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Convergence of approximations for the tracking problem in nonlinear optimization

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Abstract. In the paper the tracking problem in non-linear optimization of oscillatory processes described by integro-differential equations involving a Fredholm integral operator was researched. The study primarily focuses on the convergence of approximate solutions to the exact one. External and boundary effects are modeled as scalar functions of several variables with a non-linear dependence on vector control influences. Special attention is devoted to examining the influence of the Fredholm integral operator on the convergence rate of approximate solutions. Sufficient conditions providing the convergence of approximate solutions of the nonlinear optimization problem to the exact solution are established.

1. Introduction

The tracking problem occupies a central position in numerous applied and theoretical studies related to control systems, robotics, navigation, computer vision, biomedical engineering, and other fields [1, 2, 3]. At its core lies the necessity to monitor the evolution of a dynamic system in real time. The most general formulation requires determining control actions or parameters that ensure the system follows a specified trajectory or adapts to observed data — directly linking the tracking problem to optimization methods.

Historically, the tracking problem gained significant development in the mid-20th century within the framework of state estimation and filtering techniques, such as the Kalman filter [10]. These methods relied on linear models and Gaussian noise, which provided analytical simplicity but limited their applicability. Over time, it became evident that most real-world systems exhibit pronounced non-linear characteristics and require consideration of constraints and prior knowledge, leading to a shift towards non-linear and numerical optimization methods.

Modern approaches to solving the tracking problem increasingly rely on nonlinear optimization methods. These approaches allow for the consideration of nonlinear dynamics, various types of noise, and the integration of complex prior knowledge about the system's behavior. Different aspects of the tracking problem in both linear and nonlinear optimization have been investigated by many authors [4, 5, 6].

In the paper [2], the tracking problem was studied in the context of nonlinear vector optimization of oscillatory processes described by integro-differential partial differential equations. Scalar functions

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describing external and boundary effects are nonlinear with respect to several control inputs. It was established that this problem has a number of specific characteristics. In particular, the components of the optimal distributed and boundary vector controls satisfy a system of equalities and are determined as the solution to a system of nonlinear integral equations. An algorithm for constructing a solution to the tracking problem has been developed, and sufficient conditions for its unique solvability have been established.

This paper addresses issues related to the construction of approximate solutions for a tracking problem and analyzes their convergence, as previously studied in [2]. The aim of the present research is to investigate the influence of a Fredholm integral operator on the processes of constructing and analyzing the convergence of approximate solutions. It is demonstrated that the presence of an integral operator in the boundary value problem necessitates the development of three distinct types of approximations for the optimal process: approximations utilizing the resolvent of the integral operator's kernel, approximations based on optimal controls, and finite-dimensional approximations. Accordingly, three types of approximations of the minimal value of the functional are also considered. Sufficient conditions for the convergence of the aforementioned approximations are established, including the convergence of distributed and boundary vector optimal controls, all three types of optimal process approximations, and approximations of the minimal value of the functional.

2. On the solvability of the tracking problem in non-linear optimization

Consider the tracking problem in nonlinear optimization of oscillatory processes described by integro-differential equations. We need to minimize an integral functional

$$\begin{aligned} J[\bar{u}(t,x),\bar{\vartheta}(t,x)] = &\int_0^T\int_Q [V(t,x)-\xi(t,x)]^2\,dxdt \\ &+\int_0^T\left[\alpha\int_Q h[t,x,\bar{u}(t,x)]dx+\beta\int_\gamma b[t,x,\bar{\vartheta}(t,x)]dx\right]dt, \alpha,\beta>0, \end{aligned} \tag{1}$$

on the set of solutions to the boundary value problem

$$V_{tt}(t,x)-AV(t,x)=\lambda\int_0^T K(t,\tau)V(\tau,x)d\tau+f[t,x,\bar{u}(t,x)],\quad x\in Q\subset\mathbb{R}^n,\quad 0<t<T, \tag{2}$$

$$V(0,x)=\psi_1(x),\quad V_t(0,x)=\psi_2(x),\quad x=(x_1,x_2,\dots,x_n)\in Q, \tag{3}$$

$$\Gamma V(t,x)\equiv\sum_{i,k=1}^n a_{ik}(x)V_{x_k}(t,x)\cos(\delta,x_i)+a(x)V(t,x)=p[t,x,\bar{\vartheta}(t,x)],\quad x\in\gamma,\quad 0<t<T. \tag{4}$$

Here, the function $V(t,x)$, defined on the domain $Q_T=Q\times(0,T]$ describes the state of the controlled oscillatory process and is the sought function. Q is a bounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$ with a piecewise smooth boundary γ.

