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Completely J-positive (bi-)linear maps on Krein spaces and their Choi J-matrices

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Abstract. In this paper, we consider completely J-positive linear maps and (p, q, r)-J-positive bilinear maps for $p, q, r \in \mathbb{N}$ in the Krein space setting. We investigate relations between completely J-positive (bi-)linear maps and their Choi J-matrices. We prove that the dual cone of the set of completely J-positive linear maps, under a bilinear J-pairing, coincides with itself. Several characterizations of various J-positivity of bilinear maps are proved, including the one in terms of the J-positivity of the corresponding Choi J-matrices. Finally, we introduce a notion of a partial J-positivity of bilinear maps to clarify the relationship between J-positivity of bilinear maps and J-positivity of their linearization.

1. Introduction

The Krein space is an important generalization of Hilbert spaces where the inner product is not necessarily positive definite. Krein spaces provide a natural framework for understanding mathematical problems involving indefinite metrics, which arise in various areas of physics and engineering. For example, Krein spaces are used in the study of differential equations, operator theory, and quantum field theory. Moreover, Krein spaces have played an important role in the local formulation of gauge quantum field theories, where locality and covariance of the gauge fields are incompatible with positivity of the inner product. Krein space structure allows for deeper insights into the spectral properties of operators, stability of dynamical systems, and the formulation of physical theories with indefinite energy or probability.

A Krein space $\mathcal{K}$ is a Hilbert space equipped with an indefinite inner product that can be decomposed into a direct sum $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$of two Hilbert spaces, one with a positive definite inner product and one with a negative definite inner product. In a Krein space, the *fundamental symmetry* is a self-adjoint, unitary operator that connects the indefinite inner product of the Krein space with a positive definite inner product, which we will denote by J. That is, the operator $J = P_+ - P_-$ can be defined, where $P_\pm$ are orthogonal projections onto $\mathcal{K}_\pm$, so that the indefinite inner product have the connection as follows; $[\cdot, \cdot]_J = \langle J\cdot, \cdot\rangle$. The symmetry symmetry J is central to the structure and analysis of Krein spaces, which generalize Hilbert spaces by introducing an indefinite inner product. A classical example of a Krein space

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is the four dimensional spacetime used in special relativity where the indefinite inner product is given by

$$[(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)]_J = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$$

and $J = \mathrm{diag}(1, -1, -1, -1)$.

In the Krein spaces framework, spectral theory, Fredhom theory and numerical ranges have been studied in [1–3] and such a study is applicable for the investigation of the spectra of non-hermitian operators with PT symmetry. When passing to Krein spaces, the adjoint of a matrix with respect to the indefinite metric is defined through a J-metric and is used for the construction of states and quantum channels. A quantum J-channel is a structure-preserving transformation compatible with indefinite metrics, extending the usual framework of quantum information to Krein spaces and provides a general setting for the theory of quantum information by means of tools arising from operators theory on Krein spaces. Felipe-Sosa and Felipe [6] introduced the notions of J-states and quantum J-channels on matrix algebras and the author [9] established the equivalence of Kraus J-decompositions and Choi J-matrices and discussed the entanglement breaking condition of quantum J-channels.

We first consider completely J-positive linear maps and define (p, q, r)-J-positive bilinear maps for $p, q, r \in \mathbb{N}$ in the Krein space setting. In section two, we review the Choi J-matrices associated with linear maps and define a bilinear J-pairing of two linear maps using the corresponding Choi J-matrices. We investigate relations between completely J-positive (bi-)linear maps and their Choi J-matrices. Using a bilinear J-pairing of two linear maps, we observe that the dual cone of the set of completely J-positive linear maps with respect to a bilinear J-pairing coincide with itself. This self-duality mirrors properties of classical positive semidefinite cones and underscores the intrinsic symmetry in J-positivity frameworks. We prove the equivalence of complete J-positivity of a linear map between matrix algebras and J-positivity of Choi J-matrix associated to the matrix satisfying the Choi J-correspondence. In the third section, several characterizations of various J-positivity of bilinear maps are proved, including the one in terms of the J-positivity of the corresponding Choi J-matrices. Finally, we introduce a notion of a partial J-positivity of bilinear maps to clarify the relationship between J-positivity of bilinear maps and J-positivity of their linearization.

2. Completely *J*-positive linear maps and Choi *J*-matrices

Let $\mathcal{K}$ be a Hilbert space with a positive definite inner product $\langle \cdot, \cdot \rangle$ and J be a fundamental symmetry, i.e. $J = J^* = J^{-1}$. We denote by $[\cdot, \cdot]_J = \langle J \cdot, \cdot \rangle$ the indefinite inner product induced by J. We say that the pair $(\mathcal{K}, J)$ is a Krein space with an indefinite inner product $[\cdot, \cdot]_J$, or simply a Krein space. Let $\mathcal{B}(\mathcal{K})$ be the set of bounded linear operators on $\mathcal{K}$ and $\mathcal{K}^n$ $(n \geq 2)$ be the direct sum of n-copies of a Hilbert space $\mathcal{K}$. We denote by $(\mathcal{K}^n, J^n)$ the Krein space with an indefinite inner product

$$[\mathbf{x}, \mathbf{y}]_{J^n} = \langle J^n \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \langle Jx_i, y_i \rangle = \sum_{i=1}^{n} [x_i, y_i]_J$$

where $J^n = \mathrm{diag}(J, \ldots, J) \in M_n(\mathcal{B}(\mathcal{K}))$ and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{K}^n$. Throughout this paper, $(\mathcal{K}_i, J_i)$ $(i = 1, 2)$ denote Krein spaces with indefinite inner products $[\cdot, \cdot]_{J_i}$, unless specified otherwise

Let Φ be a linear map from $\mathcal{B}(\mathcal{K}_1)$ into $\mathcal{B}(\mathcal{K}_2)$ and $n \geq 2$ be a positive integer. The n-fold amplification of Φ is the map $\Phi^n := \mathrm{id}_n \otimes \Phi : M_n(\mathbb{C}) \otimes \mathcal{B}(\mathcal{K}_1) \to M_n(\mathbb{C}) \otimes \mathcal{B}(\mathcal{K}_2)$, which is given by applying Φ an element by an element to each matrix over $\mathcal{B}(\mathcal{K}_1)$, that is,

$$\Phi^n([T_{ij}]) = [\Phi(T_{ij})] \quad \text{for } [T_{ij}] \in M_n(\mathcal{B}(\mathcal{K}_1)).$$

For each $i = 1, 2$, we denote by $\mathcal{B}(\mathcal{K}_i)^{J+}$ the set of all J_i-positive bounded linear operators T on a Krein space $\mathcal{K}_i$, i.e.,

$$0 \leq [Tx, x]_{J_i} := \langle J_i Tx, x \rangle, \quad \text{for all } x \in \mathcal{K}_i.$$

Definition 2.1. *Let $\Phi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ be a linear map.*

(i) *Φ is J-positive if $\Phi(\mathcal{B}(\mathcal{K}_1)^{J+}) \subseteq \mathcal{B}(\mathcal{K}_2)^{J+}$.*

(ii) *For each $n \geq 2$, Φ is n-J-positive if*

$$\Phi^n(M_n(\mathcal{B}(\mathcal{K}_1))^{J+}) \subseteq M_n(\mathcal{B}(\mathcal{K}_2))^{J+}$$

where $M_n(\mathcal{B}(\mathcal{K}_i))^{J+} = \mathcal{B}(\mathcal{K}_i^n)^{J+}$ is the set of all J_i^n-positive linear operators on the Krein space $(\mathcal{K}_i^n, J_i^n)$.

