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Degenerate hyperharmonic polynomials and numbers

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Abstract. This paper explores new identities and relations for degenerate hyperharmonic numbers and degenerate hyperharmonic polynomials, which are respectively a degenerate version of hyperharmonic numbers and a polynomial extension of the degenerate hyperharmonic numbers. Key results include expressing degenerate hyperharmonic numbers and polynomials in terms of finite sums involving unsigned degenerate Stirling numbers of the first kind, degenerate Bernoulli polynomials, and other related numbers. We also derive a recurrence relation for the degenerate hyperharmonic numbers and a closed-form expression for an alternating sum involving degenerate harmonic numbers.

1. Introduction

The field of degenerate special numbers and polynomials, initially established by Carlitz's foundational work on degenerate Bernoulli and Euler polynomials (see [4]), has experienced a significant revival in recent years (see [9, 11, 12, 14–22, 27]). This renewed interest has led to the exploration of several new structures, notably the expansion of this inquiry to include transcendental functions, culminating in the development of the degenerate gamma function (see [18]). Paper [26] investigates the degenerate gamma function by using the complex delta function to establish a novel series representation. It also solves the relevant fractional kinetic equation and derives new fractional transform equations through a novel representation. This paper contributes to this line of research by presenting several new results concerning degenerate hyperharmonic numbers, degenerate hyperharmonic polynomials, and related identities.

This paper is structured as follows: Section 1 reviews the essential concepts and notations relevant to this study. We recall the definitions and properties of the degenerate exponentials, degenerate logarithms, and degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$. We then introduce the degenerate Stirling numbers of the first kind, unsigned degenerate Stirling numbers of the first kind ${n \brack k}_{\lambda}$, and degenerate Stirling numbers of the second kind ${n \brace k}_{\lambda}$. The section continues by recalling the harmonic numbers, hyperharmonic numbers, degenerate harmonic numbers $H_{n,\lambda}$, and degenerate hyperharmonic numbers $H_{n,\lambda}^{(r)}$. Finally, we state the binomial inversion relation.

Section 2 contains the principal results of this paper. We begin by recalling the generalized degenerate harmonic numbers $H_{\lambda}(n,r)$, a generalization of the degenerate harmonic numbers. Theorems in this section establish various identities and relations:

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• Theorem 2.1 expresses $H^{(r+1)}_{n+1,\lambda}$ as a finite sum involving $H_\lambda(n+1,k)$.
• Theorem 2.2 derives an expression for the generating function $\sum_{n=k-1}^{\infty}\left[{n+1 \atop k}\right]_\lambda \frac{x^n}{n!}$.
Before proceeding, we introduce the degenerate hyperharmonic polynomials $H^{(r)}_{n,\lambda}(x)$.
• Theorem 2.3 shows that $H^{(r)}_{n,\lambda}(x)$ can be represented as a finite sum involving $H^{(r)}_{k,\lambda}$ and the falling factorial $(x)_{n-k}$.
• Theorem 2.4 provides an identity for $H^{(r+1)}_{n+1,\lambda}(\lambda)$ as a finite sum involving $\left[{n+2 \atop k+2}\right]_\lambda$.
• Theorem 2.5 derives a closed-form for the alternating sum $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{k,\lambda}$.
• Theorem 2.6 expresses $H^{(r+1)}_{n+1,\lambda}$ as a finite sum involving $H_{k+1,\lambda}$.
• Theorem 2.7 proves the identity $H^{(r+1)}_{n+1,\lambda} = H^{(r+2)}_{n+1,\lambda} - H^{(r+2)}_{n,\lambda}$, $(n \geq 1)$.
• Theorem 2.8 indicates $H^{(1)}_{n+1,\lambda}(x+\lambda)$ as a finite sum involving $\left[{n+1 \atop k+1}\right]_\lambda$ and $\beta_{k,\lambda}(x)$.
• Theorem 2.9 finds an expression for the generating function $\sum_{n=k-1}^{\infty}\left\{{n+1 \atop k}\right\}_\lambda \frac{t^n}{n!}$.
• Theorem 2.10 shows that $\beta_{n,\lambda}(-r-\lambda)$ is a finite sum involving $\left\{{n+1 \atop k+1}\right\}_\lambda$ and $H^{(r+1)}_{k+1,\lambda}$.
• Theorem 2.11 and Theorem 2.12 find expressions respectively for $H^{(1)}_{n,\lambda}(x)$ and $H^{(1)}_{n+1,\lambda}(x)$.
As general references of this paper, we let the reader refer to [1, 6–8]. For the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by (see [10, 18, 26])

$$e^x_\lambda(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad e_\lambda(t) = e^1_\lambda(t), \tag{1}$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\big(x-(n-1)\lambda\big), \ (n \geq 1).$$

Note that $\lim_{\lambda\to 0} e^x_\lambda(t) = e^{xt}$.

