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Weaker forms of Menger-type properties in bitopology: Equivalences and games

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Abstract. This paper contributes to the recent researches on weaker forms of the Menger property in bitopological spaces. It carries on the investigation deeper in this area, examining various equivalent conditions and present game-theoretic observations related to weaker forms of the Menger-type properties. Furthermore we define a new type of Menger property, nearly Mengerness and discuss its properties with in the context of bitopological spaces.

1. Introduction

Selection principles play a fundamental role in topology, particularly in the study of covering properties. These principles provide a unifying framework that connects compactness, Lindelöfness, and various other notions to topological spaces. Among these, the Menger property has received significant attention due to its deep connections with Ramsey theory, infinite-dimensional topology, and game theory. A natural direction of this research is to explore Menger-type properties in bitopological spaces, which are equipped with two topologies rather than one. Understanding the behavior of Menger-type properties in such spaces enriches the theory of selection principles and extends its applicability to more general frameworks.

Weak forms of selection principles, particularly the almost Menger and weakly Menger properties in bitopological spaces, have been studied extensively in previous works [11, 12, 28]. Notably, in [11], the conditions under which (i, j)-almost Menger and (i, j)-weakly Menger bitopological spaces are equivalent were investigated. It was shown that if (X, τ_2) is a p-space and (X, τ_1, τ_2) is (i, j)-weakly Menger, then it is also (i, j)-almost Menger. Furthermore, from a bitopological perspective, if (X, τ_1, τ_2) is (i, j)-regular bitopological space and (i, j)-weakly p-space, then the almost Menger and weakly Menger properties coincide.

Selection principles and their weak forms can often be characterized in terms of infinite games. In [14], a topological game corresponding to the almost Menger property in bitopological spaces was introduced, providing a game-theoretic characterization of the property.

This paper aims to continue the discussion initiated in [11, 28] on the equivalence of almost Menger and weakly Menger properties in bitopological spaces. Additionally, these equivalence conditions will be examined within the framework of almost star Menger and weakly star Menger bitopological spaces. A

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topological game corresponding to the weakly Menger property will be introduced and analyzed. Furthermore, we propose a new weak covering property, called the nearly Menger property, and explore its implications and interrelations within the bitopological setting.

The paper is organized as follows. In Section 2, we provide necessary preliminaries and recall fundamental definitions related to selection principles, Menger-type properties, and topological games. Section 3 discusses the equivalence of the weak Menger-type properties and introduces new game-theoretic formulations of Menger-related properties and examines their equivalences. Finally, in Section 4, we define the concept of nearly Mengerness in bitopological spaces and explore its properties in relation to existing selection principles.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real number, respectively. Let X be a topological space, $\mathcal{U}$ a collection of subsets of X, then $\bigcup \mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$.

A topological space X is called Menger [25] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.

Selection Principles: Classical selection principles have been defined in a general form as follows [33]. Let $\mathcal{A}$ and $\mathcal{B}$ be collection of subsets of a space X. Then:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

If $\mathcal{O}$ denotes the family of all open covers of a topological space X, the property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ is called the *Rothberger property* [31].

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ denotes the *Menger property* [25].

We denote by $\overline{\mathcal{O}}$ the collection of all families $\mathcal{U}$ of open subsets of X such that $\{\overline{\mathcal{U}} : U \in \mathcal{U}\}$ is a cover of X, and by $\mathcal{D}$ the collection of all families $\mathcal{U}$ of open subsets of X such that $\cup\mathcal{U}$ is dense in X. A topological space X is almost Menger (weakly Menger) if it satisfies the selection hypothesis $\mathsf{S}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$ ($\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$).

Recently many papers, investigating the weaker forms of the classical selection properties have been published [6, 17, 18, 20]. Particularly Menger-type properties have been considered in [16–18, 32, 35].

Weaker forms of the Menger property in a bitopological context were first introduced by Kočinac and Özçağ in [19] and widely investigated in [11, 12, 28]. In addition to these investigations several papers on the weak versions of the Menger property in bitopological spaces have been appeared [11, 13, 21, 22, 28, 29]. The reader may refer to [26, 27] for bitopological selective versions of separability in function spaces. A bitopological space (X, τ_1, τ_2) defined by J.C. Kelly, is a set X with two topologies τ_1 and τ_2 on it [15]. Game theoretic properties related to weak forms of the Menger property in bitopological spaces were studied in [14, 23].

Bitopological spaces: Throughout the paper (X, τ_1, τ_2) (or X) denotes a bitopological space. A set X endowed with two, in general unrelated, topologies τ_1 and τ_2 is called a bitopological space (or shortly, bispace). If there is a relation between the two distinct topologies as $\tau_1 < \tau_2$ or $\tau_2 < \tau_1$, then it means certain subsets of the space are simultaneously open with respect to both topologies.

