Filomat 40:4 (2026), 1361–1376
https://doi.org/10.2298/FIL2604361S

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Harish-Chandra homomorphism and representations of quantum superalgebra $U_q(\mathrm{osp}(1,2,\mathbf{c}))$

Chunyan Sun[a], Wanxia Wang[b], Shilin Yang[c,d,*]

[a] *Department of Public Education, Xingtai Medical College, Xingtai, Hebei, 054000, P. R. China*
[b] *College of Big Data and Software Engineering, Zhejiang Wanli University, Ningbo, Zhejiang, 315100, P. R. China*
[c] *School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing, 100124, P. R. China*
[d] *Department of General Education, Shandong Xiehe University, Jinan, Shandong, 250109, P. R. China*

Abstract. In this paper, a new class of quantum algebras is constructed. It is a Hopf superalgebra and an extension of the quantized enveloping superalgebra of $\mathrm{osp}(1,2)$. For these algebras, the Harish-Chandra homomorphism are obtained. All simple modules for an arbitrary parameter q are determined up to isomorphism.

Introduction

It is well known that the usual quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ of the simple Lie algebra $\mathfrak{sl}_2$ has several generalizations. It has been extensively studied recently (see [5], [8]-[10]). Liu [8] introduced a new Hopf algebra $\mathfrak{sl}^t_{q,z}(2)$ by adding an extra central element z to $U_q(\mathfrak{sl}_2)$ with extensive relations. Then he found a family of new universal R-matrices for $\mathfrak{sl}^t_{q,z}(2)$ and proved that $\mathfrak{sl}^t_{q,z}(2)$ is a charmed Hopf algebra. In [5] the quantum algebras $U_q(f(K))$ were introduced and their finite-dimensional representations were investigated. Thereafter, in [9], the author explicitly constructed irreducible representations of the quantum groups $U_q(f_m(K))$, a special case of $U_q(f(K))$. Recently, Wu [10] introduced a new generalization of $U_q(\mathfrak{sl}_2)$ by adding central generators J, J^{-1}. The structures and representations of these generalized algebras were obtained and classified, respectively. We point out that the algebra introduced in [8] was generalized to the general case for finite dimensional simple Lie algebras (see [10, 11]).

In the case of Lie superalgebras, the situation becomes more complex except the quantum superalgebra $U_q(\mathrm{osp}(1,2n))$. The examples related to this topics are given in [12, 13]. The results in [12] state that there is a natural connection between finite-dimensional representations of $U_q(\mathrm{osp}(1,2n))$ and those of the quantum algebra $U_q(\mathrm{so}(2n+1))$. The integrable representations of $U_q(\mathrm{osp}(1,2n))$ were explored in [13]. Among $U_q(\mathrm{osp}(1,2n))$, the quantum superalgebra $U_q(\mathrm{osp}(1,2))$ is the most simple one and of particular interesting. Several results for $U_q(\mathrm{osp}(1,2))$, such as those on its representations, center or scenter, have been explored and described. See for example [1]-[4]. In [4] a new quantum algebra $U_q(\mathrm{osp}(1,2,f))$, a generalization of $U_q(\mathrm{osp}(1,2))$ by the idea in [5], was constructed, its center is also characterized.

2020 *Mathematics Subject Classification*. Primary 20G42; Secondary 17B37, 16T20.
Keywords. quantum superalgebra; simple module; Scasimir element.
Received: 27 September 2025; Revised: 19 October 2025; Accepted: 03 November 2025
Communicated by Dragan S. Djordjević
* Corresponding author: Shilin Yang
Email addresses: katechun@126.com (Chunyan Sun), wangwx@zwu.edu.cn (Wanxia Wang), slyang@bjut.edu.cn (Shilin Yang)

In the present paper, we concern a quantum superalgebra $U := U_q(\mathrm{osp}(1,2,\mathbf{c}))$, which is a generalization of $U_q(\mathrm{osp}(1,2))$ by the way in [8, 10]. This quantum algebra shares many similar properties with the quantum superalgebras $U_q(\mathrm{osp}(1,2))$. However, the algebra is more extensive, and the corresponding results are more abundant. Similar to the idea in [10], the comultiplication and the counit can be equipped with U to ensure that it is a Hopf superalgebra. Then the Harish-Chandra homomorphism is explored. Finally, its finite-dimensional simple modules without any assumption on the parameter q are determined. The results show that not all finite dimensional modules are completely reducible, even if q is not a root of unity.

The paper is organized as follows. In Section 1, we give the definition and some basic results of the generalized quantum algebra $U := U_q(\mathrm{osp}(1,2,\mathbf{c}))$. It is shown that U is a Noetherian Hopf superalgebra without non-zero divisors. In Section 2, the centre of U and the Harish-Chandra homomorphism are obtained. In Section 3, we classify all simple U-modules when q is not a root of unity. An example is given to show that not all finite dimensional modules are completely reducible. This means that the representation theory is unlike the representations of the classic quantum superalgebra $U_q(\mathrm{osp}(1,2))$. In Section 4, all simple U-modules are constructed and classified in the case when q is a d-th primitive root of unity.

Throughout, we denote by $\mathbb{K}$ the algebraically closed field with characteristic zero. The parameter $q \in \mathbb{K}$ is non-zero, $q^2 \neq 1$, and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

1. Preliminaries

First, let us introduce some notations.

For any integer n and $v \in \mathbb{K},\ v \neq 0, \pm 1$, we set

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+3} + v^{-n+1}.$$

Let $[0]_v! = 1$ and for $k > 0$, define

$$[k]_v! = [1]_v[2]_v \cdots [k]_v.$$

For any integers $0 \leq k \leq n$, let

$$\begin{bmatrix} n \\ k \end{bmatrix}_v = \frac{[n]_v!}{[k]_v![n-k]_v!}.$$

For $v = q$, we denote $[n] = [n]_q$ for the simplicity.

Definition 1.1. *The algebra $U = U_q(\mathrm{osp}(1,2,\mathbf{c}))$ is generated by $E,\ F,\ K,\ K^{-1}, \mathbf{c}$, and $\mathbf{c}^{-1}$, with the relations:*

$$KK^{-1} = K^{-1}K = 1, \quad \mathbf{c}\mathbf{c}^{-1} = \mathbf{c}^{-1}\mathbf{c} = 1, \tag{1}$$

$$\mathbf{c}x = x\mathbf{c}, \text{ for } x = E, F, K, K^{-1}, \tag{2}$$

$$KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \tag{3}$$

$$EF + FE = \frac{K - K^{-1}\mathbf{c}^r}{q - q^{-1}}, r \in \mathbb{Z}. \tag{4}$$

Set $t = \sqrt{q}\,\mathbf{i}$, and

$$[K; \mathbf{c}; m]_t = \frac{t^m K - t^{-m} K^{-1}\mathbf{c}^r}{q - q^{-1}}.$$

For $m,\ n \in \mathbb{Z}$, it is straightforward to see that

$$E^m K^n = q^{-mn} K^n E^m, \quad F^m K^n = q^{mn} K^n F^m, \tag{5}$$

$$EF^m - (-1)^m F^m E = \frac{F^{m-1}}{q-q^{-1}} \left[\frac{q^{-m} + (-1)^{m-1}}{q^{-1}+1} K - \frac{q^m + (-1)^{m-1}}{q+1} K^{-1} \mathbf{c}^r \right]. \tag{6}$$

In fact, (6) can be written as

$$EF^m - (-1)^m F^m E = (-1)^{m-1} [m]_t F^{m-1} [K; \mathbf{c}; 1-m]_t. \tag{7}$$

Similarly, the following relations hold in U:

$$E^m F - (-1)^m F E^m = \left[\frac{q^{-m} + (-1)^{m-1}}{q^{-1}+1} K - \frac{q^m + (-1)^{m-1}}{q+1} K^{-1} \mathbf{c}^r \right] \frac{E^{m-1}}{q-q^{-1}}, \tag{8}$$

and hence

$$E^m F - (-1)^m F E^m = (-1)^{m-1} [m]_t [K; \mathbf{c}; 1-m]_t E^{m-1}. \tag{9}$$

Now we apply the Ore extensions to describe the PBW basis of U. For the concept of Ore extension, the readers can refer to [6].

