Filomat 40:4 (2026), 1207–1224
https://doi.org/10.2298/FIL2604207X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Kantorovich type inequalities and Young type inequalities for the Hadamard product of matrices

Zhengwei Xie[a,*], Jianzhong Liu[a]

[a] *School of Mathematics and Physics, Jiangsu University of Technology, Changzhou, 213001, China*

Abstract. In this paper, we present some Kantorovich type inequalities and Young type inequalities for the Hadamard product of positive definite matrices using the convexity of some functions involving positive definite matrices and the properties of Hadamard products of matrices. Based on this, we get some Kantorovich type inequalities for the spectral condition numbers of Hadamard product of positive definite matrices and some Kantorovich type inequalities in the Löwner partial ordering. Secondly, we obtain some Kantorovich type inequalities involving the Hadamard product of positive matrices by the properties of positive linear functionals in matrix space. Finally, we provide some Kantorovich type inequalities for the permanent of positive matrices.

1. Introduction

Let A be an $n \times n$ positive definite matrix and $x \in C^n$ $(x \neq 0)$. In 1948, Kantorovich [8] introduced the well-known Kantorovich inequality. Mond et al. [11, 12] presented two matrix versions of the Ky Fan generalization of the Kantorovich inequality. The basic form of a matrix type Kantorovich inequality is as follows:

$$\frac{x^\dagger A x x^\dagger A^{-1} x}{(x^\dagger x)^2} \leq \frac{(\rho(A) + \sigma(A))^2}{4\rho(A)\sigma(A)}, \tag{1}$$

where $x^\dagger$ stands for the conjugate transpose of x, $\rho(A)$ and $\sigma(A)$ represent the maximum and minimum eigenvalues of A respectively. When B and A are $n \times n$ positive definite matrices with $AB = BA$, Greub and Rheinboldt [6] provided a generalized form of (1) as follows:

$$\frac{x^\dagger A^2 x x^\dagger B^2 x}{(x^\dagger ABx)^2} \leq \frac{(\rho(A)\rho(B) + \sigma(A)\sigma(B))^2}{4\rho(A)\rho(B)\sigma(A)\sigma(B)}. \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 15A39; Secondary 15A45, 15A60.
Keywords. Convex map, Kantorovich type inequality, Young type inequality, Positive linear mapping, Positive definite matrix.
Received: 04 September 2025; Revised: 27 November 2025; Accepted: 30 November 2025
Communicated by Dragan S. Djordjević
This work is supported by the Natural Science Foundation of Jiangsu Province of China [Grant No. BK20211358], the Postgraduate Education Reform Project of Jiangsu University of Technology [Grant No. YJSJG2401], and the High-level Personnel Project of Jiangsu University of Technology (Grant No. KYY20021).
* Corresponding author: Zhengwei Xie
Email addresses: xiezhengwei@jsut.edu.cn (Zhengwei Xie), ljz@jsut.edu.cn (Jianzhong Liu)
ORCID iDs: https://orcid.org/0009-0006-9897-3872 (Zhengwei Xie)

Xie et al. [16] presented a Kantorovich type inequality for positive matrices by using the properties of positive linear functionals on matrices, which provided some Kantorovich type inequalities for spectral radius, numerical radius and spectral norm of the product of positive matrices. For more generalized Kantorovich inequalities, please refer to [4, 13–15]. Ando [3] obtained an interesting Hölder type inequality for the Hadamard product of positive definite matrices in the Löwner partial ordering as follows:

$$A \circ B \leq (A^p \circ I)^{\frac{1}{p}} (B^q \circ I)^{\frac{1}{q}}, \tag{3}$$

where A, B are positive definite matrices of the same order and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $a, b > 0$ and $0 < \nu < 1$. The well-known Young inequality is as follows:

$$\nu a + (1-\nu)b \geq a^\nu b^{1-\nu}. \tag{4}$$

In some literature, (4) is also said to be the weighted arithmetic-geometric mean inequality. For positive definite matrices A, B of the same order, their weighted geometric and arithmetic mean are respectively as follows:

$$\mathcal{G}_\nu(A, B) = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{1-\nu} A^{\frac{1}{2}}, \mathcal{A}_\nu(A, B) = \nu A + (1-\nu)B.$$

As is well known, their relationship is as follows:

$$\mathcal{G}_\nu(A, B) \leq \mathcal{A}_\nu(A, B). \tag{5}$$

In recent years, researchers have investigated the extension, improvement, and inverse of inequality (5) through the improvement, generalization, and inverse of Young's inequality (4) in many literature. For example, Kittaneh and Manasrah [9] gave the improved Young inequality as follows:

$$a^\nu b^{1-\nu} + \tau \left(\sqrt{a} - \sqrt{b}\right)^2 \leq \nu a + (1-\nu)b, \tag{6}$$

where $\tau = \min\{\nu, 1-\nu\}$. Using the improved Young inequality (6), they proposed some improvements on some Young type inequalities involving matrix singular values and matrix norms. Kittaneh and Manasrah [10] presented an inverse of the Young inequality as follows:

$$\nu a + (1-\nu)b \leq a^\nu b^{1-\nu} + \sigma \left(\sqrt{a} - \sqrt{b}\right)^2, \tag{7}$$

where $\sigma = \max\{\nu, 1-\nu\}$. Using (7), Kittaneh and Manasrah provided some inverse forms of the matrix weighted arithmetic-geometric mean inequality. Alzer et al. [2] investigated the difference of weighted arithmetic and weighted geometric mean with respect to two different weights. They used differential methods, through a scalar inequality, and showed that

$$\frac{\nu}{\mu} \left(\mathcal{A}_\mu(A, B) - \mathcal{G}_\mu(A, B)\right) \leq \mathcal{A}_\nu(A, B) - \mathcal{G}_\nu(A, B) \leq \frac{1-\nu}{1-\mu} \left(\mathcal{A}_\mu(A, B) - \mathcal{G}_\mu(A, B)\right), \tag{8}$$

where A, B are positive definite matrices and $0 < \nu \leq \mu < 1$.

We use the following standard notation. $\mathbb{R}$ and $\mathbb{C}$ represent the sets of real numbers and the sets of complex numbers, respectively. $M_{m\times n}$ and M_n represent the complex linear space formed by $m \times n$ complex matrices and $n \times n$ complex matrices respectively. M_n^+ $(\widetilde{M_n})$ stands for a convex cone formed by an $n \times n$ positive definite (semidefinite) matrix. The set of $n \times n$ Hermite matrices is denoted by H_n. For $A \in M_{m\times n}$, $A^\dagger$ stands for the conjugate transpose of A. For $A, B \in H_n$, $A > 0$ $(A \geq 0)$ stands that A is positive definite (semidefinite), $A \geq B$ stands for $A - B \geq 0$, and this relationship is said to be the Löwner partial ordering. $\mathbb{R}_{m\times n}$ and $\mathbb{R}_n$ represent the real linear space formed by $m\times n$ real matrices and $n\times n$ real matrices respectively. For $A = [a_{ij}] \in \mathbb{R}_{m\times n}$, $A > 0$ stands for $a_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$. In which case, A is said to be a nonnegative matrix. In addition, if $a_{ij} > 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$, then A is said to be a positive matrix. For $A \in M_{m\times n}$, A^T stands for the transpose of A. $P_{m,n}$ and $P^+_{m,n}$ stand for convex cones formed by $m \times n$ nonnegative matrices and positive matrices, respectively. When $m = n$, corresponding symbols are abbreviated as P_n and P^+_n respectively. $A \succ B$ stands for $A - B \succ 0$. $E_{m,n}$ represents an $m \times n$ matrix with all elements being 1. When $m = n$, we denote $E_{n,n}$ by E_n. For $A = [a_{ij}] \in P^+_{m,n}$ and $\alpha \in \mathbb{R}$, $A^{(\alpha)}$ stands for $\left[a_{ij}^{\alpha}\right]$. $A \circ B$ stands for the Hadamard product of A and B. Denote $n \times n$ the identity matrix by I_n, or I for short. For $A \in M_n$, $w(A) = \sup\{|x^\dagger Ax| : x \in C^n, x^\dagger x = 1\}$ is said to the numerical radius of A. For $A \in H_n$, $\rho(A)$, $\sigma(A)$ stand for the maximum eigenvalue, minimum eigenvalue of A, respectively. If A is a positive definite matrix, then $\kappa(A) = \frac{\rho(A)}{\sigma(A)}$ represents the spectral condition number of A.