For a subset $A \subseteq X$ we denote the closure of A and interior of A with respect to τ_i by $\mathrm{Cl}_{\tau_i}(\mathrm{A})$ and $\mathrm{Int}_{\tau_i}(\mathrm{A})$ respectively for $i = 1, 2$. Always $i, j \in \{1, 2\}$ and $i \neq j$. We note that if $\mathcal{P}$ is a topological property for a bitopological space (X, τ_1, τ_2) then (i, j)-$\mathcal{P}$ denotes an analogue of this property for τ_i with respect to τ_j and p-$\mathcal{P}$ denotes the conjuction $(1, 2)$-$\mathcal{P} \wedge (2, 1)$-$\mathcal{P}$ where "p" is the abbreviation for "pairwise".

We refer the reader to [10] for topological terminology and notations, while our general references for bitopological spaces are [9, 15].

Topological games: There are games $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ and $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ which are associated to the selection principles $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$, respectively.

The symbol $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play an inning for each $n \in \mathbb{N}$. In the n-th inning ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing an element $B_n \in A_n$. A play $(A_0, B_0, A_1, B_1, \ldots, A_n, B_n, \ldots)$ is won by TWO if $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

The symbol $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play an inning for each for each $n \in \mathbb{N}$. In the n-th inning ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing a finite subset $B_n \subset A_n$. A play $(A_0, B_0, A_1, B_1, \ldots, A_n, B_n, \ldots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$), then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$) holds. The converse implication need not be always true, [30].

If ONE is a player of a game G, we denote by ONE $\uparrow$ G the fact that ONE has a winning strategy in G, and by ONE $\not\uparrow$ G the fact that ONE does not have a winning strategy in G.

In [5] Babinkostova, Pansera and Scheepers investigated game-theoretic properties of selection principles related to weaker forms of the Menger properties and characterized the almost Menger space by ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$ where $\overline{\mathcal{O}}$ denotes the collection of families $\mathcal{U}$ of open sets in X with $\bigcup\{\mathrm{Cl}(U) : U \in \mathcal{U}\} = X$.

3. Characterizing (*i*, *j*)-weak Mengerness

In this section we study the relationships between the almost Menger and weakly Menger and Menger properties in bitopological spaces.

Definition 3.1. ([11, 28]) A bitopological space (X, τ_1, τ_2) is said to be (i, j)-*almost Menger* ((i, j)-*weakly Menger*), $i, j = 1, 2$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V)$ ($X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$).

Obviously every (i, j)-almost Menger bitopological space is (i, j)-weakly Menger. Eysen and Özçağ illustrated an example [11, Example 2.3] showing that an (i, j)-weakly Menger bitopological space is not necessarily (i, j)-almost Menger. The authors investigated that under which conditions all these properties are equivalent and consequently obtained the result that if a bitopological space (X, τ_1, τ_2) is $(1, 2)$-weakly Menger and (X, τ_2) is a P-space, then (X, τ_1, τ_2) is $(1, 2)$-almost Menger. (see, [11, Proposition 2.4]). Recall that a topological space (X, τ) is a P-space if any intersection of countably many open sets of X is open.

Singh and Tyagi in [34] investigated under which the almost Menger and weakly Menger properties are equivalent. Based on their work we aim to explore analogous questions in the context of bitopological spaces.

Theorem 3.2. *Let (X, τ_1, τ_2) be (i, j)-weakly Menger bitopological space. If for each τ_j-dense set $Y \subseteq X$ the subspace $X \setminus Y$ has the (i, j)-almost Menger property, then (X, τ_1, τ_2) is (i, j)-almost Menger bitopological space.*

Proof. We only consider the case $i = 1$, $j = 2$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since X is $(1, 2)$-weakly Menger then there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$. Let

$$\bigcup_{n\in\mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\} = Y.$$

Then Y is τ_2-dense set in X. By assumption there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and for each $W \in \mathcal{W}_n$ we can find $U_W \in \mathcal{U}_n$ such that $W = U_W \cap (X \setminus Y)$.