As the inverse relation of $e_\lambda(t)$, the degenerate logarithm is given by $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$. Then we note that (see [9, 11, 15, 16, 19, 20, 27])

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!} = \sum_{n=1}^{\infty}\binom{\lambda-1}{n-1}\frac{t^n}{n}, \tag{2}$$

where $\binom{x}{n}$ denotes the binomial coefficient given by

$$\binom{x}{0} = 1, \ \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}, \ (n \geq 1).$$

Note that $\lim_{\lambda\to 0}\log_\lambda(1+t) = \log(1+t)$.

It is well known that the degenerate Bernoulli polynomials are defined by Carlitz as (see [4])

$$\frac{t}{e_\lambda(t)-1}e^x_\lambda(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$, $(n \geq 0)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$, $(n \geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by (see [6])

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

From (3), we note that (see [4])

$$\beta_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)_{n-k,\lambda},\ (n\geq 0).$$

The degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ are defined by (see [9, 11, 16, 19–21, 27])

$$(x)_n=\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda},\quad (n\geq 0),\tag{4}$$

where

$$(x)_0=1,\ (x)_n=x(x-1)(x-2)\cdots(x-n+1),\ (n\geq 1).$$

The unsigned degenerate Stirling numbers of the first kind are defined by ${n \brack k}_\lambda=(-1)^{n-k}S_{1,\lambda}(n,k),\ (n,k\geq 0)$.

From (2), we note that

$$\frac{1}{k!}\log_{-\lambda}^{k}\Big(\frac{1}{1-t}\Big)=\frac{1}{k!}\Big(-\log_\lambda(1-t)\Big)^k=\sum_{n=k}^{\infty}{n \brack k}_\lambda\frac{t^n}{n!},\tag{5}$$

where k is a nonnegative integer (see [11, 19, 20, 27]).

As the inversion relation of (4), the degenerate Stirling numbers of the second kind ${n \brace k}_\lambda$ are defined by

$$(x)_{n,\lambda}=\sum_{k=0}^{n}{n \brace k}_\lambda(x)_k,\quad (n\geq 0).\tag{6}$$

Thus, by (6), we get (see [11, 16, 19, 20, 27])

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}{n \brace k}_\lambda\frac{t^n}{n!},\quad (k\geq 0).\tag{7}$$

Note that (see [6])

$$\lim_{\lambda\to 0}{n \brack k}_\lambda={n \brack k},\quad \lim_{\lambda\to 0}{n \brace k}_\lambda={n \brace k},$$

where

$$\frac{1}{k!}\log^k\Big(\frac{1}{1-t}\Big)=\sum_{n=k}^{\infty}{n \brack k}\frac{t^n}{n!},\quad \frac{1}{k!}\Big(e^t-1\Big)^k=\sum_{n=k}^{\infty}{n \brace k}\frac{t^n}{n!},\quad (k\geq 0).\tag{8}$$

The harmonic numbers are defined by

$$H_0=0,\quad H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},\ (n\in\mathbb{N}).\tag{9}$$

Thus, by (9), we get (see [3, 5, 13, 23, 24])

$$\frac{1}{1-t}\log\Big(\frac{1}{1-t}\Big)=\sum_{n=1}^{\infty}H_nt^n.$$

Recently, Kim–Kim introduced the degenerate harmonic numbers given by (see [19])

$$H_{0,\lambda}=0,\quad H_{n,\lambda}=\frac{1}{\lambda}\sum_{k=1}^{n}\binom{\lambda}{k}(-1)^{k-1}=\sum_{k=1}^{n}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k},\tag{10}$$

where n is a positive integer. Note that

$$\lim_{\lambda\to 0} H_{n,\lambda} = H_n, \quad (n \geq 0).$$

From (10), we note that (see [9, 11, 15, 16, 21, 27])

$$\frac{1}{1-t}\log_{-\lambda}\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_{n,\lambda} t^n, \quad H_{0,\lambda} = 0. \tag{11}$$

