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Generalized derivations of algebras with bracket

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Abstract. For an algebra with bracket and two two-sided ideals, several subalgebras of the Lie algebra of derivations of an algebra with bracket are introduced and characterized. The interplay of these subalgebras with central derivations and inner derivations of tautological algebras with bracket is analyzed. Exact sequences of Wells-type are obtained, which include the various types of subalgebras of introduced derivations, and which are associated with an abelian extension of algebras with bracket.

1. Introduction

The algebraic structure of algebra with bracket (AWB for short) was introduced in [5] and its origins can be found in the physics literature (see [6]). It is an associative algebra (not necessarily commutative) endowed with a second operation, denoted by $[-,-]$, such that the following identity holds for all a, b, c:

$$[ab, c] = a[b, c] + [a, c]b$$

The main example of this structure is a Poisson algebras. Other examples and a study of homological properties of this algebraic structure can be seen in [1, 2, 4, 5].

The proposal on this article focuses on the study of central derivations of algebras with bracket and some of their subalgebras, as well as on inner derivations of tautological algebras with bracket and their relationship with central derivations. Furthermore, the role played by the different subalgebras of central derivations in the construction of Wells-type exact sequences associated with an abelian extension of AWB is analyzed [4].

The paper is structured as follows: after a review of basic notions in Section 2, actions and cohomology, necessary for the development of the paper, in Section 3, we introduce central derivations of an AWB A, that is derivations whose image is contained in the center of A, and some subalgebras of $\mathsf{Der}(\mathsf{A})$, the Lie algebra of derivations of A, such as $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) = \{d \in \mathsf{Der}(\mathsf{A}) \mid d(\mathsf{A}) \in \mathsf{M}; d(\mathsf{N}) = 0\}$ (M, N are two-sided ideals of A), studying their most important properties such as the isomorphism between central derivations and linear transformations from the abelianized AWB to its center (Proposition 3.3), the characterization

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of stem AWBs using their central derivations (Proposition 3.5), or the characterization of the subalgebra $\mathsf{C}^*(\mathsf{A}) = \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),\mathsf{Z}(\mathsf{A})}(\mathsf{A})$ when A is a finite-dimensional AWB (Theorem 3.8). We also prove several results on central derivations and stem AWB using the isoclinism equivalence relation introduced in [3] (Proposition 3.12, Corollary 3.16). In Section 4, we discuss inner derivations of tautological AWBs, since there are no inner derivations for AWB in general. We also study the relationship between inner derivations and the subalgebra $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ (Theorem 4.4), the conditions under which inner derivations coincide with central derivations (Corollary 4.7) or the equality of the center of inner derivations with $\mathsf{C}^*(\mathsf{A})$ (Theorem 4.9). Finally, Section 5 is devoted to the construction of several Wells-type exact sequences associated with an abelian extension or a central extension of AWBs in which the introduced subalgebras of derivations play a central role (Theorem 5.5, Corollary 5.6). Furthermore, the splitting of these sequences when the abelian extension or the central extension is split is proved (Theorem 5.7).

2. Preliminaries

2.1. Basic definitions

Definition 2.1 ([5]). *An algebra with bracket, or an* AWB *for short, is an associative (not necessarily commutative) algebra* A *equipped with a bilinear map (bracket operation)* $[-,-]\colon \mathsf{A}\times\mathsf{A} \to \mathsf{A}$, $(a,b) \mapsto [a,b]$ *satisfying the following identity:*

$$[ab, c] = [a, c]b + a[b, c] \tag{2.1}$$

for all $a, b, c \in \mathsf{A}$.

A homomorphism of AWB's is a homomorphism of associative algebras preserving the bracket operation. We denote by AWB the respective category of AWB's.

Example 2.2.

(a) Any vector space A *with the trivial multiplication and bracket, i.e.* $ab = 0$ *and* $[a,b] = 0$ *for all* $a, b \in \mathsf{A}$*, is an* AWB*, called* an abelian AWB.

(b) Any Poisson algebra is an AWB.

(c) Let $(\mathsf{A},\cdot)$ *be an associative algebra, then* A *endowed with the bracket* $[a,b] := ab - ba, a, b \in \mathsf{A}$*, is an algebra with bracket called the tautological algebra with bracket associated to* $(\mathsf{A},\cdot)$*. Some examples are included throughout the text.*

(d) For other examples we refer to [4, 5] and references given therein.

Let us recall the following notions from [1]. *A subalgebra* B of an AWB A is a vector subspace satisfying $\mathsf{B}\,\mathsf{B} \subseteq \mathsf{B}$ and $[\mathsf{B},\mathsf{B}] \subseteq \mathsf{B}$. A subalgebra B is said to be a *right (respectively, left) ideal* if $\mathsf{A}\,\mathsf{B} \subseteq \mathsf{B}$, $[\mathsf{A},\mathsf{B}] \subseteq \mathsf{B}$ (respectively, $\mathsf{B}\,\mathsf{A} \subseteq \mathsf{B}$, $[\mathsf{B},\mathsf{A}] \subseteq \mathsf{B}$). If B is both left and right ideal, then it is said to be a *two-sided ideal*. In this case, the quotient A/B is endowed with an AWB structure naturally induced from the operations on A.

Let A be an AWB and B,C be two-sided ideals of A. The *commutator ideal* of B and C is the two-sided ideal of B and C

$$[[\mathsf{B},\mathsf{C}]] = \langle\{bc, cb, [b,c], [c,b] \mid b \in \mathsf{B}, c \in \mathsf{C}\}\rangle.$$

Observe that $[[\mathsf{B},\mathsf{C}]]$ is not a two-sided ideal of A in general, except when $\mathsf{B} = \mathsf{A}$ or $\mathsf{C} = \mathsf{A}$. In the particular case $\mathsf{B} = \mathsf{C} = \mathsf{A}$, one obtains the definition of derived algebra of A, $[[\mathsf{A},\mathsf{A}]] = \langle\{aa', [a,a'] \mid a, a' \in \mathsf{A}\}\rangle$. Note that the quotient $\mathsf{A}/[[\mathsf{A},\mathsf{A}]]$ is an abelian AWB.

The *center* of an AWB A is the two-sided ideal

$$\mathsf{Z}(\mathsf{A}) = \{a \in \mathsf{A} \mid ab = 0 = ba, [a,b] = 0 = [b,a], \text{for all } b \in \mathsf{A}\}.$$

Note that an AWB A is abelian if and only if $\mathsf{A} = \mathsf{Z}(\mathsf{A})$. An AWB A is said to be *stem* if $\mathsf{Z}(\mathsf{A}) \subseteq [[\mathsf{A}, \mathsf{A}]]$.

Let M be a two-sided ideal of an AWB A. The two-sided ideal of A

$$\mathsf{C}_{\mathsf{A}}(\mathsf{A}, \mathsf{M}) = \{a \in \mathsf{A} \mid aa', a'a, [a, a'], [a', a] \in \mathsf{M}, \text{for all } a' \in \mathsf{A}\}$$

is said to be the *centralizer* of A and M on A (see [2]). Obviously, $\mathsf{C}_{\mathsf{A}}(\mathsf{A}, 0) = \mathsf{Z}(\mathsf{A})$.

The sequence of two-sided ideals defined recursively by

$$\mathsf{A}^{[1]} = \mathsf{A}; \qquad \mathsf{A}^{[i]} = [[\mathsf{A}^{[i+1]}, \mathsf{A}]], i \geq 2$$

is said to be the *lower central series* of an AWB A. A is said to be *nilpotent* with class of nilpotency c if and only if $\mathsf{A}^{[c+1]} = 0$ and $\mathsf{A}^{[c]} \neq 0$ (see [2]).

The upper central series (see [2]) of an AWB A is the sequence of two-sided ideals defined recursively by

$$\mathcal{Z}_0(\mathsf{A}) = 0; \qquad \mathcal{Z}_i(\mathsf{A}) = \mathsf{C}_{\mathsf{A}}(\mathsf{A}, \mathcal{Z}_{i-1}(\mathsf{A})), i \geq 1$$

Theorem 2.3. *[2] An* AWB A *is nilpotent with class of nilpotency c if and only if* $\mathcal{Z}_c(\mathsf{A}) = \mathsf{A}$ *and* $\mathcal{Z}_{c-1}(\mathsf{A}) \neq \mathsf{A}$.

2.2. Actions

Definition 2.4. *Let* A *and* M *be two* AWB*'s. An action of* A *on* M *consists of four bilinear maps*

$$\begin{array}{llll} \mathsf{A} \times \mathsf{M} \to \mathsf{M}, & (a, m) \mapsto a \cdot m, & \mathsf{M} \times \mathsf{A} \to \mathsf{M}, & (m, a) \mapsto m \cdot a, \\ \mathsf{A} \times \mathsf{M} \to \mathsf{M}, & (a, m) \mapsto \{a, m\}, & \mathsf{M} \times \mathsf{A} \to \mathsf{M}, & (m, a) \mapsto \{m, a\}, \end{array}$$

such that the following conditions hold:

$$\begin{array}{ll} (a_1a_2) \cdot m = a_1 \cdot (a_2 \cdot m), & \{a_1{\cdot}m, a_2\} = a_1{\cdot}\{m, a_2\} + [a_1, a_2] \cdot m, \\ m \cdot (a_1a_2) = (m \cdot a_1){\cdot}a_2, & \{m{\cdot}a_1, a_2\} = \{m, a_2\}{\cdot}a_1 + m{\cdot}[a_1, a_2], \\ (a_1{\cdot}m){\cdot}a_2 = a_1{\cdot}(m{\cdot}a_2), & \{a_1a_2, m\} = a_1{\cdot}\{a_2, m\} + \{a_1, m\} \cdot a_2, \\ (m_1m_2){\cdot}a = m_1(m_2{\cdot}\, a), & [m_1{\cdot}a, m_2] = m_1\{a, m_2\} + [m_1, m_2]{\cdot}a, \\ a{\cdot}(m_1m_2) = (a{\cdot}m_1)m_2, & [a{\cdot}m_1, m_2] = a \cdot [m_1, m_2] + \{a, m_2\}m_1, \\ (m_1{\cdot}a)m_2 = m_1(a{\cdot}m_2), & \{m_1m_2, a\} = m_1\{m_2, a\} + \{m_1, a\}m_2, \end{array} \tag{2.2}$$

for all $a, a_1, a_2 \in \mathsf{A}$, $m, m_1, m_2 \in \mathsf{M}$.

The action is called trivial if all these bilinear maps are trivial.

Let us remark that if an action of an AWB A on an abelian AWB M is given, then all six equations in the last three lines of (2.2) vanish and we get the definition of a *representation* M of A (see [5]).

Example 2.5.

(a) If M *is a two-sided ideal of an* AWB A*, then the structural operations in* A *yield an action of* A *on* M*, that is* $a{\cdot}m = am$, $m{\cdot}a = ma$, $\{a, m\} = [a, m]$, $\{m, a\} = [m, a]$, *for all* $m \in \mathsf{M}$, $a \in \mathsf{A}$.

(b) Let $0 \to \mathsf{M} \xrightarrow{i} \mathsf{E} \underset{p}{\overset{s}{\leftrightarrows}} \mathsf{A} \to 0$ *be a split short exact sequence of AWBs, then there is an action of* A *on* M*, given by:*

$$\begin{array}{ll} a \cdot m = i^{-1}\big(s(a)i(m)\big), & m \cdot a = i^{-1}\big(i(m)s(a)\big), \\ \{a, m\} = i^{-1}\big(\, [s(a), i(m)]\,\big), & \{m, a\} = i^{-1}\big(\, [i(m), s(a)]\,\big), \end{array} \tag{2.3}$$

for all $m \in \mathsf{M}$, $a \in \mathsf{A}$.

