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The Pexiderized Cauchy functional equation on restricted domains and its asymptotic properties

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Abstract. The aim of this paper is to investigate the Hyers–Ulam stability of the Pexiderized Cauchy functional equation on certain restricted domains in normed spaces. We also investigate the asymptotic behavior and hyperstability of this equation and show how approximate solutions may lead to exact ones under suitable conditions. The results obtained in this work extend and generalize several well-known findings in the literature.

1. Introduction

The investigation of stability in functional equations began with a pivotal question posed by S. M. Ulam [37] in 1940: *"Given an approximately linear mapping f, when does a linear mapping T estimating f exist?"* In the next year, D. H. Hyers [16] provided the first affirmative answer in the context of Banach spaces, laying the foundation for what is now recognized as Hyers–Ulam stability theory.

The field expanded significantly with contributions from various mathematicians. In 1950, T. Aoki [2] extended Hyers' results to additive mappings. In 1978, Th. M. Rassias [31] broadened these findings by introducing an unbounded form of the Cauchy difference

$$\|Cf(x,y)\| \leqslant \varepsilon\left(\|x\|^p + \|y\|^p\right),$$

where $Cf(x,y) := f(x+y) - f(x) - f(y)$, $\varepsilon > 0$ and $p \in [0,1)$. This work has been instrumental in evolving the Hyers–Ulam–Rassias stability theory, which remains fundamental in the study of functional equations. A generalization of Rassias' theorem was obtained by G.-L. Forti [14] in 1980.

In parallel, a major advance occurred in 1994 when P. Găvruta [15] introduced the concept of generalized Hyers–Ulam–Rassias stability, further broadening the theory's applicability. Over the decades, the stability of functional equations has been extensively studied, leading to numerous generalizations and applications.

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Since then, many mathematicians have studied extensively the stability problems of several functional equations. For further information on this area of research, we can refer the reader to [1, 3–13, 17, 20–22, 24–29, 32, 34, 36].

Skof's introduction of stability analysis within restricted domains marked a notable development, demonstrating that inequalities governing the Cauchy equation remain stable under conditions such as $\|x\| + \|y\| \geqslant d$. This finding illustrated that local stability can imply global validity, a principle central to ongoing research. Building on Skof's insights, researchers like S. M. Jung, J. M. Rassias and J. M. Rassias et al. explored stability problems in restricted domains for the Jensen functional equation [18], mixed type functional equations [30] and Jensen type functional equations [33]. A unified conclusion emerged: for mappings $f : X \to Y$ between real normed and Banach spaces, the satisfaction of key inequalities for pairs (x, y) where $\|x\| + \|y\| \geqslant d$ guarantees their validity throughout the space $X \times X$.

In 1983, F. Skof [35] resolved the Ulam stability problem for additive mappings within restricted domains. Her theorem established that a mapping $f : \mathbb{R} \to E$ satisfying

$$\|Cf(x, y)\| \leqslant \varepsilon \quad \text{for } |x| + |y| > d$$

admits a unique additive approximation $A : \mathbb{R} \to E$ with

$$\|f(x) - A(x)\| \leqslant 9\varepsilon \quad \text{for all } x \in \mathbb{R},$$

where E is a Banach space and $\mathbb{R}$ is the set of real numbers. This result was crucial in characterizing asymptotic behaviors, particularly illustrating the equivalence between the convergence of $Cf(x, y)$ to a fixed $z \in Y$ as $\|x\| + \|y\| \to \infty$ and the exact condition $Cf(x, y) = z$ universally.

Recent advancements by Z. Kominek [23] further extended the stability results of the Jensen equation across bounded and unbounded domains, while S. M. Jung et al. [21] dealt with the generalized Jensen equation. Jung's work [19] notably advanced the Hyers–Ulam stability for additive and quadratic equations, paving the way for modern refinements in error bounds and domain generalizations.

The main purpose of this paper is to study the stability analysis of the Pexiderized Cauchy functional equation $f(x + y) = g(x) + h(y)$ on certain restricted domains in a normed space. As an application, we consider the asymptotic behavior and hyperstability results of this functional equation.