The remainder of this paper is organized as follows. In Section 2, we give some of the definitions and lemmas that will be used in the rest of the paper. Then, in Section 3, we present two Kantorovich type inequalities and some Young type inequalities for the Hadamard product of positive definite matrices. Based on these, we obtain some Kantorovich type inequalities for the upper bound of the spectral condition number of Hadamard product of positive definite matrices. In Section 4, we provide some Kantorovich type inequalities involving the Hadamard product of two positive matrices and the permanent of positive matrices.

2. Preliminaries

In this section, we introduce some basic concepts and lemmas related to matrix theory.

Definition 2.1. *Let $D \subset M_{m,n}$ be a convex set and $f : D \to H_k$ be a map. Then f is said to be a convex map, if for any $U, V \in D, \lambda \in [0, 1]$, we have*

$$f(\lambda U + (1 - \lambda)V) \leq \lambda f(U) + (1 - \lambda) f(V).$$

When the above inequality is reversed, f is called a concave map on D.

If f is continuous, then f is a convex mapping on D if and only if

$$f\left(\frac{U + V}{2}\right) \leq \frac{f(U) + f(V)}{2}.$$

Remark 2.2. *Obviously, if f is a convex function on D and $A \in \widetilde{M_k}$, then $g(U) = f(U)A$ is a convex map on D.*

Definition 2.3. *Let $\Phi : M_k \to M_l$ be a linear map. If $C \geq 0$ implies $\Phi(C) \geq 0$, then Φ is called a positive linear map on M_k. In addition, if $C > 0$ implies $\Phi(C) > 0$, then Φ is called a strictly positive linear map on M_k. If $\Phi(I_k) = I_l$, then Φ is called a unital map. When $l = 1$, Φ is called a linear functional on M_k.*

Definition 2.4. *Let $\Phi : \mathbb{R}_{m\times n} \to \mathbb{R}$ be a linear function. If $A \succ 0$ implies $\Phi(A) \geq 0$, then Φ is called a positive linear functional on $\mathbb{R}_{m,n}$. In addition, if any $A \in P^+_{m,n}$ implies $\Phi(A) > 0$, then Φ is called a strictly positive linear functional on $\mathbb{R}_{m\times n}$. If $\Phi(E_{m,n}) = 1$, then Φ is called a unit linear functional on $\mathbb{R}_{m\times n}$.*

Definition 2.5. *Let $A, B \in M^+_n, 0 < \nu < 1$. The weighted geometric Hadamard mean and arithmetic Hadamard mean of A, B are defined respectively as follows:*

$$\mathcal{G}^{\circ}_{\nu}(A, B) = A^{\nu} \circ B^{1-\nu}, \mathcal{A}^{\circ}_{\nu}(A, B) = \nu (A \circ I_n) + (1 - \nu)(B \circ I_n).$$

Lemma 2.6. *Let $A, B \in M^+_n$ and $\Phi : M_n \to M_k$ be a positive linear map. Then*

$$G(x, y) = \Phi (A^x \circ B^y)$$

is a convex map on $\mathbb{R}^2$.

Proof. Let the spectral decomposition of A, B be $A = \sum_{i=1}^{N_1} \lambda_i P_i$, $B = \sum_{j=1}^{N_2} \mu_j Q_j$, where P_i, Q_j are the orthogonal projections, $\lambda_i, \mu_j > 0$ and $N_1, N_2 \leq n$. Then $A^x = \sum_{l=1}^{N_1} \lambda_i^x P_i$, $B^y = \sum_{j=1}^{N_2} \mu_j^y Q_j$ and $\Phi(A^x \circ B^y) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \lambda_i^x \mu_j^y \Phi\left(P_i \circ Q_j\right)$.

Using the fact that $\Phi\left(P_i \circ Q_j\right) \geq 0$ and $f(x, y) = a^x b^y$ is a convex function on $\mathbb{R}^2$ for any $a, b > 0$, the conclusion follows from Remark 2.2. □

It is easy to verify the following lemma by the definition of matrix function.

Lemma 2.7. *Let $A \in H_n$ and its eigenvalues included in the interval I. If $f, g : I \to \mathbb{R}$ satisfy $f(x) \leq g(x)$ $(x \in I)$, then $f(A) \leq g(A)$.*

Lemma 2.8. *Let $A \in M_n^+$, $f : (0, +\infty) \to \mathbb{R}$ be a convex function and Φ be a unit positive linear functional on M_n. Then*

$$f(\Phi(A)) \leq \Phi(f(A)).$$

When f is a concave function, the above inequality is reversed.

Proof. Let the spectral decomposition of A be $A = \sum_{i=1}^{k} \lambda_i P_i$, where P_i are the orthogonal projections, $\lambda_i > 0$ and $k \leq n$. Then

$$f(A) = \sum_{i=1}^{k} f(\lambda_i) P_i, \Phi(f(A)) = \sum_{i=1}^{k} f(\lambda_i) \Phi(P_i), f(\Phi(A)) = f\left(\sum_{i=1}^{k} \lambda_i \Phi(P_i)\right).$$

When f is a convex function, using the fact $\Phi(P_i) \geq 0$, $\sum_{i=1}^{k} \Phi(P_i) = \Phi\left(\sum_{i=1}^{k} P_i\right) = \Phi(I_n) = 1$, we have

$$f(\Phi(A)) = f\left(\sum_{i=1}^{k} \lambda_i \Phi(P_i)\right) \leq \sum_{i=1}^{k} \Phi(P_i) f(\lambda_i) = \Phi(f(A)).$$

When f is a concave function, the above inequality is reversed. □

Lemma 2.9. *Let $A \in M_n^+$ and Φ be a unit positive linear functional on M_n. When $0 < r < 1$, we have*

$$\Phi(A^r \circ I_n) \geq \eta_A(r) \Phi(A \circ I_n)^r, \tag{9}$$

where

$$\eta_A(r) = \begin{cases} r^{-r}(1-r)^{r-1}(\kappa(A)-1)^{-1}(\kappa(A)^r-1)^r(\kappa(A)-\kappa(A)^r)^{1-r}, & \kappa(A) > 1 \\ 1, & \kappa(A) = 1 \end{cases}.$$