(c) If $p\colon \mathsf{E} \to \mathsf{A}$ *is a surjective homomorphism of* AWB*'s with abelian kernel, then the actions (2.3) endow* $\mathsf{Ker}(\pi)$ *with a structure of* A*-representation.*

2.3. Cohomology of AWB*'s*

The Quillen cohomology of an AWB A with coefficients in a representation M of A is computed as the cohomology of an explicit cochain complex $K^*(\mathsf{A},\mathsf{M})$ in [5]. Let us recall the main constructions.

Let A be an AWB and M be a representation of A. Since M is a bimodule over the associative algebra A, we can consider the Hochschild cochain complex $C^*(\mathsf{A},\mathsf{M})$ of A with coefficients in M. Let us recall that $C^n(\mathsf{A},\mathsf{M}) = \mathsf{Hom}(\mathsf{A}^{\otimes n},\mathsf{M})$ and the coboundary map b^* is given by

$$\begin{aligned} b^n(f)(a_0,\dots,a_n) =& a_0\cdot f(a_1,\dots,a_n) + \sum_{0\le i\le n-1}(-1)^{i+1} f(a_0,\dots,a_ia_{i+1},\dots,a_n)\\ &+(-1)^{n+1}f(a_0,\dots,a_{n-1})\cdot a_n\ . \end{aligned}$$

Now we let $\overrightarrow{C}^*(\mathsf{A},\mathsf{M})$ be the cochain complex defined by

$$\overrightarrow{C}^n(\mathsf{A},\mathsf{M}) = C^{n+1}(\mathsf{A},\mathsf{M}),\quad n\ge 0 \quad \text{and} \quad \overrightarrow{C}^n(\mathsf{A},\mathsf{M}) = 0,\quad n\le 0.$$

At the same time, the vector space $\mathsf{M}^e = \mathsf{Hom}(\mathsf{A},\mathsf{M})$ has a bimodule structure over A given by

$$(a_0\cdot f)(a_1) = a_0\cdot f(a_1),\qquad (f\cdot a_0)(a_1) = f(a_1)\cdot a_0,$$

for all $a_0,a_1\in\mathsf{A}$ and $f\in\mathsf{M}^e$.

There is a cochain map

$$\alpha^n\colon \overrightarrow{C}^n(\mathsf{A},\mathsf{M})\to\overrightarrow{C}^n(\mathsf{A},\mathsf{M}^e),\quad n\ge 0$$

defined by

$$\left(\alpha^0(f)(a_0)\right)(a_1) = \{a_0,f(a_1)\} - f[a_0,a_1] + \{f(a_0),a_1\}$$

provided $n=0$ and for $n\ge 1$ by

$$\left(\alpha^n(f)(a_0,\dots,a_n)\right)(a_{n+1}) = \{f(a_0,\dots,a_n),a_{n+1}\} - \sum_{0\le i\le n} f\left(a_0,\dots,[a_i,a_{i+1}],\dots,a_{n+1}\right).$$

Then the complex $K^*(\mathsf{A},\mathsf{M})$ is defined to be the cone of the cochain map α. By definition, the n-th AWB cohomology is $H^n_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}) = H^{n-1}K^*(\mathsf{A},\mathsf{M})$, $n\ge 0$.

For future references, we specify below the cochains and coboundary maps of $K^*(\mathsf{A},\mathsf{M})$ in low dimensions:

- $K^{-1}(\mathsf{A},\mathsf{M})$ consists of all linear maps $h\colon \mathsf{A}\to\mathsf{M}$
- $K^0(\mathsf{A},\mathsf{M})$ consists of all pairs (f,g), where $f\colon \mathsf{A}^{\otimes 2}\to\mathsf{M}$ (resp. $g\colon \mathsf{A}\to\mathsf{M}^e$) is a 2-cochain (resp. 1-cochain) in the Hochschild complex $C^*(\mathsf{A},\mathsf{M})$ (resp. $C^*(\mathsf{A},\mathsf{M}^e)$).
- $K^1(\mathsf{A},\mathsf{M})$ consists of all pairs (f,g), where $\mathsf{f}\colon \mathsf{A}^{\otimes 3}\to\mathsf{M}$ (resp. $\mathsf{g}\colon \mathsf{A}^{\otimes 2}\to\mathsf{M}^e$) is a 3-cochain (resp. 2-cochain) in the Hochschild complex $C^*(\mathsf{A},\mathsf{M})$ (resp. $C^*(\mathsf{A},\mathsf{M}^e)$);
- The coboundary map ∂^{-1} is given by

$$\begin{aligned} \partial^{-1}(h)(a_0,a_1) &= \left(-b^1(h)(a_0,a_1),\ -\alpha^0(h)(a_0)(a_1)\right) \\ &= \Big(-a_0\cdot h(a_1) + h(a_0a_1) - h(a_0)\cdot a_1,\ -\{a_0,h(a_1)\} + h[a_0,a_1] - \{h(a_0),a_1\}\Big). \end{aligned} \tag{2.4}$$

- The coboundary map ∂^0 is given by

$$\partial^0(f,g) = \left(-b^2(f), b^1(g) - \alpha^1(f)\right),$$

where

$$b^2(f)(a_0,a_1,a_2) = a_0 \cdot f(a_1,a_2) - f(a_0a_1,a_2) + f(a_0,a_1a_2) - f(a_0,a_1)\cdot a_2, \tag{2.5}$$

and

$$\begin{aligned}\Big(b^1(g) - \alpha^1(f)\Big)(a_0,a_1,a_2) = {} & a_0 \cdot \Big(g(a_1)(a_2)\Big) + \Big(g(a_0)(a_2)\Big)\cdot a_1 - g(a_0a_1)(a_2) \\ & - \Big(\{f(a_0,a_1),a_2\} - f\Big([a_0,a_2],a_1\Big) - f\Big(a_0,[a_1,a_2]\Big)\Big).\end{aligned} \tag{2.6}$$

2.4. H^0_{AWB} and H^1_{AWB}

Definition 2.6. *Let* A *be an* AWB *and* M *be a representation of* A*. An* AWB-derivation *is a linear map* $d\colon \mathsf{A} \to \mathsf{M}$ *such that*

$$\begin{aligned} d(a_0a_1) &= a_0d(a_1) + d(a_0)a_1,\\ d[a_0,a_1] &= [a_0,d(a_1)] + [d(a_0),a_1], \end{aligned}$$

for all $a_0,a_1 \in \mathsf{A}$.

Let $\mathsf{Der}_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$ denote the vector space of all AWB-derivations. By [5, Lemma 4.4] there is an isomorphism of abelian AWBs

$$H^0_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}) \cong \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}). \tag{2.7}$$

In the particular case when $\mathsf{M} = \mathsf{A}$, we denote $\mathsf{Der}_{\mathsf{AWB}}(\mathsf{A},\mathsf{A})$ by $\mathsf{Der}_{\mathsf{AWB}}(\mathsf{A})$. It is easy to check that $d_1d_2 - d_2d_1 \in \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A})$ if $d_1,d_2 \in \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A})$, showing that $\mathsf{Der}_{\mathsf{AWB}}(\mathsf{A})$ is a Lie subalgebra of the Lie algebra of all derivations $\mathsf{Der}(\mathsf{A})$ of the associative algebra A.

Definition 2.7. *Any abelian extension of an* AWB A *by a representation* M *of* A *is a short exact sequence of* AWB*'s*

$$E\colon 0 \longrightarrow \mathsf{M} \stackrel{i}{\longrightarrow} \mathsf{E} \stackrel{p}{\longrightarrow} \mathsf{A} \longrightarrow 0,$$

which induces (see Example 2.5 (c)) the given representation structure on M.

It gives rise to a 0-cocycle $(f,g) \in K^0(\mathsf{A},\mathsf{M})$ by choosing a linear section $s\colon \mathsf{A} \to \mathsf{E}$ of p, and by defining $f\colon \mathsf{A}^{\otimes 2} \to \mathsf{M}$ and $g\colon \mathsf{A} \to \mathsf{Hom}(\mathsf{A},\mathsf{M})$ as follows

$$\begin{aligned} f(a_0,a_1) &= i^{-1}\Big(s(a_0)s(a_1) - s(a_0a_1)\Big),\\ g(a_0)(a_1) &= i^{-1}\Big([s(a_0),s(a_1)] - s[a_0,a_1]\Big) \end{aligned} \tag{2.8}$$

for all $a_0,a_1 \in \mathsf{A}$. This gives a well-defined bijection between the set of equivalence classes $\mathrm{Ext}_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$ of such abelian extensions of A by M and the first cohomology of A with coefficients in M [5, Lemma 4.6], i.e.

$$H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}) \cong \mathrm{Ext}_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}). \tag{2.9}$$

Let us note that this bijection allows us to endow the set $\mathrm{Ext}(\mathsf{A},\mathsf{M})$ with a vector space structure induced from the one of $H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$ (see [4]).

3. Some subalgebras of derivations

In this section, we introduce and characterize several subalgebras of the Lie algebra of derivations of an AWB, paying special attention to central derivations.

Let A be an AWB and M, N be two two-sided ideals of A. We introduce the following subalgebras of $\mathsf{Der}(\mathsf{A})$:

$$\begin{array}{lll} \mathsf{Der}^{\mathsf{M}}(\mathsf{A}) & = & \{d \in \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A}) \mid d(x) \in \mathsf{M} \text{ for all } x \in \mathsf{A}\}, \\ \mathsf{Der}_{\mathsf{N}}(\mathsf{A}) & = & \{d \in \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A}) \mid d(x) = 0 \text{ for all } x \in \mathsf{N}\}, \\ \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) & = & \mathsf{Der}^{\mathsf{M}}(\mathsf{A}) \cap \mathsf{Der}_{\mathsf{N}}(\mathsf{A}), \\ \mathsf{Der}(\mathsf{A}:\mathsf{M}) & = & \{d \in \mathsf{Der}_{\mathsf{AWB}}(\mathsf{A}) \mid d(x) \in \mathsf{M} \text{ for all } x \in \mathsf{M}\}; \\ \mathsf{C}^*(\mathsf{A}) & = & \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),\mathsf{Z}(\mathsf{A})}(\mathsf{A}). \end{array}$$

Definition 3.1. *A derivation $d : \mathsf{A} \to \mathsf{A}$ of an* AWB A *is said to be a central derivation if* $\mathsf{Im}(d) \subseteq \mathsf{Z}(\mathsf{A})$.

We denote the set of all central derivations of an AWB A by $\mathsf{Der}_z(\mathsf{A})$. Obviously, $\mathsf{Der}_z(\mathsf{A})$ is a subalgebra of $\mathsf{Der}_{\mathsf{AWB}}(\mathsf{A})$ and every element of $\mathsf{Der}_z(\mathsf{A})$ annihilates $[[\mathsf{A}, \mathsf{A}]]$.

Let A and B be two AWBs and denote by $\mathsf{T}(\mathsf{A}, \mathsf{B})$ the set of all linear transformations from A to B. Clearly, $\mathsf{T}(\mathsf{A}, \mathsf{B})$ endowed with the operations $(fg)(x) = f(x)g(x)$ and $[f, g](x) = [f(x), g(x)]$ inherits the algebraic structure that B may have.

Lemma 3.2. *Let M and N be two two-sided ideals of an* AWB A *such that $\mathsf{M} \subseteq \mathsf{Z}(\mathsf{A})$. Then the following statements hold:*

(a) The map $\Psi\colon \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \to \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right)$, which assigns to a derivation $d \in \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ the map $\chi_d\colon \frac{\mathsf{A}}{\mathsf{N}} \to \mathsf{M}$ given by $\chi_d(x + \mathsf{N}) = d(x), x \in \mathsf{A}$, is an isomorphism of abelian AWBs.

(b) If $\mathsf{M} \subseteq \mathsf{N}$, then $\Psi\colon \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \to \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right)$ is an isomorphism of AWBs*. Moreover, $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ is an abelian* AWB.