2. Main Results

Let X and Y be normed vector spaces. We denote $\mathscr{Y}$ as a Banach space. In [21], several results regarding the stability of the Pexiderized Cauchy equation on the punctured space $X_0 = X \setminus \{0\}$ were established. In this section, the stability of the Pexiderized Cauchy equation $f(x + y) = g(x) + h(y)$ is investigated and studied within certain restricted domains. As applications of the obtained results, the asymptotic behavior of the Pexiderized Cauchy equation is also examined. The results of this section complement and extend those achieved in [21].

Theorem 2.1. *Let $f, g, h : X \to \mathscr{Y}$ be mappings satisfying*

$$\|f(x + y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d \tag{2.1}$$

for some $\varepsilon \geqslant 0$ and $d > 0$. Then, there exists a unique additive mapping $A : X \to \mathscr{Y}$ such that

$$\|A(x) - f(x) + f(0)\| \leqslant 12\varepsilon, \quad x \in X, \tag{2.2}$$

$$\|A(z) - g(z) - A(a) + g(a)\| \leqslant 14\varepsilon, \quad \min\{\|a\|, \|z\|\} \geqslant d, \tag{2.3}$$

$$\|A(z) - h(z) + A(a) + h(-a)\| \leqslant 14\varepsilon, \quad \min\{\|a\|, \|z\|\} \geqslant d. \tag{2.4}$$

Moreover,

$$\|A(z) - g_o(z)\| \leqslant 3\varepsilon, \quad \|z\| \geqslant d,$$
$$\|A(z) - h_o(z)\| \leqslant 3\varepsilon, \quad \|z\| \geqslant d,$$

where g_o and h_o are the odd parts of g and h, respectively.

Proof. We use the symbol $[s,t]$ to denote the substitution in which x and y in (2.1) are replaced by s and t, respectively. Let $a \in X$ such that $\|a\| \geqslant d$. The substitutions $[x,-a]$, $[a,y]$ and $[a,-a]$ in (2.1) yield

$$\|f(x-a) - g(x) - h(-a)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.5}$$
$$\|f(a+y) - g(a) - h(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d, \tag{2.6}$$
$$\|f(0) - g(a) - h(-a)\| \leqslant \varepsilon. \tag{2.7}$$

Adding (2.5) and (2.6), we get

$$\|f(x-a) + f(a+y) - g(x) - h(y) - g(a) - h(-a)\| \leqslant 2\varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.8}$$

Combining (2.1) and (2.8), we obtain

$$\|f(x-a) + f(a+y) - f(x+y) - g(a) - h(-a)\| \leqslant 3\varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.9}$$

From (2.7) and (2.9) we obtain

$$\|f(x-a) + f(a+y) - f(x+y) - f(0)\| \leqslant 4\varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.10}$$

Letting $x = 3a$ and $y = a$ in (2.10) we obtain

$$\|2f(2a) - f(4a) - f(0)\| \leqslant 4\varepsilon.$$

Thus the recent inequality yields

$$\|f(2x) - 2f(x) + f(0)\| \leqslant 4\varepsilon, \quad \|x\| \geqslant d.$$

Hence

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{k=m}^{n} \frac{f(0)}{2^{k+1}} \right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{2^{k-1}} \tag{2.11}$$

for all $\|x\| \geqslant d$ and $n \geqslant m \geqslant 0$. As a result, the sequence $\{\frac{f(2^n x)}{2^n}\}_{n=1}^{\infty}$ forms a Cauchy sequence for $\|x\| \geqslant d$. Consequently, it is also a Cauchy sequence for all $x \in X$. By the completeness of $\mathscr{Y}$, we can define a mapping $A : X \to \mathscr{Y}$ as