In 1996, Conway and Guy introduced hyperharmonic numbers $H_n^{(r)}$, $(n, r \geq 0)$, which are given by

$$H_0^{(r)} = 0, \ (r \geq 0), \quad H_n^{(0)} = \frac{1}{n}, \ (n \geq 1), \quad H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}, \ (n, r \geq 1). \tag{12}$$

Thus, by (12), we get (see [2, 7, 13])

$$\frac{1}{(1-t)^r}\log\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_n^{(r)} t^n. \tag{13}$$

Recently, Kim-Kim introduced the degenerate hyperharmonic numbers as (see [20])

$$H_{0,\lambda}^{(r)} = 0, \ (r \geq 0), \quad H_{n,\lambda}^{(0)} = \frac{\lambda^{n-1}}{n!}(-1)^{n-1}(1)_{n,1/\lambda} = \frac{1}{\lambda}\binom{\lambda}{n}(-1)^{n-1}, \ (n \geq 1), \tag{14}$$

and

$$H_{n,\lambda}^{(r)} = \sum_{k=1}^{n} H_{k,\lambda}^{(r-1)}, \ (n, r \geq 1). \tag{15}$$

From (14), we note that (see [9, 11, 16, 21, 27])

$$\frac{1}{(1-t)^r}\log_{-\lambda}\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^r}\Big(-\log_{\lambda}(1-t)\Big) = \sum_{n=1}^{\infty} H_{n,\lambda}^{(r)} t^n. \tag{16}$$

We recall the following binomial inversion relation (see [25]):

$$a_n = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1} b_k \Longleftrightarrow b_n = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1} a_k. \tag{17}$$

The following formulas will be used in several proofs:

$$(1-t)^{-r} = \sum_{l=0}^{\infty}\binom{r+l-1}{r-1} t^l, \tag{18}$$

$$\log_{\lambda}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty}\binom{\lambda+n-1}{n-1}\frac{x^n}{n}, \tag{19}$$

$$\frac{d}{dt}e_{\lambda}(t) = e_{\lambda}^{1-\lambda}(t), \quad \frac{d}{dt}\log_{\lambda}(t) = t^{\lambda-1}. \tag{20}$$

2. Degenerate hyperharmonic polynomials and numbers

The generalized degenerate harmonic numbers are given by

$$\frac{1}{1-x}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)=\sum_{n=r+1}^{\infty}H_{\lambda}(n,r)x^{n}. \tag{21}$$

We observe, by using (1), that the left hand side of (21) is equal to

$$\begin{aligned}&\sum_{m=0}^{\infty}x^{m}\sum_{l_0=1}^{\infty}(-1)^{l_0-1}\binom{\lambda-1}{l_0-1}\frac{x^{l_0}}{l_0}\sum_{l_1=1}^{\infty}(-1)^{l_1-1}\binom{\lambda-1}{l_1-1}\frac{x^{l_1}}{l_1}\cdots\sum_{l_r=1}^{\infty}(-1)^{l_r-1}\binom{\lambda-1}{l_r-1}\frac{x^{l_r}}{l_r} \\ &=\sum_{n=r+1}^{\infty}\sum_{l_0+\cdots+l_r=r+1}^{n}\frac{(-1)^{l_0-1}\binom{\lambda-1}{l_0-1}(-1)^{l_1-1}\binom{\lambda-1}{l_1-1}\cdots(-1)^{l_r-1}\binom{\lambda-1}{l_r-1}}{l_0l_1l_2\cdots l_r}x^{n}.\end{aligned} \tag{22}$$

Thus, by (21) and (22), we get

$$H_{\lambda}(n,r)=\sum_{r+1\le l_0+\cdots+l_r\le n}\frac{(-1)^{l_0-1}\binom{\lambda-1}{l_0-1}(-1)^{l_1-1}\binom{\lambda-1}{l_1-1}\cdots(-1)^{l_r-1}\binom{\lambda-1}{l_r-1}}{l_0l_1l_2\cdots l_r},$$

where n,r are integers with $r\ge 0$, $n\ge r+1$, and the sum runs over all positive integers $l_0,l_1,\dots,l_r$, satisfying $r+1\le l_0+\cdots+l_r\le n$.