Proof. (a) χ_d is well-defined and
$\chi_d((x + \mathsf{N})(x' + \mathsf{N})) = d(xx') = d(x)x' + xd(x') = 0,$
$\chi_d(x + \mathsf{N})\chi_d(x' + \mathsf{N}) = d(x)d(x') = 0,$
$\chi_d([x + \mathsf{N}, x' + \mathsf{N}]) = d([x, x']) = [d(x), x'] + [x, d(x')] = 0,$
$[\chi_d(x + \mathsf{N}), \chi_d(x' + \mathsf{N})] = [d(x), d(x')] = 0.$

On the other hand, Ψ has trivial kernel and is surjective since for any $f \in \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right)$, define $\tau\colon \mathsf{A} \to \mathsf{A}, \tau(x) = f(x + \mathsf{N}), x \in \mathsf{A}$. It is easy to check that $\tau \in \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{A})$ and $\Psi(\tau) = \chi_\tau = f$.

(b) Direct checking. □

Proposition 3.3. *For any* AWB A, $\mathsf{Der}_z(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$ *as abelian* AWBs*. If A is also a stem* AWB*, then $\mathsf{Der}_z(\mathsf{A})$ is isomorphic to the abelian Lie algebra $\mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$.*

Proof. Let $d \in \mathsf{Der}_z(\mathsf{A})$ be, then $d(\mathsf{A}) \subseteq \mathsf{Z}(\mathsf{A})$, and d annihilates $[[\mathsf{A}, \mathsf{A}]]$. So, d induces the map $\alpha_d\colon \frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]} \longrightarrow \mathsf{Z}(\mathsf{A})$ defined by $\alpha_d(a + [[\mathsf{A}, \mathsf{A}]]) = d(a), a \in \mathsf{A}$.

Now define the map $\beta\colon \mathsf{Der}_z(\mathsf{A}) \longrightarrow \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$ by $\beta(d) = \alpha_d$. Clearly, β is a linear map, which is one-to-one by definition of α_d.

β is onto since for a given $d^* \in \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$, there exists a linear map $d : \mathsf{A} \to \mathsf{Z}(A)$, $d = d^*\pi$, where $\pi\colon \mathsf{A} \twoheadrightarrow \frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}$ is the canonical projection, such that $\beta(d) = d^*$. Finally, $d \in \mathsf{Der}_z(\mathsf{A})$ since $d(ab) = d^*(\pi(a)\pi(b)) = d^*(\overline{0}) = 0$ and $d[a, b] = d^*([\pi(a), \pi(b)]) = d^*(\overline{0}) = 0$; on the other hand, $d(a)b + ad(b) = d^*(\pi(a))b + ad^*(\pi(b)) = 0$ and $[d(a), b] + [a, d(b)] = [d^*(\pi(a)), b] + [a, d^*(\pi(b))] = 0$, since $d^*(\pi(a)), d^*(\pi(b)) \in \mathsf{Z}(\mathsf{A}), a, b \in \mathsf{A}$. Finally, we show that β is Lie algebra homomorphism. Indeed, let $a \in \mathsf{A}$. It is clear that $\beta([d_1, d_2])(\overline{a}) = \alpha_{[d_1,d_2]}(\overline{a}) = [d_1, d_2](a) = 0$ since $d_1(\mathsf{A}), d_2(\mathsf{A}) \subseteq \mathsf{Z}(\mathsf{A}) \subseteq [[\mathsf{A}, \mathsf{A}]]$ and d_1, d_2 annihilate $[[\mathsf{A}, \mathsf{A}]]$. On the other hand, $[\beta(d_1), \beta(d_2)](\overline{a}) = [\alpha_{d_1}, \alpha_{d_2}](\overline{a}) = [\alpha_{d_1}(\overline{a}), \alpha_{d_2}(\overline{a})] = 0$ since $\alpha_{d_i}(\overline{a}) \in \mathsf{Z}(\mathsf{A}), i = 1, 2$. Hence $\beta([d_1, d_2]) = [\beta(d_1), \beta(d_2)]$. This completes the proof. □

Corollary 3.4. *If* A *is a stem* AWB, *then* $\mathsf{Der}_z(\mathsf{A})$ *is an abelian Lie algebra.*

Proof. Direct consequence of Proposition 3.3. □

The converse of the above result is not true in general. Indeed, let A be the 1-dimensional abelian tautological AWB, that is A has a basis $\{e\}$ and operations $ee = [e, e] = 0$, then $\mathsf{Der}_z(\mathsf{A}) = \mathsf{Der}(\mathsf{A})$ is an abelian Lie algebra, but A is not a stem AWB since $\mathsf{Z}(\mathsf{A}) = \mathsf{A} \nsubseteq 0 = [[\mathsf{A}, \mathsf{A}]]$.

Proposition 3.5. *Let* A *be a nilpotent finite-dimensional* AWB *such that* $[[\mathsf{A}, \mathsf{A}]] \neq 0$*. Then* $\mathsf{Der}_z(\mathsf{A})$ *is abelian if and only if* A *is a stem* AWB*.*

Proof. We only need to prove the converse of Corollary 3.4. Assume that A is not a stem AWB. Then, there is some $a_1 \in \mathsf{Z}(\mathsf{A})$ such that $a_1 \notin [[\mathsf{A}, \mathsf{A}]]$. Since A is a nilpotent AWB and $[[\mathsf{A}, \mathsf{A}]] \neq 0$, it follows that $\mathsf{Z}(\mathsf{A}) \cap [[\mathsf{A}, \mathsf{A}]] \neq 0$. Now, let $H := \langle a_1 \rangle^{\perp}$ be the complement of the subspace spanned by a_1, and let $a_2 \in \mathsf{Z}(\mathsf{A}) \cap [[\mathsf{A}, \mathsf{A}]]$, $a_2 \neq 0$. Consider d_1 (d_2, respectively) as the linear transformation of A vanishing on H and mapping a_1 to a_1 (a_1 to a_2, respectively). Clearly, d_1 and d_2 are central derivations of the AWB A such that $[d_1, d_2](a_1) = d_1(d_2(a_1)) - d_2(d_1(a_1)) = -a_2 \neq 0$. Therefore $\mathsf{Der}_z(\mathsf{A})$ is not abelian. □

Corollary 3.6. *Let* A *be a finite-dimensional* AWB *and* $\mathsf{M}_1, \mathsf{M}_2, \mathsf{N}_1, \mathsf{N}_2$ *be two-sided ideals of* A *such that* $\mathsf{M}_1 \subseteq \mathsf{M}_2 \subseteq \mathsf{Z}(\mathsf{A})$ *and* $\mathsf{N}_2 \subseteq \mathsf{N}_1$*. Then* $\mathsf{Der}_{\mathsf{M}_1,\mathsf{N}_1}(\mathsf{A}) \subseteq \mathsf{Der}_{\mathsf{M}_2,\mathsf{N}_2}(\mathsf{A})$*. Also,* $\mathsf{Der}_{\mathsf{M}_1,\mathsf{N}_1}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M}_2,\mathsf{N}_2}(\mathsf{A})$ *if and only if* $\mathsf{M}_1 = \mathsf{M}_2$ *and* $\mathsf{N}_1 = \mathsf{N}_2$*.*

Proof. The inclusion is obvious. If $\mathsf{Der}_{\mathsf{M}_1,\mathsf{N}_1}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M}_2,\mathsf{N}_2}(\mathsf{A})$, then Lemma 3.2 implies $\dim\left(\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}_1}, \mathsf{M}_1\right)\right) = \dim\left(\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}_2}, \mathsf{M}_2\right)\right)$. From this equality is derived the equality between the ideals, since otherwise, that is, if $\mathsf{M}_1 \subsetneq \mathsf{M}_2$ or $\mathsf{N}_2 \subsetneq \mathsf{N}_1$, then $\dim\left(\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}_1}, \mathsf{M}_1\right)\right) < \dim\left(\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}_2}, \mathsf{M}_2\right)\right)$, which is a contradiction.

The converse is evident. □

Corollary 3.7. *Let* A *be a finite-dimensional* AWB *and* M, N *be two two-sided ideals of* A *such that* $\mathsf{M} \subseteq \mathsf{Z}(\mathsf{A})$*. Then* $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) = \mathsf{Der}_z(\mathsf{A})$ *if and only if* $\mathsf{M} = \mathsf{Z}(\mathsf{A})$ *and* $\mathsf{N} \subseteq [[\mathsf{A}, \mathsf{A}]]$*.*

Proof. By Lemma 3.2, $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}+[[\mathsf{A},\mathsf{A}]]}(\mathsf{A})$. By Corollary 3.6 and bearing in mind that $\mathsf{Der}_z(\mathsf{A}) = \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),[[\mathsf{A},\mathsf{A}]]}(\mathsf{A})$, we have $\mathsf{Der}_{\mathsf{Z}(\mathsf{A}),[[\mathsf{A},\mathsf{A}]]}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}+[[\mathsf{A},\mathsf{A}]]}(\mathsf{A})$ if and only if $\mathsf{M} = \mathsf{Z}(\mathsf{A})$ and $\mathsf{N} + [[\mathsf{A}, \mathsf{A}]] = [[\mathsf{A}, \mathsf{A}]]$ if and only if $\mathsf{M} = \mathsf{Z}(\mathsf{A})$ and $\mathsf{N} \subseteq [[\mathsf{A}, \mathsf{A}]]$. □

Theorem 3.8. *Let* A *be a finite-dimensional* AWB*. Then the following statements hold:*

(a) If A *is abelian or* $\mathsf{Z}(\mathsf{A}) = 0$*, then* $\mathsf{C}^*(\mathsf{A}) = \{0\}$*.*

(b) Let A *be a nilpotent* AWB*. Then* A *is abelian if and only if* $\mathsf{C}^*(\mathsf{A}) = \{0\}$*.*

(c) The abelian algebra with bracket $\mathsf{C}^*(\mathsf{A})$ *and* $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right)$ *are isomorphic.*

(d) Let A *be such that* $[[\mathsf{A}, \mathsf{A}]] = \mathsf{Z}(\mathsf{A})$*, then* $\mathsf{Der}_z(\mathsf{A}) = \mathsf{C}^*(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$*.*

(e) If A *is nilpotent of class 2, then* $\mathsf{Der}_z(\mathsf{A}) = \mathsf{C}^*(\mathsf{A})$ *if and only if* $[[\mathsf{A}, \mathsf{A}]] = \mathsf{Z}(\mathsf{A})$*.*

Proof. (a) If A is abelian, then $\mathsf{Z}(\mathsf{A}) = \mathsf{A}$, so $\mathsf{Der}(\mathsf{A}) = \mathsf{Der}_z(\mathsf{A})$ and every derivation $d \in \mathsf{C}^*(\mathsf{A})$ sends elements of $\mathsf{A} = \mathsf{Z}(\mathsf{A})$ to 0, hence $d = 0$.

If $\mathsf{Z}(\mathsf{A}) = 0$, then every derivation $d \in \mathsf{C}^*(\mathsf{A})$ verifies that $d(a) \in \mathsf{Z}(\mathsf{A}) = 0$, for all $a \in \mathsf{A}$, hence $d = 0$.

(b) If A is nilpotent and $\mathsf{C}^*(\mathsf{A}) = \{0\}$, then having in mind the isomorphism of abelian AWBs $\mathsf{C}^*(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right)$ provided by Lemma 3.2 (a) we have $\dim\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}\right) \cdot \dim(\mathsf{Z}(\mathsf{A})) = 0$, which implies $\mathsf{A} = \mathsf{Z}(\mathsf{A})$. The converse is given in statement (a).

(c) Consequence of Lemma 3.2 (b).

(d) By Lemma 3.2, $\mathsf{C}^*(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right)$ as abelian AWBs. By Proposition 3.3, $\mathsf{Der}_z(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$ as abelian AWBs.

(e) Since A is nilpotent of class 2, then $[[\mathsf{A}, \mathsf{A}]] \subseteq \mathsf{Z}(\mathsf{A})$. By Lemma 3.2 and Proposition 3.3, $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$, then $\dim(\mathsf{Z}(\mathsf{A})) = \dim([[\mathsf{A}, \mathsf{A}]])$.