Proof. The result is clear when $\kappa(A) = 1$. We therefore assume $\kappa(A) > 1$. Due to $f(x) = x^r$ being a concave function on $(0, +\infty)$, for $x \in [\sigma(A), \rho(A)]$, we have

$$x^r \geq \frac{\rho(A)^r - \sigma(A)^r}{\rho(A) - \sigma(A)} x + \frac{\rho(A)\sigma(A)^r - \rho(A)^r \sigma(A)}{\rho(A) - \sigma(A)}. \tag{10}$$

By (10) and Lemma 2.7, we get

$$A^r \geq \frac{\rho(A)^r - \sigma(A)^r}{\rho(A) - \sigma(A)} A + \frac{\rho(A)\sigma(A)^r - \rho(A)^r \sigma(A)}{\rho(A) - \sigma(A)} I_n. \tag{11}$$

By (11), we have

$$A^r \circ I_n \geq \frac{\rho(A)^r - \sigma(A)^r}{\rho(A) - \sigma(A)} A \circ I_n + \frac{\rho(A)\sigma(A)^r - \rho(A)^r \sigma(A)}{\rho(A) - \sigma(A)} I_n.$$

Hence we have

$$\Phi(A^r \circ I_n) \geq \frac{1}{\rho(A)-\sigma(A)}\left((\rho(A)^r-\sigma(A)^r)\,\Phi(A\circ I_n)+\rho(A)\sigma(A)^r-\rho(A)^r\sigma(A)\right). \tag{12}$$

By (12) and (4), we obtain

$$\begin{aligned}\Phi(A^r \circ I_n) &\geq \frac{1}{\rho(A)-\sigma(A)}\left\{r\left[\frac{1}{r}\left(\rho(A)^r-\sigma(A)^r\right)\Phi\left(A\circ I_n\right)\right]+(1-r)\left[\frac{1}{1-r}\left(\rho(A)\sigma(A)^r-\rho(A)^r\sigma(A)\right)\right]\right\}\\ &\geq \frac{1}{\rho(A)-\sigma(A)}\left[\frac{1}{r}(\rho(A)^r-\sigma(A)^r)\Phi(A\circ I_n)\right]^r\left[\frac{1}{1-r}\left(\rho(A)\sigma(A)^r-\rho(A)^r\sigma(A)\right)\right]^{1-r}\\ &= r^{-r}(1-r)^{r-1}\frac{\left(\rho(A)^r-\sigma(A)^r\right)^r\left(\rho(A)\sigma(A)^r-\rho(A)^r\sigma(A)\right)^{1-r}}{\rho(A)-\sigma(A)}\Phi(A\circ I_n)^r\\ &= r^{-r}(1-r)^{r-1}(\kappa(A)-1)^{-1}\left(\kappa(A)^r-1\right)^r\left(\kappa(A)-\kappa(A)^r\right)^{1-r}\Phi(A\circ I_n)^r.\end{aligned}$$

□

Lemma 2.10. *Let ξ,η be two bounded random variables on probability space (Ω,Σ,P). Hence, there exist constants $m_\xi, M_\xi, m_\eta, M_\eta$ such that $m_\xi \leq \xi \leq M_\xi, m_\eta \leq \eta \leq M_\eta$. Then their covariance satisfies*

$$|\,cov(\xi,\eta)\,| \leq \frac{1}{4}\left(M_\xi-m_\xi\right)\left(M_\eta-m_\eta\right).$$

Proof. By the Cauchy-Schwarz inequality, we have

$$|\,cov(\xi,\eta)\,| \leq \sqrt{D(\xi)D(\eta)}. \tag{13}$$

By $m_\xi \leq \xi \leq M_\xi$, we know $\left(\xi-\frac{M_\xi+m_\xi}{2}\right)^2 \leq \frac{\left(M_\xi-m_\xi\right)^2}{4}$. So, we obtain

$$D(\xi) \leq E\left(\xi-\frac{M_\xi+m_\xi}{2}\right)^2 \leq \frac{(M_\xi-m_\xi)^2}{4}. \tag{14}$$

Similarly, we have

$$D(\eta) \leq \frac{\left(M_\eta-m_\eta\right)^2}{4}. \tag{15}$$

Finally, the conclusion follows from (13), (14) and (15). □

Lemma 2.11. *(Perron Theorem, See [7]) If $A \in P_n^+$, then there exists an n-dimensional positive vector $x = (x_1, x_2, \cdots, x_n)^T$ that satisfies $\sum_{i=1}^n x_i = 1$ such that $Ax = \rho(A)x$, where $\rho(A)$ denotes the spectral radius of A and x is called the Perron vector of A.*

For $A \in P_n^+$, let $c_A = \min_{1\leq j\leq n}\sum_{i=1}^n a_{ij}$, $\|A\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^n a_{ij}$, $r_A = \min_{1\leq i\leq n}\sum_{j=1}^n a_{ij}$, $\|A\|_\infty = \max_{1\leq i\leq n}\sum_{j=1}^n a_{ij}$. Through simple calculations, we have the following lemma.

Lemma 2.12. *Let $A \in P_n^+$. Then*

$$\min\left\{E_{n,1}^T Ax : x \geq 0, \sum_{i=1}^n x_i = 1\right\} = c_A, \max\left\{E_{n,1}^T Ax : x \geq 0, \sum_{i=1}^n x_i = 1\right\} = \|A\|_1,$$
$$\min\left\{x^T AE_{n,1} : x \geq 0, \sum_{i=1}^n x_i = 1\right\} = r_A, \max\left\{x^T AE_{n,1} : x \geq 0, \sum_{i=1}^n x_i = 1\right\} = \|A\|_\infty.$$

Lemma 2.13. *(See [16]) Let $A \in P_n^+$. Then*

$$w(A) = \sup\left\{x^T Ax : x > 0, x^T x = 1\right\}.$$

3. Some Kantorovich type inequalities and Young type inequalities for Hadamard product of positive definite matrices

3.1. Two Kantorovich type inequalities for the Hadamard product of positive definite matrices

In this section, we give two Kantorovich type inequalities for the Hadamard product of positive definite matrices using the properties of positive linear functionals in matrix space.

Theorem 3.1. *If $A, B \in M_n^+$ and Φ is a unit positive linear functional on M_n, then*

$$\left[\frac{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)}+1}\right]^2 \le \frac{\Phi(A\circ B)}{\Phi(A\circ I_n)\Phi(B\circ I_n)} \le \left[\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right]^2, \tag{16}$$

$$\mid \Phi(A\circ B) - \Phi(A\circ I_n)\Phi(B\circ I_n) \mid \le \frac{1}{4}(\rho(A)-\sigma(A))(\rho(B)-\sigma(B)). \tag{17}$$

Proof. Let us first check (16). The result is clear when $\rho(A) = \sigma(A)$, i.e., when $\kappa(A) = 1$. We therefore assume $\rho(A) > \sigma(A)$. By $\sigma(A)I_n \le A \le \rho(A)I_n$, $\sigma(B)I_n \le B \le \rho(B)I_n$, we obtain

$$A - \sigma(A)I_n \ge 0, \rho(A)I_n - A \ge 0, B - \sigma(B)I_n \ge 0, \rho(B)I_n - B \ge 0.$$