The degenerate rising factorial sequence is given by

$$\langle x\rangle_{0,\lambda}=1,\quad \langle x\rangle_{n,\lambda}=x(x+\lambda)(x+2\lambda)\cdots\Big(x+(n-1)\lambda\Big),\ (n\ge 1). \tag{23}$$

From (21) and (23), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+1)}x^{n}&\overset{(16)}{=}\frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)^{r+1}}=\frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)}e_{\lambda}^{-r}\Big(\log_{\lambda}(1-x)\Big) \\ &\overset{(1)}{=}\frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)}\sum_{k=0}^{\infty}(-r)_{k,\lambda}\frac{1}{k!}\log_{\lambda}^{k}(1-x) \\ &\overset{(5)}{=}\frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)}\sum_{k=0}^{\infty}\frac{\langle r\rangle_{k,\lambda}}{k!}\log_{-\lambda}^{k}\left(\frac{1}{1-x}\right) \\ &=\frac{1}{x}\sum_{k=0}^{\infty}\frac{\langle r\rangle_{k,\lambda}}{k!}\frac{1}{1-x}\log_{-\lambda}^{k+1}\left(\frac{1}{1-x}\right) \\ &=\frac{1}{x}\sum_{k=0}^{\infty}\frac{\langle r\rangle_{k,\lambda}}{k!}\sum_{n=k+1}^{\infty}H_{\lambda}(n,k)x^{n} \\ &=\sum_{k=0}^{\infty}\frac{\langle r\rangle_{k,\lambda}}{k!}\sum_{n=k}^{\infty}H_{\lambda}(n+1,k)x^{n} \\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\langle r\rangle_{k,\lambda}}{k!}H_{\lambda}(n+1,k)x^{n},\end{aligned} \tag{24}$$

where r is a nonnegative integer.

Therefore, by (24), we obtain the following theorem.

Theorem 2.1. *For $n, r \geq 0$, we have*

$$H_{n+1,\lambda}^{(r+1)} = \sum_{k=0}^{n} \frac{\langle r \rangle_{k,\lambda}}{k!} H_\lambda(n+1,k).$$

We note that

$$\begin{aligned}\sum_{n=k-1}^{\infty} \begin{bmatrix} n+1 \\ k \end{bmatrix}_\lambda \frac{x^n}{n!} &= \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{x^{n-1}}{(n-1)!} = \frac{d}{dx} \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{x^n}{n!} \\ &\overset{(5)}{=} \frac{d}{dx} \frac{1}{k!} \log_{-\lambda}^k \left(\frac{1}{1-x} \right) \\ &\overset{(20)}{=} \frac{1}{(k-1)!} \frac{(1-x)^\lambda}{1-x} \log_{-\lambda}^{k-1} \left(\frac{1}{1-x} \right).\end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 2.2. *For $k \in \mathbb{N}$, we have*

$$\frac{1}{(k-1)!} \frac{(1-x)^\lambda}{1-x} \log_{-\lambda}^{k-1} \left(\frac{1}{1-x} \right) = \sum_{n=k-1}^{\infty} \begin{bmatrix} n+1 \\ k \end{bmatrix}_\lambda \frac{x^n}{n!}. \tag{26}$$

In view of (16), we define the *degenerate hyperharmonic polynomials* by

$$\frac{\log_{-\lambda}\left(\frac{1}{1-t}\right)}{(1-t)^r} (1-t)^x = \sum_{n=1}^{\infty} H_{n,\lambda}^{(r)}(x) t^n, \quad (r \geq 0). \tag{27}$$

When $x = 0$, $H_{n,\lambda}^{(r)}(0) = H_{n,\lambda}^{(r)}$, $(n \geq 1)$.

Thus, by (27), we get

$$H_{n,\lambda}^{(r)}(x) = \sum_{k=1}^{n} H_{k,\lambda}^{(r)} \frac{(x)_{n-k}}{(n-k)!} (-1)^{n-k}, \quad (n \geq 1), \quad H_{0,\lambda}^{(r)}(x) = 0. \tag{28}$$

Therefore, by (28), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ and $r \geq 0$, we have*

$$H_{n,\lambda}^{(r)}(x) = \sum_{k=1}^{n} H_{k,\lambda}^{(r)} \frac{(x)_{n-k}}{(n-k)!} (-1)^{n-k}, \quad H_{0,\lambda}^{(r)}(x) = 0.$$