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

It is clear that $A(0) = 0$ and $A(2x) = 2A(x)$ for all $x \in X$. Let $x, y, a \in X \setminus \{0\}$. Then there exists $m \in \mathbb{N}$ such that $\min\{\|2^n x\|, \|2^n y\|, \|2^n a\|\} \geqslant d$ for each $n \geqslant m$. By substituting x, y, and a with $2^n x$, $2^n y$, and $2^n a$ in (2.10), dividing both sides of the resulting inequality by 2^n, and using the definition of A, we obtain that

$$A(x-a) + A(a+y) = A(x+y), \quad x, y, a \in X \setminus \{0\}. \tag{2.12}$$

Letting $x = -a$ and $y = a$ in (2.12) and using $A(0) = 0$ and $A(2a) = 2A(a)$, we infer that A is an odd mapping. By setting $a = 2x$ in (2.12), we derive

$$-A(x) + A(2x+y) = A(x+y), \quad x, y \in X \setminus \{0\}. \tag{2.13}$$

It is evident that (2.13) remains valid for all $x, y \in X$. Now, substituting x with $-x$ and y with $2x + y$ in (2.13), we obtain

$$A(x) + A(y) = A(x + y), \quad x, y \in X.$$

Hence, A is additive on X. Setting $m = 0$ and taking the limit as $n \to \infty$ in (2.11), we derive

$$\left\| A(x) - f(x) + f(0) \right\| \leqslant 4\varepsilon \tag{2.14}$$

for all $\|x\| \geqslant d$. We now extend the inequality (2.14) to the entire space X. It is evident that inequality (2.14) holds for $x = 0$. Let $z \neq 0$ be an element of X. We can select $x, y, a \in X$ such that $z = x + y$, and satisfy the condition

$$\min\{\|x\|, \|y\|, \|a\|, \|x - a\|, \|y + a\|\} \geqslant d.$$

Thus, from (2.14), we conclude that

$$\left\| A(x - a) - f(x - a) + f(0) \right\| \leqslant 4\varepsilon, \tag{2.15}$$

$$\left\| A(y + a) - f(y + a) + f(0) \right\| \leqslant 4\varepsilon. \tag{2.16}$$

By adding (2.10), (2.15), and (2.16), we obtain

$$\|A(z) - f(z) + f(0)\| \leqslant 12\varepsilon.$$

This establishes (2.2). To prove (2.3), let $a, z \in X$ such that $\min\{\|a\|, \|z\|\} \geqslant d$. Then, by (2.2), we have

$$\|A(z - a) - f(z - a) + f(0)\| \leqslant 12\varepsilon.$$

Adding this inequality to (2.5), and applying (2.7), we obtain (2.3). A similar argument can be used to derive (2.4). From (2.2), we also conclude that $A(x) = \lim_{n\to\infty} \frac{f(nx)}{n}$ for all $x \in X$. This completes the proof of the uniqueness of A.

Additionally, suppose $\|z\| \geqslant d$. From inequalities (2.1) and (2.14), we have the following:

$$\|f(2z) - g(z) - h(z)\| \leqslant \varepsilon,$$
$$\|g(-z) + h(z) - f(0)\| \leqslant \varepsilon,$$
$$\|A(2z) - f(2z) + f(0)\| \leqslant 4\varepsilon.$$

By adding these three inequalities together, we obtain

$$\|A(2z) - g(z) + g(-z)\| \leqslant 6\varepsilon.$$

Since $A(2z) = 2A(z)$, we deduce $\|A(z) - g_o(z)\| \leqslant 3\varepsilon$. A similar argument can be applied to show that $\|A(z) - h_o(z)\| \leqslant 3\varepsilon$. □

Theorem 2.2. *Let $f, g, h : X \to Y$ be mappings satisfying*

$$\lim_{\min\{\|x\|,\|y\|\}\to\infty} \|f(x + y) - g(x) - h(y)\| = 0. \tag{2.17}$$

Then f is an affine mapping on X, meaning that $f - f(0)$ is additive on X.