(iii) *Φ is completely J-positive (or simply, J-CP) if Φ is n-J-positive for all $n \in \mathbb{N}$.*

For a linear map $\Phi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$, let $\Psi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ be a linear map defined by

$$\Psi(T) = J_2\Phi(J_1T) \quad \text{for } T \in \mathcal{B}(\mathcal{K}_1).$$

Then we have that, for each $n \in \mathbb{N}$ and $(T_{ij}) \in M_n(\mathcal{B}(\mathcal{K}_1))$

$$\Psi^n((T_{ij})) = (\Psi(T_{ij})) = (J_2\Phi(J_1T_{ij})) = J_2^n\Phi^n(J_1^n(T_{ij})),$$

so that $\Psi^n(\cdot) = J_2^n\Phi^n(J_1^n\,\cdot)$. Thus, we observe that a linear map $\Phi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ is completely positive if and only if the map $\Psi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ given by $\Psi(\cdot) = J_2\Phi(J_1\cdot)$ is completely J-positive. In particular, if $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ and $J_1 = J_2 = J$ and if $\Phi(JT) = J\Phi(T)$ for all $T \in \mathcal{B}(\mathcal{K})$, we see that Φ is completely positive if and only if it is completely J-positive. In [9], the author has proved a Stinespring type theorem for completely J-positive linear maps as follows: if $\Psi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ is completely J-positive, there exist a Krein space $(\mathcal{K}, J)$, a *-representation $\pi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K})$ and a bounded linear operator $V : \mathcal{K}_2 \to \mathcal{K}$ such that

$$\Psi(T) = V^{\#}\pi(T)V, \qquad (T \in \mathcal{B}(\mathcal{K}_1))$$

where $J = \pi(J_1)$ and $V^{\#} = J_2V^*J$. If, in addition, $\Psi(J_1) = J_2$, then V is an isometry.

Remark 2.2. *Kraus [10] proved that $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is completely positive if and only if*

$$\phi(X) = \sum_{i=1}^{\ell} V_i^* X V_i, \qquad X \in M_m(\mathbb{C}) \tag{1}$$

with $m \times n$ complex matrices $V_i \in M_{m,n}(\mathbb{C})$. This expression (1) is called a Kraus decomposition. Moreover, it is known that the followings are equivalent:

(i) *ϕ is completely positive.*

(ii) *ϕ has a Kraus decomposition (1).*

(iii) *the Choi matrix C_ϕ is positive (semi-definite) where C_ϕ is given by*

$$C_\phi := \sum_{i,j=1}^{m} e_{ij} \otimes \phi(e_{ij}) = (id_A \otimes \phi)\Big(\sum_{i,j=1}^{m} e_{ij} \otimes e_{ij}\Big) \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \tag{2}$$

with the canonical matrix units $\{e_{ij}\}$ in $M_m(\mathbb{C})$. □

In this section, we denote by $M_A = M_m(\mathbb{C})$ ($M_B = M_n(\mathbb{C})$, respectively) the set of all $m \times m$ complex matrices (the set of all $n \times n$ complex matrices, respectively), unless specified otherwise. We denote by J_A the fundamental symmetry in M_A which induces an indefinite inner product on $\mathcal{K}_A = \mathbb{C}^m$. Similarly, J_B denotes the fundamental symmtry in M_B.

For an $m \times n$ complex matrix V, we denote

$$V^{\#_{A,B}} := J_B V^* J_A \in M_{n,m}(\mathbb{C}).$$

Let $V_1, \dots, V_\ell$ be in $M_{m,n}(\mathbb{C})$ and we consider an elementary map $\psi : M_A \to M_B$ of the form

$$\psi(X) = \sum_{i=1}^{\ell} V_i^{\#_{A,B}} X V_i, \qquad X \in M_A. \tag{3}$$

Then such a map ψ is completely J-positive. Conversely, any completely J-positive linea map from M_A into M_B is of the form (3) (see [9] for details). We say that the decomposition (3) is *the Kraus J-decomposition of* ψ.

Definition 2.3. *[9] Let $\psi : M_A \to M_B$ be a linear map. We define C^J_ψ by*

$$C^J_\psi := \sum_{i,j=1}^{m} e_{ij} \otimes \psi(J_A e_{ij}) = (id_A \otimes \psi)\Big(\sum_{i,j=1}^{m} e_{ij} \otimes J_A e_{ij}\Big) \in M_A \otimes M_B \tag{4}$$

where $\{e_{ij} : i, j = 1, \dots, m\}$ is the set of canonical matrix units in M_A. The matrix C^J_ψ is called the Choi J-matrix of ψ.

The author [9] have showed that a linear map $\psi : M_A \to M_B$ is completely J-positive if and only if the Choi J-matrix C^J_ψ is $I_A \otimes J_B$-positive in $M_A \otimes M_B$. We say that a linear map $\phi : M_A \to M_B$ is a *quantum J-channel* if ϕ is completely J-positive and trace preserving. Then a quantum J-channel $\phi : M_A \to M_B$ maps quantum J-states into quantum J-states [9]. Here, a quantum J-state ρ means that $J\rho$ is a quantum state

Paulsen and Shultz [12] considered the matrix

$$C^{\mathfrak{A}}_\phi := \sum_{i,j=1}^{m} a_{ij} \otimes \phi(a_{ij}) = (\mathrm{id}_A \otimes \phi)\Big(\sum_{i,j=1}^{m} a_{ij} \otimes a_{ij}\Big) \in M_A \otimes M_B \tag{5}$$

which is replaced matrix units in (2) by a basis $\mathfrak{A} = \{a_{ij}\}$ of M_A. Let $C_{\mathfrak{A}}$ ($C^T_{\mathfrak{A}}$, respectively) be the linear map sending each matrix unit e_{ij} to a_{ij} (a^T_{ij}, respectively) where T denotes the transpose). We define $M_{\mathfrak{A}} := C_{\mathfrak{A}} \circ C^T_{\mathfrak{A}} \in L(M_A)$ where $L(M_A)$ is the set of all linear maps from M_A into M_A. For a given $V \in M_A$, let $\mathrm{Ad}_V : M_A \to M_A$ be defined by

$$\mathrm{Ad}_V(X) = V^* X V.$$

They [12] proved that if $M_{\mathfrak{A}} = \mathrm{Ad}_V$ for some $V \in M_A$, then $\phi : M_A \to M_B$ is completely positive if and only if $C^{\mathfrak{A}}_\phi$ is a positive semi-definite matrix in $M_A \otimes M_B$.

More generally, Kye [11] has considered the question what happens when we replace the matrix $\sum_{i,j} e_{ij} \otimes e_{ij}$ in (2) by another matrix $\Xi \in M_A \otimes M_A$, to define

$$C^{\Xi}_\phi := (\mathrm{id}_A \otimes \phi)(\Xi) \in M_A \otimes M_B.$$

For example, if $\Xi = \sum_k a_k \otimes b_k \in M_A \otimes M_A$, then we have that $C^{\Xi}_\phi = \sum_k a_k \otimes \phi(b_k)$. For the matrix $\Xi_{\mathfrak{A}} = \sum_{i,j} a_{ij} \otimes a_{ij}$ associated to some basis $\mathfrak{A} = \{a_{ij}\}$, we have $C^{\mathfrak{A}}_\phi = C^{\Xi_{\mathfrak{A}}}_\phi$.