A is an elliptic operator defined by the formula

$$AV(t,x)=\sum_{i,j=1}^n(a_{ij}(x)V_{x_j}(t,x))_{x_i}-c(x)V(t,x),\quad a_{i,j}(x)=a_{j,i}(x),\quad \sum_{i,j=1}^n a_{ij}(x)\alpha_i\alpha_j\ge c_0\sum_{i=1}^n\alpha_i^2,\quad c_0>0,$$

$a(x) \geq 0, c(x) \geq 0$ are known measurable functions; Function $K(t,\tau)$, which is the kernel of Fredholm integral operator, is defined on domain $D = \{0 \leq t \leq T, 0 \leq \tau \leq T\}$ and belongs to Hilbert space of square-integrable functions in $H(D)$, i.e., it satisfies the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty, \quad K(t,\tau) \in H(D); \tag{5}$$

λ is a parameter; the function $\psi_1(x) \in H_1(Q)$ characterizes the initial state of the controlled process at the initial time, $\psi_2(x) \in H(Q)$ is an initial velocity of the points of the controlled object; $H_1(Q)$ is a first-order Sobolev space; $H(Q)$ is a Hilbert space of square-integrable functions defined on the given domain; The given function $\xi(t,x)$, defined on domain $H(Q_T)$, describes the desired state of the controlled process throughout the entire time of the interval control; The vector δ represents the outward normal originating from the point $x \in \gamma$; T is a fixed moment of time corresponding to the end of the control period.

The scalar function $f[t,x,\bar{u}(t,x)]$ models the influence of an external source and is a non-linear scalar function of the vector-valued distributed control $\bar{u}(t,x) = (u_1(t,x), ..., u_r(t,x)) \in H^r(Q_T)$ which belongs to the functional space $H^r(Q_T) = H(Q_T) \times ... \times H(Q_T)$ with a norm $\|\bar{u}(t,x)\|^2_{H^r(Q_T)} = \int_Q \sum_{i=1}^r \bar{u}_i^2(t,x)dxdt$. Such dependence reflects the complex nature of the interaction between the external influence and the internal parameters of the controlled process, and requires the use of non-linear analysis methods in the formulation and solution of the optimal control problem.

The scalar function $p[t,x,\bar{\vartheta}(t,x)]$ describes the influence of a boundary source and depends non-linearly on the vector-valued boundary control $\bar{\vartheta}(t,x) = (\vartheta_1(t,x), ..., \vartheta_m(t,x)) \in H^m(\gamma_T)$, which belongs to a functional space $H^m(\gamma_T) = H(\gamma_T) \times ... \times H(\gamma_T)$ and $\|\bar{\vartheta}(t,x)\|^2_{H^m(\gamma_T)} = \int_\gamma \sum_{i=1}^m \bar{\vartheta}_i^2(t,x)dxdt$.

In equation (1), $h[t,x,\bar{u}(t,x)]$ and $b[t,x,\bar{\vartheta}(t,x)]$ are given scalar functions that are strictly convex with respect to their respective vector arguments $\bar{u}(t,x)$ and $\bar{\vartheta}(t,x)$. The convexity of these functions ensures the convexity of the functional (1), which, in turn, guarantees the uniqueness of the optimal controls. This fact plays a crucial role in establishing the existence and uniqueness of a complete solution to the optimization problem.

The algorithm for constructing the complete solution to the considered nonlinear optimization problem is described in detail in [2] and is formalized as a triple

$$\Big((\bar{u}(t,x),\ \bar{\vartheta}(t,x)),\ V^0(t,x),\ J[\bar{u}^0(t,x),\ \bar{\vartheta}^0(t,x)]\Big),$$

where $(\bar{u}(t,x),\ \bar{\vartheta}(t,x))$ represents a pair of vector-valued optimal controls; $V^0(t,x)$ denotes the corresponding optimal process that satisfies the integro-differential dynamics of the system; and $J[\bar{u}^0(t,x),\ \bar{\vartheta}^0(t,x)]$ is the minimal value of the given integral functional attained under all constraints of the problem.