Setting $\alpha = \Phi\left[(\rho(A)I_n - A)\circ I_n\right]$, $\beta = \Phi\left[(A-\sigma(A)I_n)\circ I_n\right]$. Obviously, $\alpha \ge 0, \beta \ge 0$ and $\alpha+\beta = (\rho(A) - \sigma(A))\Phi(I_n) = \rho(A)-\sigma(A) > 0$, so α, β are not all 0. Through simple calculation, we get

$$\Phi(A\circ I_n) = \frac{\sigma(A)\alpha+\rho(A)\beta}{\rho(A)-\sigma(A)} = \frac{\sigma(A)\alpha+\rho(A)\beta}{\alpha+\beta}. \tag{18}$$

Using the fact that $\Phi[(\rho(A)I_n - A)\circ(\rho(B)I_n - B)] \ge 0$, $\Phi\left[(A-\sigma(A)I_n)\circ(B-\sigma(B)I_n)\right] \ge 0$, we have

$$\begin{aligned}
&\sigma(B)\beta\Phi[(\rho(A)I_n - A)\circ(\rho(B)I_n - B)] + \rho(B)\alpha\Phi\left[(A-\sigma(A)I_n)\circ(B-\sigma(B)I_n)\right]\\
&= \sigma(B)\beta[\Phi(A\circ B) + \rho(B)\alpha - \rho(A)\Phi(B\circ I_n)] + \rho(B)\alpha[\Phi(A\circ B) - \sigma(B)\beta - \sigma(A)\Phi(B\circ I_n)]\\
&= (\rho(B)\alpha + \sigma(B)\beta)\Phi(A\circ B) - (\rho(B)\sigma(A)\alpha + \rho(A)\sigma(B)\beta)\Phi(B\circ I_n) \ge 0.
\end{aligned} \tag{19}$$

Similarly, using the fact that $\Phi[(\rho(A)I_n - A)\circ(B-\sigma(B)I_n)] \ge 0$ and $\Phi\left[(A-\sigma(A)I_n)\circ(\rho(B)I_n - B)\right] \ge 0$, we have

$$\begin{aligned}
&\rho(B)\beta\Phi\left[(\rho(A)I_n - A)\circ(B-\sigma(B)I_n)\right] + \sigma(B)\alpha\Phi\left[(A-\sigma(A))I_n)\circ(\rho(B)I_n - B)\right]\\
&= \rho(B)\beta[-\Phi(A\circ B) - \sigma(B)\alpha + \rho(A)\Phi(B\circ I_n] + \sigma(B)\alpha[-\Phi(A\circ B) + \rho(B)\beta + \sigma(A)\Phi(B\circ I_n)]\\
&= -(\sigma(B)\alpha + \rho(B)\beta)\Phi(A\circ B) + (\sigma(A)\sigma(B)\alpha + \rho(A)\rho(B)\beta)\Phi(B\circ I_n) \ge 0.
\end{aligned} \tag{20}$$

According to (19), it follows that

$$\frac{\Phi(A\circ B)}{\Phi(B\circ I_n)} \ge \frac{\rho(B)\sigma(A)\alpha + \rho(A)\sigma(B)\beta}{\rho(B)\alpha + \sigma(B)\beta}. \tag{21}$$

According to (20), it follows that

$$\frac{\Phi(A\circ B)}{\Phi(B\circ I_n)} \le \frac{\sigma(A)\sigma(B)\alpha + \rho(A)\rho(B)\beta}{\sigma(B)\alpha + \rho(B)\beta}. \tag{22}$$

By (18) and (21), we have

$$\begin{aligned}
\frac{\Phi(A\circ B)}{\Phi(A\circ I_n)\Phi(B\circ I_n)} &\ge \frac{(\rho(B)\sigma(A)\alpha + \rho(A)\sigma(B)\beta)(\alpha+\beta)}{(\rho(B)\alpha + \sigma(B)\beta)(\sigma(A)\alpha + \rho(A)\beta)}\\
&= \frac{(\kappa(B)\alpha + \kappa(A)\beta)(\alpha+\beta)}{(\kappa(B)\alpha + \beta)(\alpha + \kappa(A)\beta)}.
\end{aligned} \tag{23}$$

Similarly, by (18) and (22), we obtain

$$\begin{aligned}\frac{\Phi(A\circ B)}{\Phi\left(A\circ I_n\right)\Phi(B\circ I_n)} &\le \frac{(\sigma(A)\sigma(B)\alpha+\rho(A)\rho(B)\beta)(\alpha+\beta)}{(\sigma(B)\alpha+\rho(B)\beta)(\sigma(A)\alpha+\rho(A)\beta)}\\ &= \frac{(\alpha+\kappa(A)\kappa(B)\beta)(\alpha+\beta)}{(\alpha+\kappa(B)\beta)(\alpha+\kappa(A)\beta)}. \end{aligned} \tag{24}$$

As $\kappa(A), \kappa(B) \ge 1$, which implies

$$\begin{aligned}&\kappa(A)+\kappa(B)\le\kappa(A)\kappa(B)+1,\ \sqrt{\kappa(A)}+\sqrt{\kappa(B)}\le\sqrt{\kappa(A)\kappa(B)}+1,\\ &\left(\frac{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)}+1}\right)^2\le 1, \left(\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right)^2\ge 1.\end{aligned} \tag{25}$$

If $\alpha\beta = 0$, by (23) and (24), we have

$$\frac{\Phi(A\circ B)}{\Phi(A\circ I_n)\Phi(B\circ I_n)} = 1.$$

So, the conclusion holds.
If $\alpha\beta \ne 0$, setting $t = \frac{\beta}{\alpha}$, by (23), (25) and the arithmetic geometric mean inequality, we get

$$\begin{aligned}\frac{\Phi(A\circ B)}{\Phi(A\circ I_n)\Phi(B\circ I_n)} &\ge \frac{(\kappa(B)\alpha+\kappa(A)\beta)(\alpha+\beta)}{(\kappa(B)\alpha+\beta)(\alpha+\kappa(A)\beta)}\\ &= \frac{(\kappa(B)+\kappa(A)t)\left(1+\frac{1}{t}\right)}{(\kappa(B)+t)\left(\kappa(A)+\frac{1}{t}\right)}\\ &= \frac{\kappa(A)+\kappa(B)+\kappa(A)t+\frac{1}{t}\kappa(B)}{\kappa(A)\kappa(B)+1+\kappa(A)t+\frac{1}{t}\kappa(B)}\\ &= 1-\frac{\kappa(A)\kappa(B)+1-\kappa(A)-\kappa(B)}{\kappa(A)\kappa(B)+1+\kappa(A)t+\frac{1}{t}\kappa(B)}\\ &\ge 1-\frac{\kappa(A)\kappa(B)+1-\kappa(A)-\kappa(B)}{\kappa(A)\kappa(B)+1+2\sqrt{\kappa(A)\kappa(B)}}\\ &= \frac{\left(\sqrt{\kappa(A)}+\sqrt{\kappa(B)}\right)^2}{\left(\sqrt{\kappa(A)\kappa(B)}+1\right)^2}.\end{aligned} \tag{26}$$

By (26), we obtain the left side of (16). Similarly, by (24) and (25), we obtain the right side of (16).

Next, we check (17) using the spectral decomposition of matrices and Lemma 2.10.