Definition 2.4. *[11] We say that a matrix $\Xi \in M_A \otimes M_A$ satisfies the Choi correspondence when the complete positivity of a linear map $\phi : M_A \to M_B$ is equivalent to the positivity of the matrix $C^{\Xi}_\phi \in M_A \otimes M_B$.*

Remark 2.5. *Let $\psi : M_A \to M_A$ be a linear map.*

1. *Definition 2.4 says that C_ψ satisfies the Choi correspondence if and only if the complete positivity of $\phi : C_A \to C_B$ implies the complete positivity of $\phi \circ \psi : C_A \to C_B$ for all B.*
2. *Kye [11] proved that the Choi matrix $C_\psi \in M_A \otimes M_B$ satisfies the Choi correspondence if and only if ψ is of the form Ad_V for some nonsingular matrix $V \in M_A$.*

We define a *bilinear pairing* on the matrix algebra M_A or M_B as

$$(a,b) := \mathrm{Tr}(ab^T) = \sum_{i,j} a_{ij}b_{ij} \quad \text{for } a=(a_{ij}),\ b=(b_{ij})$$

where b^T denotes the transpose of b. Using their Choi matrices [4], the *bilinear pairing* of two linear maps $\phi,\psi : M_A \to M_B$ is defined by

$$(\phi,\psi) := (C_\phi, C_\psi) = \mathrm{Tr}_{AB}(C_\phi C_\psi^T) = \sum_{i,j} \mathrm{Tr}_B(\phi(e_{ij})\psi(e_{ij})^T).$$

For a linear map $\phi : M_A \to M_B$, its *adjoint map* $\phi^* : M_B \to M_A$ is defined by

$$(a, \phi^*(b))_A := (\phi(a), b)_B \quad \text{for all } a \in M_A, b \in M_B.$$

Then, it is clear that $(\phi,\psi) = (\psi^*,\phi^*)$.

We similarly define the *bilinear J-pairing* of two linear maps $\phi,\psi : M_A \to M_B$ as follows;

$$(\phi,\psi)_J := (C^J_\phi, C^J_\psi) = \mathrm{Tr}_{AB}\Big(C^J_\phi (C^J_\psi)^T\Big).$$

It is clear that we have the following identities;

1. $(\phi,\psi)_J = \sum_{i,j} \mathrm{Tr}_B\Big(\phi(J_A e_{ij})\psi(J_A e_{ij})^T\Big)$ for $\phi,\psi \in L(M_A, M_B)$,
2. $(\phi,\psi)_J = (\phi \circ M_J, \psi \circ M_J)$ for $\phi,\psi \in L(M_A, M_B)$,
3. $(\phi_2 \circ \phi_1, \psi)_J = (\phi_1, \phi_2^* \circ \psi)_J$ for $\phi_1 \in L(M_A, M_B)$, $\phi_2 \in L(M_B, M_C)$ and $\psi \in L(M_A, M_C)$

where M_J is the left multiplication map on M_A by J_A.

Let K be a subset of $L(M_A, M_B)$ where $L(M_A, M_B)$ is the set of all linear maps from M_A into M_B. We denote by K^J the *J-dual cone* of K which is the set of all linear maps $\phi \in L(M_A, L_B)$ satisfying $(\phi,\psi)_J \geq 0$ for all $\psi \in K$. Here the cone means that $\phi \in K^J \implies \lambda\phi \in K^J$ for any $\lambda > 0$ and $\phi_1, \phi_2 \in K^J \implies \phi_1 + \phi_2 \in K^J$. It is well known that $(K^J)^J$ is the smallest closed convex cone in $L(M_A, L_B)$ containing the set K. In particular, if K is a closed and convex subset in $L(M_A, L_B)$, then $K = (K^J)^J$.

Proposition 2.6. *If $\sigma : M_B \to M_B$ is a linear isomorphism and $K \subset L(M_A, M_B)$ is a convex cone, then we have that $(\sigma^* \circ K)^J = \sigma^{-1} \circ K^J$ where $\sigma^* \circ K = \{\sigma^* \circ \phi : \phi \in K\}$.*

Proof. We have that

$$\begin{aligned}\psi \in (\sigma^* \circ K)^J &\iff (\psi, \sigma^* \circ \phi)_J \geq 0 \quad \text{for all } \phi \in K\\ &\iff (\sigma \circ \psi, \phi)_J \geq 0 \quad \text{for all } \phi \in K\\ &\iff \psi \in \sigma^{-1} \circ K^J.\end{aligned}$$

This completes the proof. □

We denote by *J-CP*$[M_A, M_B]$ the set of all completely J-positive linear maps from M_A into M_B. It is clear that the set *J-CP*$[M_A, M_B]$ is a closed convex cone in $L(M_A, L_B)$.

Proposition 2.7. *Assume that J_B is a fundamental symmetry in M_B such that $J_B = J_B^T$ where J_B^T is the transpose of J_B. Then J-CP$[M_A, M_B]$ = J-CP$[M_A, M_B]^J$.*

Proof. For any $\phi, \psi \in J\text{-}CP[M_A, M_B]$, we have that

$$\begin{aligned}(\phi,\psi)_J &= (C^J_\phi, C^J_\psi) = \mathrm{Tr}\left(C^J_\phi (C^J_\psi)^T\right)\\ &= \mathrm{Tr}\Big((I_A\otimes J_B)^2\cdot C^J_\phi (C^J_\psi)^T\Big)\\ &= \mathrm{Tr}\left((I_A\otimes J_B)C^J_\phi\cdot \Big((I_A\otimes J_B)C^J_\psi\Big)^T\right)\geq 0\end{aligned}$$

where the last inequality follows from the fact that $(I_A \otimes J_B)C^J_\phi$ and $(I_A \otimes J_B)C^J_\psi$ are positive semi-definite. This implies that $J\text{-}CP[M_A, M_B] \subset J\text{-}CP[M_A, M_B]^J$.

Similarly, we see that $J\text{-}CP[M_A, M_B]^J \subset (J\text{-}CP[M_A, M_B]^J)^J$ Since $J\text{-}CP[M_A, M_B]$ is a closed convex cone in $L(M_A, L_B)$, we have that $J\text{-}CP[M_A, M_B] = (J\text{-}CP[M_A, M_B]^J)^J$, so that we get the reverse inclusion $J\text{-}CP[M_A, M_B]^J \subset J\text{-}CP[M_A, M_B]$. □

In the remaining of this section, we discuss a variant of the Choi J-matrix as a variant of Choi matrix was studied in [11]. Let $\mathfrak{A} = \{a_{ij}\}$ be a basis of M_A and $\phi : M_A \to M_B$ be a linear map. We define a variant of the Choi J-matrix of ϕ associated to $\mathfrak{A}$ as follows:

$$C^{J,\mathfrak{A}}_\phi := \sum_{i,j=1}^m a_{ij}\otimes\phi(J_A a_{ij}) = (\mathrm{id}_A\otimes\phi)\Big(\sum_{i,j=1}^m a_{ij}\otimes J_A a_{ij}\Big) \in M_A\otimes M_B. \tag{6}$$

Similarly, for a matrix $\Xi = \sum_k a_k \otimes b_k$ we define

$$C^{J,\Xi}_\phi := (\mathrm{id}_A\otimes\phi)\Big[(I_A\otimes J_A)\Xi\Big] = \sum_k a_k\otimes\phi(J_A b_k).$$

If $\Xi_{\mathfrak{A}} = \sum_{i,j} a_{ij} \otimes a_{ij}$ where $\{a_{ij}\}$ is a basis of M_A, then $C^{J,\mathfrak{A}}_\phi = C^{J,\Xi_{\mathfrak{A}}}_\phi$. Let $\mathcal{E} = \{e_{ij}\}$ be the canonical matrix units in M_A, $\mathcal{F} = \{f_{ij}\}$ be another matrix units in M_A and w be the unitary matrix such that $w^* e_{ij} w = f_{ij}$ for all i, j. For a linear map $\phi : M_A \to M_B$, we have that

$$\begin{aligned}C^{J,\mathcal{F}}_\phi &= \sum_{i,j} f_{ij}\otimes\phi(J_A f_{ij}) = \sum_{i,j} w^* e_{ij} w\otimes\phi(J_A w^* e_{ij} w)\\ &= \mathrm{Ad}_{w\otimes I}\Big(\sum_{i,j} e_{ij}\otimes\phi\circ M_J\circ\mathrm{Ad}_w(e_{ij})\Big)\\ &= \mathrm{Ad}_{w\otimes I}(C_{\phi\circ M_J\circ \mathrm{Ad}_w})\end{aligned}$$

Remark 2.8. *Let $\phi : M_A \to M_B$ be a linear map and $k = \min\{m, n\} \in \mathbb{N}$. If $\psi : M_A \to M_B$ is defined by $\psi(\cdot) = J_B\phi(J_A\cdot)$, then the followings are equivalent;*

1. *ϕ is completely J-positive.*
2. *ϕ is k-J-positive.*
3. *$C^J_\phi = C_{\phi\circ M_J}$ is $I_A \otimes J_B$-positive.*
4. *ψ is completely positive.*
5. *ψ is k-positive.*
6. *$C_\psi = C_{M_{J_B}\circ\phi\circ M_{J_A}}$ is positive.*

□

The following definition is an J-analogue of Definition 2.4 and it has some properties similar to the Choi correspondence.