Here, we present the main results of [2], which will be used in the study of the convergence of approximate solutions to the tracking problem.

1) The distributed vector optimal control $\bar{u}^0(t,x)$ and the boundary vector optimal control $\bar{\vartheta}^0(t,x)$ are defined by following formulas

$$\begin{aligned}
&\bar{u}^0(t,x) = (u_1^0(t,x), ..., u_r^0(t,x)),\\
&u_i^0(t,x) = \varphi_i[t,x,\theta_1^0(t,x),\alpha], \quad i = 1,2,...,r,\\
&\bar{\vartheta}^0(t,x) = (\vartheta_1^0(t,x), ..., \vartheta_m^0(t,x)),\\
&\vartheta_i^0(t,x) = \upsilon_i(t,x,\theta_2^0(t,x),\beta], \quad i = 1,2,...,m,
\end{aligned} \tag{6}$$

where $\varphi_i[t,x,\theta_1^0(t,x),\alpha]$ and $\upsilon_i(t,x,\theta_2^0(t,x),\beta]$ are known functions, and the vector function

$$\theta^{(0)}(t,x) = \begin{cases} \theta_1^{(0)}(t,x), & x \in Q,\\ \theta_2^{(0)}(t,x), & x \in \gamma, \end{cases}$$

is the unique solution of the operator equation

$$\theta(t,x) = F_0[\theta(t,x)] + W(t,x,\lambda). \tag{7}$$

and the solution of this operator equation is defined as the limit $\theta^{(0)}(t,x) = \lim_{n\to\infty}\theta^{(n)}(t,x)$ of vector functions

$$\theta^{(n)}(t,x) = \begin{cases} \theta_1^{(n)}(t,x), & x \in Q, \\ \theta_2^{(n)}(t,x), & x \in \gamma, \end{cases}$$

which determined by the method of successive approximations, as solutions of operator equations

$$\theta^{(n)}(t,x) = F_0[\theta^{(n-1)}(t,x)] + W(t,x,\lambda), \quad n = 1,2,3,... \tag{8}$$

and satisfy the estimate

$$\|\theta^{(0)}(t,x) - \theta^{(n)}(t,x)\|_{H(\bar{Q}_T)} \leq \frac{C^n(\alpha,\beta)}{1 - C(\alpha,\beta)}\|F_0(\theta_{(0)}(t,x)) + W(t,x,\lambda) - \theta_0(t,x)\|_{H(Q)\times H(\gamma_T)}, \tag{9}$$

$$C(\alpha,\beta) = 4T^2\left(1 + \frac{\lambda^2 K_0 T^2}{\left(\lambda_1 - |\lambda|\sqrt{K_0 T^2}\right)^2}\right)\sqrt{f_0^2\varphi_0^2(\alpha)r^2 + p_0^2 \upsilon_0^2(\beta)m^2}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}}. \tag{10}$$

2) The optimal process $V^0(t,x)$ is determined by following formula

$$V^0(t,x) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,x,\lambda)a_n^0(s)ds + a_n^0(s)\right)z_n(x), \tag{11}$$

$$\begin{aligned} a_n^0(t) &= \psi_{1n}\cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t \\ &\quad + \frac{1}{\lambda_n}\int_0^T \sin\lambda_n(t-\tau)\left(\int_Q f[t,x,\bar{u}^0(t,x)]z_n(x)dx + \int_\gamma p[t,x,\bar{\vartheta}]z_n(x)dx\right)d\tau. \end{aligned} \tag{12}$$

3) The minimum value of the functional $J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ is determined by the formula

$$\begin{aligned} J[\bar{u}^0[t,x,\bar{\vartheta}^0(t,x)] = &\int_0^T\int_Q \left[V^0(t,x) - \xi(x)\right]^2 dxdt \\ &+ \int_0^T\left[\alpha\int_Q h[t,x,\bar{u}^0(t,x)]dx + \beta\int_\gamma b[t,x,\bar{\vartheta}^0(t,x)]dx\right]dt, \\ &\alpha,\beta > 0. \end{aligned} \tag{13}$$

3. Approximations of the complete solution of the tracking problem

The main problem of this work is to study the approximate solutions to the tracking problem (1)–(4) in nonlinear optimization and to analyze their convergence. As the complete solution of the tracking problem is the triple

$$\Big((\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)),\ V^0(t,x),\ J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]\Big)$$

consisting of the optimal control, the optimal process, and the minimum value of the functional, we will consider approximations of each of these components separately.