Let the spectral decomposition of A, B be $A = \sum_{i=1}^{k}\lambda_i P_i, B = \sum_{j=1}^{l}\mu_j Q_j$, $(1 \le k, l \le n)$, where P_i, Q_j are the orthogonal projections matrices, $\sum_{i=1}^{k} P_i = I_n, \sum_{j=1}^{l} Q_j = I_n$, and λ_i, μ_j are the eigenvalues of A, B, respectively. Hence, we have

$$\begin{aligned}&A\circ B=\sum_{i=1}^{k}\sum_{j=1}^{l}\lambda_i\mu_j P_i\circ Q_j, \Phi(A\circ B)=\sum_{i=1}^{k}\sum_{j=1}^{l}\lambda_i\mu_j\Phi\left(P_i\circ Q_j\right),\\ &\sum_{i=1}^{k}\sum_{j=1}^{l}\Phi\left(P_i\circ Q_j\right)=\Phi\left(\sum_{i=1}^{k}\sum_{j=1}^{l}P_i\circ Q_j\right)=\Phi\left(I_n\right)=1.\end{aligned}$$

Taking $\Omega = \{(i,j) : i = 1,2,\cdots,k, j = 1,2,\cdots,l\}$ and Σ to be the σ-field composed of all subsets of Ω, for any $A \in \Sigma, P(A) = \sum_{(i,j)\in A} \Phi(P_i \circ Q_j), \xi(i,j) = \lambda_i, \eta(i,j) = \mu_j$, together with simple substitution, we have

$$\begin{aligned}
&E(\xi\eta) = \sum_{i=1}^{k}\sum_{j=1}^{l} \lambda_i\mu_j\Phi\left(P_i \circ Q_j\right) = \Phi\left(\sum_{i=1}^{k}\sum_{j=1}^{l} \lambda_i\mu_j P_i \circ Q_j\right) = \Phi(A \circ B),\\
&E(\xi) = \sum_{i=1}^{k}\sum_{j=1}^{l} \lambda_i\Phi\left(P_i \circ Q_j\right) = \Phi\left(\sum_{i=1}^{k}\sum_{j=1}^{l} \lambda_i P_i \circ Q_j\right) = \Phi\left(\sum_{i=1}^{k} \lambda_i P_i \circ I_n\right) = \Phi\left(A \circ I_n\right),\\
&E(\eta) = \Phi\left(B \circ I_n\right), \sigma(A) \le \xi \le \rho(A), \sigma(B) \le \eta \le \rho(B),\\
&cov(\xi,\eta) = \Phi(A \circ B) - \Phi\left(A \circ I_n\right)\Phi\left(B \circ I_n\right).
\end{aligned} \tag{27}$$

Finally, (17) follows from Lemma 2.10 and (27). □

Remark 3.2. *The bounds given in (16) and (17) are achievable. Taking $\Phi(X) = tr(WX)$ $(X \in M_n)$, where $W \in M_n^+$ and $tr(W) = 1$. It is easy to see that Φ is a unit positive linear functional on M_n. Through simple calculations, when*

$$n = 2, A = \begin{bmatrix} \rho(A) & 0 \\ 0 & \sigma(A) \end{bmatrix}, B = \begin{bmatrix} \rho(B) & 0 \\ 0 & \sigma(B) \end{bmatrix}, W = diag\{w_1, w_2\}, w_1 = \frac{1}{\sqrt{\kappa(A)\kappa(B)}+1}, w_2 = \frac{\sqrt{\kappa(A)\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)}+1},$$

the right-hand side equality of (16) holds;
When

$$n = 2, A = \begin{bmatrix} \rho(A) & 0 \\ 0 & \sigma(A) \end{bmatrix}, B = \begin{bmatrix} \sigma(B) & 0 \\ 0 & \rho(B) \end{bmatrix}, W = diag\{w_1, w_2\}, w_1 = \frac{\sqrt{\kappa(B)}}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}, w_2 = \frac{\sqrt{\kappa(A)}}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}},$$

the left-hand side equality of (16) holds;
When

$$n = 2, A = \begin{bmatrix} \rho(A) & 0 \\ 0 & \sigma(A) \end{bmatrix}, B = \begin{bmatrix} \rho(B) & 0 \\ 0 & \sigma(B) \end{bmatrix}, W = diag\left\{\frac{1}{2}, \frac{1}{2}\right\},$$

the equality of (17) holds. Furthermore, from the proof process it is easy to see that the condition for (17) can be weakened to $A, B \in H_n$ and Φ is a unit positive linear functional on M_n.

Remark 3.3. *Let $\lambda_i > 0, a_i > 0, i = 1,2,\cdots,n$. Let $m = \min_{1\le i\le n} a_i$ and $M = \max_{1\le i\le n} a_i$. Taking $A = diag\{a_1, a_2, \cdots, a_n\}, B = A^{-1}, \Phi(X) = tr(WX), (X \in M_n)$, where $W = \frac{1}{\sum_{i=1}^n \lambda_i} diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, it is easy to see that Φ is a unit positive linear functional on M_n. By the left side of (16), together with simple substitution, we have a discrete Kantorovich inequality as follows*

$$\frac{\sum_{i=1}^n \lambda_i a_i \sum_{i=1}^n \frac{\lambda_i}{a_i}}{\left(\sum_{i=1}^n \lambda_i\right)^2} \le \frac{(M+m)^2}{4Mm}. \tag{28}$$

Using the diagonalization of positive definite matrices, it is easy to verify that (28) is equivalent to (1).

For any unit vector $x \in C^n$, taking $\Phi(X) = x^\dagger X x$ $(X \in M_n)$ in Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n^+$, and $x \in C^n$ with $x^\dagger x = 1$. Then*

$$\left[\frac{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)}+1}\right]^2 \le \frac{x^\dagger A \circ Bx}{x^\dagger A \circ I_n x x^\dagger I_n \circ Bx} \le \left[\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right]^2, \tag{29}$$

$$\left|x^\dagger A \circ Bx - x^\dagger A \circ I_n x x^\dagger I_n \circ Bx\right| \le \frac{1}{4}(\rho(A) - \sigma(A))(\rho(B) - \sigma(B)). \tag{30}$$

Next, using Corollary 3.4 and Rayleigh Ritz Theorem, we provide some upper bounds on the spectrum condition number of Hadamard product of positive definite matrices.

Corollary 3.5. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n^+$. Let a, b stand for the maximum values of diagonal elements for A and B respectively. Let $\hat{a}, \hat{b}$ stand for the minimum values of diagonal elements for A and B respectively. Then*

$$\kappa(A \circ B) \le \left[\frac{\sqrt{\kappa(A)\kappa(B)} + 1}{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}} \right]^4 \frac{ab}{\hat{a}\hat{b}}. \tag{31}$$

If $\gamma < \hat{a}\hat{b}$, then

$$\kappa(A \circ B) \le \frac{ab + \gamma}{\hat{a}\hat{b} - \gamma}, \tag{32}$$

where $\gamma = \frac{1}{4}(\rho(A) - \sigma(A))(\rho(B) - \sigma(B))$.