Definition 2.9. *(cf. [11]) A matrix $\Xi \in M_A \otimes M_A$ satisfies the Choi J-correspondence when the complete J-positivity of a linear map $\phi : M_A \to M_B$ is equivalent to the $I_A \otimes J_B$-positivity of the Choi J-matrix $C^{J,\Xi}_\phi$ of ϕ associated to Ξ.*

We observe that $\Xi \in M_A \otimes M_A$ satisfies the Choi J-correspondence if the complete J-positivity of a linear map $\phi : M_A \to M_B$ is equivalent to the $I_A \otimes J_B$-positivity of the Choi J-matrix $C_\phi^{J,\Xi}$ for any $m \geq 1$. If $\Xi = C_\psi^J$ is the Choi J-matrix of a linear map $\psi : M_A \to M_A$, then we have that

$$\begin{aligned} C_\phi^{J,\Xi} &= (\mathrm{id}_A \otimes \phi)\Big[(I_A \otimes J_A)C_\psi^J\Big] \\ &= (\mathrm{id}_A \otimes \phi)\Big[(I_A \otimes J_A)\Big(\sum_{i,j} e_{ij} \otimes \psi(J_A e_{ij})\Big)\Big] \\ &= \sum_{i,j} e_{ij} \otimes \phi(J_A \psi(J_A e_{ij})) \\ &= \sum_{i,j} e_{ij} \otimes (\phi \circ M_{J_A} \circ \psi)(J_A e_{ij}) \\ &= C^J_{\phi \circ M_{J_A} \circ \psi} \end{aligned}$$

where M_{J_A} is the left multiplication map on M_A by J_A. This observation gives the following proposition.

Proposition 2.10. *For any linear map $\psi : M_A \to M_A$, the Choi J-matrix C_ψ^J satisfies the Choi J-correspondence if and only if the complete J-positivity of a linear map $\phi : M_A \to M_B$ is equivalent to the complete J-positivity of $\phi \circ M_{J_A} \circ \psi$ for any $m \geq 1$.*

3. Complete *J*-positivity of bilinear maps

In this section, we will denote unless specified otherwise. Let $\mathcal{B}(\mathcal{K}_A)$ ($\mathcal{B}(\mathcal{K}_B)$, $\mathcal{B}(\mathcal{K}_C)$, resp.) be the set of all bounded linear operators on the Krein space $\mathcal{K}_A$ ($\mathcal{K}_B$, $\mathcal{K}_C$, resp.). When J_A (J_B, resp.) is a fundamental symmetry in $\mathcal{B}(\mathcal{K}_A)$ ($\mathcal{B}(\mathcal{K}_B)$, resp.), let $J_A \otimes J_B$ be a fundamental symmetry of $\mathcal{B}(\mathcal{K}_A) \otimes \mathcal{B}(\mathcal{K}_B)$. Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map and $\widetilde{\psi} : \mathcal{B}(\mathcal{K}_A) \otimes \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be its linearization given by $\widetilde{\psi}(a \otimes b) := \psi(a, b)$.

For any $p, q \in \mathbb{N}$, we define a bilinear map $\psi_{p,q} : M_p(\mathcal{B}(\mathcal{K}_A)) \times M_q(\mathcal{B}(\mathcal{K}_B)) \to M_{pq}(\mathcal{B}(\mathcal{K}_C))$ by

$$\psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big) := \Big[\psi(a_{ij}, b_{kl})\Big] \in M_{pq}(\mathcal{B}(\mathcal{K}_C))$$

for $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A)), [b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))$. Moreover, the linearization of a bilinear map ψ is completely positive if and only if the following statement

$$[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^+,\ [b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^+ \Longrightarrow \psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big) \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^+$$

holds for every $p, q = 1, 2, \ldots$. This notion of (p, q)-fold amplification is different from the notion introduced by Christensen and Sinclair [5] which is related to the definition of matrix multiplication.

In [7], Han and Kye introduced the (p, q, r)-positivity of a bilinear map and classified tri-partite entanglement which include various kinds of bi-separability. A bilinear map $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ is (p, q, r)-positive if for any $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^+$, $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^+$ and $S \in M_{r,pq}$, we have

$$S\Big[\psi(a_{ij}, b_{kl})\Big]S^* \in M_r(\mathcal{B}(\mathcal{K}_C))^+ \tag{7}$$

where $M_{r,pq}$ denotes the $r \times pq$ matrix algebra over $\mathbb{C}$.

We denote by $\mathcal{B}(\mathcal{K}_A)^{J+}$ the set of all J-positive elements in $\mathcal{B}(\mathcal{K}_A)$ and by $J_A^p := I_p \otimes J_A$ the fundamental symmetry in $M_p(\mathcal{B}(\mathcal{K}_A))$ where I_p is the identity matrix in M_p, which induces an indefinite inner product on the Hilbert space $\mathbb{C}^p \otimes \mathcal{K}_A$. Let $M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ be the set of all J-positive elements in $M_p(\mathcal{B}(\mathcal{K}_A))$. If $[a_{ij}]_{i,j=1}^p \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ for $p = 1, 2, \ldots$, then we see that $J_A^p[a_{ij}] = [J_A a_{ij}]$ is positive semi-definite in $M_p(\mathcal{B}(\mathcal{K}_A))$.

In this section we introduce the analogue of (p, q, r)-positivity in the Krein space setting.

Definition 3.1. *Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map and let $p, q, r \in \mathbb{N}$.*

1. *ψ is J-positive if we have $\psi(a, b) \in \mathcal{B}(\mathcal{K}_C)^{J+}$ for any $a \in \mathcal{B}(\mathcal{K}_A)^{J+}$ and $b \in \mathcal{B}(\mathcal{K}_B)^{J+}$.*
2. *ψ is (p, q, r)-J-positive if for any $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$, $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{r,pq}$, we have that*

$$SJ_C^{pq}\psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big)S^* = S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* \in M_r(\mathcal{B}(\mathcal{K}_C))^+ \tag{8}$$

where $M_{r,pq}$ denotes the set of all $r \times pq$ complex matrices.

Remark 3.2. *Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map. For simplicity, we assume that $\mathcal{K}_A$, $\mathcal{K}_B$ and $\mathcal{K}_C$ are all finite dimensional.*

1. *The $(1, 1, 1)$-J-positivity of ψ implies the J-positivity of ψ.*
2. *If ψ is J-positive, then the linearization $\widetilde{\psi}$ is $(J_A \otimes J_B, J_C)$-positive.*
3. *The linearization of a bilinear map $\psi_{p,q}$ is defined by*

$$\widetilde{\psi}_{p\otimes q}\Big([a_{ij}] \otimes [b_{kl}]\Big) := \psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big) = \Big[\psi(a_{ij}, b_{kl})\Big],$$

so that $\widetilde{\psi}_{p\otimes q}$ is the map from $M_p(\mathcal{B}(\mathcal{K}_A)) \otimes M_q(\mathcal{B}(\mathcal{K}_B))$ into $M_{pq}(\mathcal{B}(\mathcal{K}_C))$ where

$$\Big[a_{ij}\Big]_{i,j} \otimes \Big[b_{kl}\Big]_{k,l} := \Big[a_{ij} \otimes \Big[b_{kl}\Big]_{k,l}\Big]_{i,j} = \begin{bmatrix} a_{11} \otimes \Big[b_{kl}\Big]_{k,l} & \cdots & a_{1p} \otimes \Big[b_{kl}\Big]_{k,l} \\ \vdots & \ddots & \vdots \\ a_{p1} \otimes \Big[b_{kl}\Big]_{k,l} & \cdots & a_{pp} \otimes \Big[b_{kl}\Big]_{k,l} \end{bmatrix}$$

can be regarded as an element in $M_{pq}(\mathcal{B}(\mathcal{K}_A) \otimes \mathcal{B}(\mathcal{K}_B))$.