3.1. Convergence of approximations of vector optimal controls

The approximate solution of operator equation (7) is used as an approximation in the construction of optimal controls. By substituting the obtained $\theta^{(k)}(t,x)$ into equations (5) and (6), we obtain approximate values of the vector optimal controls. Thus, the following approximations are formulated:

k-th approximations of distributed vector control:

$$\begin{aligned} \bar{u}^{(k)}(t,x) &= (u_1(t,x),...,u_r(t,x)),\\ u_i^0(t,x) &= \varphi_i[t,x,\theta_1^{(k)}(t,x),\alpha], \quad i=1,2,...,r,\\ \bar{\varphi}[t,x,\theta_1^{(k)}(t,x),\alpha] &= \left(\varphi_1[t,x,\theta_1^{(k)}(t,x),\alpha],...,\varphi_k[t,x,\theta_1^{(k)}(t,x),\alpha]\right). \end{aligned} \tag{14}$$

k-th approximations of vector boundary control:

$$\begin{aligned} \bar{\vartheta}^{(k)}(t,x) &= (\vartheta_1^{(k)}(t,x),...,\vartheta_m^{(k)}(t,x)),\\ \vartheta_i^{(k)}(t,x) &= \upsilon_i(t,x,\theta_2^{(k)}(t,x),\beta], \quad i=1,2,...,m,\\ \bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta] &= \left(\upsilon_1[t,x,\theta_2^{(k)}(t,x),\beta],...,\upsilon_k[t,x,\theta_2^{(k)}(t,x),\beta]\right). \end{aligned} \tag{15}$$

Lemma 3.1. *k-th approximations of the distributed and boundary vector controls, obtained under the conditions of tracking problem (1)-(4), converge to the optimal distributed and boundary vector controls, respectively, in the norms of the Hilbert spaces $H^k(Q_T)$ and $H^k(\gamma_T)$.*

Proof. The convergence of the k-th approximation of distributed vector control follows from the following inequality:

$$\begin{aligned} \|\bar{u}^0(t,x)-\bar{u}^{(k)}(t,x)\|^2_{H(\bar{Q}_T)} &= \|\bar{\varphi}[t,x,\theta_1^0(t,x),\alpha]-\bar{\varphi}[t,x,\theta_1^{(k)}(t,x),\alpha]\|^2_{H^k(Q_T)}\\ &\le \varphi_0^2(\alpha)\left(\frac{C^k(\alpha,\beta)}{1-C(\alpha,\beta)}\|F_0[\theta^{(0)}(t,x)]+W(t,x,\lambda)-\theta^{(k)}(t,x)\|_{H(\bar{Q}_T)}\right)^2 \to 0, \quad k\to\infty. \end{aligned}$$

The convergence of the k-th approximation of the boundary vector control follows from the following inequality:

$$\begin{aligned} \|\bar{\vartheta}^0(t,x)-\bar{\vartheta}^{(k)}(t,x)\|^2_{H(\bar{\gamma}_T)} &= \|\bar{\upsilon}[t,x,\theta_2^0(t,x),\beta]-\bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta]\|^2_{H^k(\gamma_T)}\\ &\le \upsilon_0^2(\beta)\left(\frac{C^k(\alpha,\beta)}{1-C(\alpha,\beta)}\|F_0[\theta^{(0)}(t,x)]+W(t,x,\lambda)-\theta^{(k)}(t,x)\|_{H(\bar{\gamma}_T)}\right)^2 \to 0, \quad k\to\infty. \end{aligned}$$

□

3.2. Approximations of the optimal process and their convergence

The presence of an integral operator in the boundary value problem (2)-(4) leads to the need to identify three main types of approximations of the optimal process, due to the specific structure of the integro-differential problem of nonlinear optimization:

