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Screen transversal Cauchy-Riemann lightlike submanifolds of semi-Riemannian product manifolds

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Abstract. We introduce screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of semi-Riemannian product manifolds. We obtain new conditions for the induced connection to be a metric connection. We investigate geometry of screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds and investigate the integrability of various distributions of such lightlike submanifolds. Finally, we find conditions for minimal STCR-lightlike submanifolds of semi-Riemannian product manifolds. Moreover, we two give examples.

1. Introduction

Given a semi-Riemannian manifold, one can consider its lightlike submanifold whose study is important from application point of view and difficult in the sense that the intersection of normal vector bundle and tangent bundle of these submanifolds is nonempty. This unique feature makes the study of lightlike submanifolds different from the study of non-degenerate submanifolds. Lightlike submanifolds have been studied widely in mathematical physics. Indeed, lightlike submanifolds appear in general relativity as some smooth parts of event horizons of the Kruskal and Kerr black holes [16]. Lightlike submanifolds of semi-Riemannian manifold have been studied by Duggal-Bejancu and Kupeli in [4] and [20], respectively. Kupeli's approach is intrinsic while Duggal-Bejancu's approach is extrinsic. Later, they were studied by many authors (see up-to date results in two books [6, 10]).

Duggal and Bejancu [4] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds. Similar to CR-lightlike submanifolds, semi-invariant lightlike submanifolds of semi-Riemannian product manifolds were introduced by Atçeken and Kılıç in [1]. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then, screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds were introduced in [7]. Screen Cauchy-Riemann (SCR)-lightlike submanifolds, analogously, screen semi-invariant lightlike submanifolds, of semi-Riemannian product manifolds were introduced by Khursheed, Thakur and Advin [17] and Kılıç, Şahin and Keleş [18], respectively. But there is no inclusion relation between CR and SCR submanifolds, so Duggal and Şahin [8] presented a new class named GCR-lightlike submanifolds of indefinite Kaehler manifolds which is an umbrella for all these types

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of submanifolds. Kumar, Kumar, Nagaich studied GCR-lightlike submanifolds of a semi-Riemannian product manifold [19]. These types of submanifolds have been studied in various manifolds by many authors [9, 13, 15, 21, 23].

But CR-lightlike, screen CR-lightlike and generalized CR-lightlike do not contain real lightlike curves. For this reason, Şahin presented screen transversal lightlike submanifolds of indefinite Kaehler manifolds and show that such submanifolds contain lightlike real curves [26]. Screen transversal lightlike submanifolds of semi-Riemannian product manifolds were introduced in [28]. Such submanifolds have studied in [11, 12, 14, 22]. On the other hand, as a generalization of CR-lightlike submanifolds and screen transversal lightlike submanifolds, Doğan, Şahin and Yaşar introduced screen transversal CR-lightlike submanifolds and studied the geometry of such lightlike submanifolds [3]. Poyraz studied contact screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of indefinite sasakian manifolds [24].

The paper is organized as follows: In Section 2, we give the basic concepts on lightlike submanifolds and semi-Riemannian product manifolds, which will be used throughout this paper. In Section 3, we introduce screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of semi-Riemannian product manifolds. We obtain new conditions for the induced connection to be a metric connection. We investigate geometry of screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds and investigate the integrability of various distributions of such lightlike submanifolds. In section 4, we find some necessary and sufficient conditions for minimal screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds. Furthermore, we give an example for minimal screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of semi-Riemannian product manifolds.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M then M is named a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\left\{u \in T_x\tilde{M} : \tilde{g}(u, v) = 0, \forall v \in T_xM, x \in M\right\} \tag{1}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, called radical distribution on M of rank $r > 0$ then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S(TM) \perp Rad(TM) \tag{2}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM))$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|M}$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{3}$$

$$T\tilde{M}\,|_M = TM \oplus tr(TM) = \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{4}$$

Theorem 2.1. *[4] Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose U is a coordinate neighbourhood of M and ξ_i, $i \in \{1, .., r\}$ is a basis of $\Gamma(Rad(TM)_{|U})$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}_{|U}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr(TM)_{|U})$ such that*

$$\tilde{g}\left(N_i, \xi_j\right) = \delta_{ij}, \quad \tilde{g}\left(N_i, N_j\right) = 0 \tag{5}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.
Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, using (4), the Gauss and Weingarten formulas are given by

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), && (6)\\ \tilde{\nabla}_X U &= -A_U X + \nabla^t_X U, && (7) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\left\{h(X, Y), \nabla^t_X U\right\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. According to (3), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (6) and (7) become

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h^l(X, Y) + h^s(X, Y), && (8)\\ \tilde{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X, N), && (9)\\ \tilde{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X, W), && (10) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla^l_X N, D^l(X, W) \in \Gamma(ltr(TM))$. Then, by using (8)-(10) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$\begin{aligned} \tilde{g}(h^s(X, Y), W) + \tilde{g}(Y, D^l(X, W)) &= g(A_W X, Y), && (11)\\ \tilde{g}(D^s(X, N), W) &= \tilde{g}(A_W X, N), && (12)\\ \tilde{g}(h^l(X, Y), \xi) + \tilde{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) &= 0. && (13) \end{aligned}$$

Let Q be a projection of TM on $S(TM)$. Thus, using (2) we can obtain

$$\begin{aligned} \nabla_X QY &= \nabla^*_X QY + h^*(X, QY)\xi, && (14)\\ \nabla_X \xi &= -A^*_\xi X - \nabla^{*t}_X \xi, && (15) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\left\{h^*(X, QY), \nabla^{*t}_X \xi\right\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.
Using the equations given above, we derive

$$\begin{aligned} \tilde{g}(h^l(X, QY), \xi) &= g(A^*_\xi X, QY), && (16)\\ \tilde{g}(h^*(X, QY), N) &= g(A_N X, QY), && (17)\\ \tilde{g}(h^l(X, \xi), \xi) &= 0, \quad A^*_\xi \xi = 0. && (18) \end{aligned}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (8), we obtain

$$(\nabla_X g)(Y, Z) = \tilde{g}(h^l(X, Y), Z) + \tilde{g}(h^l(X, Z), Y).$$

But, ∇^* is a metric connection on $S(TM)$.