4. *If a bilinear map $\psi_{p,q} : M_p(\mathcal{B}(\mathcal{K}_A)) \times M_q(\mathcal{B}(\mathcal{K}_B)) \to M_{pq}(\mathcal{B}(\mathcal{K}_C))$ is J-positive, then the linearization $\widetilde{\psi}_{p\otimes q}$ is $(J_A^p \otimes J_B^q, J_C^{pq})$-positive linear.*
5. *We denote by $\widetilde{\psi}_r$ the r-th amplification of $\widetilde{\psi}$, so that the linear map $\widetilde{\psi}_r : M_r(\mathcal{B}(\mathcal{K}_A)) \otimes M_r(\mathcal{B}(\mathcal{K}_B)) \to M_r(\mathcal{B}(\mathcal{K}_C))$ is given by $\widetilde{\psi}_r\Big([a_{ij} \otimes b_{ij}]\Big) := \Big[\widetilde{\psi}(a_{ij} \otimes b_{ij})\Big]$. After shuffling and identifying $M_p(\mathcal{B}(\mathcal{K}_A)) \otimes M_q(\mathcal{B}(\mathcal{K}_B))$ with $M_{pq}(\mathcal{B}(\mathcal{K}_A) \otimes \mathcal{B}(\mathcal{K}_B))$, we can see $\widetilde{\psi}_{p\otimes q} = \widetilde{\psi}_{pq}$.*

Proposition 3.3. *Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map. For any $p, q \in \mathbb{N}$, the followings are equivalent:*

(i) *For any $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ and $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B)B)^{J+}$, we have*

$$\psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big) \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^{J+}.$$

(ii) *ψ is (p, q, r)-J-positive for each $r = 1, 2, \ldots$.*
(iii) *ψ is (p, q, r)-J-positive for some $r \geq pq$.*
(iv) *ψ is (p, q, pq)-J-positive.*

Proof. $(i) \Rightarrow (ii)$ Let $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ and $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$. By assumption, we have that

$$\Big[\psi(a_{ij}, b_{kl})\Big] \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^{J+} \Rightarrow \Big[J_C\psi(a_{ij}, b_{kl})\Big] \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^+,$$

which implies that $S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* \in M_r(\mathcal{B}(\mathcal{K}_C))^+$ for all $r \in \mathbb{N}$ and any $r \times pq$ matrix $S \in M_{r,pq}$.

$(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (iv)$ For any $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$, $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{pq}$, we observe that

$$\begin{pmatrix} S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* & O_{pq,r-pq} \\ O_{r-pq,pq} & O_{r-pq} \end{pmatrix} = \begin{pmatrix} S \\ O_{r-pq,pq} \end{pmatrix}\Big[J_C\psi(a_{ij}, b_{kl})\Big]\begin{pmatrix} S^* & O_{pq,r-pq} \end{pmatrix},$$

which is positive semi-definite in $M_r(\mathcal{B}(\mathcal{K}_C))$. Here, M_{pq} is the set of all $pq \times pq$-matrices over $\mathbb{C}$ and $O_{m,n}$ is the $m \times n$-matrix whose entries are all 0. Thus, $S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^*$ is positive semi-definite, so that ψ is (p, q, pq)-J-positive.

$(iv) \Rightarrow (i)$ By assumption, we have

$$S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^+$$

for all $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$, $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{pq}$. By putting $S = I_{pq}$, we obtain that $\Big[\psi(a_{ij}, b_{kl})\Big] \in M_{pq}(\mathcal{B}(\mathcal{K}_C))^{J+}$. □

Corollary 3.4. *Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map and let $p \in \mathbb{N}$.*

(i) ψ is $(1, p, p)$-J-positive if and only if the linear map ${}_a\psi : M_B \to M_C$ given by

$${}_a\psi(b) := \psi(a, b)$$

is p-(J_B, J_C)-positive for any $a \in \mathcal{B}(\mathcal{K}_A)^{J+}$.

(ii) ψ is $(p, 1, p)$-J-positive if and only if the linear map $\psi_b : M_A \to M_C$ given by

$$\psi_b(a) := \psi(a, b)$$

is p-(J_A, J_C)-positive for any $b \in \mathcal{B}(\mathcal{K}_B)^{J+}$.

Proof. The proofs of (i) and (ii) immediately follow from Proposition 3.3. □

Remark 3.5. *Recall that a linear map $\phi : \mathcal{B}(\mathcal{K}_A) \to \mathcal{B}(\mathcal{K}_B)$ is (J_A, J_B)-positive if and only if the map ϕ_J given by $\phi_J(\cdot) := J_B\phi(J_A\cdot)$ is positive. Similarly, we can observe that a bilinear map $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ is J-positive if and only if the bilinear map ψ_J given by $\psi_J(\cdot, \cdot) := J_C\psi(J_A\cdot, J_B\cdot)$ is positive.* □

4. The Choi *J*-matrix associated with a bilinear map

In [7], the the Choi-Jamiolkowski isomorphism between a bilinear map and its Choi matrix was observed and the authors proved the equivalence of (p, q, r)-positivity for a bilinear map and the positive semidefiniteness of its Choi matrix. In this section, I introduce the J-analogue of the Choi matrix associted to a bilinear map and prove the equivalent conditions with (p, q, r)-J-positivity of bilinear maps on matrix algebras.

Definition 4.1. *Let M_A, M_B and M_C be matrix algebras which are not necessarily of same size and J_A, J_B and J_C be their fundamental symmetries, respectively. The Choi J-matrix C^J_ψ of a bilinear map $\psi : M_A \times M_B \to M_C$ is defined by*

$$C^J_\psi := \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} e_{ij} \otimes f_{kl} \otimes \psi(J_A e_{ij}, J_B f_{kl}) \in M_A \otimes M_B \otimes M_C$$

where $\{e_{ij}\}$ and $\{f_{kl}\}$ are canonical matrix units of M_A and M_B, respectively.

In the remaining of this section, let M_A, M_B and M_C be $m \times m$, $n \times n$ and $l \times l$-matrix algebras, respectively where m, n, l are natural numbers. We will sometimes regard a matrix in M_A (or M_B) as an operator in $\mathcal{B}(\mathcal{K}_A)$ (or $\mathcal{B}(\mathcal{K}_B)$) for some finite dimensional Krein space $\mathcal{K}_A$ (or $\mathcal{K}_B$), especially when multiplying block matrices with scalar matrices.

If $[a_{ij}]_{i,j=1}^p \in M_p(M_A)^{J+}$ and $[b_{kl}]_{k,l=1}^q \in M_q(M_B)^{J+}$ are J-positive semidefinite, then we have that $[a_{ij} \otimes b_{kl}] \in M_{pq}(M_A \otimes M_B)^{J+}$ since

$$(J_A \otimes J_B)^{pq}[a_{ij} \otimes b_{kl}] = [J_A a_{ij} \otimes J_B b_{kl}] \in M_{pq}(M_A \otimes M_B)^+.$$

Let S be an $mn \times l$-matrix in $M_{mn,l}$ and put $S^J := J_C S^*(J_A \otimes J_B) \in M_{l,mn}$. We define a bilinear map $\psi_S^J : M_A \times M_B \to M_C$ associated with S and S^J by

$$\psi_S^J(a, b) := S^J(a \otimes b)S = J_C S^*(J_A a \otimes J_B b)S, \quad (a \in M_A, b \in M_B). \tag{9}$$

We see that ψ_S^J satisfies (1) in Proposition 3.3. Indeed, for any $[a_{ij}] \in M_p(M_A)^{J+}$ and any $[b_{kl}] \in M_q(M_B)^{J+}$, we have

$$(\psi_S^J)_{p,q}\Big([a_{ij}], [b_{kl}]\Big) = \Big[\psi_S^J(a_{ij}, b_{kl})\Big] = \Big[J_C S^*(J_A a_{ij} \otimes J_B b_{kl})S\Big] \in M_{pq}(M_C)^{J+}.$$

By Proposition 3.3, the bilinear map ψ_S^J is (p, q, r)-J-positive for all $p, q, r \in \mathbb{N}$.