From (26) and (27), we have

$$\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+1)}(\lambda)x^{n}=\frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)^{r+1}}(1-x)^{\lambda}=\frac{(1-x)^{\lambda}\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)}e_{\lambda}^{-r}\Big(\log_{\lambda}(1-x)\Big) \tag{29}$$
$$\overset{(1)}{=}\frac{(1-x)^{\lambda}}{x(1-x)}\sum_{k=0}^{\infty}\langle r\rangle_{k,\lambda}\frac{\log_{-\lambda}^{k+1}\left(\frac{1}{1-x}\right)}{k!}$$
$$=\frac{1}{x}\sum_{k=0}^{\infty}\langle r\rangle_{k,\lambda}(k+1)\frac{\log_{-\lambda}^{k+1}\left(\frac{1}{1-x}\right)}{(k+1)!}\frac{(1-x)^{\lambda}}{1-x}$$
$$=\frac{1}{x}\sum_{k=0}^{\infty}\langle r\rangle_{k,\lambda}(k+1)\sum_{n=k+1}^{\infty}\begin{bmatrix}n+1\\k+2\end{bmatrix}_{\lambda}\frac{x^{n}}{n!}$$
$$=\sum_{k=0}^{\infty}\langle r\rangle_{k,\lambda}(k+1)\sum_{n=k}^{\infty}\begin{bmatrix}n+2\\k+2\end{bmatrix}_{\lambda}\frac{x^{n}}{(n+1)!}$$
$$=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{k=0}^{n}\langle r\rangle_{k,\lambda}(k+1)\begin{bmatrix}n+2\\k+2\end{bmatrix}_{\lambda}x^{n}.$$

Therefore, by (29), we obtain the following theorem.

Theorem 2.4. *For $n, r \geq 0$, we have*

$$H_{n+1,\lambda}^{(r+1)}(\lambda)=\frac{1}{(n+1)!}\sum_{k=0}^{n}\langle r\rangle_{k,\lambda}(k+1)\begin{bmatrix}n+2\\k+2\end{bmatrix}_{\lambda}. \tag{30}$$

Letting $\lambda \to 0$ in (30), we obtain

$$H_{n+1}^{(r+1)}=\frac{1}{(n+1)!}\sum_{k=0}^{n}r^{k}(k+1)\begin{bmatrix}n+2\\k+2\end{bmatrix}.$$

Now, we observe that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}H_{k,\lambda}(-1)^{k-1}x^{n}=\sum_{k=1}^{\infty}H_{k,\lambda}(-1)^{k-1}\sum_{n=k}^{\infty}\binom{n}{k}x^{n} \tag{31}$$
$$=\sum_{k=1}^{\infty}H_{k,\lambda}(-1)^{k-1}x^{k}\sum_{n=0}^{\infty}\binom{n+k}{k}x^{n}$$
$$\overset{(18)}{=}\sum_{k=1}^{\infty}H_{k,\lambda}(-1)^{k-1}x^{k}\left(\frac{1}{1-x}\right)^{k+1}$$
$$=-\frac{1}{1-x}\sum_{k=1}^{\infty}H_{k,\lambda}\left(\frac{-x}{1-x}\right)^{k}$$
$$\overset{(11)}{=}-\frac{1}{1-x}\frac{1}{1+\frac{x}{1-x}}\log_{-\lambda}\left(\frac{1}{1+\frac{x}{1-x}}\right)$$
$$=-\log_{-\lambda}(1-x)=\log_{\lambda}\left(\frac{1}{1-x}\right)$$
$$\overset{(19)}{=}\sum_{n=1}^{\infty}\binom{\lambda+n-1}{n-1}\frac{x^{n}}{n}.$$

Therefore, by (31), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$\frac{1}{n}\binom{\lambda+n-1}{n-1} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{k,\lambda}.$$

Moreover, by inversion we also have (see (17)*)*

$$H_{n,\lambda} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}\binom{\lambda+k-1}{k-1}.$$

We note that

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+1)}x^{n} &\overset{(16)}{=} \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{(1-x)^{r+1}x} = \frac{1}{x}(1-x)^{-r}\frac{1}{1-x}\log_{-\lambda}\left(\frac{1}{1-x}\right) \\ &\overset{(18)}{=} \frac{1}{x}\sum_{l=0}^{\infty}\binom{r+l-1}{r-1}x^{l}\sum_{k=1}^{\infty}H_{k,\lambda}x^{k} \\ &= \frac{1}{x}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{r+n-k-1}{r-1}H_{k,\lambda}x^{n} \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1}\binom{r+n-k}{r-1}H_{k,\lambda}x^{n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{r+n-k-1}{r-1}H_{k+1,\lambda}x^{n}.\end{aligned} \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.6. *For $n, r \geq 0$, we have*

$$H_{n+1,\lambda}^{(r+1)} = \sum_{k=0}^{n}\binom{n-k+r-1}{r-1}H_{k+1,\lambda}.$$

Noting that $H_{1,\lambda}^{(r+2)} = H_{1,\lambda}^{(r+1)}$ (see (15)), we have

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+1)}x^{n} &\overset{(16)}{=} \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)^{r+1}} = \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)^{r+2}} - \frac{x\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)^{r+2}} \\ &\overset{(16)}{=} \sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+2)}x^{n} - x\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(r+2)}x^{n} \\ &= H_{1,\lambda}^{(r+2)} + \sum_{n=1}^{\infty}\left(H_{n+1,\lambda}^{(r+2)} - H_{n,\lambda}^{(r+2)}\right)x^{n}.\end{aligned} \tag{33}$$