Definition 2.2. *A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be totally umbilical in $\tilde{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that*

$$h(X, Y) = H\tilde{g}(X, Y) \tag{19}$$

for any $X, Y \in \Gamma(TM)$. In case $H = 0$, M is called totally geodesic [5].

Using (8) and (19) it is easy to see that M is totally umbilical iff on each coordinate neighborhood U there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma(S\,(TM^{\perp}))$ such that

$$h^l(X,Y) = \bar{g}(X,Y)H^l,\ h^s(X,Y) = \bar{g}(X,Y)H^s \text{ and } D^l(X,W) = 0 \tag{20}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\,(TM^{\perp}))$.

Theorem 2.3. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [4].*

Let $\tilde{M}$ be an $n-$dimensional differentiable manifold with a tensor field F of type $(1,1)$ on $\tilde{M}$ such that

$$F^2 = I. \tag{21}$$

Then $\tilde{M}$ is called an almost product manifold with almost product structure F. If we put

$$\pi = \frac{1}{2}(I+F),\ \sigma = \frac{1}{2}(I-F)$$

then we have

$$\pi + \sigma = I,\ \pi^2 = \pi\ \sigma^2 = \sigma, \pi\sigma = \sigma\pi = 0,\ F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and F has the eigenvalue of $+1$ or -1.

If an almost product manifold $\tilde{M}$ admits a semi-Riemannian metric $\bar{g}$ such that

$$\bar{g}(FX,FY) = \bar{g}(X,Y) \tag{22}$$

for any vector fields X, Y on $\tilde{M}$, then $\tilde{M}$ is called a semi-Riemannian almost product manifold.
From (21) and (22), we have

$$\bar{g}(FX,Y) = \bar{g}(X,FY). \tag{23}$$

If, for any vector fields X, Y on $\tilde{M}$,

$$\tilde{\nabla}F = 0,\ \text{that is } (\tilde{\nabla}_X F)Y = 0, \tag{24}$$

then $\tilde{M}$ is called a semi-Riemannian product manifold, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ [25].

3. Screen transversal Cauchy-Riemann (STCR)-Lightlike Submanifolds

Definition 3.1. *Let M be a real r-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then we say that M is a screen transversal Cauchy-Riemann (STCR)-lightlike submanifold if the following conditions are satisfied:*

(A) There exist two subbundles σ_1 and σ_2 of $Rad(TM)$ such that

$$Rad(TM) = \sigma_1 \oplus \sigma_2,\ \ F(\sigma_1) \subset S(TM),\ F(\sigma_2) \subset S(TM^{\perp}). \tag{25}$$

(B) There exist two subbundles σ_0 and σ' of $S\,(TM)$ such that

$$S(TM) = \{F(\sigma_1) \oplus \sigma'\} \perp \sigma_0,\ \tilde{P}(\sigma_0) = \sigma_0,\ F(\sigma') = L_1 \perp S \tag{26}$$

where σ_0 is a non-degenerate distribution on M, L_1 and S are vector subbundles of $ltr\,(TM)$ and $S\,(TM^{\perp})$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = \sigma \oplus \bar{\sigma} \tag{27}$$

where

$$\sigma = \sigma_0 \oplus \sigma_1 \oplus F(\sigma_1) \tag{28}$$

and

$$\bar{\sigma} = \sigma_2 \oplus F(L_1) \oplus F(S). \tag{29}$$

It is clear that σ is invariant and $\bar{\sigma}$ is anti-invariant. Furthermore, we have

$$ltr(TM) = L_1 \oplus L_2, F(L_1) \subset S(TM), F(L_2) \subset S(TM^{\perp}) \tag{30}$$

and

$$S(TM^{\perp}) = \{F(\sigma_2) \oplus F(L_2)\} \perp S. \tag{31}$$

If $\sigma_1 \neq \{0\}$, $\sigma_2 \neq \{0\}$, $\sigma_0 \neq \{0\}$ and $S \neq \{0\}$, then M is called a proper screen transversal Cauchy-Riemann lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$.

Proposition 3.2. *A STCR-lightlike submanifold M of a semi-Riemannian product manifold $\tilde{M}$ is semi-invariant lightlike submanifold (respectively, screen transversal lightlike submanifold) iff $\sigma_2 = \{0\}$ (respectively, $\sigma_1 = \{0\}$).*

Proof. If M is a semi-invariant lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$, then $F(Rad(TM))$ is a distribution on $S(TM)$. Hence we get $Rad(TM) = \sigma_1$ and $\sigma_2 = \{0\}$. Then it follows that $F(ltr(TM))$ is a distribution on $S(TM)$. Conversely, if M is a STCR-lightlike submanifold such that $\sigma_2 = \{0\}$, then we have $Rad(TM) = \sigma_1$. Thus $F(Rad(TM))$ is a vector subbundle of $S(TM)$. Hence M is a semi-invariant lightlike submanifold. Similarly other assertion follows. □