We denote by $\{|i\rangle\}$ a canonical basis vector in $\mathcal{K}_A = \mathbb{C}^m$ or $\mathcal{K}_B = \mathbb{C}^n$, which may be understood as a column vector. For any vector $\mathcal{K}_A$, we can write the ket notation $|a\rangle$, so that $|a\rangle = \sum_{i=1}^m a_i|i\rangle$. The *adjoint* of a ket $|a\rangle$ is denoted by a bra $\langle a|$, which is a row vector whose entries are obtained by complex conjugation of entries of $|a\rangle$, that is, $\langle a| = (\overline{a_1}, \dots, \overline{a_m})$.

Proposition 4.2. *Let $\psi_S^J : M_A \times M_B \to M_C$ be the bilinear map associated with S and S^J where S is an $mn \times l$ complex matrix. The Choi J-matrix $C^J_{\psi_S^J}$ of the bilinear map ψ_S^J is $(I_A \otimes I_B) \otimes J_C$-positive semidefinite and $(I_A \otimes I_B) \otimes J_C \cdot C^J_{\psi_S^J}$ is a positive rank one matrix.*

Proof. Let $\{e_{ij}\}$ and $\{f_{kl}\}$ be canonical matrix units of M_A and M_B, respectively. Then we compute the Choi J-matrix $C^J_{\psi_S^J}$ as follows;

$$\begin{aligned}
C^J_{\psi_S^J} &= \sum_{i,j=1}^m \sum_{k,l=1}^n e_{ij} \otimes f_{kl} \otimes \psi_S^J(J_A e_{ij}, J_B f_{kl}) \\
&= \sum_{i,j=1}^m \sum_{k,l=1}^n e_{ij} \otimes f_{kl} \otimes S^J(J_A e_{ij} \otimes J_B f_{kl})S \\
&= \sum_{i,j=1}^m \sum_{k,l=1}^n e_{ij} \otimes f_{kl} \otimes J_C S^*(e_{ij} \otimes f_{kl})S \\
&= \sum_{i,j=1}^m \sum_{k,l=1}^n |v_{i,k}\rangle\langle v_{j,l}| \otimes J_C|s^*_{i,k}\rangle\langle s^*_{j,l}|
\end{aligned}$$

where $|v_{i,k}\rangle = |i\rangle \otimes |k\rangle \in \mathcal{K}_A \otimes \mathcal{K}_B$ and $|s^*_{i,k}\rangle = S^*|v_{i,k}\rangle \in \mathcal{K}_C$. We have that

$$\begin{aligned}
(I_A \otimes I_B) \otimes J_C \cdot C^J_{\psi_S^J} &= \sum_{i,j=1}^m \sum_{k,l=1}^n |v_{i,k}\rangle\langle v_{j,l}| \otimes |s^*_{i,k}\rangle\langle s^*_{j,l}| \\
&= \left(\sum_{i,k=1}^{m,n} |v_{i,k}\rangle \otimes |s^*_{i,k}\rangle\right)\left(\sum_{j,l=1}^{m,n} |v_{j,l}\rangle \otimes |s^*_{j,l}\rangle\right)^*,
\end{aligned}$$

which is a positive matrix of rank one whose range vector is given by $\sum_{i,k=1}^{m,n} |v_{i,k}\rangle \otimes |s^*_{i,k}\rangle$. □

Remark 4.3. *In the proof of Proposition 4.2, we observe that*

$$C^J_{\psi_S^J} = (I_A \otimes I_B \otimes J_C) \cdot C_{\psi_S} = C_{J_C \cdot \psi_S}$$

where $\psi_S : M_A \times M_B \to M_C$ is a bilinear map given by $\psi_S(a, b) = S^*(a \otimes b)S$. □

Theorem 4.4. *Let $\psi : M_A \times M_B \to M_C$ be a bilinear map. The followings are equivalent:*

(i) *the following holds for any* $p, q \in \mathbb{N}$*;*
if $[a_{ij}] \in M_p(M_A)^{J+}$ *and* $[b_{kl}] \in M_q(M_B)^{J+}$*, then* $\psi_{p,q}\big([a_{ij}], [b_{kl}]\big) \in M_{pq}(M_C)^{J+}$*.*

(ii) ψ *is* (p, q, r)*-J-positive for each* $p, q, r \in \mathbb{N}$*.*

(iii) ψ *is* (m, n, mn)*-J-positive.*

(iv) ψ *satisfies (i) for* $p = m$ *and* $q = n$*.*

(v) *the Choi J-matrix* C^J_ψ *of* ψ *is* $I_A \otimes I_B \otimes J_C$*-positive.*

(vi) *there are* $mn\times l$*-matrices* S_i *such that* $\psi = \sum_i \psi^J_{S_j}$ *where* $\psi^J_{S_j} : M_A\times M_B \to M_C$ *is given by* $\psi^J_{S_j}(a, b) = S_j{}^J(a\otimes b)S_j$*.*

Proof. $(i) \Leftrightarrow (ii)$ and $(iii) \Leftrightarrow (iv)$ have already proved in Proposition 3.3.

$(iv) \Rightarrow (v)$ Assume that ψ satisfies (i) for $p = m$ and $q = n$. For $[a_{ij}] \in M_m(M_A)^+$ and $[b_{kl}] \in M_n(M_B)^+$, we write

$$[a_{ij}] = \sum_{i,j=1}^{m} e_{ij} \otimes a_{ij} \in M_m \otimes M_A, \quad [b_{kl}] = \sum_{k,l=1}^{n} f_{kl} \otimes b_{kl} \in M_n \otimes M_B.$$

Thus, we have that

$$\Big[\psi(a_{ij}, b_{kl})\Big] = \sum_{i,j=1}^{m}\sum_{k,l=1}^{n} e_{ij} \otimes f_{kl} \otimes \psi(a_{ij}, b_{kl}).$$

We observe that $J^m_A[a_{ij}] = [J_A a_{ij}] \in M_m(M_A)^{J+}$ and $J^n_B[b_{kl}] = [J_B b_{kl}] \in M_n(M_B)^{J+}$. Since $\sum_{i,j=1}^m e_{ij} \otimes e_{ij}$ and $\sum_{k,l=1}^n f_{kl} \otimes f_{kl}$ are positive, the Choi J-matrix

$$C^J_\psi = \sum_{i,j=1}^{m}\sum_{k,l=1}^{n} e_{ij} \otimes f_{kl} \otimes \psi(J_A e_{ij}, J_B f_{kl}) = \Big[\psi(J_A e_{ij}, J_B f_{kl})\Big]$$

is $I_A \otimes I_B \otimes J_C$-positive.