Proof. Let us first check (31). For any unit vector $x \in \mathbb{C}^n$, by (29), we have

$$\left(\frac{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)} + 1} \right)^2 x^\dagger (A \circ I) x x^\dagger (B \circ I) x \le x^\dagger A \circ B x \le \left(\frac{\sqrt{\kappa(A)\kappa(B)} + 1}{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}} \right)^2 x^\dagger (A \circ I) x x^\dagger (B \circ I) x. \tag{33}$$

Because of $x^\dagger (A \circ I) x = \sum_{i=1}^n a_{ii} |x_i|^2$, we get

$$\hat{a} = \sum_{i=1}^n \hat{a} |x_i|^2 \le x^\dagger (A \circ I) x \le \sum_{i=1}^n a |x_i|^2 = a. \tag{34}$$

Similarly,

$$\hat{b} = \sum_{i=1}^n \hat{b} |x_i|^2 \le x^\dagger (B \circ I) x \le \sum_{i=1}^n b |x_i|^2 = b. \tag{35}$$

According to the Rayleigh-Ritz Theorem, we have $\rho(A \circ B) = \max_{\|x\|_2=1} x^\dagger (A \circ B) x$, $\sigma(A \circ B) = \min_{\|x\|_2=1} x^\dagger (A \circ B) x$. By (33), (34) and (35), we have

$$\left[\frac{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}}{\sqrt{\kappa(A)\kappa(B)} + 1} \right]^2 \hat{a}\hat{b} \le \sigma(A \circ B) \le \rho(A \circ B) \le \left[\frac{\sqrt{\kappa(A)\kappa(B)} + 1}{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}} \right]^2 ab. \tag{36}$$

By (36), we have

$$\kappa(A \circ B) = \frac{\rho(A \circ B)}{\sigma(A \circ B)} \le \left[\frac{\sqrt{\kappa(A)\kappa(B)} + 1}{\sqrt{\kappa(A)} + \sqrt{\kappa(B)}} \right]^4 \frac{ab}{\hat{a}\hat{b}}.$$

Let us check (32). For any unit vector $x \in C^n$ (i.e., $\|x\|_2 = 1$), by (30), we have

$$\left| x^\dagger (A \circ B) x - x^\dagger (A \circ I_n) x x^\dagger (I_n \circ B) x \right| \le \gamma. \tag{37}$$

By (37), we have

$$x^\dagger (A \circ I_n) x x^\dagger (I_n \circ B) x - \gamma \le x^\dagger (A \circ B) x \le x^\dagger (A \circ I_n) x x^\dagger (I_n \circ B) x + \gamma. \tag{38}$$

By (34), (35) and (38), we have

$$\hat{a}\hat{b} - \gamma \le x^\dagger (A \circ B) x \le ab + \gamma. \tag{39}$$

By (39), together with the Rayleigh-Ritz Theorem, we get

$$\hat{a}\hat{b} - \gamma \le \sigma(A \circ B) \le \rho(A \circ B) \le ab + \gamma. \tag{40}$$

Finally, (32) follows from (40). □

Remark 3.6. *Let $A = [a_{ij}] \in M_n^+$, then $A^T \in M_n^+$, $A \circ A^T = \left[|a_{ij}|^2\right]$, $\rho(A^T) = \rho(A), \sigma(A^T) = \sigma(A)$ and $\kappa(A^T) = \kappa(A)$. Let $a, \hat{a}$ stand for the maximum values, the minimum values of diagonal elements for A respectively, then $a^2, \hat{a}^2$ stand for the maximum values, the minimum values of diagonal elements for $A \circ A^T$ respectively. By the Rayleigh-Ritz Theorem, we obtain $\frac{a^2}{\hat{a}^2} \leq \rho(A \circ A^T)$. Taking $B = A^T$ in (31), we obtain*

$$\frac{a^2}{\hat{a}^2} \leq \kappa\left(A \circ A^T\right) \leq \frac{(\kappa(A)+1)^4}{16\kappa(A)^2}\frac{a^2}{\hat{a}^2}. \tag{41}$$

Let $A, B \in M_n^+$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Taking Φ to be the identity map in Lemma 2.6, we know that

$$g(x, y) = A^x \circ B^y$$

is a convex map on $\mathbb{R}^2$. Based on this, we can obtain the Young type inequality for Hadamard product of A and B as follows:

$$A \circ B = (A^p)^{\frac{1}{p}} \circ (B^q)^{\frac{1}{q}} \leq \frac{1}{p}(A^p \circ I) + \frac{1}{q}(B^q \circ I).$$

One can naturally ask whether

$$A \circ B \leq \frac{1}{p}(A \circ I)^p + \frac{1}{q}(B \circ I)^q$$

holds. The following example provides a negative answer. Setting

$$A = B = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$

through simple calculation, we can obtain

$$\frac{1}{p}(A \circ I)^p + \frac{1}{q}(B \circ I)^q - A \circ B = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix}.$$

It is thus clear that $\frac{1}{p}(A \circ I)^p + \frac{1}{q}(B \circ I)^q - A \circ B \geq 0$ does not hold. Next, using Corollary 3.4, we present a relationship between $\frac{1}{p}(A \circ I)^p + \frac{1}{q}(B \circ I)^q$ and $A \circ B$.

Corollary 3.7. *Let $A, B \in M_n^+$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$A \circ B \leq \left[\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right]^2 \left[\frac{1}{p}(A \circ I)^p + \frac{1}{q}(B \circ I)^q\right].$$

Proof. For any unit vector $x \in C^n$, by Corollary 3.4 and the Young inequality, we have

$$\begin{aligned} x^\dagger A \circ Bx &\leq \left(\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right)^2 x^\dagger(A \circ I)xx^\dagger(B \circ I)x \\ &\leq \left(\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right)^2 \left[\frac{1}{p}\left(x^\dagger(A \circ I)x\right)^p + \frac{1}{q}\left(x^\dagger(B \circ I)x\right)^q\right]. \end{aligned} \tag{42}$$

It is easy to see that $\Phi(X) = x^\dagger X x$ $(X \in M_n)$ is a unit positive linear functional on M_n and x^p, x^q are both convex functions on $(0, +\infty)$ for $p, q > 1$. By Lemma 2.8, we have

$$(x^\dagger(A \circ I)x)^p \leq x^\dagger(A \circ I)^p x, (x^\dagger(A \circ I)x)^q \leq x^\dagger(A \circ I)^q x. \tag{43}$$

By (42) and (43), we have

$$x^{\dagger}A\circ Bx\leq\left(\frac{\sqrt{\kappa(A)\kappa(B)}+1}{\sqrt{\kappa(A)}+\sqrt{\kappa(B)}}\right)^{2}\left[\frac{1}{p}x^{\dagger}(A\circ I)^{p}x+\frac{1}{q}x^{\dagger}(B\circ I)^{q}x\right].\tag{44}$$

So, the conclusion follows from (44). □

3.2. Some Young type inequalities and their inverse for Hadamard product of positive definite matrices

In this section, we give some Young type inequalities and their inverse for Hadamard product of positive definite matrices.

Let $A,B\in M_n^+$ and $0<\nu<1$. Taking Φ to be the identity map in Lemma 2.6, we know that

$$f(x,y)=A^{x}\circ B^{y}$$

is a convex map on $\mathbb{R}^2$. So, we can obtain $A^{\nu}\circ B^{1-\nu}\leq \nu A\circ I_n+(1-\nu)B\circ I_n$. That is, we have the following weighted geometric-arithmetic mean inequality involving the Hadamard product of matrices.

Corollary 3.8. *Let $A,B\in M_n^+$ and $0<\nu<1$. Then*

$$\mathcal{G}_{\nu}^{\circ}(A,B)\leq\mathcal{A}_{\nu}^{\circ}(A,B).\tag{45}$$

By Lemma 2.9, Corollary 3.8 and Theorem 3.1, we have the following theorem.

