=O\left(\frac{1}{n^2}\right),$$

the proof is therefore finished. □

4. Conclusion

In this study, we introduced a generalized class of Szász-type operators constructed through U-Charlier-Poisson polynomials and sequences satisfying prescribed conditions. Within this framework, we established a range of approximation properties, thereby extending and enriching the theory of positive linear operators. Furthermore, we demonstrated that the same methodology allows for the development of generalized forms of well-known operators, including the Kantorovich, Stancu and Stancu-Kantorovich variants. These results not only unify diverse operator families under a common setting but also provide new tools for advancing approximation theory. Future research may focus on investigating quantitative estimates, rates of convergence, and potential applications of these generalized operators in numerical analysis and applied mathematics.

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