1) Resolvent approximations of the optimal process and their convergence

Definition 3.2. *A truncated series of the form*

$$R_n^q(t,s,\lambda) = \sum_{i=1}^{q} \lambda^{i-1} K_{n,i}(t,s), \quad n=1,2,3,..., \tag{16}$$

is called the q-th approximation of the resolvent $R_n(t,s,\lambda)$ of the kernel $K(t,x)$ for each fixed $n=1,2,3,....$

Definition 3.3. *A function defined by the formula*

$$V^{(q)}(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n^q(t,s,\lambda)a_n^0(s)ds+a_n^0(t)\right)z_n(x),\quad n=1,2,3,...,$$

is called a q-th approximation or a "resolvent" approximation of the optimal process corresponding to each fixed value of $n=1,2,3,....$

Lemma 3.4. *"Resolvent" approximations $V^{(q)}(t,x)$ of the optimal process $V^0(t,x)$, obtained under the conditions of the tracking problem (1)-(4), converge to the optimal process in the norm of the corresponding Hilbert space.*

Proof. By direct calculations establish the relation

$$\|V^0(t,x)-V^{(q)}(t,x)\|^2_{H(Q_T)}\leq\lambda^2\left(\frac{|\lambda|\sqrt{T^2K_0}}{\lambda_n}\right)\left(1-\frac{1}{\ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1^2}}\right)^2$$
$$\times 3\left(\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}+\frac{2}{\lambda_1^2}T\left[\|f[t,x,\bar u^0(t,x]\|^2_{H(Q_T)}+\|p[t,x,\bar\vartheta^0(t,x)]\|^2_{H(\gamma_T)}\right]\right)$$
$$=(C^q(\lambda))^2\left(\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}\right)^{2q},$$

$$C^q(\lambda)=\left(1-\frac{1}{\ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1^2}}\right)$$
$$\times\left\{3\left(\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}+\frac{2}{\lambda_1^2}T\left[\|f[t,x,\bar u^0(t,x)]\|^2_{H(Q_T)}+\|p[t,x,\bar\vartheta^0(t,x)]\|^2_{H(\gamma_T)}\right]\right)\right\}^{\frac{1}{2}}.$$

That is, the following inequality holds,

$$\|V^0(t,x)-V^{(q)}(t,x)\|^2_{H(Q_T)}\leq(C^q(\lambda))^2\left(\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}\right)^q\to 0,\quad q\to\infty,$$

as $\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}<1$, from which, due to the fulfilment of the condition, the assertion of the lemma follows directly. □

2) q,k-th approximations of the optimal process and their convergence

Definition 3.5. *The functions defined by the formula*

$$V_k^q(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n^q(t,x,\lambda)a_n^{(k)}(s)ds+a_n^{(k)}(s)\right)z_n(x),\tag{17}$$

is called a q,k-th approximations of the optimal process with respect to the controls, where $\bar u^k(x)$ is k-th approximation of the vector distributed control and $\bar u^k(x)$ is a k-th approximation of vector boundary control.

Lemma 3.6. *Under the conditions of the nonlinear optimization problem (1)-(4), the q,k-th approximations $V_k^{(q)}(t,x)$ of the optimal process converge to the corresponding intermediate approximations $V^{(q)}(t,x)$ as $k\to\infty$ for any fixed $q=1,2,3,...$ in the norm of the corresponding space $H(Q_T)$.*

Proof. The assertion of the lemma follows directly from the following relation

$$\|V^{(q)}(t,x) - V_k^{(q)}(t,x)\|^2_{H(Q_T)} \leq 4T\left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\sqrt{K_0 T^2}\right)^2}\right)\times$$

$$\times \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\left(f_0^2\|\bar{u}_0(t,x) - \bar{u}_{(k)}(t,x)\|^2_{H(Q_T)} + p_0^2\|\bar{\vartheta}_0(t,x) - \bar{\vartheta}_{(k)}(t,x)\|^2_{H(\gamma_T)}\right) \to 0, \quad k \to \infty, \quad q = 1,2,3,....$$

□

3) Finite-dimensional approximations of the optimal process and their convergence

Definition 3.7. *The functions defined by the following formulas*

$$V^q_{k,r}(t,x) = \sum_{n=1}^{r}\left(\lambda \int_0^T R_n^q(t,x,\lambda) a_n^{(k)}(s)ds + a_n^{(k)}(t)\right) z_n(x), \tag{18}$$

are called q,k,l-th approximations, or finite-dimensional approximations of the optimal process.