Now we consider $\mathcal{H}_n = \{U_W : W \in \mathcal{W}_n\}$. Apparently $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and

$$X \setminus Y \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2}(U_W) : W \in \mathcal{W}_n\}.$$

Let $\mathcal{G}_n = \mathcal{V}_n \cup \mathcal{H}_n$, we have for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$.

$$
\begin{aligned}
X &= (X \setminus Y) \bigcup Y \\
&\subseteq \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2}(U_W) : W \in \mathcal{W}_n\} \cup \bigcup_{n\in\mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\} \\
&\subseteq \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2}(U_W) : W \in \mathcal{W}_n\} \cup \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2} V : V \in \mathcal{V}_n\} \\
&\subseteq \bigcup_{n\in\mathbb{N}} \bigcup \{\{\mathrm{Cl}_{\tau_2}(U_W) : W \in \mathcal{W}_n\} \cup \{\mathrm{Cl}_{\tau_2} V : V \in \mathcal{V}_n\}\} \\
&= \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2} G : G \in \mathcal{G}_n\}.
\end{aligned}
$$

Therefore (X, τ_1, τ_2) is $(1, 2)$-almost Menger bitopological space. □

Recall that a family of sets in a topological space is called *conservative* [10] if the closure of the union of any of its subfamilies is equal to the union of the closures of the elements of that subfamily, i.e. for every subfamily $\mathcal{F}_0$ we have $\mathrm{Cl}(\bigcup\{F : F \in \mathcal{F}_0\} = \bigcup\{\mathrm{Cl}(F) : F \in \mathcal{F}_0\}$.

Theorem 3.3. *Let $(X, \tau_1 \leq \tau_2)$ be a bitopological space. If every countable family of τ_j-open sets is conservative, then (i, j)-weakly Menger property implies (i, j)-almost Menger property.*

Proof. Similarly we only consider the case $i = 1$, $j = 2$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since X is $(1, 2)$-weakly Menger there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and

$$X = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\}).$$

As $\tau_1 \leq \tau_2$ the set $\bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\}$ is τ_2-open and every countable family of τ_2-open sets is conservative we have

$$X = \bigcup_{n\in\mathbb{N}} \bigcup \{\mathrm{Cl}_{\tau_2} V : V \in \mathcal{V}_n\}.$$

Thus $(X, \tau_1 \leq \tau_2)$ is (1.2)-almost Menger bitopological space. □

In [28] it was shown that if a topological space (X, τ_1) is Menger then the bitopological space (X, τ_1, τ_2) is $(1, 2)$-almost Menger and illustrated an example [28, Example 2.2] which shows that there exists $(1, 2)$-almost Menger bitopological space (X, τ_1, τ_2) such that $(X, \tau_1)$is not Menger. Another example was given in [11, Example 2.2]. We now give a condition to state the equivalence of these spaces. A topological space (X, τ) is called *locally indisrete* [8] if every open set in X is closed in X.

Theorem 3.4. *Let $(X, \tau_1 \leq \tau_2)$ be a bitopological space with (X, τ_1) is locally indiscrete. The following are equivalent:*

(1) (X, τ_1) *is Menger;*

(2) (X, τ_1, τ_2) *is* $(1, 2)$*-almost Menger;*

(3) (X, τ_1, τ_2) *is* $(1, 2)$*-weakly Menger.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are straightforward.

(3) ⇒ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since X is $(1, 2)$-weakly Menger there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\})$. Let $Y = \bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\}$. Since (X, τ_1) is locally indiscrete, the set $\bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\}$ is closed in (X, τ_1). As $\tau_1 \leq \tau_2$ we obtain

$$X = \mathrm{Cl}_{\tau_2}\left(\bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\}\right) \subseteq \mathrm{Cl}_{\tau_1}\left(\bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\}\right)$$
$$= \bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\}.$$

Thus (X, τ_1) is Menger. □

In the previous theorem we have already proved the equivalence of the spaces by taking the space (X, τ_1) as a locally indiscrete. We now consider Theorem 3.4 under the more bitopological context which leads to the following definition.

Definition 3.5. A bitopological space (X, τ_1, τ_2) is called (i, j)-locally indiscrete if every τ_i-open set is τ_j-closed.

Theorem 3.6. *Let (X, τ_1, τ_2) be $(1, 2)$-locally indiscrete bitopological space. Then the following are equivalent:*

(1) *(X, τ_1) is Menger;*

(2) *(X, τ_1, τ_2) is $(1, 2)$-almost Menger;*

(3) *(X, τ_1, τ_2) is $(1, 2)$-weakly Menger.*

Proof. (3) ⇒ (1) The proof follows the same lines as the proof of Theorem 3.4. In that proof at one point we apply the fact that (X, τ_1, τ_2) is $(1, 2)$-locally indiscrete to the sequence to the τ_1-open set $\bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\}$. Then $\bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\}$ is a τ_2-closed set. Thus $\bigcup_{n\in\mathbb{N}}\bigcup\{V : V \in \mathcal{V}_n\} = X$ then (X, τ_1) is Menger. □

Now let us examine another condition under which the properties of almost Menger and weakly Menger are equivalent in bitopological spaces. We recall that [17] X is d-paracompact if every dense family of subsets of X has a locally finite refinement. A family $\mathcal{V}$ of subsets of a topological space X is locally finite refinement of a family $\mathcal{U}$ of subsets of X if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$ and for every point $x \in X$ there exists a neighborhood of x which intersects only finitely many elements of $\mathcal{V}$.