Therefore, by (33), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}$, we have*

$$H_{n+1,\lambda}^{(r+1)} = H_{n+1,\lambda}^{(r+2)} - H_{n,\lambda}^{(r+2)}.$$

From (26) and (27), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{n!}\begin{bmatrix}n+1\\k+1\end{bmatrix}_\lambda \beta_{k,\lambda}(x)t^n=\sum_{k=0}^{\infty}(-1)^k\beta_{k,\lambda}(x)\sum_{n=k}^{\infty}\frac{1}{n!}\begin{bmatrix}n+1\\k+1\end{bmatrix}_\lambda t^n \\
&=\sum_{k=0}^{\infty}(-1)^k\beta_{k,\lambda}(x)\frac{1}{k!}\frac{(1-t)^\lambda}{1-t}\log_{-\lambda}^k\Big(\frac{1}{1-t}\Big) \\
&\overset{(5)}{=}\sum_{k=0}^{\infty}\beta_{k,\lambda}(x)\frac{(1-t)^\lambda}{1-t}\frac{1}{k!}\log_\lambda^k(1-t) \\
&=\frac{(1-t)^\lambda}{1-t}\sum_{k=0}^{\infty}\beta_{k,\lambda}(x)\frac{\log_\lambda^k(1-t)}{k!} \\
&\overset{(3)}{=}\frac{(1-t)^\lambda}{1-t}\frac{\log_\lambda(1-t)}{e_\lambda\Big(\log_\lambda(1-t)\Big)-1}e_\lambda^x\Big(\log_\lambda(1-t)\Big) \\
&=\frac{(1-t)^\lambda}{1-t}\frac{\log_\lambda(1-t)}{1-t-1}(1-t)^x=\frac{\log_{-\lambda}\Big(\frac{1}{1-t}\Big)}{t(1-t)}(1-t)^{\lambda+x} \\
&=\frac{1}{t}\sum_{n=1}^{\infty}H_{n,\lambda}^{(1)}(\lambda+x)t^n=\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(1)}(x+\lambda)t^n.
\end{aligned} \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 2.8. *For $n\geq 0$, we have*

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+1\\k+1\end{bmatrix}_\lambda\beta_{k,\lambda}(x)=H_{n+1,\lambda}^{(1)}(x+\lambda).$$

We note that

$$\begin{aligned}
&\sum_{n=k-1}^{\infty}\begin{Bmatrix}n+1\\k\end{Bmatrix}_\lambda\frac{t^n}{n!}=\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}_\lambda\frac{t^{n-1}}{(n-1)!}=\frac{d}{dt}\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}_\lambda\frac{t^n}{n!} \\
&\overset{(7)}{=}\frac{1}{k!}\frac{d}{dt}\Big(e_\lambda(t)-1\Big)^k\overset{(20)}{=}\frac{1}{(k-1)!}\Big(e_\lambda(t)-1\Big)^{k-1}e_\lambda^{1-\lambda}(t),\quad (k\in\mathbb{N}).
\end{aligned} \tag{35}$$

Therefore, by (35), we obtain the following theorem.

Theorem 2.9. *For $k\geq 1$, we have*

$$\frac{1}{(k-1)!}\Big(e_\lambda(t)-1\Big)^{k-1}e_\lambda^{1-\lambda}(t)=\sum_{n=k-1}^{\infty}\begin{Bmatrix}n+1\\k\end{Bmatrix}_\lambda\frac{t^n}{n!}. \tag{36}$$

Letting $\lambda\to 0$ in (36) and using (8), we get the following recurrence relation for the Stirling numbers of the second kind:

$$\begin{Bmatrix}n+1\\k\end{Bmatrix}=\begin{Bmatrix}n\\k-1\end{Bmatrix}+k\begin{Bmatrix}n\\k\end{Bmatrix},\quad (n\geq k).$$