Proposition 1.2. *The algebra U is Noetherian and has no zero divisors and the set*

$$\{E^i F^j K^l \mathbf{c}^s \mid i, j \in \mathbb{N}, l, s \in \mathbb{Z}\}$$

is a basis of U.

Proof. Let $A_0 = \mathbb{K}[K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}]$. Then A_0 is Noetherian with no zero divisor and $\{K^l \mathbf{c}^s \mid l, s \in \mathbb{Z}\}$ is a basis of A_0. Consider the automorphism α_1 of A_0 determined by $\alpha_1(K) = qK, \alpha_1(\mathbf{c}) = \mathbf{c}$, and the corresponding Ore extension $A_1 = A_0[F, \alpha_1, 0]$, we have A_1 has a basis consisting of the monomials $\{F^j K^l \mathbf{c}^s \mid j \in \mathbb{N}, l, s \in \mathbb{Z}\}$. It is easy to prove that A_1 is the algebra generated by F, F^{-1}, K, K^{-1}, $\mathbf{c}$, $\mathbf{c}^{-1}$ satisfying the relations $FK = qKF$, $F\mathbf{c} = \mathbf{c}F$. It follows that A_1 is Noetherian and has no zero divisor.

We define

$$\alpha_2(F^j K^l \mathbf{c}^s) = (-1)^j q^{-l} F^j K^l \mathbf{c}^s,$$
$$\delta(\mathbf{c}^s) = \delta(K^l) = 0, \quad \delta(F^j K^l \mathbf{c}^s) = \left(EF^j - (-1)^j F^j E \right) K^l \mathbf{c}^s,$$

where $j > 0$ and $l, s \in \mathbb{Z}$. It is straightforward to check that α_2 is an automorphism of A_1 and δ is an α_2-derivation of A_1. Therefore, we have the Ore extension $A_2 = A_1[E, \alpha_2, \delta]$ and A_2 is Noetherian with no zero divisor. The set $\{E^i F^j K^l \mathbf{c}^s \mid i, j \in \mathbb{N}, l, s \in \mathbb{Z}\}$ is a basis of A_2. On the other hand, the following relations hold in A_2:

$$EK = \alpha(K)E + \delta(K) = q^{-1} KE,$$

$$E\mathbf{c} = \alpha(\mathbf{c})E + \delta(\mathbf{c}) = \mathbf{c}E,$$

$$EF = \alpha(F)E + \delta(F) = -FE + \frac{K - K^{-1}\mathbf{c}^r}{q - q^{-1}}.$$

Hence, A_2 is isomorphic to U and U has the required properties. □

Recall that a super bialgebra A is defined to be a super vector space together with four maps $m : A \otimes A \longrightarrow A$, $u : \mathbb{K} \longrightarrow A$, $\Delta : A \longrightarrow A \otimes A$, and $\varepsilon : A \longrightarrow \mathbb{K}$ enjoying the following axioms:

1. (A, m, u) is a $\mathbb{Z}_2-$graded associative algebra with multiplication m and unit u;
2. (A, Δ, ε) is a super coalgebra with comultiplication Δ and counit ε;
3. The maps Δ and ε are morphisms of algebras.

If there exists a $\mathbb{K}$-map $S \in \mathrm{Hom}_{\mathbb{K}}(A, A)$ such that

$$\sum_{(a)} a_1 S(a_2) = \sum_{(a)} S(a_1)a_2 = \varepsilon(a)1, \text{ for any } a \in A,$$

A is called a Hopf superalgebra and S is said to be the antipode of A. Note that the multiplication of $A \otimes A$ satisfies

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(a_2)p(b_1)} a_1 a_2 \otimes b_1 b_2,$$

where $a_i,\ b_i \in A$ ($i = 1,\ 2$), and $p:\ A \longrightarrow \{0,\ 1\}$ is the map of graded degree. The antipode $S:\ A \longrightarrow A$ is a graded anti-morphism

$$S(a_1 a_2) = (-1)^{p(a_2)p(a_1)} S(a_2)S(a_1)$$

for homogenous elements $a_1,\ a_2 \in A$.

One can see that U is a $\mathbb{Z}_2$-graded algebra with the map of graded degree defined by

$$p(E^i F^j K^l \mathbf{c}^s) = i + j \mod 2.$$

Define the maps

$$\Delta(K) = K \otimes K, \qquad \Delta(\mathbf{c}) = \mathbf{c} \otimes \mathbf{c},$$
$$\Delta(E) = \mathbf{c}^{-r} \otimes E + E \otimes K\mathbf{c}^r, \Delta(F) = K^{-1}\mathbf{c}^{2r} \otimes F + F \otimes \mathbf{c}^{-r},$$
$$\varepsilon(K) = \varepsilon(\mathbf{c}) = 1, \varepsilon(E) = \varepsilon(F) = 0,$$

and

$$S(E) = -EK^{-1},\ S(F) = -KF\mathbf{c}^{-r},\ S(\mathbf{c}) = \mathbf{c}^{-1},\ S(K) = K^{-1}.$$

We have

Theorem 1.3. *The relations above endow U with a Hopf superalgebra.*

Proof. The proof is straightforward and we give the sketch here. Firstly, one should check that the maps Δ and ε keeps the relations (1)-(4), hence they can be extended to the algebraic homomorphisms of U. Secondly, we should show (U, Δ, ε) is a coalgebra. Finally, we show that S keeps the relations (1)-(4) from U into U^{opp} and that the identities

$$\sum_{(x)} x_1 S(x_2) = \sum_{(x)} S(x_1)x_2 = \varepsilon(x)1$$

hold for $x = E,\ F,\ K,\ K^{-1},\ \mathbf{c}^{-1}$. □

2. The centre and Harish-Chandra homomorphism

In this section, we always assume that q is not a root of unity, and we explore the centre of $U := U_q(\mathrm{osp}(1, 2, \mathbf{c}))$.

It is obvious that U has another gradation: U is a $\mathbb{Z}$−graded algebra with

$$\deg K^{\pm 1} = \deg \mathbf{c}^{\pm 1} = 0,\ \deg E = 1,\ \deg F = -1.$$

Let

$$U_m = \left\{ \sum_{i,l,s} \mathbb{K} F^i K^l E^{i+m} \mathbf{c}^s \mid i, m \in \mathbb{N};\ s, l \in \mathbb{Z} \right\}.$$

In particular,

$$U_0 = \left\{ \sum_{i,l,s} \mathbb{K} F^i K^l E^i \mathbf{c}^s \mid i \in \mathbb{N};\ s, l \in \mathbb{Z} \right\}.$$

Then $U = \bigoplus_{m \in \mathbb{Z}} U_m$.

Firstly, we set $U^0 = \mathbb{K}[K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}]$ and denote the centre of U by $Z(U)$.

Lemma 2.1. *Elements of $Z(U)$ belong to U_0.*

Proof. Let

$$x=\sum x_m\in Z(U),$$

where the summation is finite and $x_m\in U_m$. Then we have

$$x=KxK^{-1}=\sum_{m\in\mathbb{Z}}q^m x_m=\sum_{m\in\mathbb{Z}}x_m,$$

which implies that $q^m=1$, and $m=0$ and $x\in U_0$ since q is not a root of the unity. □

By Lemma 2.1, any element $z\in Z(U)$ has a unique form $z=\sum_{\text{some } i\in\mathbb{N}}F^i h_i E^i$, where $h_i\in U^0$. For $0\neq x\in\mathbb{K}$ and $f(K,\mathbf{c})\in U^0$, we define an algebra isomorphism $\gamma_x: U^0\longrightarrow U^0$ as $\gamma_x(f(K,\mathbf{c}))=f((xK),\mathbf{c})$, which is denoted by $\gamma_x f$.