Theorem 3.7. *If a bitopological space (X, τ_1, τ_2) is (i, j)-weakly Menger and (X, τ_j) is d-paracompact, then (X, τ_1, τ_2) is (i, j)-almost Menger.*

Proof. (For $i = 1, j = 2$) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Then there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n)$, i.e., $\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n$ is τ_2-dense. Since (X, τ_2) is d-paracompact $\{\mathcal{V}_n : n \in \mathbb{N}\}$ has a locally finite refinement $\mathcal{F}_n$. Then $\bigcup\mathcal{F}_n = \bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n$ and $\mathrm{Cl}_{\tau_2}(\bigcup\mathcal{F}_n) = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n)$. Since $\mathcal{F}_n$ is a locally finite family

$$\mathrm{Cl}_{\tau_2}(\bigcup\mathcal{F}_n) = \bigcup_{F\in\mathcal{F}_n}\mathrm{Cl}_{\tau_2}F$$

and for every $F \in \mathcal{F}$ there exists $n \in \mathbb{N}$ with $V_F \in \mathcal{V}_n$ so that $F \subseteq V_F$. Thus we obtain that

$$X = \bigcup_{n\in\mathbb{N}}\bigcup_{V\in\mathcal{V}_n}\mathrm{Cl}_{\tau_2}(V)$$

and (X, τ_1, τ_2) is $(1, 2)$-almost Menger. □

In [14] it has been proven that the almost Menger property $\mathsf{S}_{fin}(\mathcal{O}_i, \overline{\mathcal{O}}_j)$ is equivalent to ONE not having a winning strategy in the game $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_\mathsf{i}, \overline{\mathcal{O}}_\mathsf{j})$. To conclude this section we discuss the weakly Menger game on bitopological spaces and relate this notion to (i, j)-weakly Menger property.

Definition 3.8. The (i, j)-weakly Menger game on a bitopological space (X, τ_1, τ_2) is played as follows: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-open cover $\mathcal{U}_n$ of X and then TWO chooses a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. The play is won by TWO if

$$X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$$

otherwise, ONE is the winner.

The game associated with the (i, j)-almost Menger property is denoted by $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_{\mathsf{i}}, \overline{\mathcal{O}_{\mathsf{j}}})$, while the representation of the game corresponding to the (i, j)-weakly Menger property is expressed as $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_{\mathsf{i}}, \mathcal{D}_{\mathsf{j}})$.

Now this definition raises the question of when the property (i, j) weakly Mengerness is equivalent to player ONE not having a winning strategy in the game $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_{\mathsf{i}}, \mathcal{D}_{\mathsf{j}})$.

Theorem 3.9. *Let $\left(X, \tau_1, \tau_2\right)$ be a bitopological space and $\left(X, \tau_i\right)$ be a Lindelöf space. The following are equivalent:*

(1) $\left(X, \tau_1, \tau_2\right)$ *has the (i, j)-weakly Menger property.*

(2) ONE *does not have a winning strategy in the game (i, j)-weakly Menger.*

Proof. (1) ⇒ (2): (For $i = 1, j = 2$) A Lindelöf space $\left(X, \tau_1\right)$ satisfies the $(1, 2)$-weakly Menger property then ONE has no winning strategy in the game $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_1, \mathcal{D}_2)$ requires proof.

The proof proceeds like for Theorem 2.4 in [14]. In that proof at the point where we apply the fact that $\left(X, \tau_1, \tau_2\right)$ is $(1, 2)$-almost Menger to the sequence $\mathcal{U}_n$, we apply $(1, 2)$-weakly Menger property to the sequence $\mathcal{U}_n$. For the convenience of the readers we will remind some important steps of the proof.

Let σ be a strategy for ONE. We may assume that each move of ONE according to the strategy σ, is an ascending ω-chain of τ_1-open sets covering X. Set $\sigma(\emptyset) = \{U_{(n)} : n \in \mathbb{N}\}$, listed in $\subset$-increasing order. For each n, list $\sigma(U_{(n)}) = \{U_{(n,m)} : m \in \mathbb{N}\}$ in $\subset$-increasing order. Supposing that U_τ has been defined for each finite sequences τ of length at most k of positive integers, we now define $\sigma(U_{(n_1,n_2,\ldots,n_k)}) = \{U_{(n_1,n_2,\ldots,n_k,m)} : m \in \mathbb{N}\}$ for each $(n_1, n_2, \ldots, n_k)$.