Theorem 3.9. *Let $A,B\in M_n^+$, $0<r<1$ and Φ be a unit positive linear functional on M_n. Then*

$$\varpi(r)\Phi(A\circ I_n)^{r}\Phi(B\circ I_n)^{1-r}\leq\Phi\left(A^{r}\circ B^{1-r}\right)\leq\Phi(A\circ I_n)^{r}\Phi(B\circ I_n)^{1-r},\tag{46}$$

where

$$\varpi(r)=\eta_A(r)\eta_B(1-r)\frac{\left(\sqrt{\kappa(A)^{r}}+\sqrt{\kappa(B)^{1-r}}\right)^{2}}{\left(\sqrt{\kappa(A)^{r}\kappa(B)^{1-r}}+1\right)^{2}}$$

and

$$\eta_X(\theta)=\begin{cases}\theta^{-\theta}(1-\theta)^{\theta-1}(\kappa(X)-1)^{-1}\left(\kappa(X)^{\theta}-1\right)^{\theta}\left(\kappa(X)-\kappa(X)^{\theta}\right)^{1-\theta}, & \kappa(X)>1\\ 1, & \kappa(X)=1\end{cases}$$

for $0<\theta<1$ and positive definite matrix X.

Proof. By Corollary 3.8, we have

$$\Phi\left(A^{r}\circ B^{1-r}\right)\leq r\Phi(A\circ I_n)+(1-r)\Phi(B\circ I_n).\tag{47}$$

Replacing A with tA $(t>0)$ in (47), we obtain

$$\Phi\left(A^{r}\circ B^{1-r}\right)\leq rt^{1-r}\Phi(A\circ I_n)+(1-r)t^{-r}\Phi(B\circ I_n).\tag{48}$$

Using differential method, it is easy to prove that

$$\min\left\{rt^{1-r}\Phi(A\circ I_n)+(1-r)t^{-r}\Phi(B\circ I_n):t>0\right\}=\Phi(A\circ I_n)^{r}\Phi(B\circ I_n)^{1-r}.\tag{49}$$

By (48) and (49), we have

$$\Phi\left(A^{r}\circ B^{1-r}\right)\leq\min\left\{rt^{1-r}\Phi(A\circ I_n)+(1-r)t^{-r}\Phi(B\circ I_n):t>0\right\}=\Phi(A\circ I_n)^{r}\Phi(B\circ I_n)^{1-r}.\tag{50}$$

By (50), we know that the right side of (46) holds.

Using the fact that $\kappa(A^r) = \kappa(A)^r, \kappa(B^{1-r}) = \kappa(B)^{1-r}$, together with the left side of (16) and Lemma 2.9, we have

$$\begin{aligned}\Phi\left(A^r \circ B^{1-r}\right) &\geq \left(\frac{\sqrt{\kappa(A)^r} + \sqrt{\kappa(B)^{1-r}}}{\sqrt{\kappa(A)^r\kappa(B)^{1-r}} + 1}\right)^2 \Phi\left(A^r \circ I_n\right)\Phi\left(B^{1-r} \circ I_n\right) \\ &\geq \varpi(r)\left(\frac{\sqrt{\kappa(A)^r} + \sqrt{\kappa(B)^{1-r}}}{\sqrt{\kappa(A)^r\kappa(B)^{1-r}} + 1}\right)^2 \Phi\left(A \circ I_n\right)^r \Phi\left(B \circ I_n\right)^{1-r}.\end{aligned}$$

□

Remark 3.10. *Through simple computations, it is easy to see that the right side equality of (46) holds when $\kappa(A) = 1$ or $\kappa(B) = 1$. When $\kappa(A) = \kappa(B) = 1$, the left side equality of (46) holds.*

Next, using Lemma 2.6 we provide a relationship between the weighted Hadamard geometric-arithmetic mean of matrices under different weights.

Theorem 3.11. *Let $A, B \in M_n^+$ and $0 < \nu \leq \mu < 1$. Then*

$$\frac{\nu}{\mu}\left(\mathcal{A}_\mu^\circ(A,B) - \mathcal{G}_\mu^\circ(A,B)\right) \leq \mathcal{A}_\nu^\circ(A,B) - \mathcal{G}_\nu^\circ(A,B) \leq \frac{1-\nu}{1-\mu}(\mathcal{A}_\mu^\circ(A,B) - \mathcal{G}_\mu^\circ(A,B)). \tag{51}$$

Proof. Our proof is based on the following three steps.

Step 1. If $\mathbb{R}_+^2 = \{(x,y) : x \geq 0, y \geq 0\} - \{(0,0)\}$ and $G : \mathbb{R}_+^2 \to M_n^+$ is a convex (concave) map, then $g : [0,1] \to M_n^+$ is a convex (concave) map, where $g(x) = G(x, 1-x)$. This result can be directly verified by definition.

Step 2. If $g : [0,1] \to M_n^+$ is a concave map, $g(0) = g(1) = 0$ and $0 < \nu \leq \mu < 1$, then

$$\frac{\nu}{\mu}g(\mu) \leq g(\nu) \leq \frac{1-\nu}{1-\mu}g(\mu). \tag{52}$$

If $g : [0,1] \to M_n^+$ is a convex map, (52) is reversed.

We only verify the concave case. If $g : [0,1] \to M_n^+$ is a concave map, then

$$\begin{aligned}g(\nu) &= g\left(\frac{\nu}{\mu}\mu + \left(1 - \frac{\nu}{\mu}\right)0\right) \geq \frac{\nu}{\mu}g(\mu) + \left(1 - \frac{\nu}{\mu}\right)g(0) = \frac{\nu}{\mu}g(\mu), \\ g(\mu) &= g\left(\frac{1-\mu}{1-\nu}\nu + \frac{\mu-\nu}{1-\nu}1\right) \geq \frac{1-\mu}{1-\nu}g(\nu) + \frac{\mu-\nu}{1-\nu}g(1) = \frac{1-\mu}{1-\nu}g(\nu).\end{aligned}$$

From this, the conclusion follows from a simple transformation. Similarly, we can verify the convex case.

Step 3. By Lemma 2.6, we know that $G(x,y) = A^x \circ B^y$ is a convex map on $\mathbb{R}^2$. Therefore, from the first step mentioned above, we get $A^x \circ B^{1-x} : [0,1] \to M_n^+$ is a convex map, which implies that $g(x) : [0,1] \to M_n^+$ is a concave map satisfied $g(0) = g(1) = 0$, $g(\nu) = \mathcal{A}_\nu^\circ(A,B) - \mathcal{G}_\nu^\circ(A,B)$ and $g(\mu) = \mathcal{A}_\mu^\circ(A,B) - \mathcal{G}_\mu^\circ(A,B)$, where $g(x) = xA \circ I_n + (1-x)B \circ I_n - A^x \circ B^{1-x}$. Finally, (51) follows from (52). □

By Theorem 3.11, we get the following corollary.