$(v) \Rightarrow (vi)$ If the Choi J-matrix C^J_ψ of ψ is $I_A \otimes I_B \otimes J_C$-positive, then $(I_A \otimes I_B \otimes J_C) \cdot C^J_\psi$ is positive semidefinite in $M_A \otimes M_B \otimes M_C$. By the spectral decomposition, the matrix $(I_A \otimes I_B \otimes J_C) \cdot C^J_\psi$ is the sum of rank one positive semidefinite matrices. We define a bilinear map $\phi : M_A \times M_B \to M_C$ by

$$\phi(\cdot, \cdot) = J_C \cdot \psi(J_A\cdot, J_B\cdot).$$

Since $(I_A \otimes I_B \otimes J_C) \cdot C^J_\psi = C_\phi$ is positive semidefinite, we observe that ϕ is of the form $\sum_j \phi_{S_j}$ where each S_j is a $mn \times l$-matrix and $\phi_{S_j}(a, b) = S^*_j(a \otimes b)S_j$ $(a \in M_A, b \in M_B)$. Thus, we obtain that $\psi(J_A\cdot, J_B\cdot) = \sum_j J_C \cdot \phi_{S_j}(\cdot, \cdot)$. For any $a \in M_A$ and $b \in M_B$, we have that

$$\begin{aligned}\psi(a, b) &= \sum_j J_C \cdot \phi_{S_j}(J_A a, J_B b) = \sum_j J_C S^*_j(J_A \otimes J_B)(a \otimes b)S_j \\ &= \sum_j S^J_j(a \otimes b)S_j = \sum_j \psi^J_{S_j}(a, b).\end{aligned}$$

(6) $\Rightarrow$ (1) We consider a bilinear map ψ^J_S given by (9) for some matrix $S \in M_{mn,l}(\mathbb{C})$. Let $[a_{ij}] \in M_p(M_A)^{J+}$ and $[b_{kl}] \in M_q(M_B)^{J+}$ be J-positive. Then we see that $[J_A a_{ij}] \in M_p(M_A)^+$ and $[J_B b_{kl}] \in M_q(M_B)^+$ and that

$$\Big[\psi^J_S(a_{ij}, b_{kl})\Big] = (I_{pq} \otimes J_C)(I_{pq} \otimes S)^*\Big[J_A a_{ij} \otimes J_B b_{kl}\Big](I_{pq} \otimes S).$$

Since $(I_{pq} \otimes S)^*\Big[J_A a_{ij} \otimes J_B b_{kl}\Big](I_{pq} \otimes S)$ is positive semidefinite, $\Big[\psi^J_S(a_{ij}, b_{kl})\Big]$ is $I_{pq} \otimes J_C$-positive in $M_{pq}(M_C)$. Since $\psi = \sum_i \psi^J_{S_j}$ for some $S_j \in M_{mn,l}(\mathbb{C})$,

$$\psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big) = \sum_j \Big[\psi^J_{S_j}(a_{ij}, b_{kl})\Big]$$

is $I_{pq} \otimes J_C$-positive, which completes the proof. □

Let $\mathcal{B}(\mathcal{K}_B)$ be the set of bounded linear operators on a finite dimensional Krein space $\mathcal{K}_B$. We consider the product between $[w_{ij}] \in M_p(\mathbb{C})$ and $[b_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_B))$ given by

$$\Big[w_{ij}\Big] \cdot \Big[b_{ij}\Big] = \Big[\sum_{k=1}^{p} w_{ik}b_{kj}\Big], \quad \Big[b_{ij}\Big] \cdot \Big[w_{ij}\Big] = \Big[\sum_{k=1}^{p} b_{ik}w_{kj}\Big]$$

where wb and bw denote the product between a scalar $w \in \mathbb{C}$ and an operator $b \in \mathcal{B}(\mathcal{K}_B)$. Since J_B^p is the operator diagonal matrix in $M_p(\mathcal{B}(\mathcal{K}_B))$ with each diagonal entry J_B, we observe that $J_B w_{ij} = w_{ij} J_B$ for each scalar w_{ij}, so that $J_B^p[w_{ij}] = [w_{ij}]J_B^p$. By the associative law of matrix products, we have that

$$J_B^p\Big[w_{ij}\Big] \cdot \Big[b_{ij}\Big] \cdot \Big[w_{ij}\Big]^* = \Big[w_{ij}\Big] \cdot \Big[J_B b_{ij}\Big] \cdot \Big[w_{ij}\Big]^* \tag{10}$$

for all $[w_{ij}] \in M_p(\mathbb{C})$ and $[b_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_B))$.

In the following proposition, we will identify $\mathcal{B}(\mathcal{K}_A)$, $\mathcal{B}(\mathcal{K}_B)$ and $\mathcal{B}(\mathcal{K}_C)$ with M_A, M_B and M_C, respectively.

Proposition 4.5. *For any $p \in \mathbb{N}$, a bilinear map $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ is $(p, p, 1)$-J-positive if and only if the following holds;*

$$[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+},\ [b_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+} \Longrightarrow \sum_{i,j=1}^{p} \psi(a_{ij}, b_{ij}) \in \mathcal{B}(\mathcal{K}_C)^{J+}. \tag{11}$$

Proof. If a bilinear map ψ is $(p, p, 1)$-J-positive, then for all $S \in M_{1,p^2}(\mathbb{C})$, $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ and $[b_{kl}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$, we have

$$S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* \in \mathcal{B}(\mathcal{K}_C)^+$$

Let $\{e_j : 1 \le j \le p\}$ be a canonical basis of $\mathbb{C}^p$ written as column vectors. We denote by e_j^T the transpose of e_j, that is, a row vector, so that $S := (e_1^T \ \cdots \ e_p^T) \in M_{1,p^2}(\mathbb{C})$. Thus, we have that

$$J_C \sum_{i,j=1}^{p} \psi(a_{ij}, b_{ij}) = (e_1^T \ \cdots \ e_p^T)\Big[J_C\psi(a_{ij}, b_{kl})\Big]\begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix} = S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^*$$

is positive semidefinite in $\mathcal{B}(\mathcal{K}_C)$, which implies that $\sum_{i,j=1}^{p} \psi(a_{ij}, b_{ij})$ is J_C-positive in $\mathcal{B}(\mathcal{K}_C)$.

Conversely, assume that (11) holds. We take any element $V \in M_{1,p^2}(\mathbb{C})$ with

$$V = (v_{1,1}, v_{1,2}, \ldots, v_{1,p^2}).$$

We denote by $\widetilde{V} = \Big[\widetilde{v}_{ij}\Big]$ the $p \times p$-matrix whose entries are given by $\widetilde{v}_{ij} = v_{1,(i-1)p+j}$, that is,

$$\widetilde{V} = \Big[\widetilde{v}_{ij}\Big] = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,p} \\ v_{1,p+1} & v_{1,p+2} & \cdots & v_{1,2p} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,p(p-1)+1} & v_{1,p(p-1)+2} & \cdots & v_{1,p^2}. \end{pmatrix}$$

For any $[b_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$, we observe that $J_B^p \cdot \widetilde{V}\Big[b_{ij}\Big]\widetilde{V}^*$ is positive semidefinite in $M_p(\mathcal{B}(\mathcal{K}_B))$.

Let $\{e_j : 1 \le j \le p\}$ be the canonical basis of $\mathbb{C}^p$, written as column vectors. Then we can decompose $V \in M_{1,p^2}(\mathbb{C})$ as follows;

$$V = (e_1^T \ \cdots \ e_p^T)\begin{pmatrix} \widetilde{V} & & \\ & \ddots & \\ & & \widetilde{V} \end{pmatrix} = (e_1^T \ \cdots \ e_p^T)(I_p \otimes \widetilde{V})$$

where e_j^T is the transpose of e_j, which is a row vector. We observe that

$$[b_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+} \Longrightarrow \widetilde{V}\big[b_{ij}\big]\widetilde{V}^* \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}.$$