From (36), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k k!\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}_{\lambda} H_{k+1,\lambda}^{(r+1)}\frac{t^n}{n!} = \sum_{k=0}^{\infty}(-1)^k k! H_{k+1,\lambda}^{(r+1)}\sum_{n=k}^{\infty}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}_{\lambda}\frac{t^n}{n!} \tag{37}$$

$$= \sum_{k=0}^{\infty}(-1)^k k! H_{k+1,\lambda}^{(r+1)}\frac{1}{k!}\Big(e_\lambda(t)-1\Big)^k e_\lambda^{1-\lambda}(t)$$

$$= e_\lambda^{1-\lambda}(t)\sum_{k=0}^{\infty}H_{k+1,\lambda}^{(r+1)}\Big(1-e_\lambda(t)\Big)^k$$

$$\overset{(16)}{=} e_\lambda^{1-\lambda}(t)\frac{-\log_\lambda\Big(e_\lambda(t)\Big)}{\Big(1-e_\lambda(t)\Big)e_\lambda^{r+1}(t)}$$

$$= \frac{t}{e_\lambda(t)-1}e_\lambda^{-\lambda-r}(t) \overset{(3)}{=} \sum_{n=0}^{\infty}\beta_{n,\lambda}(-r-\lambda)\frac{t^n}{n!}.$$

Therefore, by (37), we obtain the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}(-r-\lambda) = \sum_{k=0}^{n}(-1)^k k!\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}_{\lambda} H_{k+1,\lambda}^{(r+1)}.$$

From (27), we note that

$$\sum_{n=1}^{\infty}H_{n,\lambda}^{(1)}(x)t^n = \frac{\log_{-\lambda}\left(\frac{1}{1-t}\right)}{1-t}(1-t)^x \overset{(2),(18)}{=} \sum_{k=1}^{\infty}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}t^k\sum_{n=0}^{\infty}\binom{n-x}{n}t^n \tag{38}$$

$$= \sum_{k=1}^{\infty}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}\sum_{n=k}^{\infty}\binom{n-x-k}{n-k}t^n$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}\binom{n-x-k}{n-k}t^n.$$

Therefore, by (38), we obtain the following theorem.

Theorem 2.11. *For $n \in \mathbb{N}$, we have*

$$H_{n,\lambda}^{(1)}(x) = \sum_{k=1}^{n}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}\binom{n-x-k}{n-k}.$$

By (27), we get

$$\sum_{n=0}^{\infty}H_{n+1,\lambda}^{(1)}(x)t^n \overset{(16)}{=} \frac{\log_{-\lambda}\left(\frac{1}{1-t}\right)}{t(1-t)}(1-t)^x = \frac{1}{1-t}\frac{\log_{-\lambda}\left(\frac{1}{1-t}\right)}{t(1-t)}(1-t)^{x+1} \tag{39}$$

$$= \sum_{k=0}^{\infty}H_{k+1,\lambda}^{(1)}(x+1)t^k\sum_{l=0}^{\infty}t^l = \sum_{n=0}^{\infty}\sum_{k=0}^{n}H_{k+1,\lambda}^{(1)}(x+1)t^n.$$

Therefore, by (39), we obtain the following theorem.

Theorem 2.12. *For $n \geq 0$, we have*

$$H_{n+1,\lambda}^{(1)}(x) = \sum_{k=0}^{n}H_{k+1,\lambda}^{(1)}(x+1).$$

3. Conclusion

In this paper, we successfully investigated new properties and relationships for degenerate hyperharmonic numbers $H_{n,\lambda}^{(r)}$ and degenerate hyperharmonic polynomials $H_{n,\lambda}^{(r)}(x)$, significantly contributing to the evolving field of degenerate special numbers and polynomials. Our primary findings established novel identities, notably expressing these numbers and polynomials as finite sums involving unsigned degenerate Stirling numbers of the first kind, degenerate Bernoulli polynomials and other related numbers. We also derived a useful recurrence relation for $H_{n,\lambda}^{(r)}$ and provided a closed-form expression for an alternating sum of degenerate harmonic numbers. These findings contribute to the expanding field of degenerate special numbers and their applications.

We would like to continue to explore degenerate versions of many special numbers and polynomials (see [9, 11, 12, 14–22, 27]), and find their applications to science, physics, engineering and mathematics (see [26]).

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Conflict of Interest
The authors declare that they have no conflict of interest.