Lemma 3.8. *Under the conditions of the boundary value problem (2)-(4), the finite-dimensional approximations of the optimal process converge to the corresponding intermediate approximations $V_k^q(t,x)$ as $r \to \infty$, for any fixed $q,k = 1,2,3,...$, in the norm of the space $H(Q_T)$.*

Proof. The assertion of the lemma follows directly from the relation

$$\begin{aligned}
&\|V_k^{(q)}(t,x) - V_{k,r}^{(q)}(t,x)\|^2_{H(Q_T)} \\
&\leq 6T\left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\sqrt{K_0 T^2}\right)^2}\right)\left\{\sum_{n=1}^{\infty}\psi_{1n}^2 + \frac{1}{\lambda_1}\sum_{n=1}^{\infty}\psi_{2n}^2 + \sum_{n=r+1}^{\infty}\int_0^T\left(\int_Q f[\tau,x,\bar{u}^{(k)}(t,x)]z_n dx\right)^2 d\tau\right. \\
&\quad \left. + \sum_{n=r+1}^{\infty}\int_0^T\left(\int_\gamma p[\tau,x,\bar{\vartheta}^k(t,x)]z_n dx\right)^2 d\tau\right\} \to 0,\ r \to \infty,\ q,k = 1,2,3,....
\end{aligned}$$

and this holds due to the convergence of the remainder terms of the corresponding convergent series for each fixed q,k. □

4) Convergence of finite-dimensional approximations to the optimal process

Theorem 3.9. *Let the following conditions be satisfied:*

1. *The functions of the external and boundary effects satisfy the Lipschitz condition on the functional variables (on the controls)*

$$\|f[\eta,\xi,\hat{u}(\eta,\xi)] - f[\eta,\xi,\tilde{u}(\eta,\xi)]\|^2_{H(Q_T)} \leq f_0^2\|\hat{u}(\eta,\xi) - \tilde{u}(\eta,\xi)\|^2_{H(Q_T)}, \quad f_0^2 = const,$$

$$\|p[\eta,\xi,\hat{\vartheta}(\eta,\xi)] - p[\eta,\xi,\tilde{\vartheta}(\eta,\xi)]\|^2_{H(Q_T)} \leq p_0^2\|\hat{\vartheta}(\eta,\xi) - \tilde{\vartheta}(\eta,\xi)\|^2_{H(Q_T)}, \quad p_0^2 = const.$$

2. *Intermediate vector functions $\bar{\varphi}[t,x,\theta_1(t,x),\alpha], x \in Q$, and $\bar{v}[t,x,\theta_2(t,x),\beta], x \in \gamma$, satisfy the Lipschitz condition for functional variables:*

$$\|\bar{\varphi}[t,x,\hat{\theta}_1(t,x),\alpha] - \bar{\varphi}[t,x,\tilde{\theta}_1(t,x),\alpha]\|_{H(Q_T)} \leq \varphi_0(\alpha)\|\hat{\theta}_1(t,x) - \tilde{\theta}_1(t,x)\|_{H(Q_T)}, \quad \varphi_0(\alpha) > 0,$$

$$\|\bar{v}[t,x,\hat{\theta}_2(t,x),\beta] - \bar{v}[t,x,\tilde{\theta}_2(t,x),\beta]\|_{H(Q_T)} \leq v_0(\beta)\|\hat{\theta}_2(t,x) - \tilde{\theta}_2(t,x)\|_{H(Q_T)}, \quad v_0(\beta) > 0.$$

3. *With respect to the parameters of the nonlinear optimization problem (1)-(4), the inequality holds*

$$C(\alpha,\beta) = 4T^2\left(1+\frac{\lambda^2 K_0 T^2}{\left(\lambda_1-|\lambda|\sqrt{K_0T^2}\right)^2}\right)\sqrt{f_0^2\varphi_0^2(\alpha)r^2+p_0^2 v_0^2(\beta)m^2}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}} < 1.$$

Then the finite-dimensional approximations $V_{k,l}^{(q)}(t,x)$ of the optimal process under the conditions of tracking problem (1)-(4) converge to the optimal process $V^0(t,x)$ for $q,k,l\to\infty$ in the space norm $H(Q_T)$.