Now we define for each n and k:

$$U_k^n = \begin{cases} U_{(k)} & , n = 0 \\ U_k^{n-1} \cap \Big(\bigcap\limits_{\tau\in\mathbb{N}^n} U_{\tau\frown(k)} \Big) & , n \neq 0. \end{cases}$$

Then for each n the set $\mathcal{U}_n = \left\{U_k^n : k \in \mathbb{N}\right\}$ is an increasing chain of τ_1-open sets with each $\mathcal{U}_n$ is an τ_1-open cover of X.

Now we apply the fact that $\left(X, \tau_1, \tau_2\right)$ is $(1, 2)$-weakly Menger to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $X = Cl_{\tau_2}\Big(\bigcup\limits_{n\in\mathbb{N}} U_{f(n)}^n \Big)$ so the sequence of moves

$$(U_{(f(0))}, U_{(f(0),f(1))}, \ldots, U_{(f(0),f(1),\ldots,f(n))}, \ldots)$$

by TWO defeats ONE's strategy σ. □

We note that these games have either TWO has a winning strategy or ONE has a winning strategy or neither player has a winning strategy. The following implications always hold in the general case [4].

$$\text{TWO} \uparrow \mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$$
$$\Downarrow$$
$$\text{ONE} \not\uparrow \mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$$
$$\Downarrow$$
$$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$$

Conditions under which ONE has no winning strategy in games are better understood. There is still no characterizations for the bitopological spaces for which TWO has a winning strategy in the game $\mathsf{G}_{\mathsf{fin}}(\mathcal{O}_\mathsf{i}, \mathcal{D}_\mathsf{j})$.

4. Weak star-Menger properties

Let us recall [18] that if A is a subset of a topological space X, $x \in X$ and $\mathcal{U}$ is a family of subsets of X, then

$$\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\} \text{ and } \mathrm{St}(x, \mathcal{U}) = \mathrm{St}(\{x\}, \mathcal{U})$$

denote the *star* of A and x with respect to $\mathcal{U}$.

In this section we introduce almost star Menger and weakly star Menger bitopological spaces and discuss relationships between these spaces and other related spaces. furthermore, several authors have investigated bitopological star-covering properties from a different perspective, where pairs of τ_1-open and τ_2-open covers are regarded as the fundamental bi-covering units. The work [7] develop a systematic framework for studying such bi-coverings and explore how classical star-covering properties behave when lifted to the bitopological setting.

Definition 4.1. ([12]) A bitopological space (X, τ_1, τ_2) is said to be (i, j)-*almost star Menger*, $i, j = 1, 2$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) = X$.

It is obvious that if (X, τ_1) is star Menger, then (X, τ_1, τ_2) is $(1, 2)$-almost star Menger. Now we introduce a weakly version of star Menger bitopological spaces.

Definition 4.2. A bitopological space (X, τ_1, τ_2) is said to be (i, j)-*weakly star Menger*, $i, j = 1, 2$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and

$$\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}\,(\cup\mathcal{V}_n, \mathcal{U}_n)) = X.$$

Clearly, every (i, j)-almost star Menger bitopological space is (i, j)-weakly star Menger. Now we investigate conditions under which these spaces are equivalent.

Theorem 4.3. *Let (X, τ_1, τ_2) be (i, j)-weakly star Menger bitopological space. If for each τ_j-dense set $Y \subseteq X$ the subspace $X \setminus Y$ has the (i, j)-almost star Menger property, then (X, τ_1, τ_2) is (i, j)-almost star Menger bitopological space.*

Proof. We consider only the case $i = 1$, $j = 2$.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since X is $(1, 2)$-weakly star Menger there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) = X$. Now let $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) = Y$

It is obvious that the set Y is τ_2-dense in X. Since $X \setminus Y$ has the $(1, 2)$-almost star Menger property, there exists a sequence $(\mathcal{H}_n : n \in \mathbb{N})$(similar to the proof of Theorem 3.2) such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and $\mathrm{Cl}_{\tau_2}\{\mathrm{St}(\cup\mathcal{H}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a cover of $X \setminus Y$.