Lemma 2.2. *The element $x=\sum_{i\in\mathbb{N}}F^i h_i E^i\in Z(U)$ if and only if*

$$h_i=(-1)^i[i+1]_t[K;\mathbf{c};-i]_t h_{i+1}+(-1)^i\gamma_{q^{-1}}h_i,\ \textit{for all } i\in\mathbb{N}. \tag{10}$$

Proof. If $x=\sum_{i\in\mathbb{N}}F^i h_i E^i\in Z(U)$, then

$$\begin{aligned}
Ex &= \sum_{i\in\mathbb{N}}EF^i h_i E^i=\sum_{i\in\mathbb{N}}\Big((-1)^{i-1}[i]_tF^{i-1}[K;\mathbf{c};1-i]_t+(-1)^iF^iE\Big)h_iE^i\\
&= \sum_{i\in\mathbb{N}}[i+1]_tF^i[K;\mathbf{c};-i]_th_{i+1}E^{i+1}+\sum_{i\in\mathbb{N}}(-1)^iF^i\gamma_{q^{-1}}h_iE^{i+1}=xE\\
&= \sum_{i\in\mathbb{N}}F^ih_iE^{i+1}.
\end{aligned}$$

Hence the relation (10) holds.

Conversely, if (10) holds, then $Ex=xE, Fx=xF$ and $Kx=xK$ for any $x\in U_0$. Hence $x\in Z(U)$. □

By Proposition 1.2 and Lemma 2.2, $h_1,h_2,\cdots$ are uniquely determined by h_0.

Now, if $0\neq h_2\in\mathbb{K}$, in particular $h_2=1$, we see that $h_i=0$ for all $i>2$ by (10). In this case, let us determine h_0,h_1.

By (10) we have

$$\begin{aligned}
h_0-\gamma_{q^{-1}}h_0 &= \frac{K-K^{-1}\mathbf{c}^r}{q-q^{-1}}h_1,\\
h_1+\gamma_{q^{-1}}h_1 &= \frac{q^{-1}K-qK^{-1}\mathbf{c}^r}{q-q^{-1}}h_2-\frac{K-K^{-1}\mathbf{c}^r}{q-q^{-1}}h_2=\frac{K+K^{-1}\mathbf{c}^r}{(t-t^{-1})^2}h_2+\frac{q^{-1}K+qK^{-1}\mathbf{c}^r}{(t-t^{-1})^2}h_2.
\end{aligned}$$

Assume that $0\neq h_2\in\mathbb{K}$, we can choose

$$h_1=\frac{K+K^{-1}\mathbf{c}^r}{(t-t^{-1})^2}h_2.$$

It yields that

$$h_0-\gamma_{q^{-1}}h_0 = \frac{K-K^{-1}c^r}{q-q^{-1}}h_1=\frac{qK^2+q^{-1}K^{-2}c^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}h_2-\frac{q^{-1}K^2+qK^{-2}c^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}h_2.$$

Thus

$$h_0=\frac{qK^2+q^{-1}K^{-2}\mathbf{c}^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}h_2.$$

Thus, if we fix that $h_2 = 1 \in \mathbb{K}$, we get that

$$C_q = F^2E^2 + F\frac{K + K^{-1}\mathbf{c}^r}{(t - t^{-1})^2}E + \frac{qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r}}{(q - q^{-1})^2(t - t^{-1})^2}.$$

The element C_q is called the quantum Casimir element.

Let π be the map from U_0 to U^0 defined by

$$\pi(\sum_{i\in\mathbb{N}} F^ih_iE^i) = h_0, \text{ for all } h_i \in U^0.$$

Then π is a linear projection and an algebra homomorphism, which is called the Harish-Chandra homomorphism. It is obvious that

$$\ker\pi = \mathbb{K}\left\langle F^iK^tE^i\mathbf{c}^s \mid i \in \mathbb{N} - \{0\}, t, s \in \mathbb{Z}\right\rangle$$

is an ideal of U_0. In fact, each element $z \in U_0$ can be written as $\pi(z) + \sum_{i>0} F^ih_iE^i$.

Lemma 2.3. *$\pi\mid_{Z(U)}$ is injective from $Z(U)$ to U^0.*

Proof. Suppose that $u = \sum_{i\geq 0} F^ih_iE^i \in Z(U)$, and note that $\pi(u) = h_0$.

If $\pi(u) = 0$, then $h_0 = 0$, and hence $h_i = 0$ for all $i \geq 1$ by Lemma 2.2. Hence $u = 0$ and $\pi|_{Z(U)}$ is injective. □

Let us fix (λ, α) with $\lambda\alpha \neq 0$ and consider $\mathbb{K}$-vector space $V(\lambda, \alpha)$ with a basis $\{v_i \mid i \in \mathbb{N}\}$. For $n \geq 0$, set

$$Kv_n = (-1)^n\lambda t^{-2n}v_n, \quad \mathbf{c}v_n = \alpha^2v_n, \tag{11}$$

$$Ev_{n+1} = (-1)^n\frac{t^{-n}\lambda - t^n\lambda^{-1}\alpha^{2r}}{q - q^{-1}}v_n, \quad Ev_0 = 0, \tag{12}$$

$$Fv_n = [n+1]_tv_{n+1}. \tag{13}$$

It is straightforward to see that $V(\lambda, \alpha)$ is a U-module with the above relations. Such a U-module $V(\lambda, \alpha)$ is called a Verma module of (highest) weight (λ, α) and a (highest) weight vector v_0.

Recall that $\pi(x)$ is in fact a Laurent polynomial in $K, \mathbf{c}$. We denote $\pi(z)(\lambda, \alpha^2)$ the value at $K = \lambda, \mathbf{c} = \alpha^2$.

Lemma 2.4. *Let $V(\lambda, \alpha)$ be the Verma module of weight (λ, α). Then for any central element z of U and $v \in V(\lambda, \alpha)$, we have $zv = \pi(z)(\lambda, \alpha^2)v$.*

Proof. Let v_0 be the highest weight vector of $V(\lambda, \alpha)$ and z a central element of U: $z = \pi(z) + \sum_{i>0} F^ih_iE^i$ where $h_i \in U^0$. Since $Ev_0 = 0, \mathbf{c}v_0 = \alpha^2v_0, Kv_0 = \lambda v_0$, we get $zv_0 = \pi(z)(\lambda, \alpha^2)v_0$. If v is an arbitrary element in $V(\lambda, \alpha)$, then $v = xv_0$ for some $x \in U$, and hence $zv = zxv_0 = \pi(z)(\lambda, \alpha^2)v$. □

Lemma 2.5. *Suppose that $z \in Z(U)$ and $(\gamma_{t^{-1}} \circ \pi)(z) = \sum_{i\in\mathbb{Z}} a_iK^i$, where $a_i \in \mathbb{K}[\mathbf{c}, \mathbf{c}^{-1}]$ has at most finite many non-zero monomials. Then $a_i = 0$ if i is odd.*

Proof. By the assumption, we write $\pi(z) = \sum_{i\in\mathbb{Z}} a_it^iK^i$. Now we choose $h_j \in U^0$, such that

$$z = \sum_{i\in\mathbb{Z}} a_it^iK^i + \sum_{j>0} F^jh_jE^j.$$

For any $\lambda, \alpha \in \mathbb{K}^* = \mathbb{K} - \{0\}$, and the Verma module $V(\lambda, \alpha)$, z acts on $V(\lambda, \alpha)$ as a scalar $a_i(\alpha^2)t^i\lambda^i$, where $a_i(\alpha^2)$ is the value of a_i at $\mathbf{c} = \alpha^2$. For simplicity, we write a_i for $a_i(\alpha^2)$. If $\lambda = \alpha^r t^{n-1}$, then $\sum_{i\in\mathbb{Z}} a_i t^i \lambda^i = \sum_{i\in\mathbb{Z}} a_i t^{ni}\alpha^{ir}$. By the relations (11) and (20), we have

$$\begin{aligned} Kv_n &= (-1)^n \lambda t^{-2n} v_n = (-1)^n t^{-n-1}\alpha^r v_n, \\ Ev_n &= (-1)^{n-1}\frac{t^{1-n}\lambda - t^{n-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} v_{n-1} \\ &= (-1)^{n-1}\frac{t^{1-n}\alpha^r t^{n-1} - t^{n-1}\alpha^{-r}t^{-n+1}\alpha^{2r}}{q-q^{-1}} v_{n-1} \\ &= 0. \end{aligned}$$