The converse is provided by statement (d). □

Example 3.9. *Consider the two-dimensional complex associative algebra As_2^1 with basis $\{e_1, e_2\}$ and operation given by $e_1e_1 = e_2$ and zero elsewhere [7]. Then the tautological* AWB A *associated to As_2^1 satisfies the requirements of statement* (d) *in Theorem 3.8, since* $[[\mathsf{A},\mathsf{A}]] = \mathsf{Z}(A) = \langle\{e_2\}\rangle$.

Following [3], two AWBs A and B are said to be isoclinic, if there exist a pair of isomorphisms $\eta\colon \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \to \frac{\mathsf{B}}{\mathsf{Z}(\mathsf{B})}$ and $\xi\colon [[\mathsf{A},\mathsf{A}]] \to [[\mathsf{B},\mathsf{B}]]$ such that the following diagram is commutative

$$\begin{array}{ccccc} [[\mathsf{A},\mathsf{A}]] & \xleftarrow{P_{\mathsf{A}}} & \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \times \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} & \xrightarrow{C_{\mathsf{A}}} & [[\mathsf{A},\mathsf{A}]] \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta\times\eta} & & \downarrow{\scriptstyle \xi} \\ [[\mathsf{B},\mathsf{B}]] & \xleftarrow{P_{\mathsf{B}}} & \mathsf{B}\times\mathsf{B} & \xrightarrow{C_{\mathsf{B}}} & [[\mathsf{B},\mathsf{B}]] \end{array} \tag{3.1}$$

where $P_{\mathsf{A}}\colon \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \times \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \to [[\mathsf{A},\mathsf{A}]]$ is given by $P_{\mathsf{A}}(a_1 + \mathsf{Z}(\mathsf{A}), a_2 + \mathsf{Z}(\mathsf{A})) = a_1a_2$ and $C_{\mathsf{A}}\colon \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \times \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \to [[\mathsf{A},\mathsf{A}]]$ is given by $C_{\mathsf{A}}(a_1 + \mathsf{Z}(\mathsf{A}), a_2 + \mathsf{Z}(\mathsf{A})) = [a_1, a_2]$, $a_1, a_2 \in \mathsf{A}$ (correspondingly, $P_{\mathsf{B}}, C_{\mathsf{B}}$).

The pair (η, ξ) is called an isoclinism from A to B and will be denoted by $(\eta, \xi) : \mathsf{A} \sim \mathsf{B}$.

Lemma 3.10. *Let A and B be two isoclinic* AWB*'s. If A is nilpotent of class c, then so is B.*

Proof. For any $a \in \mathsf{A}$, denote $\bar{a} = a + \mathsf{Z}(\mathsf{A})$. Then for $a, a_i \in \mathsf{A}, i = 1, \ldots, n$, we have $(((\bar{a}\bar{a}_1)\bar{a}_2)\ldots\bar{a}_n) = (((aa_1)a_2)\ldots a_n) + \mathsf{Z}(\mathsf{A})$, so $a \in \mathcal{Z}_{i+1}(\mathsf{A})$ if and only if $\bar{a} \in \mathcal{Z}_i\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}\right)$. Thus $\frac{\mathcal{Z}_{i+1}(\mathsf{A})}{\mathsf{Z}(\mathsf{A})} = \mathcal{Z}_i\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}\right)$. If (η, ξ) denotes the isoclinism between A and B, then we have

$$\eta\left(\frac{\mathcal{Z}_{i+1}(\mathsf{A})}{\mathsf{Z}(\mathsf{A})}\right) = \eta\left(\mathcal{Z}_i\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}\right)\right) = \mathcal{Z}_i\left(\frac{\mathsf{B}}{\mathsf{Z}(\mathsf{B})}\right) = \frac{\mathcal{Z}_{i+1}(\mathsf{B})}{\mathsf{Z}(\mathsf{B})}.$$

It follows that

$$\frac{\mathsf{A}}{\mathcal{Z}_{i+1}(\mathsf{A})} \cong \frac{\mathsf{A}\,/\,\mathsf{Z}(\mathsf{A})}{\mathcal{Z}_{i+1}(\mathsf{A})\,/\,\mathsf{Z}(\mathsf{A})} \cong \frac{\mathsf{B}\,/\,\mathsf{Z}(\mathsf{B})}{\mathcal{Z}_{i+1}(\mathsf{B})\,/\,\mathsf{Z}(\mathsf{B})} \cong \frac{\mathsf{B}}{\mathcal{Z}_{i+1}(\mathsf{B})}$$

Now assume that A is nilpotent of class c, that is $\mathcal{Z}_c(\mathsf{A}) = \mathsf{A}$ (see Theorem 2.3), then $\frac{\mathsf{B}}{\mathcal{Z}_c(\mathsf{B})} \cong \frac{\mathsf{A}}{\mathcal{Z}_c(\mathsf{A})} = 0$, hence $\mathsf{B} = \mathcal{Z}_c(\mathsf{B})$.

Also, $\frac{\mathsf{A}}{\mathcal{Z}_{c-1}(\mathsf{A})} \neq 0$ if and only if $\frac{\mathsf{B}}{\mathcal{Z}_{c-1}(\mathsf{B})} \neq 0$. Consequently, B is nilpotent of class c. □

Lemma 3.11. *Let $(\eta, \xi) : \mathsf{A} \sim \mathsf{B}$ be. If A is a stem AWB, then ξ maps $\mathsf{Z}(\mathsf{A})$ onto $\mathsf{Z}(\mathsf{B}) \cap [[\mathsf{B},\mathsf{B}]]$.*

Proof. As $\mathsf{Z}(\mathsf{A}) \subseteq [[\mathsf{A},\mathsf{A}]]$, then any element $z \in \mathsf{Z}(\mathsf{A})$ can be written as $z = \sum\limits_i \lambda_i x_i y_i + \mu_i[a_i, b_i], \lambda_i, \mu_i \in \mathbb{K}, x_i, y_i, a_i, b_i \in \mathsf{A}$. Let $\eta(x_i + \mathsf{Z}(\mathsf{A})) = x_i' + \mathsf{Z}(\mathsf{B}), \eta(y_i + \mathsf{Z}(\mathsf{A})) = y_i' + \mathsf{Z}(\mathsf{B}), \eta(a_i + \mathsf{Z}(\mathsf{A})) = a_i' + \mathsf{Z}(\mathsf{B}), \eta(b_i + \mathsf{Z}(\mathsf{A})) = b_i' + \mathsf{Z}(\mathsf{B})$ be, then

$$\begin{array}{rcl} \xi(z) + \mathsf{Z}(\mathsf{B}) & = & \xi\left(\sum\limits_i \lambda_i x_i y_i + \mu_i[a_i, b_i]\right) + \mathsf{Z}(\mathsf{B}) = \sum\limits_i \lambda_i \xi(x_i y_i) + \mu_i \xi[a_i, b_i] + \mathsf{Z}(\mathsf{B}) \\ & \overset{(*)}{=} & \sum\limits_i \lambda_i(x_i' y_i') + \mu_i[a_i', b_i'] + \mathsf{Z}(\mathsf{B}) \\ & = & \sum\limits_i \lambda_i((x_i' + \mathsf{Z}(\mathsf{B}))(y_i' + \mathsf{Z}(\mathsf{B}))) + \mu_i[a_i' + \mathsf{Z}(\mathsf{B}), b_i' + \mathsf{Z}(\mathsf{B})] \\ & = & \sum\limits_i \lambda_i \eta\left((x_i y_i) + \mathsf{Z}(\mathsf{A})\right) + \mu_i\left([a_i, b_i] + \mathsf{Z}(\mathsf{A})\right) \\ & = & \eta\left(\sum\limits_i \lambda_i\,(x_i y_i) + \mu_i\,([a_i, b_i]) + \mathsf{Z}(\mathsf{A})\right) = \eta(\mathsf{Z}(\mathsf{A})) = \mathsf{Z}(\mathsf{B}). \end{array}$$

hence $\xi(z) \in \mathsf{Z}(\mathsf{B})$.

(∗)

$$\begin{aligned}\xi(x_i y_i) &= \xi(x_i)\xi(y_i) = \xi\left(P_{\mathsf{A}}(x_i + \mathsf{Z}(\mathsf{A}))\xi P_{\mathsf{A}}(y_i + \mathsf{Z}(\mathsf{A}))\right)\\ &= P_{\mathsf{B}}\left(\eta \times \eta\left((x_i + \mathsf{Z}(\mathsf{A}))(y_i + \mathsf{Z}(\mathsf{A}))\right)\right)\\ &= P_{\mathsf{B}}\left(((x_i' + \mathsf{Z}(\mathsf{B}))((y_i' + \mathsf{Z}(\mathsf{B})))\right) = x_i' y_i';\end{aligned}$$

Similarly, $\xi[x_i, y_i] = [x_i', y_i']$.

Obviously, $\xi[[\mathsf{A},\mathsf{A}]] \subseteq [[\mathsf{B},\mathsf{B}]]$, therefore $\xi(\mathsf{Z}(\mathsf{A})) \subseteq \mathsf{Z}(\mathsf{B})$ and $\xi(\mathsf{Z}(\mathsf{A})) \subseteq \xi[[\mathsf{A},\mathsf{A}]] \subseteq [[\mathsf{B},\mathsf{B}]]$, so $\xi(\mathsf{Z}(\mathsf{A})) \subseteq \mathsf{Z}(\mathsf{B}) \cap [[\mathsf{B},\mathsf{B}]]$.

Finally, it is easy to check that ξ is surjective. □

Proposition 3.12. *Let* A *and* B *be two isoclinic* AWB *where* A *is a stem* AWB*. Then every* $d \in \mathsf{Der}_z(\mathsf{A})$ *induces a central derivation* d^* *of* B*. Moreover, the map* $\phi\colon \mathsf{Der}_z(\mathsf{A}) \to \mathsf{Der}_z(\mathsf{B}), d \mapsto d^*$ *is an injective Lie algebra homomorphism.*

Proof. Let $(\eta, \xi)\colon \mathsf{A} \sim \mathsf{B}$ be. For $d \in \mathsf{Der}_z(\mathsf{A}), d(a) \in \mathsf{Z}(\mathsf{A})$ for all $a \in \mathsf{A}$, then $\xi(d(a)) \in \mathsf{Z}(\mathsf{B}) \cap [[\mathsf{B},\mathsf{B}]]$ by Lemma 3.11. Define $d^*\colon \mathsf{B} \to \mathsf{B}$ by $d^*(b) = d(\xi(a))$, for all $b \in \mathsf{B}$ and $b + \mathsf{Z}(B) = \eta(a + \mathsf{Z}(\mathsf{A}))$. Now the statements of the proposition are easy to verify. □

Lemma 3.13. *An* AWB A *is a stem* AWB *if and only if* $\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] \neq 0$ *for all non-zero two-sided ideal* J *of* A*.*

Proof. Assume that A is not a stem AWB and let $a \in \mathsf{Z}(\mathsf{A})$ such that $a \notin [[\mathsf{A},\mathsf{A}]]$. Consider the two-sided ideal $\mathsf{J} := \langle\{a\}\rangle$. Then J is a non-zero two-sided ideal of A such that $\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0$.