Proof. Based on Lemmas 1-4, the assertion of the theorem follows from the relation:

$$\begin{aligned}\|V^0(t,x)-V_{k,l}^{(q)}(t,x)\|_{H(Q_T)} &\le \|V^0(t,x)-V^{(q)}(t,x)\|_{H(Q_T)}+\|V^{(q)}(t,x)-V_k^{(q)}(t,x)\|_{H(Q_T)}\\ &+\|V_k^{(q)}(t,x)-V_{k,l}^{(q)}(t,x)\|_{H(Q_T)}\to 0,\quad q,k,l\to\infty.\end{aligned}$$

□

3.3. Approximations of the minimum value of the functional and their convergence

The minimum value of the functional (1) in accordance with the approximations of the optimal process has three different types of approximations: resolvent approximations of the minimum value of the functional, q,k-th approximations of the minimum value of functional, finite-dimensional approximations of the minimum value of the functional.

1) Resolvent approximations of the minimum value of the functional and their convergence

q-th resolvent approximations of the minimum value of the functional, taking into account the resolvent approximations of the optimal process, are calculated by the formulas

$$\begin{aligned}J^{(q)}[\bar u^0(t,x),\bar\vartheta^0(t,x)] &= \int_0^T\int_Q\left[V^{(q)}(t,x)-\xi(t,x)\right]^2 dxdt\\ &+\int_0^T\left[\alpha\int_Q h(t,x,\bar u^0(t,x))\,dx+\beta\int_\gamma b(t,x,\bar\vartheta^0(t,x))\,dx\right]dt.\end{aligned}\tag{19}$$

Lemma 3.10. *Under the conditions of the nonlinear optimization problem (1)-(4), the resolvent approximations $J^{(q)}[\bar u^0(t,x),\bar\vartheta^0(t,x)]$ of the minimal value of the functional converge to the exact minimal value $J[\bar u^0(t,x),\bar\vartheta^0(t,x)]$ for $q\to\infty$ in the norm of real numbers space $\mathbb{R}$.*

Proof. The assertion follows directly from the inequality:

$$\begin{aligned}\left|J[\bar u^0(t,x),\bar\vartheta^0(t,x)]-J^{(q)}[\bar u^0(t,x),\bar\vartheta^0(t,x)]\right| &\le \left|\int_0^T\int_Q\left[V^0(t,x)-\xi(t,x)\right]^2 dxdt-\int_0^T\int_Q[V^q(t,x)-\xi(t,x)]^2\,dxdt\right|\\ &\le \left\|V^0(t,x)-V^q(t,x)-2\xi(t,x)\right\|_{H(Q_T)}\left\|V^0(t,x)-V^q(t,x)\right\|_{H(Q_T)}\\ &\to 0,\quad q\to\infty.\end{aligned}$$

□

2) q,k-th approximations of the minimal value of the functional and their convergence

q,k-th approximations of the minimal value of the functional are determined by following formulas:

$$\begin{aligned}J_k^{(q)}[\bar u^{(k)}(t,x),\bar\vartheta^{(k)}(t,x)] &= \int_0^T\int_Q\left[V_k^{(q)}(t,x)-\xi(t,x)\right]^2 dxdt\\ &+\int_0^T\left[\alpha\int_Q h(t,x,\bar u^{(k)}(t,x))dx+\beta\int_\gamma b(t,x,\bar\vartheta^{(k)}(t,x))\,dx\right]dt,\\ &\alpha,\beta>0.\end{aligned}\tag{20}$$

Lemma 3.11. *Under the conditions of the nonlinear optimization problem (1)-(4), the q,k-th approximations $J_k^{(q)}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]$ of the minimal value of the functional converge to the q-th approximations $J^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]$ for $k\to\infty$ for all fixed $q=1,2,3,\ldots$, in the norm of real numbers space $\mathbb{R}$.*