Thus there exists a submodule of $V(\alpha^r t^{n-1}, \alpha)$ which is isomorphic to $V((-1)^n\alpha^r t^{-n-1}, \alpha)$, and z acts on the two modules as the same scalar multiplication. Note that z acts on $V((-1)^n\alpha^r t^{-n-1}, \alpha)$ as scalar multiplication by

$$\sum_{i\in\mathbb{Z}} (-1)^{ni} a_i t^{-ni-i}\alpha^{ir} t^i = \sum_{i\in\mathbb{Z}} (-1)^{ni} a_i \alpha^{ir} t^{-ni},$$

we have

$$\sum_{i\in\mathbb{Z}} a_i t^{ni}\alpha^{ir} = \sum_{i\in\mathbb{Z}} (-1)^{ni} a_i\alpha^{ir}t^{-ni} = \sum_{i\in\mathbb{Z}} (-1)^{-ni} a_{-i}\alpha^{-ir}t^{ni}.$$

If we choose a even number n, then $\sum_{i\in\mathbb{Z}} (a_i\alpha^{ir} - a_{-i}\alpha^{-ir})t^{ni} = 0$. If we choose an odd number n, then $\sum_{i\in\mathbb{Z}} (a_i\alpha^{ir} - (-1)^i a_{-i}\alpha^{-ir})t^{ni} = 0$. Both of them have infinite many roots t^n. Therefore, $a_i\alpha^{ir} - a_{-i}\alpha^{-ir} = 0$ and $a_i\alpha^{ir} - (-1)^i a_{-i}\alpha^{-ir} = 0$ for all i. It follows that $a_i\alpha^{ir} = 0$, and hence $a_i = a_i(\alpha^2) = 0$ for any α if i is odd. □

Theorem 2.6. *Suppose that q is not a root of unity. Then the centre $Z(U)$ of U is a commutative algebra generated by the elements $C_q, \mathbf{c}, \mathbf{c}^{-1}$ over the field $\mathbb{K}$, and the restriction of Harish-Chandra homomorphism to $Z(U)$ is an isomorphism onto the subalgebra of $\mathbb{K}[K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}]$ generated by $\mathbf{c}, \mathbf{c}^{-1}$, and $qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r}$.*

Proof. Firstly, assume that $n = 2m (m \geq 1)$, we consider the Verma module $V(t^{n-1}\alpha^r, \alpha)$ for $\alpha \neq 0$. By the relations (11) and (20), $Ev_n = 0$, $Kv_n = t^{-n-1}\alpha^r v_n$. v_n is the highest weight vector of weight $(t^{n-1}\alpha^r, \alpha)$. By Lemma 2.4, a central element z acts on the module generated by v_n as the multiplication by scalar $\pi(z)(t^{-n-1}\alpha^r, \alpha^2)$, but since v_n is in $V(t^{n-1}\alpha^r, \alpha)$, the element z also acts as the scalar $\pi(z)(t^{n-1}\alpha^r, \alpha^2)$. Thus

$$\pi(z)(t^{-n-1}\alpha^r, \alpha^2) = \pi(z)(t^{n-1}\alpha^r, \alpha^2), \tag{14}$$

where $\alpha \neq 0$ and $n = 2m > 0 (m = 1, 2, 3, \cdots)$.

Suppose that $\pi(z) = P(K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1})$, (14) implies that

$$P(t^{n-1}\alpha^r, t^{-n+1}\alpha^{-r}, \alpha^2, \alpha^{-2}) = P(t^{-n-1}\alpha^r, t^{n+1}\alpha^{-r}, \alpha^2, \alpha^{-2}). \tag{15}$$

Let $\psi_\alpha(x) = P(t^{-1}\alpha^r x, t\alpha^{-r}x^{-1}, \alpha^2, \alpha^{-2})$. By (15), $\psi_\alpha(t^n) = \psi_\alpha(t^{-n})$ for all $n = 2m (m = 1, 2, 3, \cdots)$. Hence we can write $\psi_\alpha(x) = \sum_{i\geq 0} a_i(\alpha)(x + x^{-1})^i$, where $a_i(\alpha) \in \mathbb{K}[\alpha, \alpha^{-1}]$. Therefore

$$\psi_\alpha(t\alpha^{-r}K) = \sum_{i\geq 0} a_i(\alpha)\left(t\alpha^{-r}K + t^{-1}\alpha^r K^{-1}\right)^i = P\left(K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}\right).$$

Since $P(K, K^{-1}, (-\alpha)^2, (-\alpha)^{-2}) = P(K, K^{-1}, \alpha^2, \alpha^{-2})$, we have

$$\sum_{i\geq 0} a_i(\alpha)(t\alpha^{-r}K + t^{-1}\alpha^r K^{-1})^i = \sum_{i\geq 0} a_i(-\alpha)(t(-\alpha)^{-r}K + t^{-1}(-\alpha)^r K^{-1})^i,$$

That is

$$\sum_{i\geq 0} a_i(\alpha)\alpha^{-ri}(tK + t^{-1}\alpha^{2r}K^{-1})^i = \sum_{i\geq 0} a_i(-\alpha)(-\alpha)^{-ir}(tK + t^{-1}(-\alpha)^{2r}K^{-1})^i,$$

which implies that $a_i(\alpha)\alpha^{-ri} = a_i(-\alpha)(-\alpha)^{-ir}$. We can write $a_i(\alpha) = \alpha^{ir}b_i(\alpha^2)$, where $b_i(\alpha^2) \in \mathbb{K}[\alpha, \alpha^{-1}]$. Therefore

$$\sum_{i\geq 0} b_i(\alpha^2)(tK + t^{-1}\alpha^{2r}K^{-1})^i = P(K, K^{-1}, \alpha^2, \alpha^{-2}).$$

Consequently,

$$\pi(z) = \sum_{i\geq 0} c_i(\mathbf{c}, \mathbf{c}^{-1})(tK + t^{-1}K^{-1}\mathbf{c}^r)^i,$$

where $c_i(\mathbf{c}, \mathbf{c}^{-1}) = \sum_{\text{finite sum}} a_i\mathbf{c}^i \in \mathbb{K}[\mathbf{c}, \mathbf{c}^{-1}]$ and $i \in \mathbb{Z}$. Moreover, $c_i(\mathbf{c}, \mathbf{c}^{-1}) = 0$ for all odd numbers i by Lemma 2.5. So

$$\begin{aligned}
\pi(z) &= \sum_{2j\geq 0} c_{2j}(\mathbf{c}, \mathbf{c}^{-1})(-1)^j(qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r} - 2\mathbf{c}^r)^j \\
&= \sum_{2j\geq 0} c_{2j}(\mathbf{c}, \mathbf{c}^{-1})(-1)^j \sum_{i=0}^{j} d_i(\mathbf{c})(qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r})^i \\
&= \sum_{i\geq 0} f_i(\mathbf{c}, \mathbf{c}^{-1})(qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r})^i,
\end{aligned}$$

where $d_i(\mathbf{c}) \in \mathbb{K}[\mathbf{c}]$, $f_i(\mathbf{c}, \mathbf{c}^{-1}) \in \mathbb{K}[\mathbf{c}, \mathbf{c}^{-1}]$.

Let $\mathcal{Z}$ be the subalgebra of $\mathbb{K}[K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}]$ generated by $\mathbf{c}, \mathbf{c}^{-1}$, and $qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r}$. For any $f \in \mathcal{Z}$, note that

$$f = \sum_{\text{finite sum}} a_{ij}\mathbf{c}^i(qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r})^j, \ \ i \in \mathbb{Z}, j \in \mathbb{N}.$$

The map π, which enjoys

$$\pi(\mathbf{c}) = \mathbf{c}, \ \pi(\mathbf{c}^{-1}) = \mathbf{c}^{-1}, \pi(C_q) = \frac{qK^2 + q^{-1}K^{-2}\mathbf{c}^{2r}}{(q - q^{-1})^2(t - t^{-1})^2},$$

is a surjective map obviously from $Z(U)$ to the subalgebra $\mathcal{Z}$. Hence $\pi|_{Z(U)}$ from $Z(U)$ to $\mathcal{Z}$ is an isomorphism by Lemma 2.3. Consequently, $Z(U)$ is a commutative algebra generated by the central elements $C_q, \mathbf{c}, \mathbf{c}^{-1}$ (with $\mathbf{c}\mathbf{c}^{-1} = \mathbf{c}^{-1}\mathbf{c} = 1$). It is remarked that C_q, $\mathbf{c}$ are algebraic independent by Proposition 1.2. □

3. Representations when q is not a root of unity

In this section, we always assume that q is not a root of unity. All finite-dimensional simple U-modules are determined.