Conversely, suppose that A is a stem AWB and let J be a non-zero two-sided ideal of A such that $\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0$. Then $\mathsf{J} \subseteq \mathsf{Z}(\mathsf{A})$, so $\mathsf{J} = \mathsf{J} \cap \mathsf{Z}(\mathsf{A}) \subseteq \mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0$. This leads to the conclusion that $\mathsf{J} = 0$, which is a contradiction. □

Lemma 3.14. *Let* A *be an* AWB *and* J *be a two-sided ideal of* A*, then* $\frac{\mathsf{A}}{\mathsf{J}} \sim \frac{\mathsf{A}}{\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]]}$*. Consequently, if* $\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0$*, then* $\mathsf{A} \sim \frac{\mathsf{A}}{\mathsf{J}}$*.*

Proof. Consider the map $\eta\colon \frac{\mathsf{A}_1}{\mathsf{Z}(\mathsf{A}_1)} \to \frac{\mathsf{A}_2}{\mathsf{Z}(\mathsf{A}_2)}$, where $\mathsf{A}_1 = \frac{\mathsf{A}}{\mathsf{J}}$ and $\mathsf{A}_2 = \frac{\mathsf{A}}{\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]]}$, defined by $\eta(\bar{a} + \mathsf{Z}(\mathsf{A}_1)) = \tilde{a} + \mathsf{Z}(\mathsf{A}_2)$ with $\bar{a} = a + \mathsf{J}$ and $\tilde{a} = a + \mathsf{J} \cap [[\mathsf{A},\mathsf{A}]]$. Then η is clearly an isomorphism. Also, the map $\xi\colon [[\mathsf{A}_1,\mathsf{A}_1]] \to [[\mathsf{A}_2,\mathsf{A}_2]]$ defined by $\xi(\bar{a}_1\bar{a}_2) = \tilde{a}_1\tilde{a}_2; \xi[\bar{a}_1,\bar{a}_2] = [\tilde{a}_1,\tilde{a}_2], a_1, a_2 \in \mathsf{A}$ is a well-defined isomorphism and diagram (3.1) is commutative. Therefore, (η, ξ) is an isoclinism. □

Proposition 3.15. *Every* AWB *is isoclinic to some stem* AWB*.*

Proof. Let A be an AWB. Consider the set

$$\mathcal{M} = \{\mathsf{J} \mid \mathsf{J} \text{ is a two-sided ideal of } \mathsf{A} \text{ satisfying } \mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0\}.$$

Obviously, $\mathsf{J} = 0$ belongs to $\mathcal{M}$, so $\mathcal{M}$ is non-empty and partially ordered by the set inclusion. By Zorn's lemma, it contains a maximal two-sided ideal, denoted by M. By Lemma 3.14, it follows that A is isoclinic to $\frac{\mathsf{A}}{\mathsf{M}}$, since $\mathsf{M} \cap [[\mathsf{A},\mathsf{A}]] = 0$.

It remains to show that $\frac{\mathsf{A}}{\mathsf{M}}$ is a stem AWB. To do so, let J be an arbitrary two-sided ideal of A containing M and satisfying $\frac{\mathsf{J}}{\mathsf{M}} \cap [[\frac{\mathsf{A}}{\mathsf{M}}, \frac{\mathsf{A}}{\mathsf{M}}]] = \mathsf{M}$. By Lemma 3.13 it is enough to show that $\mathsf{J} \subseteq \mathsf{M}$. Indeed, it is easy to show that $\mathsf{J} \cap [[\mathsf{A},\mathsf{A}]] = 0$, then $\mathsf{J} \in \mathcal{M}$, so $\mathsf{J} \subseteq \mathsf{M}$ by the maximality of M. □

Corollary 3.16. *For any arbitrary* AWB A*, the Lie algebra* $\mathsf{Der}_z(\mathsf{A})$ *has a central subalgebra* N *isomorphic to* $\mathsf{T}\left(\frac{\mathsf{B}}{[[\mathsf{B},\mathsf{B}]]}, \mathsf{Z}(\mathsf{B})\right)$ *for some stem* AWB *algebra* B *isoclinic to* A*.*

Moreover, each element of N *sends* Z(A) *to zero.*

Proof. By Proposition 3.15 there is a stem AWB B isoclinic to A. Denote this isoclinism by (η, ξ). Now, by the proof of Proposition 3.12, $\mathsf{N} := \{d^* \colon \mathsf{B} \to \mathsf{B} \mid d \in \mathsf{Der}_z(\mathsf{A})\}$ is a subalgebra of $\mathsf{Der}_z(\mathsf{B})$ isomorphic to $\mathsf{Der}_z(\mathsf{A})$.

Moreover, N is a central subalgebra of $\mathsf{Der}_z(\mathsf{B})$. Indeed, if $d_0 \in \mathsf{N}$ and $d_1 \in \mathsf{Der}_z(\mathsf{B})$, then for any $b \in \mathsf{B}$, we have $d_0^*(b) = \xi(d_0(a))$ with $b + \mathsf{Z}(\mathsf{B}) = \eta(a + \mathsf{Z}(\mathsf{A}))$, so $d_1(d_0^*(b)) = 0$ since $d_0^*(\mathsf{B}) \subseteq \mathsf{Z}(\mathsf{B}) \cap [[\mathsf{B}, \mathsf{B}]]$ due to Lemma 3.11. Also, $d_0^*(\mathsf{Z}(\mathsf{B})) = 0$ since η is one-to-one and ξ is an isomorphism. In particular, $d_0^*(d_1(b)) = 0$, since $d_1(\mathsf{B}) \subseteq \mathsf{Z}(\mathsf{B})$. Therefore, $[d_0^*, d_1] = 0$. Moreover, for any $d_0^* \in \mathsf{N}$ we have $d_0^*(\mathsf{Z}(\mathsf{B})) = 0$ as mentioned above.

To complete the proof, notice that $\mathsf{Der}_z(\mathsf{B}) \cong \mathsf{T}\left(\frac{\mathsf{B}}{[[\mathsf{B},\mathsf{B}]]}, \mathsf{Z}(\mathsf{B})\right)$ by Proposition 3.3. □

4. On inner derivations

In this section we study inner derivations of tautological AWBs and analyze their relationship with the different subalgebras of the Lie algebra of the derivations of AWBs introduced in Section 3.

Note that the right and left multiplication operators of an AWB are in general not derivations. However, if we consider tautological AWBs A (see Example 2.2 (c)), then the operators $d_x \colon \mathsf{A} \to \mathsf{A}, d_x(a) = [x, a] = xa - ax, a \in \mathsf{A}$, for a fix $x \in \mathsf{A}$, are derivations. This type of derivations is called an inner derivation of the tautological algebra with bracket A, and we denote by $\mathsf{IDer}(\mathsf{A})$ the Lie subalgebra of all inner derivations of A.

From now on we consider tautological algebras with bracket. Clearly, $\mathsf{Z}^{Ass}(\mathsf{A}) = \bigcap_{x \in \mathsf{A}} \mathsf{Ker}(d_x)$ and $[\mathsf{A}, \mathsf{A}]_{Ass} = \sum_{x \in \mathsf{A}} \mathsf{Im}(d_x)$, where $\mathsf{Z}^{Ass}(\mathsf{A}) = \{x \in \mathsf{A} \mid xy = yx, \text{ for all } y \in \mathsf{A}\} = \{x \in \mathsf{A} \mid [x, y] = 0, \text{ for all } y \in \mathsf{A}\}$ and $[\mathsf{A}, \mathsf{A}]_{Ass} = \langle \{[x, y] \mid x, y \in \mathsf{A}\} \rangle = \langle \{xy - yx \mid x, y \in \mathsf{A}\} \rangle$.

Let $\mathfrak{D}$ be a subalgebra of $\mathsf{Der}(\mathsf{A})$ such that $\mathsf{IDer}(\mathsf{A}) \subseteq \mathfrak{D}$. Define $E(\mathsf{A}) = \bigcap_{d \in \mathfrak{D}} \mathsf{Ker}(d)$. $E(\mathsf{A})$ is a two-sided ideal of A whenever $E(\mathsf{A}) \subseteq \mathsf{Z}(\mathsf{A})$. Examples of the subalgebra $\mathfrak{D}$ are $\mathsf{Der}(\mathsf{A})$ or $\mathsf{Der}_{\mathsf{Z}^{Ass}(\mathsf{A})}(\mathsf{A})$. We will denote by $\mathsf{Der}_e(\mathsf{A})$ the subalgebra $\mathsf{Der}^{E(\mathsf{A})}(\mathsf{A})$.

Proposition 4.1. *Let* A *be a tautological* AWB *such that* $E(\mathsf{A}) \subseteq \mathsf{Z}(\mathsf{A})$. *Then*

(a) $\mathsf{Der}_e(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, E(\mathsf{A})\right)$ *as abelian* AWBs *and it is an* AWB *isomorphism whenever* $E(\mathsf{A}) \subseteq [[\mathsf{A}, \mathsf{A}]]$.

(b) If $\mathfrak{D} = \mathsf{Der}(\mathsf{A})$, *then* $\mathsf{Der}_e(\mathsf{A})$ *is isomorphic to the abelian* AWB $\mathsf{T}\left(\frac{\mathsf{A}}{E(\mathsf{A})}, E(\mathsf{A})\right)$.

Proof. (a) Direct consequence of Lemma 3.2 bearing in mind that $\mathsf{Der}_e(\mathsf{A}) = \mathsf{Der}_{E(\mathsf{A}),[[\mathsf{A},\mathsf{A}]]}(\mathsf{A})$, since every $d \in \mathsf{Der}_e(\mathsf{A})$ satisfies that $d(a) \in E(\mathsf{A}) \subseteq \mathsf{Z}(\mathsf{A})$, that is, d is a central derivation, therefore it vanishes over $[[\mathsf{A}, \mathsf{A}]]$.

(b) By Lemma 3.2, $\mathsf{T}\left(\frac{\mathsf{A}}{E(\mathsf{A})}, E(\mathsf{A})\right) \cong \mathsf{Der}_{E(\mathsf{A}),E(\mathsf{A})}(\mathsf{A}) = \mathsf{Der}_e(\mathsf{A})$. □

Example 4.2. *Consider the three-dimensional complex associative algebra* As_3^2 *with basis* $\{e_1, e_2, e_3\}$ *and operation given by* $e_1e_3 = e_2, e_3e_1 = \alpha e_2, \alpha \in \mathbb{C} \setminus \{1\}$, *and zero elsewhere [7]. Let* A *be the tautological* AWB *associated to* As_3^2 *and* $\mathfrak{D} = \mathsf{IDer}(\mathsf{A})$. *Then* A *satisfies the conditions in Proposition 4.1 since* $E(\mathsf{A}) = \mathsf{Z}(\mathsf{A}) = \langle \{e_2\} \rangle$.

Lemma 4.3. *Let* M *and* N *be two two-sided ideals of a tautological* AWB A *such that* $\mathsf{Z}^{Ass}\left(\frac{\mathsf{A}}{\mathsf{M}}\right) = \frac{\mathsf{H}}{\mathsf{M}}$ *for some* H. *Then*

$$\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cap \mathsf{IDer}(\mathsf{A}) \cong \frac{\mathsf{H} \cap \mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N})}{\mathsf{Z}^{Ass}(\mathsf{A})}$$

as abelian AWBs, *where* $\mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) = \{a \in \mathsf{A} \mid [a, n] = 0 \text{ for all } n \in \mathsf{N}\}$. *In particular, if* $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \subseteq \mathsf{IDer}(\mathsf{A})$, *then*

$$\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cong \frac{\mathsf{H} \cap \mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N})}{\mathsf{Z}^{Ass}(\mathsf{A})}$$

as abelian AWBs.

Proof. Define the surjective linear map $\varphi\colon \mathsf{H} \cap \mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) \to \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cap \mathsf{IDer}(\mathsf{A})$ by $\varphi(x) = d_x, d_x(y) = [x, y]$, for all $y \in \mathsf{A}$. Since $\mathsf{Ker}(\varphi) = \mathsf{Z}^{Ass}(\mathsf{A})$, then the isomorphism is obvious. □

Theorem 4.4. *Let* M *and* N *be two two-sided ideals of a tautological* AWB A *such that* $\mathsf{Z}^{Ass}\left(\frac{\mathsf{A}}{\mathsf{M}}\right) = \frac{\mathsf{H}}{\mathsf{M}}$ *for some* H*, then*

(a) $\mathsf{IDer}(\mathsf{A}) \subseteq \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ *if and only if* $\mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) = \mathsf{A} = \mathsf{H}$.