Proposition 3.3. *There exist no coisotropic, isotropic or totally lightlike proper STCR-lightlike submanifolds M of a semi-Riemannian product manifold. Any isotropic STCR-lightlike submanifold is a screen transversal lightlike submanifold. Also, a coisotropic STCR-lightlike submanifold is a semi-invariant lightlike submanifold.*

Proof. Assume that M is a proper STCR-lightlike submanifold. From definition of proper STCR-lightlike submanifold, we know that $\sigma_1 \neq \{0\}$, $\sigma_2 \neq \{0\}$, $\sigma_0 \neq \{0\}$ and $S \neq \{0\}$, that is both $S(TM)$ and $S(TM^{\perp})$ are non-zero. Hence, M can not be a coisotropic, isotropic or totally lightlike submanifold. On the other hand, if M be a isotropic STCR-lightlike submanifold, then $S(TM) = \{0\}$, i.e., $F(\sigma_1) = \{0\}$ and $Rad(TM) = \sigma_2$. Therefore, we obtain $F(Rad(TM)) = F(\sigma_2) \subset \Gamma(S(TM^{\perp}))$ and M is a screen transversal lightlike submanifold. Similarly, if M is a coisotropic STCR-lightlike submanifold, then $S(TM^{\perp}) = \{0\}$, i.e., $F(\sigma_2) = \{0\}$ and $Rad(TM) = \sigma_1$. Since, $F(Rad(TM)) = F(\sigma_1) \subset \Gamma(S(TM))$ then M is a semi-invariant lightlike submanifold. □

Example 3.4. *Let $\tilde{M} = (\mathbb{R}^{12}_2, \tilde{g})$ be a semi-Euclidean space, where $\tilde{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12})$. If we define a mapping F by*

$$F(x_1, x_2, x_3, ..., x_{11}, x_{12}) = (x_4, x_3, x_2, x_1, x_5, x_6, x_8, x_7, x_{10}, -x_{11}, -x_{12})$$

then $F^2 = I$ and F is an almost product structure on $\mathbb{R}^{12}_2$. Assume that M is a submanifold of $\mathbb{R}^{12}_2$ defined by

$$\begin{aligned} x_1 &= u_1,\ x_2 = u_2,\ x_3 = 0,\ x_4 = u_3 - \frac{u_4}{2},\ \ x_5 = u_5 + u_6, \\ x_6 &= u_5 - u_6,\ x_7 = u_3 + \frac{u_4}{2},\ x_8 = u_1,\ x_9 = u_2,\ \ x_{10} = 0, \\ x_{11} &= u_5 + u_7,\ x_{12} = u_5 - u_7. \end{aligned}$$

Then, TM is spanned by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, *where*

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_8, \;\; Z_2 = \partial x_2 + \partial x_9, \;\; Z_3 = \partial x_4 + \partial x_7, \\ Z_4 &= \frac{1}{2}(-\partial x_4 + \partial x_7), \;\; Z_5 = \partial x_5 + \partial x_6 + \partial x_{11} + \partial x_{12}, \\ Z_6 &= \partial x_5 - \partial x_6, \; Z_7 = \partial x_{11} - \partial x_{12}. \end{aligned}$$

Hence M is a $2-$ *lightlike submanifold of* $\mathbb{R}^{12}_2$ *with* $Rad(TM) = Span\{Z_1, Z_2\}$. *It is easy to see* $F(Z_1) = Z_3 \in \Gamma(S(TM))$, *thus* $\sigma_1 = Span\{Z_1\}$ *and* $\sigma_2 = Span\{Z_2\}$. *On the other hand, since* $F(Z_6) = Z_6$ *and* $F(Z_7) = Z_7$, *we derive* $\sigma_0 = Span\{Z_6, Z_7\}$ *and by direct calculations, we get the lightlike transversal bundle spanned by*

$$N_1 = \frac{1}{2}(-\partial x_1 + \partial x_8), \;\; N_2 = \frac{1}{2}(-\partial x_2 + \partial x_9).$$

Then we see that $L_1 = Span\{N_1\}$, $L_2 = Span\{N_2\}$, $S(TM^\perp) = Span\{F(Z_2), F(N_2), F(Z_5)\}$ *and* $S = Span\{F(Z_5) = W\}$. *Thus,* $\sigma' = Span\{F(N_1) = Z_4, F(W) = Z_5\}$ *and M is a proper STCR-lightlike submanifold of* $\mathbb{R}^{12}_2$.

Now, we denote the projections from $\Gamma(TM)$ to $\Gamma(\sigma_0)$, $\Gamma(F\sigma_1)$, $\Gamma(F(L_1))$, $\Gamma(F(S))$, $\Gamma(\sigma_1)$ and $\Gamma(\sigma_2)$ by P_0, P_1 , P_2, P_3, R_1 and R_2, respectively. We also denote the projections from $\Gamma(tr(TM))$ to $\Gamma(F(\sigma_2))$, $\Gamma(F(L_2))$, $\Gamma(S)$, $\Gamma(L_1)$ and $\Gamma(L_2)$ by S_1, S_2, S_3, Q_1 and Q_2, respectively. Hence, we can write

$$X = PX + RX = P_0X + P_1X + P_2X + P_3X + R_1X + R_2X \tag{32}$$

and

$$FX = TX + \omega X \tag{33}$$

for any $X \in \Gamma(TM)$, where $PX \in \Gamma(\sigma)$, $RX \in \Gamma(\bar{\sigma})$ and TX and ωX are the tangential parts and the transversal parts of FX, respectively. Applying F to (32) and denoting FP_0, FP_1, FP_2, FP_3, FR_1, FR_2 by T_0, T_1, ω_L, ω_S, $T_{\bar{1}}$, $\omega_{\bar{2}}$, respectively, we derive