Let V be a $U-$module. For $(\lambda, \alpha) \in (\mathbb{K}^*)^2$, we set

$$V^{\lambda,\alpha} = \{v \in V \mid Kv = \lambda v, \mathbf{c}v = \alpha^2 v\}.$$

The pair (λ, α) is called a weight of V if $V^{\lambda,\alpha} \neq 0$. Here, the action $\mathbf{c}v = \alpha^2 v$ is possible since $\mathbf{c}$ is a central element and $\mathbb{K}$ is an algebraic closed field.

It is obvious that $EV^{\lambda,\alpha} \subseteq V^{q\lambda,\alpha}$ and $FV^{\lambda,\alpha} \subseteq V^{q^{-1}\lambda,\alpha}$.

Definition 3.1. *Let V be a U-module and $(\lambda, \alpha) \in (\mathbb{K}^*)^2$. If $Ev = 0$, $Kv = \lambda v$ and $\mathbf{c}v = \alpha^2 v$, then $0 \neq v \in V$ is a highest weight vector of weight (λ, α). And a U-module V is the highest weight module of highest weight (λ, α) if V is generated by the highest weight vector of weight (λ, α).*

The analogous proof of [6, Proposition VI.3.3] shows that any non-zero finite-dimensional U-module contains a highest weight vector on which E and F acting are nilpotent. Thus, any simple finite-dimensional U-module is generated by the highest weight vector. Now assume that $v \in V$ is a highest weight vector of weight (λ, α). Set $v_0 = v$ and $v_p = \frac{1}{[p]_t!}F^p v$ for $p > 0$, then

$$Kv_p = q^{-p}\lambda v_p, \mathbf{c}v_p = \alpha^2 v_p, Fv_{p-1} = [p]_t v_p,$$

and

$$Ev_p = (-1)^{p-1}\frac{t^{1-p}\lambda - t^{p-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_{p-1}, \tag{16}$$

where $t = \sqrt{q}\,\mathbf{i}$.

Let V be a finite-dimensional U-module generated by a highest weight vector v of weight (λ, α). The vectors $\{v_p \mid p \geq 0\}$ are defined as above. By the assumption, there is an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. By (16), we obtain $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. Since

$$0 = Ev_{n+1} = (-1)^n\frac{t^{-n}\lambda - t^n\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_n,$$

we have $\lambda^2 = t^{2n}\alpha^{2r}$, which implies that $\lambda = \varepsilon t^n\alpha^r$, where $\varepsilon = \pm 1$.

Theorem 3.2. *Suppose that q is not a root of unity, and V is a finite-dimensional U-module generated by a highest weight vector v of weight (λ, α). Then*

1. *$\lambda = \varepsilon t^n\alpha^r$, where $\varepsilon = \pm 1$ and n is determined by* $\dim V = n+1$;
2. *Let $v_p = \frac{1}{[p]_t!}F^p v$, the set $\{v_0, v_1, ..., v_n\}$ is a basis of V;*
3. *The action of K on V is diagonalizable with $(n+1)$ distinct eigenvalues*

$$\{\varepsilon t^n\alpha^r, -\varepsilon t^{n-2}\alpha^r, \cdots, (-1)^{n-1}\varepsilon t^{-n+2}\alpha^r, (-1)^n\varepsilon t^{-n}\alpha^r\},$$

and $\mathbf{c}$ acts on V by α^2;

4. *Any other highest weight vector in V is a scalar multiple of v and is of weight (λ, α);*
5. *V is simple.*

Proof. The proof of (1)-(3) is easy.

(4) Let v' be another highest weight vector in V. It is an eigenvector for the action of K and $\mathbf{c}$, hence it is a scalar multiple of some vector v_i. But the vector v_i is killed by E if and only if $i = 0$. The result follows.

(5) Let V' be a non-zero U-submodule of V and $0 \neq v' \in V'$. Without loss of generality, we assume that v' is a highest weight vector. Thus, v' is a non-zero scalar multiple of v. Hence $V = V'$ and V is simple. □

Theorem 3.2 implies that, up to isomorphism, there exist two isoclasses of simple U-modules of dimension $n+1$, which are generated by highest weight vectors v of weight $(\varepsilon t^n\alpha^r, \alpha)$. We denote this module by $V_{\varepsilon,n,\alpha}$. The action of U on $V_{\varepsilon,n,\alpha}$ is as follows:

$$Kv_p = (-1)^p t^{n-2p}\varepsilon\alpha^r v_p, \quad \mathbf{c}v = \alpha^2 v_p, \quad Fv_{p-1} = [p]_t v_p, \tag{17}$$

$$Ev_p = (-1)^p\varepsilon\alpha^r\frac{[n-p+1]_t}{t+t^{-1}}v_{p-1}. \tag{18}$$

The following example implies that not all finite-dimensional representations are completely reducible.

Example 3.3. Let V be a 2-dimensional vector space with a basis $\{v_1, v_2\}$. Define

$$E \cdot v_i = F \cdot v_i = 0, \quad i = 1, 2;$$
$$K \cdot v_1 = \varepsilon\alpha^r v_1, \; K \cdot v_2 = \frac{1}{2}\varepsilon r\alpha^{r-2}\beta v_1 + \varepsilon\alpha^r v_2,$$
$$\mathbf{c} \cdot v_1 = \alpha^2 v_1, \; \mathbf{c} \cdot v_2 = \beta v_1 + \alpha^2 v_2,$$

where $\beta \in \mathbb{K}^*$. Then V is indecomposable but not simple.

Proof. Indeed, the matrices of generators E, F, K and $\mathbf{c}$ acting on the basis v_1, v_2 are

$$M_E = M_F = 0, \quad M_{\mathbf{c}} = \begin{pmatrix} \alpha^2 & \beta \\ 0 & \alpha^2 \end{pmatrix}, \quad M_K = \varepsilon\alpha^{r-2}\begin{pmatrix} \alpha^2 & \frac{1}{2}r\beta \\ 0 & \alpha^2 \end{pmatrix},$$

respectively.

It is easy to see that

$$M_K M_{\mathbf{c}} = M_{\mathbf{c}} M_K,\ M_K^2 = M_{\mathbf{c}^r},\ \text{and } M_K, M_{\mathbf{c}} \text{ are invertible.}$$

This implies that V is a U-module. Furthermore, V is not simple since V contains a proper submodule $\mathbb{K}v_1$. V is indecomposable since M_K and $M_{\mathbf{c}}$ can not be diagonalizable simultaneously. Of course, it is not semisimple. □

Define

$$C = C_q + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r} \in Z(U),$$

where

$$C_q = F^2E^2 + F\frac{K + K^{-1}\mathbf{c}^r}{(t-t^{-1})^2}E + \frac{qK^2 + qK^{-2}\mathbf{c}^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}$$

is the quantum Casimir element of U.

Let $v_0 \in V_{\varepsilon,0,\alpha}$, we see that

$$\begin{aligned} Cv_0 &= F^2E^2v_0 + F\frac{K + K^{-1}\mathbf{c}^r}{(t-t^{-1})^2}Ev_0 + \frac{qK^2 + qK^{-2}\mathbf{c}^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}v_0 + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}v_0 \\ &= \frac{q\lambda^2 + q\lambda^{-2}\alpha^{4r}}{(q-q^{-1})^2(t-t^{-1})^2}v_0 + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}v_0 \\ &= 0. \end{aligned}$$

In general, if $\dim V_{\varepsilon',n,\alpha} > 1$, choosing the highest weight vector $v_n \in V_{\varepsilon',n,\alpha}$, we have

$$\begin{aligned} C \cdot v_n &= F^2E^2v_m + F\frac{K + K^{-1}\mathbf{c}^r}{(t-t^{-1})^2}Ev_n + \frac{qK^2 + qK^{-2}\mathbf{c}^{2r}}{(q-q^{-1})^2(t-t^{-1})^2}v_n + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}v_n \\ &= \left(\frac{-t^{2n+2}\alpha^{2r} - t^{-2n-2}\alpha^{2r}}{(q-q^{-1})^2(t-t^{-1})^2} + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}\right)v_n \neq 0. \end{aligned}$$

The last step is due to that q and t are not roots of unity. This implies that for all $v \in V_{\varepsilon',n,\alpha}$,

$$C \cdot v = \left(\frac{-t^{2n+2}\alpha^{2r} - t^{-2n-2}\alpha^{2r}}{(q-q^{-1})^2(t-t^{-1})^2} + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}\right)v \neq 0.$$

Suppose that V' is a simple submodule of V with $\dim V' > 1$, and $\dim V/V' = 1$. We claim that there exists a one-dimensional module V_2 such that $V = V' \oplus V_2$.