Now let $\mathcal{G}_n = \mathcal{V}_n \cup \mathcal{H}_n$. For each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$. Then:

$$\begin{aligned} X &= (X \setminus Y) \cup Y \\ &= \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{H}_n, \mathcal{U}_n) \cup \bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) \\ &\subseteq \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{H}_n, \mathcal{U}_n)) \cup \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) \end{aligned}$$

$$= \bigcup_{n\in\mathbb{N}} (\mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{H}_n, \mathcal{U}_n)) \cup \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)))$$

$$= \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{G}_n, \mathcal{U}_n)).$$

Therefore (X, τ_1, τ_2) is $(1, 2)$-almost star Menger bitopological space. □

Theorem 4.4. *Let (X, τ_1, τ_2) be $(1, 2)$-locally indiscrete bitopological space. Then the following are equivalent:*

(1) (X, τ_1) *has the star Menger property;*

(2) (X, τ_1, τ_2) *has the $(1, 2)$-almost star Menger property;*

(3) (X, τ_1, τ_2) *has the $(1, 2)$-weakly star Menger property.*

Proof. We will show (3) ⇒ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since X is $(1, 2)$-weakly star Menger there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) = X$. The set $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$ is τ_1-open and since (X, τ_1, τ_2) is $(1, 2)$-locally indiscrete, $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$ is τ_2-closed. Therefore $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) = X$. Hence (X, τ_1) has the star Menger property. □

Theorem 4.5. *Let $(X, \tau_1 \le \tau_2)$ be a bitopological space. If every countable family of τ_j-open sets is conservative, then (i, j)-weakly star Menger property implies (i, j)-almost star Menger property.*

Proof. We consider only the case $i = 1$, $j = 2$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X.There exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) = X$. The set $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$ is τ_1-open and since $\tau_1 \le \tau_2$ the set is τ_2-open and every countable family of τ_2-open sets is conservative then we have

$$\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)) = X.$$

Thus $(X, \tau_1 \le \tau_2)$ is (i, j)-almost star Menger bitopological space. □

5. Nearly Menger bitopological spaces

Nearly Menger spaces was introduced by Kočinac in [20] by utilizing the interior of the closure of an open set. On the other hand Aqsa and Khan [3] defined nearly Menger spaces by taking the semi-closure of an open set. In [3] the authors mentioned that both notions of nearly Menger spaces coincide in the presence of open covers as seen in Lemma 5.3.

In this section we aim to define nearly Mengerness in a bitopological context. It is worth to mention here that when there is a relationship between the topologies in a bitopological space it provides a structured way to understand how the open sets from one topology affect the properties and behaviour of the space concerning the other topology.

Definition 5.1. A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-nearly Menger*, $(i, j = 1, 2)$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}_{\tau_j}(\mathrm{Cl}_{\tau_j}(V)) : V \in \mathcal{V}_n\}$.

In the recently published paper [1] the authors provided a different definition of nearly Mengerness in bitopological spaces.

Lemma 5.2. ([2]) *For any subset A of a space X,* $\mathrm{A} \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{A})) = \mathrm{scl}(\mathrm{A})$.

Lemma 5.3. ([24]) *For an open set O in a space X,* $\mathrm{Int}(\mathrm{Cl}(\mathrm{O})) = \mathrm{scl}(\mathrm{O})$.

In bitopological spaces there are two different closure and interior operations, one associated with each topology. Hence Lemma 5.3 may not be true in bitopological spaces, however we have the following lemma.

Lemma 5.4. *Let $(X, \tau_1 < \tau_2)$ be a bitopological space. For τ_1-open set $A \subseteq X$;* $\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(\mathrm{A})) = \mathrm{scl}_{\tau_2}(\mathrm{A})$.

Proposition 5.5. *Let (X, τ_1, τ_2) be a bitopological space, then:*

(1) *If (X, τ_2) is nearly Menger, then $(X, \tau_1 < \tau_2)$ is $(1, 2)$-nearly Menger.*

(2) *If $(X, \tau_1 > \tau_2)$ is $(1, 2)$-nearly Menger, then (X, τ_2) is nearly Menger.*

Proof. (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since $\tau_1 < \tau_2$ the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ is τ_2-open covers of X. If (X, τ_2) is nearly Menger, then there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(V)) : V \in \mathcal{V}_n\}$ thus $(X, \tau_1 < \tau_2)$ is $(1, 2)$-nearly Menger.

(2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_2-open covers of X and since $\tau_1 > \tau_2$ $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of τ_1-open covers of X. Then there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(V)) : V \in \mathcal{V}_n\}$ thus (X, τ_2) is nearly Menger. □

Proposition 5.6. *If (X, τ_1) is a Menger space, then the bitopological space $(X, \tau_1 < \tau_2)$ is $(1, 2)$-nearly Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since (X, τ_1) is Menger, then for every $n \in \mathbb{N}$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that , $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup\mathcal{V}_n$ is a cover of X. Since for each $V \in \mathcal{V}_n$ we have $V = \mathrm{Int}_{\tau_2} V \subseteq \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(V))$ and we obtain $(X, \tau_1 < \tau_2)$ is $(1, 2)$-nearly Menger. □