$$FX = T_0X + T_1X + T_{\bar{1}}X + \omega_LX + \omega_SX + \omega_{\bar{2}}X \tag{34}$$

for any $X \in \Gamma(TM)$, where $T_0X \in \Gamma(\sigma_0)$, $T_1X \in \Gamma(\sigma_1)$, $T_{\bar{1}}X \in \Gamma(F(\sigma_1))$, $\omega_LX \in \Gamma(L_1)$, $\omega_SX \in \Gamma(S)$, and $\omega_{\bar{2}}X \in \Gamma(F(L_2))$. Similarly we can write

$$U = S_1U + S_2U + S_3U + Q_1U + Q_2U \tag{35}$$

for any $U \in \Gamma(tr(TM))$ and we denote FS_1, FS_2, FS_3, FQ_1, FQ_2 by B_2, C_L, $B_{\bar{S}}$, $B_{\bar{L}}$, $C_{\bar{L}}$, respectively. Thus we get

$$FU = B_2U + B_{\bar{S}}U + B_{\bar{L}}U + C_LU + C_{\bar{L}}U \tag{36}$$

and

$$FU = BU + CU \tag{37}$$

where BU and CU are sections of TM and $tr(TM)$, respectively. Differentiating (34) and using (8)-(10), (24), (34) and (37), we get

$$\begin{aligned} &\nabla_X TY + h^l(X, TY) + h^s(X, TY) + \{-A_{\omega_L Y}X + \nabla^l_X(\omega_L Y) + D^s(X, \omega_L Y)\} \\ &+\{-A_{\omega_S Y}X + \nabla^s_X(\omega_S Y) + D^l(X, \omega_S Y)\} \\ &+\{-A_{\omega_{\bar{2}} Y}X + \nabla^s_X(\omega_{\bar{2}} Y) + D^l(X, \omega_{\bar{2}} Y)\} \\ = \;& T\nabla_X Y + \omega_L \nabla_X Y + \omega_S \nabla_X Y + \omega_{\bar{2}} \nabla_X Y + Bh^l(X, Y) + Ch^l(X, Y) \\ &+Bh^s(X, Y) + Ch^s(X, Y) \end{aligned} \tag{38}$$

for any $X, Y \in \Gamma(TM)$. Taking the tangential, lightlike transversal and screen transversal parts of (38) we obtain

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y = A_{\omega_L Y}X + A_{\omega_S Y}X + A_{\omega_{\bar{2}} Y}X + Bh(X,Y), \tag{39}$$

$$D^l(X, \omega_S Y) + D^l(X, \omega_{\bar{2}} Y) = \omega_L \nabla_X Y - \nabla^l_X(\omega_L Y) - h^l(X, TY) + Ch^s(X,Y) \tag{40}$$

and

$$D^s(X, \omega_L Y) = \omega_S \nabla_X Y + \omega_{\bar{2}} \nabla_X Y - \nabla^s_X(\omega_S Y) - \nabla^s_X(\omega_{\bar{2}} Y) - h^s(X, TY) + Ch^l(X,Y) + Ch^s(X,Y) \tag{41}$$

respectively.

Theorem 3.5. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then ∇ is a metric connection iff for any $X \in \Gamma(TM)$, the following conditions are holded*

$$\nabla^*_X F\xi + h^*(X, F\xi) \in \Gamma(F(\sigma_1)), Bh(X, F\xi) = 0,\ \xi \in \Gamma(\sigma_1), \tag{42}$$

and

$$A_{F\xi}X \in \Gamma(F(\sigma_1)), B(\nabla^s_X F\xi +\ D^l(X, F\xi)) = 0,\ \xi \in \Gamma(\sigma_2). \tag{43}$$

Proof. Suppose that $\tilde{\nabla}$ is a metric connection. Then from Theorem 2.3, the radical distribution is parallel with respect to ∇, i.e., $\nabla_X \xi \in \Gamma(Rad(TM))$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. From (21) and (24) we obtain

$$\tilde{\nabla}_X \xi = F\tilde{\nabla}_X F\xi \tag{44}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus from (6), (14), (33), (37) and (44) we obtain

$$\nabla_X \xi + h(X, \xi) = T\nabla^*_X F\xi + Th^*(X, F\xi) + w\nabla^*_X F\xi + wh^*(X, F\xi) + Bh(X, F\xi) + Ch(X, F\xi), \tag{45}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Taking the tangential parts of the above equation we derive

$$\nabla_X \xi = T(\nabla^*_X F\xi + h^*(X, F\xi)) - Bh(X, F\xi) \tag{46}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\sigma_1)$. Letting $\xi \in \Gamma(\sigma_2)$ in (44) and using (6), (10), (33) and (37) we get

$$\nabla_X \xi = TA_{F\xi}X - B(\nabla^s_X F\xi +\ D^l(X, F\xi)). \tag{47}$$

Thus using Theorem 2.3, the proof follows from (46) and (47). □

Theorem 3.6. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then*
(i) The distribution $\bar{\sigma}$ is integrable iff

$$A_{FX}Y = A_{FY}X.$$

(ii) The distribution σ is integrable iff

$$h(X, FY) = h(FX, Y).$$

Proof. Suppose that M is a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Considering (39) we obtain

$$-T\nabla_X Y = A_{\omega_L Y}X + A_{\omega_S Y}X + A_{\omega_{\bar{2}} Y}X + Bh(X,Y)$$