Proof. Let $\varepsilon > 0$ be any given real number. From (2.17), there exists a $d_\varepsilon > 0$ such that

$$\|f(x + y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d_\varepsilon.$$

According to Theorem 2.1, there exists an additive mapping $A_\varepsilon : X \to \mathscr{Y}$ ($\mathscr{Y}$ is the completion of Y) such that

$$\|A_\varepsilon(z) - f(z) + f(0)\| \leqslant 12\varepsilon, \quad z \in X.$$

Thus, for all $x, y \in X$, we have

$$\begin{aligned}\|f(x+y) - f(x) - f(y) + f(0)\| &\leqslant \|f(x+y) - A_\varepsilon(x+y) - f(0)\| \\ &\quad + \|A_\varepsilon(x) - f(x) + f(0)\| + \|A_\varepsilon(y) - f(y) + f(0)\| \\ &\leqslant 36\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$f(x+y) = f(x) + f(y) - f(0), \quad x, y \in X.$$

Therefore, it follows that $f - f(0)$ is additive on X. □

The following example demonstrates that, based on the previous theorem, we cannot conclude that g or h are affine.

Example 2.3. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = x + 2, \quad g(x) = 1 + x + \frac{1}{1+|x|}, \quad h(x) = 1 + x + \frac{x}{1+x^2}, \quad x \in \mathbb{R}.$$

It is clear that

$$\lim_{\min\{|x|,|y|\}\to\infty} |f(x+y) - g(x) - h(y)| = 0.$$

However, g and h are not affine mappings on $\mathbb{R}$.

It is worth noting that Theorem 2.1 remains valid even when inequality (2.1) is considered for $\|x\| + \|y\| \geqslant d$. However, in this case, a different and simpler proof can be provided, yielding better bounds than those stated in Theorem 2.1. What follows generalizes Theorem 3 and Corollary 4 from [18], as well as Theorem 3.1 from [21].To clarify, Theorem 3 of [18] studied the Hyers-Ulam stability of the Jensen equation, while Theorem 3.1 of [21] dealt with the Hyers-Ulam stability of a generalized Jensen equation on a restricted domain. Both of these are special cases of the Pexiderized Cauchy functional equation considered in Theorem 2.4. Similarly, Corollary 4 of [18] establishes the asymptotic behavior of additive mappings, which is a special case of the asymptotic results for the Pexiderized Cauchy functional equation proved in Corollary 2.5.

Theorem 2.4. *Let $f, g, h : X \to \mathscr{Y}$ be mappings satisfying*

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d \tag{2.18}$$

for some $\varepsilon \geqslant 0$ and $d > 0$. Then, there exists a unique additive mapping $A : X \to \mathscr{Y}$ such that

$$\|A(x) - f(x) + g(0) + h(0)\| \leqslant 9\varepsilon, \quad x \in X, \tag{2.19}$$

$$\|A(x) - g(x) + g(0)\| \leqslant 8\varepsilon, \quad x \in X, \tag{2.20}$$

$$\|A(x) - h(x) + h(0)\| \leqslant 8\varepsilon, \quad x \in X. \tag{2.21}$$

Proof. The substitutions $[x, 0]$ and $[0, y]$ in (2.18) yield

$$\|f(x) - g(x) - h(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.22}$$

$$\|f(y) - g(0) - h(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d. \tag{2.23}$$

Combining (2.18), (2.22) and (2.23), we obtain

$$\|f(x+y) - f(x) - f(y) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.24}$$

Thus,

$$\|f(2x) - 2f(x) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d.$$

Hence

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{k=m}^{n} \frac{g(0)+h(0)}{2^{k+1}} \right\| \leqslant \sum_{k=m}^{n} \frac{3\varepsilon}{2^{k+1}} \tag{2.25}$$

for all $\|x\| \geqslant d$ and $n \geqslant m \geqslant 0$. Using a similar reasoning as in the proof of Theorem 2.1, we can define the mapping $A : X \to \mathscr{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

It is evident that $A(0) = 0$. Using (2.24) along with the definition of A, we conclude

$$A(x+y) = A(x) + A(y), \quad x, y \in X \setminus \{0\}.$$

Since $A(0) = 0$, we deduce that A is an additive mapping on X. By setting $m = 0$ and taking the limit as n approaches infinity in (2.25), we derive

$$\|A(x) - f(x) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d. \tag{2.26}$$