Corollary 5.7. *If (X, τ_2) is Menger, then $(X, \tau_1 < \tau_2)$ is $(1, 2)$-nearly Menger.*

Theorem 5.8. *If $(X, \tau_1 > \tau_2)$ is $(1, 2)$-nearly Menger and (X, τ_2) is a regular space, then (X, τ_2) is Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_2-open covers of X and τ_1-open covers of X, since $\tau_1 > \tau_2$. In the hypothesis (X, τ_2) is regular so for each $n \in \mathbb{N}$, there exists a sequence of τ_2-open covers $(\mathcal{V}_n : n \in \mathbb{N})$ of X such that $\mathcal{V}'_n = \{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(V)) : V \in \mathcal{V}_n\}$ form a refinement of $\mathcal{U}_n$. Since X is $(1, 2)$ nearly Menger, there exists a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ with $\mathcal{F}_n$ is a finite subset of $\mathcal{V}'_n$ and $\cup(\mathcal{F}'_n : n \in \mathbb{N})$ is a cover of X where $\mathcal{F}'_n = \{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(F)) : F \in \mathcal{F}_n\}$. For every $F \in \mathcal{F}_n$ and $n \in \mathbb{N}$ we have $U_F \in \mathcal{U}_n$ such that $\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(F) \subseteq U_F$.

Now we set $\mathcal{U}'_n = \{U_F : F \in \mathcal{F}_n\}$ and we prove that $\cup\{\mathcal{U}'_n : n \in \mathbb{N}\}$ is a τ_2 open cover of X. Let $x \in X$. There is $n \in \mathbb{N}$ and $\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(F)) \subseteq \mathcal{F}'_n$ with $x \in \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(F))$. There exists $U_F \in \mathcal{U}_n$ such that $\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(F)) \subseteq U_F$ and we obtain $x \in U_F$. □

Example 5.9. Let the real line $\mathbb{R}$ be endowed with the Euclidean topology τ_1 and the Sorgenfrey topology τ_2. Since $(\mathbb{R}, \tau_1)$ is Menger, the bitopological space $(\mathbb{R}, \tau_1 \leq \tau_2)$ is $(1, 2)$-nearly Menger by Proposition 5.6. On the other hand, $(\mathbb{R}, \tau_2)$ is not Menger.

We immediately note that:

Proposition 5.10. *If the bitopological space (X, τ_1, τ_2) is (i, j)-nearly Menger, then (X, τ_1, τ_2) is (i, j)-almost Menger.*

Definition 5.11. Let (X, τ_1, τ_2) be a bitopological space. A subset $A \subseteq X$ is said to be (i, j)-clopen if A is τ_i-closed and τ_j-open set. A is clopen if it is both (i, j)-clopen and (j, i)-clopen in X.

Theorem 5.12. *A bitopological space (X, τ_1, τ_2) is $(1, 2)$ nearly Menger if and only if every clopen subspace $(K, \tau_{1\diagup K}, \tau_{2\diagup K})$ is $(\tau_{1\diagup K}, \tau_{2\diagup K})$ nearly Menger.*

Proof. (⇒) Let $K \subseteq X$ be a clopen subset of a $(1, 2)$ nearly Menger bispace. Suppose that $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of $\tau_{1\diagup K}$-open covers of K. Then for each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$ there exists a τ_1-open set B_U in X such that $U = K \cap B_U$.

Now set $\mathcal{B}_n = \{B_U : U \in \mathcal{U}_n\} \cup \{X - K\}$ and $(\mathcal{B}_n : n \in \mathbb{N})$ is a sequence of τ_1-open covers of (X, τ_1, τ_2). Since X is $(1,2)$ nearly Menger there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ with $\mathcal{V}_n$ is a finite subset of $\mathcal{B}_n$ and for each $n \in \mathbb{N}$,

$$\bigcup\{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(B) : B \in \mathcal{B}_n\}$$

is a cover of X. The set $X - K$ is clopen, then $X - K$ is τ_1-open and τ_2-open set, hence $\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(X - K)) = X - K$ so

$$\bigcup\{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(B)) : B \in \mathcal{B}_n : B \neq X - K\}$$

covers K. ($\Leftarrow$) This part is trivially true, because X itself is a clopen subspace of X. □

Definition 5.13. A subset A of a space X is s-regular open (s-regular closed) if $A = \mathrm{Int(scl(A))}(A = \mathrm{Cl(sInt(A))})$.

We note that an s-regular open set is regular open, open as well as semi closed, and if A is open, then cl(A) is an s-regular closed set.

In [20] Kočinac characterized nearly Menger spaces by regular open sets. However in [3] Aqsa and Khan obtained the following result. Denote by $\mathcal{SRO}$ the set of all s-regular open subsets of a space X.