Indeed, V/V' has weight $(\varepsilon\alpha^r, \alpha)$ and let

$$C = C_q + \frac{t^2 + t^{-2}}{(q-q^{-1})^2(t-t^{-1})^2}\alpha^{2r}.$$

Since C acts by zero on V/V', we have $CV \subseteq V'$. On the other hand, C acts on V' as some scalar $\beta \neq 0$. Hence $\beta^{-1}C|_{V'}$ is the identity on V' and therefore is a projector of V to V'. This projector is U-linear since C is central. Let $V_2 = \ker(\beta^{-1}C)$, one has $V = V' \oplus V_2$.

4. Representations when q is a root of unity

Let
$$\mathcal{S}_q = q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}\mathbf{c}^r - \eta FE,$$
where $\eta = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q - q^{-1})$, $\mathcal{S}_q$ is called the Scasimir element of U. The Scasimir element $\mathcal{S}_q$ is useful for exploring the representations of U in the case that q is a root of unity. It is straightforward to check that

$$\mathcal{S}_qF = -F\mathcal{S}_q,\ \ \mathcal{S}_qE = -E\mathcal{S}_q,\ \ \mathcal{S}_qK = K\mathcal{S}_q,\ \ \mathcal{S}_q^2 - 2\mathbf{c}^r = \left(q - q^{-1}\right)^2\left(t - t^{-1}\right)^2 C_q. \tag{19}$$

Let q be a d-th primitive root of unity ($d > 2$), and

$$e = \begin{cases} d, & \text{if } d \text{ is even;} \\ 2d, & \text{if } d \text{ is odd;} \end{cases} \qquad e' = \begin{cases} \frac{d}{2}, & \text{if } d \text{ is twice an odd integer;} \\ e, & \text{otherwise.} \end{cases}$$

It is easy to see that e is always even, $q^e = 1$ and $[e]_q = 0$. Also, $[e']_q = 0$ and $-q$ is the e'-th root of unity.

Lemma 4.1. 1. *The elements E^e, F^e and K^e belong to the center of U.*
2. *If d is twice an odd integer, then*
$$E^{e'}F = -FE^{e'},\ F^{e'}E = -EF^{e'},\ KE^{e'} = -E^{e'}K,\ KF^{e'} = -F^{e'}K.$$

Proof. The proof is straightforward.
(1) For example, since
$$E^eF - FE^e = \left[\frac{q^{-e} + (-1)^{e-1}}{1+q^{-1}}K - \frac{q^e + (-1)^{e-1}}{1+q}K^{-1}\mathbf{c}^r\right]\frac{E^{e-1}}{q-q^{-1}} = 0,$$
and $KE^eK^{-1} = (KEK^{-1})^e = (qE)^e = q^eE^e = E^e$. So E^e belongs to the center of U.
(2) Similarly,
$$E^{e'}F + FE^{e'} = \left[\frac{q^{-e'} + (-1)^{e'-1}}{1+q^{-1}}K - \frac{q^{e'} + (-1)^{e'-1}}{1+q}K^{-1}\mathbf{c}^r\right]\frac{E^{e'-1}}{q-q^{-1}} = 0.$$
Here we use the fact that $q^{e'} = -1$ and $(-1)^{e'-1} = 1$. □

Recall that if V is a finite-dimensional simple U-module, then $\mathrm{End}_U(V) = \mathbb{K}$. This means that the central element acting on V is scalar.

Considering the Verma module $V(\lambda, \alpha)$ of U with a basis $\{v_i \mid i \in \mathbb{N}\}$:
$$\begin{aligned} &Kv_n = (-1)^n\lambda t^{-2n}v_n, \quad \mathbf{c}v_n = \alpha^2 v_n, \\ &Ev_{n+1} = (-1)^n\frac{t^{-n}\lambda - t^n\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_n, \quad Ev_0 = 0, \\ &Fv_n = [n+1]_t v_{n+1}, \end{aligned}$$

Proposition 4.2. *Keeping notations as above. Then we have*
$$\mathcal{S}_qv_0 = bv_0, \quad \mathcal{S}_qv_p = (-1)^p bv_p,$$
for some $b \in \mathbb{K}$ and all $0 \le p \le n$.

Proof. Let

$$b = q^{\frac{1}{2}}\lambda - q^{-\frac{1}{2}}\lambda^{-1}\alpha^{2r} \in \mathbb{K}.$$

Since v_0 is the highest weight vector of $V_{\varepsilon,n,\alpha}$, we have

$$\mathcal{S}_q v_0 = q^{\frac{1}{2}}Kv_0 - q^{-\frac{1}{2}}K^{-1}\mathbf{c}^r v_0 - \eta FEv_0 = \left(q^{\frac{1}{2}}\lambda - q^{-\frac{1}{2}}\lambda^{-1}\alpha^{2r}\right)v_0 = bv_0.$$

Therefore,

$$\mathcal{S}_q v_p = \mathcal{S}_q F^p v_0 = (-1)^p F^p \mathcal{S}_q v_0 = (-1)^p F^p bv_0 = (-1)^p bv_p.$$

The result follows. □

Lemma 4.3. *There is no finite-dimensional simple U-module of dimension greater than e.*

Proof. Suppose that V is a U-module V of dimension greater than e. Then there exists a non-zero submodule of dimension less than or equal to e.

If d is even, then $\mathbf{c}, K^{\frac{d}{2}}\mathcal{S}_q, E^e, F^e$ belong to the center of U, and thus $K, K^{\frac{d}{2}}\mathcal{S}_q, \mathbf{c}, E^e, F^e$ commute with each other.

If d is odd, then $\mathbf{c}, E^e, F^e$ belong to the center of U, and $K, \mathbf{c}, \mathcal{S}_q, E^e, F^e$ commute with each other.

In any case, by the knowledge of linear algebras there exists a non-zero vector $v \in V$ such that

$$Kv = \lambda v,\ \mathbf{c}v = \alpha^2 v,\ \mathcal{S}_q v = cv,\ E^e v = \varrho v, \text{ and } F^e v = bv,$$

where $\varrho, b \in \mathbb{K}$ and c is determined by ϱ, b.

Two cases should be discussed.

Case 1. If $F^e v = 0$, then there exists a integer $0 \le \ell < e$ such that $F^\ell v \neq 0$ and $F^{\ell+1}v = 0$. Replacing v by $F^\ell v$, without loss of generality, we can assume that $Fv = 0$. In this case, we claim that the subspace V' spanned by $v, Ev, \cdots, E^{e-1}v$ is a submodule of V.

To see this, it is enough to check that V' is stable under the action of $E, F, K, \mathbf{c}$. It is obvious for $E, K, \mathbf{c}$. For example, if $\ell < e-1$, then $E(E^\ell v) = E^{\ell+1}v \in V'$ and $E(E^{e-1}v) = E^e v = av \in V'$. Finally, V' is stable under the action of F. Indeed, recall that

$$FE^\ell = (-1)^\ell E^\ell F + [\ell]_t[K;\mathbf{c};1-\ell]_t E^{\ell-1}, \text{ for all } \ell > 0.$$

Therefore, we have

$$\begin{aligned} FE^\ell v &= (-1)^\ell E^\ell Fv + [d]_t[K;\mathbf{c};1-\ell]_t E^{\ell-1}v \\ &= [\ell]_t \frac{q^{\ell-1}t^{1-\ell}\lambda - q^{1-\ell}t^{\ell-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} E^{\ell-1}v \in V', \text{ for all } \ell > 0. \end{aligned}$$

Case 2. If $F^e v = bv \neq 0$ for some $b \neq 0$, then the space V' spanned by $v, Fv, \cdots, F^{e-1}v$ is a submodule of V. To see this, it is enough to check that V' is stable under the actions of $E, F, K, \mathbf{c}$.