For any elements $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$ and $[b_{kl}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$, we have that

$$\begin{aligned}
V\Big[J_C\psi(a_{ij}, b_{kl})\Big]V^* &= (e_1^T \ \cdots \ e_p^T)(I_p \otimes \widetilde{V})\Big[J_C\psi(a_{ij}, b_{kl})\Big](I_p \otimes \widetilde{V})^*(e_1^T \ \cdots \ e_p^T)^* \\
&= (e_1^T \ \cdots \ e_p^T)(I_p \otimes \widetilde{V})(I_p \otimes J_C^p)\Big[\psi(a_{ij}, b_{kl})\Big](I_p \otimes \widetilde{V})^*(e_1^T \ \cdots \ e_p^T)^* \\
&= (e_1^T \ \cdots \ e_p^T)(I_p \otimes J_C^p)(I_p \otimes \widetilde{V})\Big[\psi(a_{ij}, b_{kl})\Big](I_p \otimes \widetilde{V})^*(e_1^T \ \cdots \ e_p^T)^* \\
&= (e_1^T \ \cdots \ e_p^T)(I_p \otimes J_C^p)\widetilde{\psi}_{p\otimes p}\Big((a_{ij}) \otimes \widetilde{V}(b_{kl})\widetilde{V}^*\Big)(e_1^T \ \cdots \ e_p^T)^* \\
&= (e_1^T \ \cdots \ e_p^T)\Big[J_C\psi(a_{ij}, b'_{kl})\Big]\begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix} \\
&= J_C \sum_{i,j=1}^{p} \psi(a_{ij}, b'_{ij}).
\end{aligned}$$

Since $[b'_{ij}] = \widetilde{V}[b_{ij}]\widetilde{V}^*$ is J_B^p-positive in $M_p(\mathcal{B}(\mathcal{K}_B))$ by (10), It follows from the assumption (11) that ψ is $(p, p, 1)$-J-positive. □

Definition 4.6. *Let $\psi : \mathcal{B}(\mathcal{K}_A) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ be a bilinear map and let $p, q, r \in \mathbb{N}$.*

1. *ψ is partial J-positive if $\psi(a, b) \in M_C^{J+}$ for any $a \in \mathcal{B}(\mathcal{K}_A)^+$ and $b \in \mathcal{B}(\mathcal{K}_B)^{J+}$.*
2. *ψ is partial (p, q, r)-J-positive if*

$$SJ_C^{pq}\psi_{p,q}\Big([a_{ij}], [b_{kl}]\Big)S^* = S\Big[J_C\psi(a_{ij}, b_{kl})\Big]S^* \in M_r(\mathcal{B}(\mathcal{K}_C))^+$$

for any $[a_{ij}] \in M_p(\mathcal{B}(\mathcal{K}_A))^+$, $[b_{kl}] \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{r,pq}(\mathbb{C})$.

Proposition 4.7. *A bilinear map $\psi : M_p(\mathbb{C}) \times \mathcal{B}(\mathcal{K}_B) \to \mathcal{B}(\mathcal{K}_C)$ is partial $(p, p, 1)$-J-positive if and only if the linearization $\widetilde{\psi} : M_p(\mathcal{B}(\mathcal{K}_B)) \to M_C$ is (J_B^p, J_C)-positive.*

Proof. Suppose that ψ is partial $(p, p, 1)$-J-positive. Let $\{e_{ij} : 1 \le i, j \le p\}$ be the set of canonical matrix units in $M_p(\mathbb{C})$. Since $[e_{ij}]_{i,j=1}^p \in M_p(M_p(\mathbb{C}))$ is positive semidefinite, it follows from the definition that for any $[b_{ij}]_{i,j=1}^p \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$,

$$\sum_{i,j=1}^{n} \psi(e_{ij}, b_{ij}) = S\psi_{p,p}\Big([e_{ij}], [b_{kl}]\Big)S^* \in \mathcal{B}(\mathcal{K}_C)^{J+}$$

where $S = (e_1^T \ \cdots \ e_p^T) \in M_{1,p^2}(\mathbb{C})$ with a canonical basis $\{e_i : 1 \le i \le p\}$ in $\mathbb{C}^p$. Thus, for any $[b_{ij}]_{i,j=1}^p \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$ we have that

$$\widetilde{\psi}\Big(J_B^p[b_{ij}]\Big) = \widetilde{\psi}\Big(\sum_{i,j=1}^{n} e_{ij} \otimes J_B b_{ij}\Big) = \sum_{i,j=1}^{n} \widetilde{\psi}\Big(e_{ij} \otimes J_B b_{ij}\Big) = \sum_{i,j=1}^{n} \psi(e_{ij}, J_B b_{ij}),$$

which is J_C-positive in $\mathcal{B}(\mathcal{K}_C)$. This implies that $\widetilde{\psi}$ is (J_B^p, J_C)-positive.

Conversely, assume that the linearization $\widetilde{\psi} : M_p(\mathcal{B}(\mathcal{K}_B)) \to \mathcal{B}(\mathcal{K}_C)$ is (J_B^p, J_C)-positive. For any $[a_{ij}] \in M_p(M_p(\mathbb{C}))^+$, $[b_{kl}] \in M_p(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{1,p^2}(\mathbb{C})$, we observe that

$$S([a_{ij}] \otimes [b_{kl}])S^* \in M_p(\mathcal{B}(\mathcal{K}_B))$$

is J_B^p-positive. Hence we have that

$$S\psi_{p,p}\Big([a_{ij}],[b_{kl}]\Big)S^* = S\widetilde{\psi}_{p\otimes p}\Big([a_{ij}]\otimes[b_{kl}]\Big)S^* = \widetilde{\psi}\Big(S\Big([a_{ij}]\otimes[b_{kl}]\Big)S^*\Big)$$

is J_C-positive in $\mathcal{B}(\mathcal{K}_C)$, which completes the proof. □

Remark 4.8. *There is a relation between a bilinear map and the amplification of its linearization. We observe that if the linearization $\widetilde{\psi}: M_p(\mathcal{B}(\mathcal{K}_B)) \to \mathcal{B}(\mathcal{K}_C)$ of a bilinear map ψ is r-J-positive, then ψ is (p,q,r)-J-positive for any $p,q \in \mathbb{N}$. Indeed, since*

$$(J_A\otimes J_B)^r(S(x\otimes y)S^*) = S(J_A^p\otimes J_B^q)(x\otimes y)S^* = S(J_A^px\otimes J_B^qy)S^*$$

for any $x \in M_p(\mathcal{B}(\mathcal{K}_A))^{J+}$, $y \in M_q(\mathcal{B}(\mathcal{K}_B))^{J+}$ and $S \in M_{r,pq}(\mathbb{C})$, we see that $S(x\otimes y)S^*$ is $(J_A\otimes J_B)^r$-positive. If $\widetilde{\psi}$ is r-J-positive, we have that

$$S\psi_{p,q}(x\otimes y)S^* = S\widetilde{\psi}_{p\otimes q}(x\otimes y)S^* = \widetilde{\psi}_r(S(x\otimes y)S^*) \in M_r(\mathcal{B}(\mathcal{K}_C))^{J+},$$

which implies that ψ is (p,q,r)-J-positive. □

5. Conclusions

We explored the structural aspects of completely J-positive (bi-)linear maps through their associated Choi J-matrices. This will give foundational insights into the interplay between completely J-positive (bi-)linear maps and their Choi J-matrices, advancing the understanding of operator-theoretic positivity structures. By employing a bilinear J-pairing framework, the dual cone of the set of completely J-positive linear maps coincides with itself. This self-duality mirrors properties of classical positive semidefinite cones and underscores the intrinsic symmetry in J-positivity frameworks. Multiple criteria for J-positivity of bilinear maps are proved, with a central result linking the J-positivity of a bilinear map to the J-positivity of its Choi J-matrix. This bridges abstract operator-theoretic properties to concrete matrix analysis. A notion of partial J-positivity is offering a refined perspective on the linearization of bilinear maps in the context of J-structures. This concept resolves ambiguities in extending positivity properties from bilinear forms to their linear counterparts. These results unify and generalize tools for studying positivity in operator algebras, with applications in quantum information theory, matrix analysis, and functional analysis. The Choi J-matrix characterization offers a practical method to verify J-positivity, while partial J-positivity provides a nuanced framework for analyzing linearized systems. This work sets the stage for further exploration of (complete) J-positivity in noncommutative settings and operator-valued mappings.

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