Proof. The assertion follows from the relation:

$$\begin{aligned}&\left|J^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]-J_k^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\right|\le\\&\le\left\|V^{(q)}(t,x)-V_k^{(q)}(t,x)-2\xi(t,x)\right\|_{H(Q_T)}\left\|V_k^q(t,x)-V^q(t,x)\right\|_{H(Q_T)}\\&+\alpha h_0\left\|\bar{u}^0(t,x)-\bar{u}^k(t,x)\right\|_{H(Q_T)}+\beta b_0\left\|\bar{\vartheta}^0(t,x)-\bar{\vartheta}^k(t,x)\right\|_{H(\gamma_T)}\\&\to 0,\quad k\to\infty.\end{aligned}$$

□

3) Convergence of finite-dimensional approximations of the minimal value of the functional

According to formulas (19) and (24), finite-dimensional approximations of the minimal value of the functional are calculated using following formulas:

$$\begin{aligned}J_{k,r}^{(q)}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]&=\int_0^T\int_Q\left[V_{k,r}^{(q)}(t,x)-\xi(t,x)\right]^2dxdt\\&+\int_0^T\left[\alpha\int_Q h(t,x,\bar{u}^{(k)}(t,x))dx+\beta\int_\gamma b(t,x,\bar{\vartheta}^{(k)}(t,x))\,dx\right]dt,\\&\alpha,\beta>0.\end{aligned}\tag{21}$$

Lemma 3.12. *Under the conditions of tracking problem (1)-(4), finite-dimensional approximations $J_{k,r}^{(q)}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]$ of the minimal value of the functional converge to the corresponding q,k-th approximations $J_k^{(q)}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]$ for $r\to\infty$ for all fixed $q,k=1,2,3,\ldots$, in the norm of numerical space $\mathbb{R}$ of real numbers.*

Proof. According to formulas (66) and (67), we obtain:

$$\begin{aligned}&\left|J_k^{(q)}\left[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)\right]-J_{k,r}^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\right|\le\\&\le\left\|V_k^{(q)}(t,x)-V_{k,r}^{(q)}(t,x)-2\xi(t,x)\right\|_{H(Q_T)}\left\|V_k^q(t,x)-V_{k,r}^q(t,x)\right\|_{H(Q_T)}\\&\to 0,\quad r\to\infty,\quad q,k=1,2,3,\ldots,\end{aligned}$$

which confirms the validity of the lemma. □

Theorem 3.13. *Let the conditions of Theorem 1 be satisfied. Then finite-dimensional approximations $J_{k,r}^{(q)}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]$ of the minimal value of the functional under the conditions of tracking problem (1)-(4) converge to the minimal value $J[\bar{u}^{(0)}(t,x),\bar{\vartheta}^{(0)}(t,x)]$ of functional for $q,k,r\to\infty$ in the norm of numerical space $\mathbb{R}$ of real numbers.*

Proof. Based on Lemmas 8-10, the assertion follows from:

$$\begin{aligned}&\left|J\left[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)\right]-J_{k,r}^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\right|\le\\&\le\left|J\left[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)\right]-J^{(q)}\left[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)\right]\right|+\\&+\left|J^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]-J_k^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\right|+\\&+\left|J_k^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]-J_{k,r}^{(q)}[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\right|\\&\to 0,\quad q,k,r\to\infty.\end{aligned}$$

□

4. Conclusion

In this paper, we have investigated the convergence of various types of approximations for the tracking problem in nonlinear optimization of oscillatory processes described by integro-differential equations with a Fredholm integral operator. The main results can be summarized as follows:

1. We established sufficient conditions for the convergence of approximations of vector optimal controls (distributed and boundary) to their exact values in the corresponding Hilbert spaces.

2. Three types of approximations for the optimal process were analyzed: - Resolvent approximations based on the kernel of the integral operator - Approximations with respect to optimal controls - Finite-dimensional approximations

3. Corresponding approximations for the minimal value of the functional were studied and their convergence was proved.

4. The key technical tools included: - Method of successive approximations - Estimates in Sobolev and Hilbert spaces - Analysis of resolvent convergence

The results obtained can be applied to various problems of optimal control for distributed parameter systems with integral terms in the dynamics. Future research directions may include: - Extension to stochastic systems - Consideration of more general nonlinearities - Development of numerical algorithms based on the proposed approximations - Applications to specific physical systems described by integro-differential equations

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