for any $X, Y \in \Gamma(\bar{\sigma})$. Therefore we get

$$T[X,Y] = -A_{\omega_L Y}X + A_{\omega_L X}Y - A_{\omega_S Y}X + A_{\omega_S X}Y - A_{\omega_{\bar{2}} Y}X + A_{\omega_{\bar{2}} X}Y$$

which proves assertion (i). Using (40) and (41) we derive

$$h(X,TY) = \omega_L \nabla_X Y + \omega_S \nabla_X Y + \omega_{\bar{2}} \nabla_X Y + Ch(X,Y)$$

for any $X, Y \in \Gamma(\sigma)$. Thus we obtain

$$h(X,TY) - h(Y,TX) = \omega_L [X,Y] + \omega_S [X,Y] + \omega_{\bar{2}} [X,Y]$$

which proves the assertion (ii). □

Theorem 3.7. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then, σ is integrable iff the following conditions are holded*

$$h^s(X,FY) - h^s(Y,FX) \in \Gamma(F(L_2))$$

and

$$h^l(X,FY) - h^l(Y,FX) \in \Gamma(L_2)$$

for any $X, Y \in \Gamma(\sigma)$.

Proof. Definition of STCR-lightlike submanifolds, σ is integrable iff for any $X, Y \in \Gamma(\sigma)$, $[X,Y] \in \Gamma(\sigma)$, i.e.,

$$\tilde{g}([X,Y], N_2) = g([X,Y], F\xi_1) = g([X,Y], FW) = 0$$

for any $X, Y \in \Gamma(\sigma)$, $N_2 \in \Gamma(L_2)$, $\xi_1 \in \Gamma(\sigma_1)$ and $W \in \Gamma(S)$. Thus, using (8), (22), (23) and (24) we have

$$\tilde{g}([X,Y], N_2) = \tilde{g}(h^s(X,FY) - h^s(Y,FX), FN_2), \tag{48}$$

$$g([X,Y], F\xi_1) = \tilde{g}(h^l(X,FY) - h^l(Y,FX), \xi_1), \tag{49}$$

$$g([X,Y], FW) = \tilde{g}(h^s(X,FY) - h^s(Y,FX), W). \tag{50}$$

Therefore, the proof follows from (48)-(50). □

Theorem 3.8. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then, $\bar{\sigma}$ is integrable iff*

$$A_{FX}Y - A_{FY}X \in \Gamma(\bar{\sigma})$$

for any $X, Y \in \Gamma(\bar{\sigma})$.

Proof. Suppose that M is a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Note that $\bar{\sigma}$ is integrable iff for any $X, Y \in \Gamma(\bar{\sigma})$, $[X,Y] \in \Gamma(\bar{\sigma})$, i.e.,

$$\tilde{g}([X,Y], N_1) = g([X,Y], FN_1) = g([X,Y], Z) = 0$$

for any $Z \in \Gamma(\sigma_0)$ and $N_1 \in \Gamma(L_1)$. Hence, considering (7), (22) and (24) we have

$$\tilde{g}([X,Y], N_1) = g(A_{FX}Y - A_{FY}X, FN_1) \tag{51}$$

for any $X, Y \in \Gamma(\bar{\sigma})$ and $N_1 \in \Gamma(L_1)$. Similarly, using again (7), (22), (23) and (24) we obtain

$$g([X,Y], FN_1) = \tilde{g}(A_{FX}Y - A_{FY}X, N_1), \tag{52}$$

$$g([X,Y], Z) = g(A_{FX}Y - A_{FY}X, FZ), \tag{53}$$

for any $X, Y \in \Gamma(\bar{\sigma})$, $Z \in \Gamma(\sigma_0)$ and $N_1 \in \Gamma(L_1)$. Thus the proof comes from (51)-(53). □

4. STCR-Lightlike product manifold

Definition 4.1. *A STCR-lightlike submanifold M of a semi-Riemannian product manifold $\tilde{M}$ is called STCR-lightlike product manifold if both the distributions σ and $\bar{\sigma}$ define totally geodesic foliation in M.*

Theorem 4.2. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then, σ defines a totally geodesic foliation in $\tilde{M}$ iff*

$$Bh(X, FY) = 0$$

for any $X, Y \in \Gamma(\sigma)$.

Proof. Suppose that M is a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Thus σ defines a totally geodesic foliation iff

$$g(\nabla_X Y, F\xi_1) = \tilde{g}(\nabla_X Y, N_2) = g(\nabla_X Y, FW) = 0$$

for any $X, Y \in \Gamma(\sigma)$, $\xi_1 \in \Gamma(\sigma_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$. From (8), (22), (23) and (24) we obtain

$$g(\nabla_X Y, F\xi_1) = \tilde{g}(h^l(X, FY), \xi_1), \tag{54}$$
$$\tilde{g}(\nabla_X Y, N_2) = \tilde{g}(h^s(X, FY), FN_2), \tag{55}$$
$$g(\nabla_X Y, FW) = \tilde{g}(h^s(X, FY), W). \tag{56}$$

Hence from (54) we say that $h^l(X, FY)$ has no components in L_1 and from (55) and (56) we say that $h^s(X, FY)$ has no components in $F(\sigma_2) \perp S$. This completes the proof. □

Theorem 4.3. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then, $\bar{\sigma}$ defines a totally geodesic foliation in M iff*
(i) $A_{N_1}X$ has no components in $F(\sigma_1) \perp F(S)$.
(ii) $A_{FY}X$ has no components in $\sigma_o \perp \sigma_1$,
for any $X, Y \in \Gamma(\bar{\sigma})$.