Theorem 5.14. *For a topological space X the following are equivalent:*

(1) *X is nearly Menger;*

(2) *X satisfies* $\mathsf{S}_{fin}(\mathcal{SRO}, \mathcal{SRO})$.

Theorem 5.15. *If* $(X, \tau_1 > \tau_2)$ *is* $(1,2)$ *nearly Menger, then* (X, τ_2) *satisfies* $\mathsf{S}_{fin}(\mathcal{SRO}_{\tau_2}, \mathcal{SRO}_{\tau_2})$.

Proof. If $(X, \tau_1 > \tau_2)$ is $(1,2)$ nearly Menger, then by Proposition 5.5, (X, τ_2) is nearly Menger and by Theorem 5.14, $(X, \tau_1 > \tau_2)$ satisfies $\mathsf{S}_{fin}(\mathcal{SRO}_{\tau_2}, \mathcal{SRO}_{\tau_2})$. □

Theorem 5.16. *If* (X, τ_2) *satisfies* $\mathsf{S}_{fin}(\mathcal{SRO}_{\tau_2}, \mathcal{SRO}_{\tau_2})$*, then* $(X, \tau_1 < \tau_2)$ *is* $(1,2)$ *nearly Menger.*

Proof. Let (X, τ_2) satisfies $\mathsf{S}_{fin}(\mathcal{SRO}_{\tau_2}, \mathcal{SRO}_{\tau_2})$, then by Theorem 5.14 , (X, τ_2) is nearly Menger. We obtain by Proposition 5.5 , $(X, \tau_1 < \tau_2)$ is $(1,2)$ nearly Menger. □

Call a mapping $f : X \to Y$ nearly continuous if for every s-regular open set $A \subseteq Y$, $f^{\leftarrow}(A)$ is an open set in X.

Clearly every continuous mapping is almost continuous and every almost continuous mapping is nearly continuous.

Lemma 5.17. ([3]) *If* $f : X \to Y$ *is nearly continuous and open mapping, then for every s-regular open set* $A \subseteq Y$, $\mathrm{scl}(f^{\leftarrow}(A)) \subseteq f^{\leftarrow}(\mathrm{scl}(A))$.

Theorem 5.18. *Let* $(X, \tau_1 > \tau_2)$ *be* $(1,2)$ *nearly Menger bitopological space and* $(Y, \sigma_1 < \sigma_2)$ *be a bitopological space. If* $f : X \to Y$ *is* $(\tau_2 - \sigma_2)$ *nearly continuous and open surjection, then* $(Y, \sigma_1 < \sigma_2)$ *is* $(1,2)$ *nearly Menger.*

Proof. Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence of covers of Y by s-regular σ_2-open sets. As f is $(\tau_2 - \sigma_2)$ nearly continuous and open surjection $\mathcal{A}'_n = \{f^{\leftarrow}(A) : A \in \mathcal{A}_n\}$ is a sequence of τ_2-open covers of X. ($\mathcal{A}'_n$ is also a sequence of τ_1 open covers of X) By the hypothesis $(X, \tau_1 > \tau_2)$ is $(1,2)$ nearly Menger then there exists a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(\mathcal{B}_n : n \in \mathbb{N})$ is a finite subset of $(\mathcal{A}'_n : n \in \mathbb{N})$ and $\cup(\mathcal{B}'_n : n \in \mathbb{N})$ covers X where $\mathcal{B}'_n = \{\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(B)) : B \in \mathcal{B}_n\}$. For every $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ we can choose a member A_B in $\mathcal{A}_n$ such that $B = f^{\leftarrow}(A_B)$.

Now let $\mathcal{C}_n = \{A_B : B \in \mathcal{B}_n\}$. We will show $\cup(\mathcal{C}_n : n \in \mathbb{N})$ covers Y. Suppose $y \in Y$ arbitrary. Then there exists $x \in X$ such that $y = f(x)$. For $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $x \in \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(B))$. As $B = f^{\leftarrow}(A_B)$,

$$x \in \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(f^{\leftarrow}(A_B))) \subseteq f^{\leftarrow}(\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(A_B))).$$

We obtain that $(Y, \sigma_1 < \sigma_2)$ is $(1,2)$ nearly Menger. □

6. Conclusion

In this paper, we continued the recent developments on weaker Menger-type properties in bitopological spaces by establishing several equivalent conditions and providing related game-theoretic observations. The game-theoretic approach investigate interactions between the corresponding covering properties and the classical selection principles.

In addition, we introduced the notion of nearly Mengerness and studied its fundamental behavior within the bitopological settings. As a further direction, it would be of interest to investigate the interactions among these properties and to develop their corresponding game-theoretic characterizations within the bitopological setting.

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