Let $z \in X$ and choose $x, y \in X$ such that $z = x + y$ and $\min\{\|x\|, \|y\|\} \geqslant d$. Using (2.24) and (2.26), we arrive at:

$$\begin{aligned}\|A(z) - f(z) + g(0) + h(0)\| &\leqslant \|A(x) - f(x) + g(0) + h(0)\| \\ &\quad + \|A(y) - f(y) + g(0) + h(0)\| \\ &\quad + \|f(x) + f(y) - f(x+y) - g(0) - h(0)\| \\ &\leqslant 9\varepsilon.\end{aligned}$$

This establishes (2.19). Let $x \in X$ and choose $y \in X$ such that

$$\min\{\|y\|, \|x+y\|\} \geqslant d.$$

From (2.26), we derive the following inequalities:

$$\|A(x+y) - f(x+y) + g(0) + h(0)\| \leqslant 3\varepsilon, \tag{2.27}$$
$$\|f(y) - A(y) - g(0) - h(0)\| \leqslant 3\varepsilon. \tag{2.28}$$

By summing (2.18), (2.27) and (2.28), we arrive at

$$\|A(x) + f(y) - g(x) - h(y)\| \leqslant 7\varepsilon.$$

Combining this result with (2.23), we establish (2.20). A similar reasoning can be applied to derive (2.21), and the details are omitted for brevity. □

As a direct application of Theorem 2.4, we now turn our attention to investigating the asymptotic behavior of the Pexiderized Cauchy functional equation.

Corollary 2.5. *Let $f, g, h : X \to Y$ be mappings satisfying*

$$\lim_{\|x\|+\|y\|\to\infty} \|f(x+y) - g(x) - h(y)\| = 0. \tag{2.29}$$

Then f, g and h are affine mappings on X.

Proof. Let $\varepsilon > 0$ be an arbitrary positive real number. From (2.29), there exists a constant $d_\varepsilon > 0$ such that

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d_\varepsilon.$$

Using Theorem 2.4, we can establish the existence of an additive mapping $A_\varepsilon : X \to \mathscr{Y}$ ($\mathscr{Y}$ is the completion of Y) that satisfies

$$\begin{aligned}
\|A_\varepsilon(x) - f(x) + g(0) + h(0)\| &\leqslant 9\varepsilon, \quad x \in X,\\
\|A_\varepsilon(x) - g(x) + g(0)\| &\leqslant 8\varepsilon, \quad x \in X,\\
\|A_\varepsilon(x) - h(x) + h(0)\| &\leqslant 8\varepsilon, \quad x \in X.
\end{aligned}$$

As a result, for any $x, y \in X$, the following inequality holds:

$$\begin{aligned}
\|f(x+y) - f(x) - f(y) + g(0) + h(0)\| &\leqslant \|f(x+y) - A_\varepsilon(x+y) - g(0) - h(0)\|\\
&\quad + \|A_\varepsilon(x) - f(x) + g(0) + h(0)\|\\
&\quad + \|A_\varepsilon(y) - f(y) + g(0) + h(0)\|\\
&\leqslant 27\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$f(x+y) = f(x) + f(y) - g(0) - h(0), \quad x, y \in X.$$

This further implies that $f(x+y) = f(x) + f(y) - f(0)$ for all $x, y \in X$. From this, it follows that $f - f(0)$ is an additive mapping on X, and thus f is affine on X. Using a similar approach, it can be shown that g and h are also affine on X. □

Theorem 2.6. *Let $f, g, h : X \to \mathscr{Y}$ be mappings satisfying*

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d \tag{2.30}$$

for some $\varepsilon \geqslant 0$ and $d > 0$. Then, there exists a unique additive mapping $A : X \to \mathscr{Y}$ such that

$$\|A(x) - f(x) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d, \tag{2.31}$$

$$\|A(x) - g(x) + g(0)\| \leqslant 8\varepsilon, \quad x \in X, \tag{2.32}$$

$$\|A(x) - h(x) + h(0)\| \leqslant 8\varepsilon, \quad x \in X. \tag{2.33}$$

Proof. By setting $x = 0$ and $y = 0$ separately in (2.30), we obtain the following inequalities:

$$\|f(x) - g(x) - h(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.34}$$

$$\|f(y) - g(0) - h(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d. \tag{2.35}$$