References

[1] Abramowitz, M.; Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables,* National Bureau of Standards Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, DC, 1964.
[2] Benjamin, A. T.; Gaebler, D.; Gaebler, R. *A combinatorial approach to hyperharmonic numbers,* Integers **3** (2003), A15, 9 pp.
[3] Benjamin, A. T.; Preston, G. O.; Quinn, J. J. *A Stirling Encounter with Harmonic Numbers,* Math. Mag. **75** (2002), no. 2, 95-103.
[4] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.
[5] Cheon, G.-S.; El-Mikkawy, M. E. A. *Generalized harmonic numbers with Riordan arrays,* J. Number Theory **128** (2008), no. 2, 413-425.
[6] Comtet, L. *Advanced combinatorics.* The art of finite and infinite expansions, Revised and enlarged ed., D. Reidel Publishing Co., Dordrecht, 1974.
[7] Conway, J. H. ; Guy, R. K. *The book of numbers,* Copernicus, New York, 1996.
[8] Djordjevic, G. B.; Milovanovic, G. V. *Special classes of polynomials,*University of Nis, Faculty of Technology Leskovac, 2014.
[9] Dolgy, D. V.; Kim, D. S.; Kim, H.K.; Kim, T. *Degenerate harmonic and hyperharmonic numbers,* Proc. Jangjeon Math. Soc. **26** (2023), no. 3, 259-268.
[10] Kim, D. S.; Bu, S.; Lee, H.; Khalil, M.; Kim, T. *Spivey's type recurrence relation for Lah-Bell polynomials,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, 2547873.
[11] Kim, D. S.; Kim, T. *Some numbers and polynomials related to degenerate harmonic and degenerate hyperharmonic numbers,* Appl. Anal. Discrete Math.**19** (2025), no. 2, 284–5298 https://doi.org/10.2298/AADM240124014K
[12] Kim, D. S.; Kim, T.; Lee, S.-H. *Combinatorial identities involving harmonic and hyperharmonic numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 3, 393-413.
[13] Kim, T.; Kim, D. s. *Several expressions for moments of sums of hyperbolic secant random variables* , Electron. Res. Arch. **33** (2025), no. 9, 5457–5470.
[14] Kim, T.; Kim, D. S. *Heterogeneous Stirling Numbers and Heterogeneous Bell Polynomials,* Russ. J. Math. Phys **32** (2025), no. 3, 498–509.
[15] Kim, T.; Kim, D. S. *Some identities on degenerate harmonic and degenerate higher-order harmonic numbers,* Appl. Math. Comput. **486** (2025), Paper No. 129045, 10 pp.
[16] Kim, T.; Kim, D. S. *Combinatorial identities involving degenerate harmonic and hyperharmonic numbers,* Adv. in Appl. Math. **148** (2023), Paper No. 102535, 15 pp.
[17] Kim, T.; Kim, D. S. *Spivey-type recurrence relations for degenerate Bell and Dowling polynomials,* Russ. J. Math. Phys. **32** (2025), no. 2, 288-296.
[18] Kim, T.; Kim, D. S. *Note on the degenerate gamma function,* Russ. J. Math. Phys. **27** (2020), no. 3, 352–358.
[19] Kim, T.; Kim, D. S. *Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials,* Symmetry **12** (2020), no. 6, Paper No. 905, 16 pp. https://doi.org/10.3390/sym12060905
[20] Kim, T.; Kim, D. S. *Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers,* Russ. J. Math. Phys. **30** (2023), no. 1, 62-75.
[21] Kim, T.; Kim, D. S.; Kim, H. K. *Some identities of degenerate harmonic and degenerate hyperharmonic numbers arising from umbral calculus,* Open Math. **21** (2023), no. 1, Paper No. 20230124, 12 pp.
[22] Kurt, B. *Notes on the degenerate harmonic numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **33** (2023), no. 3, 213-219.

[23] Li, C.; Chu, W. *Fifteen series on harmonic numbers and quintic central binomial coefficients,* Discrete Math. Lett. **16** (2025), 15-20.
[24] Li, W. *Multiscale estimates for the condition number of non-harmonic Fourier matrices,* Math. Comp. **94** (2025), no. 356, 2895-2929.
[25] Riordan, J. *Combinatorial Identities,* Wiley, New York, 1968.
[26] Tassaddiqa, A.; Alharbib, R.; Kasmanic, R. M.; Nantomah, K. *Novel series representation of degenerate gamma function with formulation of new generalized kinetic equation,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, 2466541.
[27] Wang, Q.; Hei, Y.; Dolgy, D. V.; Kim, T.; Kim, D. S. *Some identities associated with degenerate harmonic and hyperharmonic numbers,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, 2545196.