(b) If $\mathsf{M} \subseteq \mathsf{N} \cap \mathsf{Z}(\mathsf{A})$ *and* $\dim\left(\frac{A}{\mathsf{Z}^{Ass}(\mathsf{A})}\right) < \infty$*, then* $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ *if and only if* $\mathsf{N} \subseteq \mathsf{Z}^{Ass}(\mathsf{A}), [\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$ *and* $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right) \cong \frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}$ *as abelian* AWBs*.*

Proof. (a) Suppose that $\mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) = \mathsf{A} = \mathsf{H}$. Let $d_x \in \mathsf{IDer}(\mathsf{A})$, for any $x \in \mathsf{A}$, then $d_x(y) = [x, y] \in [\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$, hence $\mathsf{Im}(d_x) \subseteq \mathsf{M}$. On the other hand, every d_x vanishes on N. Therefore, $d_x \in \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$.

Conversely, for any $x \in \mathsf{A}, d_x \in \mathsf{IDer}(\mathsf{A}) \subseteq \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$, which implies that $[\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$, therefore $\frac{\mathsf{A}}{\mathsf{M}} \subseteq \mathsf{Z}^{Ass}\left(\frac{\mathsf{A}}{\mathsf{M}}\right) = \frac{\mathsf{H}}{\mathsf{M}}$, so $\mathsf{A} \subseteq \mathsf{H}$, and thus $\mathsf{A} = \mathsf{H}$. On the other hand, $[x, n] = d_x(n) = 0$ for all $x \in \mathsf{A}, n \in \mathsf{N}$, that is $x \in \mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N})$, hence $\mathsf{A} \subseteq \mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N})$.

(b) Suppose that $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$. By statement (a), $\mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) = \mathsf{A} = \mathsf{H}$. By Lemma 3.2 we have $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right)$. Moreover, the kernel of the surjective linear map $\varphi\colon \mathsf{A} \twoheadrightarrow \mathsf{IDer}(\mathsf{A}), \varphi(x) = d_x$, is $\mathsf{Z}^{Ass}(\mathsf{A})$, hence $\frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})} \cong \mathsf{IDer}(\mathsf{A})$ as abelian AWBs.

Since $[\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M} \subseteq \mathsf{N}$ and $\mathsf{N} \subseteq \mathsf{Z}^{Ass}(\mathsf{A})$, then we have

$$\frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})} \cong \mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right).$$

Conversely, $\mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right) \cong \frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})} \cong \mathsf{IDer}(\mathsf{A})$.

Since $\mathsf{N} \subseteq \mathsf{Z}^{Ass}(\mathsf{A})$, then $\mathsf{C}_{\mathsf{A}}^{Ass}(\mathsf{N}) = \mathsf{A}$ and statement (a) implies $\mathsf{IDer}(\mathsf{A}) \subseteq \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$. Now the finite dimension of $\frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}$ implies the equality. □

Corollary 4.5. *Let* M *and* N *be two two-sided ideals of a finite-dimensional tautological* AWB A *such that* $\mathsf{M} \subseteq \mathsf{Z}(\mathsf{A}) \subseteq \mathsf{Z}^{Ass}(\mathsf{A}) \subseteq \mathsf{N}$*. Then* $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$ *if and only if* $\mathsf{N} = \mathsf{Z}^{Ass}(\mathsf{A}), [\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$ *and* $\dim(\mathsf{M}) = 1$.

Proof. If $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{M},\mathsf{N}}(\mathsf{A})$, then $\mathsf{N} \subseteq \mathsf{Z}^{Ass}(\mathsf{A}), [\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$ and $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right) \cong \frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}$ by Theorem 4.4 (b). Thus $\mathsf{N} = \mathsf{Z}^{Ass}(\mathsf{A}), [\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{M}$ and $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right) \cong \frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}$, which implies $\dim(\mathsf{M}) = 1$.

Conversely, $\dim(\mathsf{M}) = 1$ implies $\dim\left(\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right)\right) = \dim\left(\frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}\right)$, hence $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{N}}, \mathsf{M}\right) \cong \frac{\mathsf{A}}{\mathsf{Z}^{Ass}(\mathsf{A})}$ as abelian AWBs, now Theorem 4.4 (b) concludes the proof. □

Corollary 4.6. *Let* A *be a finite-dimensional tautological* AWB *such that* $\mathsf{Z}(\mathsf{A}) = \mathsf{Z}^{Ass}(\mathsf{A})$*. Then* $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),\mathsf{Z}(\mathsf{A})}(\mathsf{A})$ *if and only if* $[\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{Z}(\mathsf{A})$ *and* $\dim(\mathsf{Z}(\mathsf{A})) = 1$.

Proof. Take $\mathsf{M} = \mathsf{Z}(\mathsf{A}), \mathsf{N} = \mathsf{Z}^{Ass}(\mathsf{A})$ in Corollary 4.5. □

Corollary 4.7. *Let* A *be a finite-dimensional tautological* AWB *such that* $\mathsf{Z}^{Ass}(\mathsf{A}) = [[\mathsf{A}, \mathsf{A}]]$*. Then* $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_z(\mathsf{A})$ *if and only if* $[\mathsf{A}, \mathsf{A}]_{Ass} = \mathsf{Z}(\mathsf{A})$ *and* $\dim(\mathsf{Z}(\mathsf{A})) = 1$.

Proof. Consider $\mathsf{M} = \mathsf{Z}(\mathsf{A})$ and $\mathsf{N} = [[\mathsf{A}, \mathsf{A}]]$ in Corollary 4.5, then $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),[[\mathsf{A},\mathsf{A}]]}(\mathsf{A})$. Now Corollary 4.5 implies that $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_z(\mathsf{A})$.

Conversely, if $\mathsf{IDer}(\mathsf{A}) = \mathsf{Der}_z(\mathsf{A})$, then $[\mathsf{A}, \mathsf{A}]_{Ass} \subseteq \mathsf{Z}(\mathsf{A})$. Hence, $[[\mathsf{A}, \mathsf{A}]] = [\mathsf{A}, \mathsf{A}]_{Ass} = \mathsf{Z}(\mathsf{A}) = \mathsf{Z}^{Ass}(\mathsf{A})$. Now Corollary 4.5 implies $\dim(\mathsf{Z}(\mathsf{A})) = 1$. □

Example 4.8. *Example 4.2 satisfies the requirements of Corollaries 4.5, 4.6 and 4.7.*

Consider the two-dimensional complex associative algebra As_2^4 *with basis* $\{e_1, e_2\}$ *and operation given by* $e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2$ *and zero elsewhere [7]. Then the tautological* AWB A *associated to* As_2^4 *satisfies the requirements of Corollaries 4.5 and 4.7, but no those of Corollary 4.6, since* $[[\mathsf{A},\mathsf{A}]] = \mathsf{Z}^{Ass}(\mathsf{A}) = \mathsf{A}$ *and* $\mathsf{Z}(\mathsf{A}) = 0$.

It is an easy task to check that, for tautological AWB, the central derivations commute with the inner derivations. Furthermore, the following facts hold for any tautological AWB A:

$$\mathsf{Z}\left(\mathsf{IDer}(\mathsf{A})\right) \subseteq \mathsf{Der}_z(\mathsf{A}) = \mathsf{C}^{Ass}_{\mathsf{Der}(\mathsf{A})}(\mathsf{IDer}(\mathsf{A})). \tag{4.1}$$

Next, we characterize tautological algebras with bracket for which the inclusion in (4.1) is actually an equality.

Theorem 4.9. *Let* A *be a tautological* AWB*. Then the following statements hold:*

(a) $\mathsf{Z}(\mathsf{IDer}(\mathsf{A})) \subseteq \mathsf{C}^*(\mathsf{A})$.
(b) $\mathsf{Z}(\mathsf{IDer}(\mathsf{A})) = \mathsf{C}^*(\mathsf{A})$ *if and only if* $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right) \cong \mathsf{Z}(\mathsf{IDer}(\mathsf{A}))$.
(c) If A *is nilpotent of class 2, then* $\mathsf{IDer}(\mathsf{A}) \subseteq \mathsf{C}^*(\mathsf{A})$.

Proof. (a) Let $d_a \in \mathsf{Z}(\mathsf{IDer}(\mathsf{A}))$. By (4.1), $\mathsf{Im}(d_a) \subseteq \mathsf{Z}(\mathsf{A})$.

On the other hand, if $a \in \mathsf{Z}(\mathsf{A})$, then $d_a(b) = [a, b] = 0$, for all $b \in \mathsf{A}$, i.e. $d_a(\mathsf{Z}(\mathsf{A})) = 0$.

(b) By Lemma 3.2, $\mathsf{C}^*(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right)$.

(c) Since A is nilpotent of class 2, then $[[\mathsf{A},\mathsf{A}]] \subseteq \mathsf{Z}(\mathsf{A})$, which implies that every inner derivation d_a belongs to $\mathsf{Z}(\mathsf{IDer}(\mathsf{A}))$. Now statement (a) concludes the proof. □

Theorem 4.10. *Let* A *be a finite-dimensional tautological* AWB *such that* $\mathsf{Z}(\mathsf{A}) \neq 0$*. If* $\mathsf{Z}\left(\mathsf{IDer}(\mathsf{A})\right) = \mathsf{Der}_z(\mathsf{A})$*, then* $\mathsf{Z}(\mathsf{A}) = [[\mathsf{A},\mathsf{A}]]$.

Proof. If $\mathsf{Z}\left(\mathsf{IDer}(\mathsf{A})\right) = \mathsf{Der}_z(\mathsf{A})$, then $\mathsf{Der}_z(\mathsf{A}) = \mathsf{Der}_{\mathsf{Z}(\mathsf{A}),\mathsf{Z}(\mathsf{A})}(\mathsf{A})$. Lemma 3.2 provides the isomorphism of abelian AWBs $\mathsf{Der}_{\mathsf{Z}(\mathsf{A}),\mathsf{Z}(\mathsf{A})}(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, \mathsf{Z}(\mathsf{A})\right)$ and Proposition 3.3 provides the isomorphism of abelian AWBs $\mathsf{Der}_z(\mathsf{A}) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$, hence $\mathsf{Z}(\mathsf{A}) = [[\mathsf{A},\mathsf{A}]]$. □

Theorem 4.11. *Let* A *be a nilpotent* AWB *of class 2. Then* $\mathsf{Der}_z(\mathsf{A})$ *has a central subalgebra isomorphic to* $\mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A}))}, [[\mathsf{A},\mathsf{A}]]\right)$.

Moreover, if A *is a tautological* AWB*, then* $\mathsf{IDer}(\mathsf{A}) \subseteq \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A}))}, [[\mathsf{A},\mathsf{A}]]\right)$.

Proof. By Proposition 3.15 there is a stem AWB B isoclinic to A. Since A is nilpotent of class 2, then so is B by Lemma 3.10. Then $\mathsf{Z}(\mathsf{B}) = [[\mathsf{B},\mathsf{B}]] \cong [[\mathsf{A},\mathsf{A}]]$ and $\frac{\mathsf{B}}{[[\mathsf{B},\mathsf{B}]]} \cong \frac{\mathsf{B}}{\mathsf{Z}(\mathsf{B})} \cong \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}$. So $\mathsf{T}\left(\frac{\mathsf{B}}{[[\mathsf{B},\mathsf{B}]]}, \mathsf{Z}(\mathsf{B})\right) \cong \mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$. Therefore $\mathsf{Der}_z(\mathsf{A})$ has a central subalgebra N isomorphic to $\mathsf{T}\left(\frac{\mathsf{A}}{[[\mathsf{A},\mathsf{A}]]}, \mathsf{Z}(\mathsf{A})\right)$ due to Corollary 3.16.