By combining (2.30), (2.34) and (2.35), we arrive at

$$\|f(x+y) - f(x) - f(y) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \min\{\|x\|, \|y\|, \|x+y\|\} \geqslant d.$$

Thus,

$$\|f(2x) - 2f(x) + g(0) + h(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d.$$

Hence

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{k=m}^{n} \frac{g(0)+h(0)}{2^{k+1}}\right\| \leqslant \sum_{k=m}^{n} \frac{3\varepsilon}{2^{k+1}} \tag{2.36}$$

for all $\|x\| \geqslant d$ and $n \geqslant m \geqslant 0$. Using a similar reasoning as in the proof of Theorem 2.1, we can define the mapping $A : X \to \mathscr{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

It is evident that $A(0) = 0$ and $A(2x) = 2A(x)$ for all $x \in X$. It follows from (2.34) and (2.35) that

$$\lim_{n\to\infty} \frac{g(2^n x)}{2^n} = A(x), \quad \lim_{n\to\infty} \frac{h(2^n x)}{2^n} = A(x), \quad x \in X.$$

We now aim to prove that A is additive on X. From (2.30), we obtain

$$A(x+y) = A(x) + A(y), \quad x, y \in X,\ x+y \neq 0. \tag{2.37}$$

By setting $y = -2x$ in (2.37), we conclude that $A(-x) + A(x) = 0$ for all $x \in X \setminus \{0\}$. Since $A(0) = 0$, it follows that A is odd on X. Consequently, it follows that (2.37) holds true for every $x, y \in X$, which implies that the mapping A is additive on X. By substituting $m = 0$ and allowing n to approach infinity in (2.36), we obtain (2.31).

By adding inequality (2.31) to inequality (2.34), we arrive at

$$\|g(x) - A(x) - g(0)\| \leqslant 4\varepsilon, \quad \|x\| \geqslant d. \tag{2.38}$$

Similarly, using (2.31) and (2.35), we obtain

$$\|h(y) - A(y) - h(0)\| \leqslant 4\varepsilon, \quad \|y\| \geqslant d. \tag{2.39}$$

Let $x \in X$ and choose $y \in X$ such that $\min\{\|y\|, \|x+y\|\} \geqslant d$. Using (2.31), we get

$$\|A(x+y) - f(x+y) + g(0) + h(0)\| \leqslant 3\varepsilon. \tag{2.40}$$

By combining (2.30), (2.39) and (2.40), and applying the principle of additivity for A, we arrive at (2.32). Similarly, one can use a comparable proof to establish (2.33), and we will omit the details. □

Theorem 2.7. *Let $f, g, h : X \to Y$ be mappings satisfying*

$$\lim_{\|x+y\|\to\infty} \|f(x+y) - g(x) - h(y)\| = 0. \tag{2.41}$$

Then g and h are affine mappings on X.

Proof. Let $\varepsilon > 0$ be an arbitrary positive real number. Based on equation (2.41), we can find a constant $d_\varepsilon > 0$ such that

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d_\varepsilon.$$

By applying Theorem 2.6, there exists an additive mapping $A_\varepsilon : X \to \mathscr{Y}$ ($\mathscr{Y}$ is the completion of Y) satisfying

$$\|A(z) - g(z) + g(0)\| \leqslant 8\varepsilon, \quad \|A(z) - h(z) + h(0)\| \leqslant 8\varepsilon, \quad z \in X.$$

Consequently, for any $x, y \in X$, the following inequality holds:

$$\begin{aligned}\|g(x+y) - g(x) - g(y) + g(0)\| &\leqslant \|g(x+y) - A_\varepsilon(x+y) - g(0)\| \\ &\quad + \|A_\varepsilon(x) - g(x) + g(0)\| + \|A_\varepsilon(y) - g(y) + g(0)\| \\ &\leqslant 24\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$g(x+y) = g(x) + g(y) - g(0), \quad x, y \in X.$$

This implies that $g - g(0)$ is an additive mapping on X, and therefore g is affine on X. A similar argument shows that h is also affine on X. $\square$

The example below illustrates that, even when the conditions of Theorem 2.7 are satisfied, it does not necessarily follow that f is affine.