Indeed, it is obvious that V' is stable under the actions of F, K and $\mathbf{c}$. To see that V' is stable under the action of E, we recall that

$$\mathcal{S}_q = q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}\mathbf{c}^r - \eta FE,$$

where $\eta = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q - q^{-1})$. Note that $\mathcal{S}_q F^{e-1}v = (-1)^{e-1}cF^{e-1}v$, we get

$$\begin{aligned} Ev &= b^{-1}EF^e v = b^{-1}(EF)F^{e-1}v \\ &= b^{-1}\left[\frac{K - K^{-1}\mathbf{c}^r}{q-q^{-1}} - \frac{1}{\eta}(q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}\mathbf{c}^r - \mathcal{S}_q)\right]F^{e-1}v \\ &= \left(b^{-1}f\right)F^{e-1}v, \end{aligned}$$

where

$$f = \frac{q^{1-e}\lambda - q^{e-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} - \frac{1}{\eta}\left(q^{\frac{3}{2}-e}\lambda - q^{-\frac{3}{2}+e}\lambda^{-1}\alpha^{2r} + (-1)^{e-1}c\right). \tag{20}$$

That is

$$c = (-1)^e\left[ab - \frac{\eta\left(q^{1-e}\lambda - q^{e-1}\lambda^{-1}\alpha^{2r}\right)}{q-q^{-1}} + \left(q^{\frac{3}{2}-e}\lambda - q^{-\frac{3}{2}+e}\lambda^{-1}\alpha^{2r}\right)\right],$$

where $a = b^{-1}f$. For any $0 \le \ell \le e-1$, we have

$$\begin{aligned} EF^{\ell+1}v &= (-1)^{\ell}[\ell+1]_t F^{\ell}[K;\mathbf{c};-\ell]_t v + (-1)^{\ell+1}F^{\ell+1}Ev \\ &= (-1)^{\ell}\left([\ell+1]_t\frac{t^{-\ell}\lambda - t^{\ell}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} - f\right)F^{\ell}v. \end{aligned}$$

This implies that V' is stable under the action of E.

The proof is finished. □

Recall that q is a d-th primitive root of unity ($d > 2$). Let $W_{\lambda,\alpha,a}$ be a d-dimensional U-module with the basis $\{v_0, v_1, \cdots, v_{d-1}\}$. The action of U is given by

$$\begin{aligned} &Kv_p = q^p\lambda v_p, \quad \mathbf{c}v_p = \alpha^2 v_p, \\ &Ev_p = v_{p+1}, \quad Ev_{d-1} = av_0; \\ &Fv_{p+1} = [p+1]_t\frac{q^p t^{-p}\lambda - q^{-p}t^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_p, \quad Fv_0 = 0, \end{aligned}$$

where $0 \le p < d-1$. Obviously, the matrix of F under the basis $v_0, \cdots, v_{d-1}$ can not be diagonalizable, and $W_{\lambda,\alpha,a}$ is indecomposable. In particular, $W_{\lambda,\alpha,a}$ is simple if $a \ne 0$.

Let $W_{\lambda,\alpha,a,b}$ be a d-dimensional U-module with the basis $\{v_0, v_1, \cdots, v_{d-1}\}$. The action of U is given by

$$\begin{aligned} &Kv_p = q^{-p}\lambda v_p, \quad \mathbf{c}v_p = \alpha^2 v_p, \\ &Ev_{p+1} = (-1)^p\left([p+1]_t\frac{t^{-p}\lambda - t^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}} - ab\right)v_p, \text{ and } Ev_0 = av_{d-1}, \\ &Fv_p = v_{p+1}, \text{ and } Fv_{d-1} = bv_0, \end{aligned}$$

where $0 \le p < d-1$. If $b \ne 0$, the matrices of E and F under the basis $v_0, \cdots, v_{d-1}$ can not be diagonalized simultaneously, and $W_{\lambda,\alpha,a,b}$ is indecomposable. In particular, $W_{\lambda,\alpha,a,b}$ is simple if $b \ne 0$.

Theorem 4.4. *Any non-zero finite-dimensional simple U-module up to isomorphism is one of the following lists:*

1. $V_{\varepsilon,n,\alpha}, 0 \le n \le e'-1$;
2. $W_{\lambda,\alpha,a},\ a \ne 0$;
3. $W_{\lambda,\alpha,a,b},\ b \ne 0$.

Proof. By Lemma 4.3, the dimension of a simple U-module V is less than e or equal to e.

Recall that $V_{\varepsilon,n,\alpha}$ is a U-module with another basis $v_0, \cdots, v_p = F^p v_0, \cdots, v_n = F^n v_0$. The actions on this basis can be written as

$$\begin{aligned} &Kv_p = (-1)^p t^{n-2p}\varepsilon\alpha^r v_p, \quad \mathbf{c}v = \alpha^2 v_p, \\ &Fv_{p-1} = v_p,\ Fv_n = 0, \\ &Ev_0 = 0,\ Ev_p = (-1)^p\varepsilon\alpha^r\frac{[p]_t[n-p+1]_t}{t+t^{-1}}v_{p-1}, \end{aligned}$$

where v_0 is the highest weight vector of $V_{\varepsilon,n,\alpha}$ for $0 \le p \le n$.

If $n \geq e'$, then $V_{\varepsilon,n,\alpha}$ is not simple since $v_{e'}, \cdots, v_n$ span a proper submodule of $V_{\varepsilon,n,\alpha}$.

If $n < e'$, it is easy to see that $V_{\varepsilon,n,\alpha}$ is simple.

Now we suppose that V is a simple U-module with $\dim V = \ell \leq e$. By the proof of Lemma 4.3, we see that V with a basis $\{v_0, v_1, \cdots, v_{\ell-1}\}$ has to be the following forms:

$$Kv_p = q^p\lambda v_p, \qquad \mathbf{c}v_p = \alpha^2 v_p,$$
$$Ev_p = v_{p+1}, \text{ and } Ev_{\ell-1} \in V;$$
$$Fv_{p+1} = [p+1]_t\frac{q^pt^{-p}\lambda - q^{-p}t^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_p, \qquad Fv_0 = 0,$$

for some λ, where $0 \leq p < \ell - 1$.

Let $Ev_{\ell-1} = \sum_{0\leq i\leq \ell-1} a_iv_i$. Since $KEv_{\ell-1} = q^\ell\lambda Ev_{\ell-1}$, we have

$$q^\ell\lambda\sum_{0\leq i\leq\ell-1}a_iv_i = \lambda\sum_{0\leq i\leq\ell-1}q^ia_iv_i.$$

It deduces that $v_\ell = Ev_{\ell-1} = 0$ or $Ev_{\ell-1} \neq 0$.

In the previous case, we have

$$[\ell]_t\frac{q^{\ell-1}t^{-(\ell-1)}\lambda - q^{-(\ell-1)}t^{\ell-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} = 0.$$

Hence

$$t^\ell - t^{-\ell} = 0 \quad \text{ or } \quad q^{\ell-1}t^{-(\ell-1)}\lambda - q^{-(\ell-1)}t^{\ell-1}\lambda^{-1}\alpha^{2r} = 0.$$

So, $\ell = d$ is even, or $\ell = \frac{d}{2}$ is odd, and hence d is a twice an odd integer, or $\ell = 2d$ when d is odd; or $\lambda = \varepsilon q^{-(\ell-1)}t^{\ell-1}\alpha^r$. Hence, $\ell = e'$ or $\lambda = \varepsilon q^{-(\ell-1)}t^{\ell-1}\alpha^r (1 \leq \ell < e')$. Therefore, we get the simple modules $V = V_{\varepsilon,n,\alpha}$ for $0 \leq n \leq e' - 1$ up to isomorphism as the proof of Theorem 3.2.

In the latter case, if $\ell < e$, then $\ell = d$ is an odd number, and we must have that $Ev_{\ell-1} = av_0$ for $a \neq 0$. In this case, we get a simple U-module of dimension d:

$$Kv_p = q^p\lambda v_p, \qquad \mathbf{c}v_p = \alpha^2 v_p,$$
$$Ev_p = v_{p+1}, \qquad Ev_{d-1} = av_0 \ (a \neq 0);$$
$$Fv_{p+1} = [p+1]_t\frac{q^pt^{-p}\lambda - q^{-p}t^p\lambda^{-1}}{q-q^{-1}}\alpha^r v_p, \qquad Fv_0 = 0,$$

where $0 \leq p < d - 1$, which is just $W_{\lambda,\alpha,a}$, $a \neq 0$.