Now assume that A is a tautological AWB. The map $\zeta\colon \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \to \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, [[\mathsf{A},\mathsf{A}]]\right)$ given by $\zeta(a+\mathsf{Z}(\mathsf{A}))\colon \frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})} \to [[\mathsf{A},\mathsf{A}]]$, $\zeta(a + \mathsf{Z}(\mathsf{A}))(a' + \mathsf{Z}(\mathsf{A})) = [a, a'] = aa' - a'a, a, a' \in \mathsf{A}$, is a well-defined one-to-one linear map. Finally, $\mathsf{IDer}(\mathsf{A}) = \mathsf{Im}(\zeta) \subseteq \mathsf{T}\left(\frac{\mathsf{A}}{\mathsf{Z}(\mathsf{A})}, [[\mathsf{A},\mathsf{A}]]\right)$ since $\zeta(a + \mathsf{Z}(\mathsf{A}))(a' + \mathsf{Z}(\mathsf{A})) = [a, a'] = d_a(a')$. □

5. Exact sequences associated with an abelian extensions

In this section we will construct exact sequences similar to the Wells-type exact sequence associated with an abelian extension of AWB (see [4, Theorem 3.9]) in which the subalgebras of derivations of AWB play a central role.

Let $E\colon 0 \longrightarrow \mathsf{M} \stackrel{i}{\longrightarrow} \mathsf{E} \stackrel{p}{\longrightarrow} \mathsf{A} \longrightarrow 0$ be an abelian extension of AWB with a linear section $s : \mathsf{A} \to \mathsf{E}$. Then every element in E can be written as $i(m)+s(a)$, for some $a \in \mathsf{A}, m \in \mathsf{M}$. Moreover, any $d \in \mathsf{Der}(\mathsf{E} : \mathsf{M})$ induces a derivation $\delta = d_{|\mathsf{M}} \in \mathsf{Der}(\mathsf{M})$ and a derivation $\partial \in \mathsf{Der}(\mathsf{A})$ defined by $\partial(a) = p(d(s(a))), a \in \mathsf{A}$. Therefore, the map $\omega\colon \mathsf{Der}(\mathsf{E} : \mathsf{M}) \to \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$ given by $\omega(d) = (\delta, \partial)$ is a Lie algebra homomorphism (see [4, Subsection 3.2]).

Lemma 5.1. *Let* $E\colon 0 \longrightarrow \mathsf{M} \xrightarrow{i} \mathsf{E} \xrightarrow{p} \mathsf{A} \longrightarrow 0$ *be an abelian extension of* AWB *and* $s\colon \mathsf{A} \to \mathsf{E}$ *a linear section of* p. *If* $d \in \mathsf{Der}(\mathsf{E} : \mathsf{M})$, *then there exists* $(\delta, \partial, \chi) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \oplus \mathsf{Hom}(\mathsf{A}, \mathsf{M})$ *such that:*

1.
   (a) $f(\partial(a_1), a_2) + f(a_1, \partial(a_2)) - \delta(f(a_1, a_2)) = \chi(a_1a_2) - \chi(a_1)s(a_2) - s(a_1)\chi(a_2)$;
   (b) $g(\partial(a_1)(a_2)) + g(a_1)(\partial(a_2)) - \delta(g(a_1)(a_2)) = \chi[a_1, a_2] - [\chi(a_1), s(a_2)] - [s(a_1), \chi(a_2)]$;
2.

$$\begin{aligned}\delta(s(a)m) &= s(\partial(a))m + s(a)\delta(m);\\ \delta(ms(a)) &= \delta(m)s(a) + ms(\partial(a));\\ \delta[s(a), m] &= [s(\partial(a)), m] + [s(a), \delta(m)];\\ \delta[m, s(a)] &= [\delta(m), s(a)] + [m, s(\partial(a))],\end{aligned}$$

for all $a, a_1, a_2 \in \mathsf{A}, m \in \mathsf{M}$ and $(f, g) \in K^0(\mathsf{A}, \mathsf{M})$ is the 0-cocycle given by (2.8).

Conversely, if $(\delta, \partial, \chi) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \oplus \mathsf{Hom}(\mathsf{A}, \mathsf{M})$ satisfying 1. and 2., then $d\colon \mathsf{E} \to \mathsf{E}$ given by $d(e) = d(i(m) + s(a)) = s(\partial(a)) + \chi(a) + \delta(m)$ belongs to $\mathsf{Der}(\mathsf{E} : \mathsf{M})$.

Proof. Given $d \in \mathsf{Der}(\mathsf{E} : \mathsf{M})$, then the homomorphism ω provides the pair $(\delta, \partial) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$. On the other hand, for any $a \in \mathsf{A}$, $\partial(a) = pds(a)$, hence $ds(a) = s\partial(a) + m$, for some $m \in \mathsf{M}$, so we have defined a linear map $\chi\colon \mathsf{A} \to \mathsf{M}$ given by $\partial(a) = ds(a) - s\partial(a)$.

Applying d to the equalities (2.8) we obtain:

$$s(\partial(a_1, a_2)) + \chi(a_1a_2) + \delta(f(a_1, a_2)) = d(s(a_1))s(a_2) + s(a_1)d(s(a_2)),$$

and

$$\delta(g(a_1)(a_2)) + \chi[a_1, a_2] + s(\partial[a_1, a_2]) = [d(s(a_1)), s(a_2)] + [s(a_1), d(s(a_2))].$$

From these equations, the equalities 1. (a) and (b) are easily derived.

To obtain 2. just reproduce a calculation like the following with each of the operations:

$$\delta(s(a)m) = d(s(a)m) = (s(\partial(a)) + \chi(a))m + s(a)\delta(m) = s(\partial(a))m + s(a)\delta(m).$$

Conversely, let $(\delta, \partial, \chi) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \oplus \mathsf{Hom}(\mathsf{A}, \mathsf{M})$ satisfying 1. and 2.

For any $e_i = s(a_i) + m_i; p(e_i) = a_i, a_i \in \mathsf{A}, m_i \in \mathsf{M}, i = 1, 2$, it is easy to check that $d(e_1e_2) = d(e_1)e_2 + e_1d(e_2)$ and $d[e_1, e_2] = [d(e_1), e_2] + [e_1, d(e_2)]$, simply by using the identities (2.8), 1. and 2. So $d \in \mathsf{Der}(\mathsf{E})$. Finally, $d(m) = \delta(m) \in \mathsf{M}$, so $d \in \mathsf{Der}(\mathsf{E} : \mathsf{M})$ □

Definition 5.2. *Let* M *and* A *be* AWBs *with an action of* A *over* M. *A pair* $(\delta, \partial) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$ *is said to be compatible if the following identities hold for any* $a \in \mathsf{A}, m \in \mathsf{M}$*:*

$$\begin{aligned}\partial(a) \cdot m &= \delta(a \cdot m) - a \cdot \delta(m),\\ m \cdot \partial(a) &= \delta(m \cdot a) - \delta(m) \cdot a,\\ \{\partial(a), m\} &= \delta\{a, m\} - \{a, \delta(m)\},\\ \{m, \partial(a)\} &= \delta\{m, a\} - \{\delta(m), a\}.\end{aligned} \tag{5.1}$$

Example 5.3.

(a) *If* A *acts trivially on* M, *then any pair* $(\delta, \partial) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$ *is compatible. An example of trivial action is given by the induced action provided by a split central extension (*$\mathsf{M} \subseteq \mathsf{Z}(\mathsf{E})$*) (see Example 2.5 (b)).*

(b) *The pair* (δ, ∂) *constructed in Lemma 5.1 from* $d \in \mathsf{Der}(\mathsf{E} : \mathsf{M})$ *is compatible.*

Let $\mathsf{C}_\alpha = \{(\delta, \partial) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \mid (\delta, \partial) \text{ is a compatible pair}\}$ be the subalgebra of compatible pairs in $\mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$. Let be

$$\mathsf{C}_1 = \{\delta \in \mathsf{Der}(\mathsf{M}) \mid (\delta, 0) \in \mathsf{C}_\alpha\}$$

$$\mathsf{C}_2 = \{\partial \in \mathsf{Der}(\mathsf{A}) \mid (0, \partial) \in \mathsf{C}_\alpha\}$$

For $\delta \in \mathsf{C}_1, \partial \in \mathsf{C}_2$, define for any $a_i \in \mathsf{A}$ the maps $k^i_\delta, k^i_\partial, k^i_{\delta,\partial} \colon \mathsf{A} \oplus \mathsf{A} \to \mathsf{M}, i = 1,2$, by

$$\begin{array}{l} k^1_\delta(a_1,a_2) = \delta(f(a_1,a_2));\\ k^2_\delta(a_1,a_2) = \delta(g(a_1)(a_2));\\ k^1_\partial(a_1,a_2) = f(\partial(a_1),a_2) + f(a_1,\partial(a_2));\\ k^2_\delta(a_1,a_2) = g(\partial(a_1))(a_2) + g(a_1)(\partial(a_2));\\ k^1_{\delta,\partial} = f(\partial(a_1),a_2) + f(a_1,\partial(a_2)) - \delta(f(a_1,a_2));\\ k^2_{\delta,\partial} = g(\partial(a_1))(a_2) + g(a_1)(\partial(a_2)) - \delta(g(a_1)(a_2)), \end{array} \tag{5.2}$$

where $(f,g) \in K^0(\mathsf{A},\mathsf{M})$ is the 0-cocycle given by (2.8).

Lemma 5.4. *The pairs $(k^1_\delta, k^2_\delta), (k^1_\partial, k^2_\partial), (k^1_{\delta,\partial}, k^2_{\delta,\partial})$ are 0-cocycles.*

Proof. A simple but tedious computation allows to check the 0-cocycle conditions:

$$a_0 \cdot k^1_\delta(a_1,a_2) - k^1_\delta(a_0a_1,a_2) + k^1_\delta(a_0,a_1a_2) - k^1_\delta(a_0,a_1)\cdot a_2 = 0$$

$$a_0 \cdot k^2_\delta(a_1,a_2) + k^2_\delta(a_0,a_2)\cdot a_1 - k^2_\delta(a_0a_1,a_2) - \{k^1_\delta(a_0,a_1),a_2\} + k^1_\delta([a_0,a_2],a_1) + k^1_\delta(a_0,[a_1,a_2]) = 0$$

for all $a_0,a_1,a_2 \in \mathsf{A}$. Similarly for $(k^1_\partial, k^2_\partial), (k^1_{\delta,\partial}, k^2_{\delta,\partial})$. □

Next, we define the linear maps:

$$\begin{array}{rcl} \lambda_1 \colon \mathsf{C}_1 & \to & H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}); \\ \delta & \mapsto & [(k^1_\delta, k^2_\delta)]; \end{array} \quad \begin{array}{rcl} \lambda_2 \colon \mathsf{C}_2 & \to & H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}); \\ \partial & \mapsto & [(k^1_\partial, k^2_\partial)]; \end{array} \quad \begin{array}{rcl} \lambda_\alpha \colon \mathsf{C}_\alpha & \to & H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M}) \\ (\delta,\partial) & \mapsto & [(k^1_{(\delta,\partial)}, k^2_{(\delta,\partial)})] \end{array}$$

where $[(k^1_\delta, k^2_\delta)]$, $[(k^1_\partial k^2_\partial)]$ and $[(k^1_{(\delta,\partial)}, k^2_{(\delta,\partial)})]$ denote the corresponding cohomology cosets.

Now, we check that the definitions are consistent, that is, they do not depend on the linear section of $p \colon \mathsf{E} \to \mathsf{A}$. To do so, consider two linear sections $s, s' \colon \mathsf{A} \to \mathsf{E}$. Then there exist $f, f' \colon \mathsf{A}^{\otimes 2} \to \mathsf{M}$ and $g, g' \colon \mathsf{A} \to \mathsf{Hom}(\mathsf{A},\mathsf{M})$ such that (f,g) and (f',g') satisfy the corresponding identities (2.8) relative to s and s'.