Proof. Definition of STCR-lightlike submanifolds, $\bar{\sigma}$ defines a totally geodesic foliation iff

$$\tilde{g}(\nabla_X Y, N_1) = g(\nabla_X Y, FN_1) = g(\nabla_X Y, Z) = 0$$

for any $X, Y \in \Gamma(\bar{\sigma})$, $N_1 \in \Gamma(L_1)$ and $Z \in \Gamma(\sigma)$. Using that $\tilde{\nabla}$ is a metric connection and (6), (7), (22), (23) and (24) imply

$$\tilde{g}(\nabla_X Y, N_1) = -g(A_{FY}X, FN_1). \tag{57}$$

From (6), (7), (22), (23) and (24) we derive

$$g(\nabla_X Y, FN_1) = -\tilde{g}(A_{FY}X, N_1), \tag{58}$$

$$g(\nabla_X Y, Z) = -g(A_{FY}X, FZ). \tag{59}$$

Thus the proof comes from (57)-(59). □

Theorem 4.4. *Let M be a STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. If $(\nabla_X T)Y = 0$, then M is a STCR-lightlike product manifold.*

Proof. Let $X, Y \in \Gamma(\bar{\sigma})$, hence $TY = 0$. Then using (39) with the hypothesis, we get $T\nabla_X Y = 0$. Thus $\nabla_X Y \in \Gamma(\bar{\sigma})$ the distribution $\bar{\sigma}$ defines a totally geodesic foliation. Next, let $X, Y \in \Gamma(\sigma)$; therefore $\omega Y = 0$. Then using (39), we get $Bh(X, FY) = 0$. From Theorem 4.2, σ defines a totally geodesic foliation in M. Hence, M is a STCR-lightlike product manifold. □

Theorem 4.5. *Let M be an irrotational STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then, M is a STCR-lightlike product manifold if the following conditions are satisfied:*
i) $\nabla_X U \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$,
ii) $A^*_{\xi} Y \in \Gamma(F(\sigma_1) \perp F(S))$, for any $Y \in \Gamma(\sigma)$.

Proof. Let (i) holds, then from (9) and (10) we get $A_N X = 0$, $A_W X = 0$, $D^l(X, W) = 0$ and $\nabla^l_X N = 0$ for any $X \in \Gamma(TM)$. Hence for any $X, Y \in \Gamma(\sigma)$ and $W \in \Gamma(S(TM^{\perp}))$ and using (11), we obtain $\tilde{g}(h^s(X, Y), W) = 0$. Hence, the non degeneracy of $S(TM^{\perp})$ implies that $h^s(X, Y) = 0$. Therefore, $Bh^s(X, Y) = 0$. Now, for any $X, Y \in \Gamma(\sigma)$ and $\xi \in \Gamma(Rad(TM))$, then using (13) and (ii) and the hypothesis that M is irrotational, we have $\tilde{g}(h^l(X, Y), \xi) = \tilde{g}(Y, A^*_{\xi} X) = 0$. Thus, we get $h^l(X, Y) = 0$. Therefore $Bh^l(X, Y) = 0$. Then, from Theorem 4.2 the distribution σ defines a totally geodesic foliation in M.

Next, for any $X, Y \in \Gamma(\bar{\sigma})$, then $FY = \omega Y \in \Gamma(L_1 \perp S \perp F(\sigma_2)) \subset \Gamma(tr(TM))$. Considering (39) we get $T\nabla_X Y = -Bh(X, Y)$, comparing the components along $\bar{\sigma}$, we derive $T\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(\bar{\sigma})$. Thus the distribution $\bar{\sigma}$ defines a totally geodesic foliation in M. Thus M is a STCR-lightlike product manifold. □

5. Minimal STCR lightlike submanifolds

Definition 5.1. *We say that a lightlike submanifold M of a semi-Riemannian manifold $\tilde{M}$ is minimal if:*
(i) $h^s = 0$ on $Rad(TM)$ and
(ii) $trh = 0$, where trace is written with respect to g restricted to $S(TM)$.
It has been shown in [2] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $tr(TM)$.

Example 5.2. *Let $(\tilde{M} = \mathbb{R}^{14}_4, \tilde{g})$ be a 14-dimensional semi-Euclidean space with signature $(-,-,-,-,+,+,+,+,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14})$ be the standard coordinate system of $\mathbb{R}^{14}_4$. If we define a mapping F by*

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}) = (x_2, x_1, x_4, x_3, x_6, x_5, x_8, x_7, x_{10}, x_9, x_{13}, x_{14}, x_{11}, x_{12})$$

then $F^2 = I$ and F is an almost product structure on $\mathbb{R}^{14}_4$. Let M be a submanifold of $\tilde{M}$ defined by

$$\begin{aligned}
x_1 &= u_1 \sinh\alpha + u_2 \cosh\alpha,\ x_2 = u_3 \sinh\alpha + \frac{u_4}{2}\sinh\alpha,\\
x_3 &= u_1,\ x_4 = u_3 - \frac{u_4}{2},\ x_5 = u_2,\ x_6 = 0,\\
x_7 &= u_1 \cosh\alpha + u_2 \sinh\alpha,\ x_8 = u_3 \cosh\alpha + \frac{u_4}{2}\cosh\alpha,\\
x_9 &= \sinh u_5 \sinh u_6,\ x_{10} = \cosh u_5 \cosh u_6,\ x_{11} = \sin u_7 \cosh u_8,\\
x_{12} &= -\cos u_7 \sinh u_8,\ x_{13} = -u_7 + u_8,\ \ x_{14} = u_7 + u_8.
\end{aligned}$$