Example 2.8. *Consider the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x) = 3 + x + \frac{1}{1+|x|}, \quad g(x) = 1 + x, \quad h(x) = 2 + x, \quad x \in \mathbb{R}.$$

It is evident that

$$\lim_{|x+y|\to\infty} |f(x+y) - g(x) - h(y)| = 0.$$

Nevertheless, f is not an affine function on $\mathbb{R}$.

3. Hyperstability results

In this section, we discuss the hyperstability results for the functional equation introduced in Section 2.

Theorem 3.1. *Let $p, q, r, s < 0$ and $f, g, h : X \to Y$ be mappings that satisfy the following inequality*

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q) + \theta\|x\|^r\|y\|^s, \quad x, y \in X \setminus \{0\} \tag{3.1}$$

for some constants $\varepsilon, \theta \geqslant 0$. Then, f is an affine mapping on X.

Proof. From inequality (3.1), we can deduce that

$$\lim_{\min\{\|x\|,\|y\|\}\to\infty} \|f(x+y) - g(x) - h(y)\| = 0.$$

Consequently, according to Theorem 2.2, we conclude that f must be an affine mapping on X. $\square$

Corollary 3.2. *A mapping $f : X \to Y$ is additive on X if and only if there exist constants $p, q, r, s < 0$ and $\varepsilon, \theta \geqslant 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q) + \theta\|x\|^r\|y\|^s, \quad x, y \in X \setminus \{0\}. \tag{3.2}$$

Proof. It is clear that if f is an additive mapping on X, then the equation (3.2) holds true. Now, consider the case where f fulfills the condition stated in (3.2). According to Theorem 3.1, the mapping f is classified as an affine mapping on X. We define $F(x) := f(x) - f(0)$ for all $x \in X$. Given that F is an additive mapping on X, from (3.2), we can derive the following

$$\begin{aligned}\|f(0)\| &= \|F(x+y) - f(0) - F(x) + f(0) - F(y) + f(0)\| \\ &= \|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q) + \theta\|x\|^r\|y\|^s\end{aligned}$$

for all $x, y \in X\setminus\{0\}$. This leads us to conclude that $\|f(0)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q) + \theta\|x\|^r\|y\|^s$ holds for any $x, y \in X\setminus\{0\}$. Consequently, we find that $f(0) = 0$, which indicates that f must indeed be an additive mapping on X. $\square$

Theorem 3.3. *Let $p < 0$ and $f, g, h : X \to Y$ be mappings that satisfy the following inequality*

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon\|x+y\|^p, \quad x + y \neq 0 \tag{3.3}$$

for some constant $\varepsilon \geqslant 0$. Then g and h are affine mappings on X.

Proof. From inequality (3.3), we can deduce that

$$\lim_{\|x+y\|\to\infty} \|f(x+y) - g(x) - h(y)\| = 0.$$

Consequently, according to Theorem 2.7, we conclude that g and h must be affine mappings on X. □

Corollary 3.4. *A mapping $f : X \to Y$ is additive on X if and only if there exist constants $p < 0$ and $\varepsilon, \geqslant 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \|x+y\|^p, \quad x+y \neq 0. \tag{3.4}$$

Proof. It is clear that if f is an additive mapping on X, then the equation (3.4) holds true. Now, consider the case where f fulfills the condition stated in (3.4). According to Theorem 3.3, the mapping f is classified as an affine mapping on X. We define $F(x) := f(x) - f(0)$ for all $x \in X$. Given that F is an additive mapping on X, from (3.4), we get

$$\begin{aligned} \|f(0)\| &= \|F(x+y) - f(0) - F(x) + f(0) - F(y) + f(0)\| \\ &= \|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \|x+y\|^p \end{aligned}$$

for all $x, y \in X$ with $x + y \neq 0$. This leads us to conclude that $\|f(0)\| \leqslant \varepsilon \|x+y\|^p$ holds for any $x, y \in X$ with $x + y \neq 0$. Consequently, we find that $f(0) = 0$, which indicates that f must indeed be an additive mapping on X. □

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