Since for any $a \in \mathsf{A}$, $ps(a) = ps'(a)$, then there exists some $m \in \mathsf{M}$, denoted by $k(a)$, such that $s(a) - s'(a) = k(a)$. Thus, we have a linear map $k \colon \mathsf{A} \to \mathsf{M}, k(a) = s(a) - s'(a), a \in \mathsf{A}$, which gives:

$$\begin{array}{rcl} f(a_1,a_2) - f'(a_1,a_2) & = & s(a_1)s(a_2) - s(a_1a_2) - s'(a_1)s'(a_2) + s'(a_1a_2)\\ & = & s(a_1)s(a_2) - ((s(a_1) - k(a_1))(s(a_2) - k(a_2))) - k(a_1a_2)\\ & = & s(a_1)k(a_2) + k(a_1)s(a_2) - k(a_1a_2)\\ & = & a_1 \cdot k(a_2) + k(a_1)\cdot a_2 - k(a_1a_2). \end{array}$$

$$\begin{array}{rcl} g(a_1)(a_2) - g'(a_1)(a_2) & = & [s(a_1),s(a_2)] - s[a_1,a_2] - [s'(a_1),s'(a_2)] + s'[a_1,a_2]\\ & = & [s(a_1),s(a_2)] - [s(a_1) - k(a_1), s(a_2) - k(a_2)] - k[a_1,a_2]\\ & = & [s(a_1),k(a_2)] + [k(a_1),s(a_2)] - k[a_1,a_2]\\ & = & \{a_1,k(a_2)\} + \{k(a_1),a_2\} - k[a_1,a_2]. \end{array}$$

Since $\delta \in \mathsf{C}_1$, then we have:

$$\begin{array}{rcl} \delta(f(a_1,a_2) - f'(a_1,a_2)) & = & \delta(a_1 \cdot k(a_2) + k(a_1)\cdot a_2 - k(a_1a_2))\\ & = & a_1 \cdot (\delta k)(a_2) + (\delta k)(a_1)\cdot a_2 - (\delta k)(a_1a_2). \end{array}$$

$$\begin{array}{rcl} \delta(g(a_1)(a_2) - g'(a_1)(a_2)) & = & \delta(\{a_1,k(a_2)\} + \{k(a_1),a_2\} - k[a_1,a_2])\\ & = & \{a_1,(\delta k)(a_2)\} + \{(\delta k)(a_1),a_2\} - (\delta k)[a_1,a_2] \end{array}$$

therefore

$$(k^1_\delta, k^2_\delta) = \delta(f,g) = \delta(f',g') + \delta(\partial^{-1}(k))) = \delta(f',g') + \partial^{-1}(\delta(k))$$

thus $\lambda_1(\delta) = [(k^1_\delta, k^2_\delta)] = [\delta(f,g)] = [\delta(f',g')]$.

The proof for λ_2 is similar to the previous one. It is simply necessary to use that $\partial \in \mathsf{C}_2$ and that it is a derivation. Finally, λ_α is well-defined, since $\lambda_\alpha = \lambda_2 - \lambda_1$.

Theorem 5.5. *Let* $E\colon 0 \longrightarrow \mathsf{M} \xrightarrow{i} \mathsf{E} \xrightarrow{p} \mathsf{A} \longrightarrow 0$ *be an abelian extension of* AWBs *with a linear section* $s\colon \mathsf{A} \to \mathsf{E}$. *Then there exist the following exact sequences:*

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}^{\mathsf{M}}(\mathsf{E}) \xrightarrow{\tau_1} \mathsf{C}_1 \xrightarrow{\lambda_1} H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$$
$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}_{\mathsf{M}}(\mathsf{E}) \xrightarrow{\tau_2} \mathsf{C}_2 \xrightarrow{\lambda_2} H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$$
$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}(\mathsf{E} : \mathsf{M}) \xrightarrow{\tau} \mathsf{C}_\alpha \xrightarrow{\lambda_\alpha} H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$$

Proof. First sequence: $\iota\colon \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \longrightarrow \mathsf{Der}^{\mathsf{M}}(\mathsf{E})$ is the inclusion; $\tau_1\colon \mathsf{Der}^{\mathsf{M}}(\mathsf{E}) \longrightarrow \mathsf{C}_1$ is given by $\tau_1(d) = (d_{|\mathsf{M}}, 0)$. An easy computation shows that $\mathsf{Ker}(\tau_1) = \mathsf{Im}(\iota)$.

$\mathsf{Im}(\tau_1) \subseteq \mathsf{Ker}(\lambda_1)$ since $\lambda_1\tau_1(d) = [(k^1_{d_{|\mathsf{M}}}, k^2_{d_{|\mathsf{M}}})]$ and

$k^1_{d_{|\mathsf{M}}} = d_{|\mathsf{M}}(f(a_1,a_2)) = -\chi(a_1a_2) + \chi(a_1)\cdot a_2 + a_1 \cdot \chi(a_2)$ by Lemma 5.1 1.(a).

$k^2_{d_{|\mathsf{M}}} = d_{|\mathsf{M}}(g(a_1)(a_2)) = -\chi[a_1,a_2] + \{\chi(a_1),a_2\} + \{a_1,\chi(a_2)\}$ by Lemma 5.1 1.(b).

Therefore $(k^1_{d_{|\mathsf{M}}}, k^2_{d_{|\mathsf{M}}})$ is a coboundary, so $\lambda_1\tau_1(d) = \bar{0}$.

Now we prove that $\mathsf{Ker}(\lambda_1) \subseteq \mathsf{Im}(\tau_1)$. Let $(d,0) \in \mathsf{Ker}(\lambda_1)$, then (k^1_d, k^2_d) is a coboundary. Therefore, there exists a linear map $\chi\colon \mathsf{A} \to \mathsf{M}$ such that

$d(f(a_1,a_2)) = \chi(a_1a_2) - \chi(a_1)\cdot a_2 - a_1\cdot\chi(a_2)$

$d(g(a_1)(a_2)) = \chi[a_1,a_2] - \{\chi(a_1),a_2\} - \{a_1,\chi(a_2)\}$

Since $(d,0) \in \mathsf{C}_1$ and by the converse of Lemma 5.1, $d(e) = d(i(m)+s(a)) \in \mathsf{M}$, then $d \in \mathsf{Der}^{\mathsf{M}}(\mathsf{E})$ and $\tau_1(d) = (d_{|\mathsf{M}}, 0) = (d,0)$, i.e. $(d,0) \in \mathsf{Im}(\tau_1)$.

The exactness of the other two sequences is proved with parallel arguments to the first sequence. □

Corollary 5.6. *Let* $E\colon 0 \longrightarrow \mathsf{M} \xrightarrow{i} \mathsf{E} \xrightarrow{p} \mathsf{A} \longrightarrow 0$ *be a central extension of algebras with bracket with a linear section* $s\colon \mathsf{A} \to \mathsf{E}$. *Then the following sequence is exact*

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}(\mathsf{E} : \mathsf{M}) \xrightarrow{\tau} \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \xrightarrow{\lambda} H^1_{\mathsf{AWB}}(\mathsf{A},\mathsf{M})$$

Proof. In case of central extensions, induced actions are trivial, hence $\mathsf{C}_\alpha \cong \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A})$ in the third sequence of Theorem 5.5. □

Let $E\colon 0 \longrightarrow \mathsf{M} \xrightarrow{i} \mathsf{E} \xrightarrow{p} \mathsf{A} \longrightarrow 0$ be an abelian extension of algebras with bracket with a linear section $s\colon \mathsf{A} \to \mathsf{E}$. Let be $\mathsf{C}^*_1 = \{\delta \in \mathsf{C}_1 \mid \lambda_1(\delta) = 0\}$, $\mathsf{C}^*_2 = \{\partial \in \mathsf{C}_2 \mid \lambda_2(\partial) = 0\}$ and $\mathsf{C}^*_\alpha = \{(\delta,\partial) \in \mathsf{C}_\alpha \mid \lambda_\alpha(\delta,\partial) = 0\}$. Then it follows from Theorem 5.5 that the following sequences are exact:

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}^{\mathsf{M}}(\mathsf{E}) \xrightarrow{\tau_1} \mathsf{C}^*_1 \longrightarrow 0 \tag{5.3}$$

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}_{\mathsf{M}}(\mathsf{E}) \xrightarrow{\tau_2} \mathsf{C}^*_2 \longrightarrow 0 \tag{5.4}$$

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}(\mathsf{E} : \mathsf{M}) \xrightarrow{\tau} \mathsf{C}^*_\alpha \longrightarrow 0 \tag{5.5}$$

If $E\colon 0 \longrightarrow \mathsf{M} \xrightarrow{i} \mathsf{E} \xrightarrow{p} \mathsf{A} \longrightarrow 0$ is a central extension of AWBs with a linear section $s\colon \mathsf{A} \to \mathsf{E}$ and $\mathsf{C}^* = \{(\delta,\partial) \in \mathsf{Der}(\mathsf{M}) \oplus \mathsf{Der}(\mathsf{A}) \mid \lambda(\delta,\partial) = 0\}$, then Corollary 5.6 implies that the following sequence is exact

$$0 \longrightarrow \mathsf{Der}_{\mathsf{M},\mathsf{M}}(\mathsf{E}) \xrightarrow{\iota} \mathsf{Der}(\mathsf{E} : \mathsf{M}) \xrightarrow{\tau} \mathsf{C}^* \longrightarrow 0 \tag{5.6}$$

Theorem 5.7. *Let* $E\colon 0 \longrightarrow \mathsf{M} \stackrel{i}{\longrightarrow} \mathsf{E} \stackrel{p}{\longrightarrow} \mathsf{A} \longrightarrow 0$ *be a split abelian extension of* AWBs*. Then sequences (5.3), (5.4) and (5.5) are also split.*

Furthermore, if E is a central extension, then sequence (5.6) is also split.

Proof. We only prove that the sequence (5.5) is split. The other cases are similar.

Consider the semi-direct product $\mathsf{M} \rtimes \mathsf{A}$ given in [1, Definition 2.5], then the split extension $E\colon 0 \longrightarrow \mathsf{M} \stackrel{i}{\longrightarrow} \mathsf{E} \stackrel{p}{\longrightarrow} \mathsf{A} \longrightarrow 0$ is equivalent to the trivial extension $0 \longrightarrow \mathsf{M} \stackrel{\kappa}{\longrightarrow} \mathsf{M} \rtimes \mathsf{A} \stackrel{\pi}{\longrightarrow} \mathsf{A} \longrightarrow 0$, hence $\mathsf{Der}(\mathsf{M} \rtimes \mathsf{A} : \mathsf{M}) \cong \mathsf{Der}(\mathsf{E} : \mathsf{M})$.

Note that for a split extension, the corresponding 0-cocycle $(f, g) \in K^0(\mathsf{A}, \mathsf{M})$ is trivial, hence $\mathsf{C}_\alpha^* \cong \mathsf{C}_\alpha$. Now we define $\beta\colon \mathsf{C}_\alpha^* \to \mathsf{Der}(\mathsf{M} \rtimes \mathsf{A} : \mathsf{M}), (\theta, \varphi) \mapsto \gamma$, with $\gamma(m, a) = (\theta(m), \varphi(a)), m \in \mathsf{M}, a \in \mathsf{A}$. β is well-defined, since γ is indeed a derivation due to the compatibility conditions:

$$\begin{aligned}
\gamma((m,a)(m',a')) &= \gamma(m \cdot a' + a \cdot m', aa') \\
&= (\theta(m) \cdot a' + m \cdot \varphi(a') + a \cdot \theta(m') + \varphi(a) \cdot m', \varphi(a)a' + a\varphi(a')) \\
&= (\theta(m), \varphi(a))(m', a') + (m, a)(\theta(m'), \varphi(a')) \\
&= \gamma(m,a)(m',a') + (m,a)\gamma(m',a')
\end{aligned}$$

and

$$\begin{aligned}
\gamma[(m,a)(m',a')] &= \gamma(\{m,a'\} + \{a,m'\}, [a,a']) \\
&= (\{\theta(m), a'\} + \{m, \varphi(a')\} + \{a, \theta(m')\} + \{\varphi(a), m'\}, [\varphi(a), a'] + [a, \varphi(a')]) \\
&= [\theta(m), \varphi(a)), (m', a')] + [(m,a), (\theta(m'), \varphi(a')] \\
&= [\gamma(m,a)(m',a')] + [(m,a), \gamma(m',a')]
\end{aligned}$$

Finally, it is easy to check that β is a homomorphism of AWBs and a splitting of τ. □

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