Then it is easy to see that a local frame of TM is given by

$$\begin{aligned}
Z_1 &= \sinh\alpha\partial x_1 + \partial x_3 + \cosh\alpha\partial x_7,\ Z_2 = \cosh\alpha\partial x_1 + \partial x_5 + \sinh\alpha\partial x_7,\\
Z_3 &= \sinh\alpha\partial x_2 + \partial x_4 + \cosh\alpha\partial x_8,\ Z_4 = \frac{1}{2}(\sinh\alpha\partial x_2 - \partial x_4 + \cosh\alpha\partial x_8),\\
Z_5 &= \cosh u_5 \sinh u_6 \partial x_9 + \sinh u_5 \cosh u_6 \partial x_{10},\ Z_6 = \sinh u_5 \cosh u_6 \partial x_9 + \cosh u_5 \sinh u_6 \partial x_{10},\\
Z_7 &= \cos u_7 \cosh u_8 \partial x_{11} + \sin u_7 \sinh u_8 \partial x_{12} - \partial x_{13} + \partial x_{14},\\
Z_8 &= \sin u_7 \sinh u_8 \partial x_{11} - \cos u_7 \cosh u_8 \partial x_{12} + \partial x_{13} + \partial x_{14}.
\end{aligned}$$

We see that M is a 2-lightlike submanifold with $Rad(TM) = Sp\{Z_1, Z_2\}$, $F(\sigma_1) = Sp\{F(Z_1) = Z_3\}$, $\sigma_0 = Sp\{Z_5, Z_6\}$ and it is easy to say that

$$\begin{aligned} ltr(TM) &= Sp\{N_1 = \frac{1}{2}(\sinh\alpha\partial x_1 - \partial x_3 + \cosh\alpha\partial x_7), \\ N_2 &= \frac{1}{2}(-\cosh\alpha\partial x_1 + \partial x_5 - \sinh\alpha\partial x_7)\}, \\ F(N_1) &= Z_4, S(TM^{\perp}) = Sp\{F(Z_2), F(N_2), F(Z_7), F(Z_8)\}. \end{aligned}$$

On the other hand, by direct computations and using Gauss and Weingarten formulas, we obtain

$$\bar{\nabla}_{Z_i} Z_T = 0\,, \quad i = 1, 2\,, 3, 4, \quad 1 \le T \le 8$$

and

$$\begin{aligned} h(Z_5, Z_5) &= 0\,,\ h(Z_6, Z_6) = 0,\ h^l(Z_7, Z_7) = 0\,,\ h^l(Z_8, Z_8) = 0, \\ h^s(Z_7, Z_7) &= \frac{(\sin u_7 \cosh u_8 + \cos u_7 \sinh u_8)F(Z_7) + (-\sin u_7 \cosh u_8 + \cos u_7 \sinh u_8)F(Z_8)}{\cos^2 u_7 + \cosh^2 u_8 + 1}, \\ h^s(Z_8, Z_8) &= \frac{-(\sin u_7 \cosh u_8 + \cos u_7 \sinh u_8)F(Z_7) + (\sin u_7 \cosh u_8 - \cos u_7 \sinh u_8)F(Z_8)}{\cos^2 u_7 + \cosh^2 u_8 + 1}, \end{aligned}$$

that is, $h^s = 0$ on $Rad(TM)$ and

$$trace\ |_{S(TM)}\ h = 0.$$

Then, it is clear that M is not totally geodesic and, but it is a minimal STCR-lightlike submanifold of $\tilde{M} = \mathbb{R}^{14}_4$.

Theorem 5.3. *Let M be a proper STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then M is minimal iff*

$$traceA^*_{\xi_q|S(TM)} = 0, traceA_{W_\alpha|S(TM)} = 0, \tilde{g}(Y, D^l(X, W)) = 0 \tag{60}$$

for any $X, Y, \xi_q \in \Gamma(Rad(TM))$ and $W_\alpha, W \in \Gamma(S(TM^{\perp}))$, where $q \in \{1, 2, ..., r\}$ and $\alpha \in \{1, 2, ..., n - r\}$.

Proof. We know that $h^l = 0$ on $Rad(TM)$ [2]. Definition of a STCR-lightlike submanifold, M is minimal iff

$$\sum_{i=1}^{2l} \epsilon_i h(e_i, e_i) + \sum_{j=1}^{p} h(F\xi_j, F\xi_j) + \sum_{j=1}^{p} h(FN_j, FN_j) + \sum_{l=1}^{k} \epsilon_l h(FW_l, FW_l) = 0$$

and $h^s = 0$ on $Rad(TM)$. From (11), we have $h^s = 0$ on $Rad(TM)$ iff $\tilde{g}(Y, D^l(X, W)) = 0$, for any $X, Y \in \Gamma(Rad(TM))$, and $W \in \Gamma(S(TM^{\perp}))$. On the other hand, we have

$$\begin{aligned} traceh\ |\ _{S(TM)} = &\frac{1}{r}\sum_{q=1}^{r}\sum_{j=1}^{p}\{\tilde{g}(h^l(F\xi_j, F\xi_j), \xi_q)N_q + \tilde{g}(h^l(FN_j, FN_j), \xi_q)N_q\} \\ &+ \frac{1}{n-r}\sum_{j=1}^{p}\sum_{\alpha=1}^{n-r}\epsilon_\alpha\{\tilde{g}(h^s(F\xi_j, F\xi_j), W_\alpha)W_\alpha + \tilde{g}(h^s(FN_j, FN_j), W_\alpha)W_\alpha\} \\ &+ \sum_{\alpha=1}^{n-r}\epsilon_\alpha\frac{1}{n-r}\{\sum_{i=1}^{2l}\tilde{g}(h^s(e_i, e_i), W_\alpha)W_\alpha + \sum_{l=1}^{k}\tilde{g}(h^s(FW_l, FW_l), W_\alpha)W_\alpha\} \\ &+ \sum_{q=1}^{r}\frac{1}{r}\{\sum_{i=1}^{2l}\tilde{g}(h^l(e_i, e_i), \xi_q)N_q + \sum_{l=1}^{k}\tilde{g}(h^l(FW_l, FW_l), \xi_q)N_q\}. \end{aligned} \tag{61}$$