By Lemma 4.3, V is of basis $\{v_0, v_1, \cdots, v_{e-1}\}$ with

$$Kv_p = q^p\lambda v_p, \qquad \mathbf{c}v_p = \alpha^2 v_p,$$
$$Ev_p = v_{p+1}, \qquad Ev_{e-1} = av_0;$$
$$Fv_{p+1} = [p+1]_t\frac{q^pt^{-p}\lambda - q^{-p}t^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_p, \qquad Fv_0 = 0$$

where $0 \leq p < e - 1$.

If d is odd, then $e = 2d$, then V is not simple since $\{v_0+v_d, v_1+_{d+1}, \cdots, v_{d-1}+v_{e-1}\}$ span a simple submodule $W_{\lambda,\alpha,a}$ of V with the dimension d.

So, we can assume that $\ell = d = e$ is even. If $Fv_i \neq 0$ for all $1 \leq i \leq e-1$, hence $\lambda \neq \varepsilon q^{i-1}t^{1-i}\alpha^r$, then $V = W_{\lambda,\alpha,a}$ is simple. Indeed, we can choose another basis

$$\left\{y_0 = v_{e-1}, y_1 = Fy_0, \cdots, \ y_{e-1} = F^{e-1}y_0\right\}$$

and we have

$$K \cdot y_p = \lambda' q^{-p}y_p, \ Ey_{p+1} = \kappa_py_p, \ Fy_{e-1} = 0,$$

for some $\lambda' \neq 0$, $\kappa_p \neq 0$, and $0 \leq p \leq e-1$ by the assumptions. Let

$$v = \sum_{\text{some } i} a_i y_i \neq 0, \text{ where } a_i \neq 0.$$

Then we have

$$K^l \cdot v = \sum_{\text{some } i} a_i \lambda'^l q^{-li} y_i.$$

Choosing suitable numbers l, we can rewrite y_i as a combination of $K^l \cdot v$ by Cramer's Rule and Vandermonde determinant. In particular, y_0 can be generated by v over $\mathbb{K}[K, K^{-1}, \mathbf{c}, \mathbf{c}^{-1}]$. So $W_{\lambda,\alpha,a} = U \cdot v$ is simple.

If $Fv_i = 0$ for some $1 \leq i \leq e-1$, then $\lambda = \varepsilon q^{i-1} t^{1-i} \alpha^r$, $W_{\lambda,\alpha,0}$ is not simple since the space spanned by $v_i, v_{i+1}, \cdots, v_{e-1}$ is its proper submodule. But if $a \neq 0$, even if $\lambda = \varepsilon q^{i-1} t^{1-i} \alpha^r$, $W_{\lambda,\alpha,a}$ is still simple by the analogous statements as above. Consequently, $W_{\lambda,\alpha,a}$, $(a \neq 0)$ is simple whenever d is even or odd.

If $F^e v = bv$ for some $b \neq 0$, then V is spanned by linearly independent vectors $v, Fv, \cdots, F^{e-1}v$. Let $v_0 = v$ and $v_p = F^p v_0 (0 \leq p \leq e-1)$. Then

$$Kv_p = q^{-p}\lambda v_p, \ \mathbf{c}v_p = \alpha^2 v_p, \ E^e v_0 = \varrho v_0, \text{ and } Fv_{e-1} = bv_0,$$

where $\varrho, b \in \mathbb{K}$.

We get that

$$\begin{aligned} Ev_0 &= b^{-1}EF^e v_0 = b^{-1}(EF)F^{e-1}v \\ &= b^{-1}\left[\frac{K - K^{-1}\mathbf{c}^r}{q - q^{-1}} - \frac{1}{\eta}(q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}\mathbf{c}^r - \mathcal{S}_q)\right]F^{e-1}v \\ &= \left(b^{-1}f\right)v_{e-1} := av_{e-1}, \end{aligned}$$

where $a = b^{-1}f$, hence $f = ab$, and f as in (20).

For any $0 \leq p < e-1$, we have

$$\begin{aligned} Ev_{p+1} &= EF^{p+1}v_0 = (-1)^p[p+1]_t F^p[K; \mathbf{c}; -p]_t v_0 + (-1)^{p+1}F^{p+1}Ev_0 \\ &= (-1)^p\left([p+1]_t \frac{t^{-p}\lambda - t^p\lambda^{-1}\alpha^{2r}}{q - q^{-1}} - f\right)F^p v_0 \\ &= (-1)^p\left([p+1]_t \frac{t^{-p}\lambda - t^p\lambda^{-1}\alpha^{2r}}{q - q^{-1}} - ab\right)v_p. \end{aligned}$$

Hence V:

$$\begin{aligned} &Kv_p = q^{-p}\lambda v_p, \qquad \mathbf{c}v_p = \alpha^2 v_p, \\ &Ev_{p+1} = (-1)^p\left([p+1]_t \frac{t^{-p}\lambda - t^p\lambda^{-1}\alpha^{2r}}{q - q^{-1}} - ab\right)v_p, \text{ and } Ev_0 = av_{e-1}, \\ &Fv_p = v_{p+1}, \ \text{ and } Fv_{e-1} = bv_0, \end{aligned}$$

where $0 \leq p < e-1$.

If d is odd, then $e = 2d$, V is not simple since $\{v_0 + v_d, v_1 + v_{d+1}, \cdots, v_{d-1} + v_{e-1}\}$ span a simple submodule $W_{\lambda,\alpha,a,b}$ of dimension d.

If d is even, then $e = d$, and $V = W_{\lambda,\alpha,a,b}$ is simple by the analogous statement as the previous cases.

The results are followed. □

Acknowledgments

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their insightful comments and constructive suggestions, which have greatly improved the quality of this manuscript. This work was supported by the National Natural Science Foundation of China (Grants No. 12471038 and 12561007).

References

[1] Arnaudon D., Centre and representations of small quantum superalgebras at root of unity, Czech. J. Physics, 46(1996), 1121-1129.
[2] Arnaudon D., Bauer M., Scasimir operator, scentre and representations of $U_q(\mathrm{osp}(1|2))$, Lett. Math. Phys., 40(1997), 307-320.
[3] Gomes X., Projective indecomposable modules over the small quantum osp(1|2), Czech. J. Physics, 50(2000), 71-79.
[4] Hou B., Zhang Z. L., Cai B. L., The structure of quantum group $U_q(\mathrm{osp}(1,2,f))$, J. Math Res. and Exp., 31(2011), 451-461.
[5] Ji Q., Wang D., Zhou X., Finite-dimensional representations of quantum groups $U_q(f(K))$, East-West J Math., 2(2000), 201-203.
[6] Kassel C., Quantum groups, Graduated Texts in Mathematics, Springer-Verlag, 1995, vol. 155.
[7] Kobabayashi K. I., Representation theory of $\mathrm{osp}(1|2)_q$, Z. Phys. C, 59(1993), 155-158.
[8] Liu K., A family of new universal R-matrices, Proc. of AMS, 125(1997), 987-999.
[9] Tang X., Constructing irreducible representations of quantum groups $U_q(f_m(K))$, Front Math China, 3(2008), 371-397.
[10] Wu Z., Extended quantum enveloping algebras of $sl(2)$, Glasgow Math. J., 51(2009), 441-465.
[11] Wu Z., Extension of a quantized enveloping algebra by a Hopf algebra, Sci. China Math., 53(2010), 1337-1344.
[12] Zhang R. B., Finite-dimensional representations of $U_q(\mathrm{osp}(1|2n))$ and its connection with quantum $so(2n+1)$, Lett. Math. Phys., 25(1992), 317.
[13] Zou Y. M., Integrable representations of $U_q(\mathrm{osp}(1,2n))$, J. Pure Appl. Algebra, 130(1998), 99-112.
[14] Zhang Y., Li L., The centre of quantum group $U_q(\mathrm{osp}(1,2))$, J. of Math.(Chinese), 26(2006), 569-573.