Considering (11) and (16), we get

$$
\begin{aligned}
traceh \mid_{S(TM)} &= \frac{1}{r}\sum_{q=1}^{r}\sum_{j=1}^{p}\{\tilde{g}(A^{*}_{\xi_q}F\xi_j, F\xi_j)N_q + \tilde{g}(A^{*}_{\xi_q}FN_j, FN_j)N_q\} \\
&+ \frac{1}{n-r}\sum_{j=1}^{p}\sum_{\alpha=1}^{n-r}\epsilon_\alpha\{\tilde{g}(A_{W_\alpha}F\xi_j, F\xi_j)W_\alpha + \tilde{g}(A_{W_\alpha}FN_j, FN_j)W_\alpha\} \\
&+ \sum_{\alpha=1}^{n-r}\epsilon_\alpha\frac{1}{n-r}\{\sum_{i=1}^{2l}\tilde{g}(A_{W_\alpha}e_i, e_i)W_\alpha + \sum_{l=1}^{k}\tilde{g}(A_{W_\alpha}FW_l, FW_l)W_\alpha\} \\
&+ \sum_{q=1}^{r}\frac{1}{r}\{\sum_{i=1}^{2l}\tilde{g}(A^{*}_{\xi_q}e_i, e_i)N_q + \sum_{l=1}^{k}\tilde{g}(A^{*}_{\xi_q}FW_l, FW_l)N_q\}.
\end{aligned} \tag{62}
$$

This completes the proof. □

Theorem 5.4. *Let M be a totally umbilical STCR-lightlike submanifold of a semi-Riemannian product manifold $\tilde{M}$. Then M is minimal iff*

$$traceA^{*}_{\xi_q} \mid_{\sigma_0 \perp F(S)} = traceA_{W_\alpha} \mid_{\sigma_0 \perp F(S)} = 0 \tag{63}$$

for any $\xi_q \in \Gamma(Rad(TM))$ and $W_\alpha \in \Gamma(S(TM^\perp))$, where $q \in \{1, 2, ..., r\}$ and $\alpha \in \{1, 2, ..., n-r\}$.

Proof. We know that M is minimal iff $h^s = 0$ on $Rad(TM)$ and $traceh = 0$ on $S(TM)$. We derive

$$
\begin{aligned}
traceh \mid_{S(TM)} &= traceh \mid_{\sigma_0} + traceh \mid_{F(\sigma_1)} + traceh \mid_{F(L_1)} + traceh \mid_{F(S)} \\
&= \sum_{i=1}^{2l}\epsilon_i h(e_i, e_i) + \sum_{j=1}^{p}h(F\xi_j, F\xi_j) + \sum_{j=1}^{p}h(FN_j, FN_j) + \sum_{l=1}^{k}\epsilon_l h(FW_l, FW_l).
\end{aligned} \tag{64}
$$

Since M is totally umbilical then from (20) $h^s = 0$ on $Rad(TM)$ and we obtain

$$
\begin{aligned}
traceh \mid_{S(TM)} &= traceh \mid_{\sigma_0} + traceh \mid_{F(S)} \\
&= \sum_{i=1}^{2l}\epsilon_i(h^l(e_i, e_i) + h^s(e_i, e_i)) + \sum_{l=1}^{k}\epsilon_l(h^l(FW_l, FW_l) + h^s(FW_l, FW_l)) \\
&= \sum_{q=1}^{r}\frac{1}{r}\{\sum_{i=1}^{2l}\tilde{g}(h^l(e_i, e_i), \xi_q)N_q + \sum_{l=1}^{k}\tilde{g}(h^l(FW_l, FW_l), \xi_q)N_q\} \\
&\quad \sum_{\alpha=1}^{n-r}\epsilon_\alpha\frac{1}{n-r}\{\sum_{i=1}^{2l}\tilde{g}(h^s(e_i, e_i), W_\alpha)W_\alpha + \sum_{l=1}^{k}\tilde{g}(h^s(FW_l, FW_l), W_\alpha)W_\alpha\}.
\end{aligned} \tag{65}
$$

Moreover, if we use (11) and (16) in (65), we get

$$
\begin{aligned}
traceh \mid_{S(TM)} &= \sum_{q=1}^{r}\frac{1}{r}\{\sum_{i=1}^{2l}\tilde{g}(A^{*}_{\xi_q}e_i, e_i)N_q + \sum_{l=1}^{k}\tilde{g}(A^{*}_{\xi_q}FW_l, FW_l)N_q\} \\
&+ \sum_{\alpha=1}^{n-r}\epsilon_\alpha\frac{1}{n-r}\{\sum_{i=1}^{2l}\tilde{g}(A_{W_\alpha}e_i, e_i)W_\alpha + \sum_{l=1}^{k}\tilde{g}(A_{W_\alpha}FW_l, FW_l)W_\alpha\} \\
&= 0
\end{aligned}
$$

which completes the proof. □

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