Corollary 3.12. *Let $A, B \in M_n^+$ and $0 < \nu < 1$, $\tau = \min\{\nu, 1-\nu\}$ and $\sigma = \max\{\nu, 1-\nu\}$. Then*

$$2\tau\left(\mathcal{A}_{\frac{1}{2}}^\circ(A,B) - \mathcal{G}_{\frac{1}{2}}^\circ(A,B)\right) \leq \mathcal{A}_\nu^\circ(A,B) - \mathcal{G}_\nu^\circ(A,B) \leq 2\sigma\left(\mathcal{A}_{\frac{1}{2}}^\circ(A,B) - \mathcal{G}_{\frac{1}{2}}^\circ(A,B)\right). \tag{53}$$

Proof. The result is clear when $\nu = \frac{1}{2}$. We therefore assume $\nu \neq \frac{1}{2}$. When $0 < \nu < \frac{1}{2}$, by (51), we have

$$2\nu\left(\mathcal{A}^{\circ}_{\frac{1}{2}}(A,B) - \mathcal{G}^{\circ}_{\frac{1}{2}}(A,B)\right) \le \mathcal{A}^{\circ}_{\nu}(A,B) - \mathcal{G}^{\circ}_{\nu}(A,B) \le 2(1-\nu)\left(\mathcal{A}^{\circ}_{\frac{1}{2}}(A,B) - \mathcal{G}^{\circ}_{\frac{1}{2}}(A,B)\right). \tag{54}$$

When $\frac{1}{2} < \nu < 1$, we have

$$\frac{1}{2\nu}\left(\mathcal{A}^{\circ}_{\nu}(A,B) - \mathcal{G}^{\circ}_{\nu}(A,B)\right) \le \mathcal{A}^{\circ}_{\frac{1}{2}}(A,B) - \mathcal{G}^{\circ}_{\frac{1}{2}}(A,B) \le \frac{1}{2(1-\nu)}\left(\mathcal{A}^{\circ}_{\nu}(A,B) - \mathcal{G}^{\circ}_{\nu}(A,B)\right). \tag{55}$$

Through simple deformations, the conclusion follows from of (54) and (55). □

Remark 3.13. *By (3), we have*

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \le (A \circ I)^{\frac{1}{2}} (B \circ I)^{\frac{1}{2}}.$$

So, we get

$$\mathcal{A}^{\circ}_{\frac{1}{2}}(A,B) - \mathcal{G}^{\circ}_{\frac{1}{2}}(A,B) \ge \frac{1}{2}A \circ I + \frac{1}{2}B \circ I - (A \circ I)^{\frac{1}{2}} (B \circ I)^{\frac{1}{2}} = \frac{1}{2}\left((A \circ I)^{\frac{1}{2}} - (B \circ I)^{\frac{1}{2}}\right)^2. \tag{56}$$

The improvement for (45) can be obtained from the left side of (53) and (56) as follows:

$$\mathcal{A}^{\circ}_{\nu}(A,B) \ge \mathcal{G}^{\circ}_{\nu}(A,B) + \tau\left((A \circ I)^{\frac{1}{2}} - (B \circ I)^{\frac{1}{2}}\right)^2. \tag{57}$$

4. Some Kantorovich type inequalities for Hadamard product of positive matrices

With the rapid development of data acquisition technology, large amounts of data, such as online documents, medical images, traffic data, health data, and other high-dimensional data, are accumulating. Nonnegative matrices have received a lot of attention in the recent years, since they provide a powerful tool in analysing nonnegative data. For example, Adam et al. [1] presented sequences of lower and upper bounds for the spectral radius of a nonnegative matrix. Bui [5] gave an effective bound of the joint spectral radius for a finite set of nonnegative matrices. Xie et al. [16] presented a Kantorovich type inequality for positive matrices, which provided some generalized forms of discrete Kantorovich inequality. In this section, we investigate some Kantorovich type inequalities for Hadamard product of positive matrices.

4.1. Two Kantorovich type inequalities for Hadamard product of positive matrices

Theorem 4.1. *Let $A = [a_{ij}], B = [b_{ij}] \in P^{+}_{m,n}$ and Φ be a strictly positive linear functional on $\mathbb{R}_{m,n}$. Let $m_A = \min_{1\le i\le m,1\le j\le n} a_{ij}$, $M_A = \max_{1\le i\le m,1\le j\le n} a_{ij}$, $m_B = \min_{1\le i\le m,1\le j\le n} b_{ij}$, $M_B = \max_{1\le i\le m,1\le j\le n} b_{ij}$,$R_A = \frac{M_A}{m_A}$, $R_B = \frac{M_B}{m_B}$. Then*

$$1 \le \frac{\Phi\left(A^{(2)}\right)\Phi\left(B^{(2)}\right)}{[\Phi(A\circ B)]^2} \le \frac{(R_A R_B + 1)^2}{4R_A R_B}, \tag{58}$$

$$0 \le \frac{\Phi\left(A^{(2)}\right)}{\Phi(A\circ B)} - \frac{\Phi(A\circ B)}{\Phi\left(B^{(2)}\right)} \le \frac{\left(\sqrt{M_A M_B} - \sqrt{m_A m_B}\right)^2}{M_B m_B}. \tag{59}$$

Proof. Let us first check (58). For any positive number $t > 0$, using the fact that $A\circ B = B\circ A$, $(A-tB)\circ(A-tB) = A^{(2)} + t^2B^{(2)} - 2tA\circ B > 0$, we obtain

$$\Phi\left(A^{(2)}\right) + t^2\Phi\left(B^{(2)}\right) - 2t\Phi(A\circ B) \ge 0.$$

That is

$$\Phi(A\circ B)\le\frac{1}{2}\left(\frac{1}{t}\Phi\left(A^{(2)}\right)+t\Phi\left(B^{(2)}\right)\right). \tag{60}$$

By (60), we have

$$\Phi(A\circ B)\le\min_{t>0}\frac{1}{2}\left(\frac{1}{t}\Phi\left(A^{(2)}\right)+t\Phi\left(B^{(2)}\right)\right)=\sqrt{\Phi\left(A^{(2)}\right)\Phi\left(B^{(2)}\right)}. \tag{61}$$

Then (60) implies the left side of (58).

On the other hand, we have

$$\frac{m_A}{M_B}E_{m,n}\prec A\circ B^{(-1)}\prec\frac{M_A}{m_B}E_{m,n},\ \frac{m_B}{M_A}E_{m,n}\prec A^{(-1)}\circ B\prec\frac{M_B}{m_A}E_{m,n}. \tag{62}$$

By (58), we obtain

$$\begin{aligned}&A\circ B\circ\left(\frac{M_A}{m_B}E_{m,n}-A\circ B^{(-1)}\right)\circ\left(\frac{M_B}{m_A}E_{m,n}-A^{(-1)}\circ B\right)\\&=\left(\frac{M_AM_B}{m_Am_B}+1\right)A\circ B-\frac{M_B}{m_A}A^{(2)}-\frac{M_A}{m_B}B^{(2)}\succ 0.\end{aligned} \tag{63}$$

By (63), we obtain

$$\left(\frac{M_AM_B+m_Am_B}{m_Am_B}\right)\Phi(A\circ B)-\frac{M_B}{m_A}\Phi\left(A^{(2)}\right)-\frac{M_A}{m_B}\Phi\left(B^{(2)}\right)\ge 0.$$

That is

$$\frac{M_B}{m_A}\Phi\left(A^{(2)}\right)+\frac{M_A}{m_B}\Phi\left(B^{(2)}\right)\le\left(\frac{M_AM_B+m_Am_B}{m_Am_B}\right)\Phi(A\circ B). \tag{64}$$

By (64) and the arithmetic geometric mean inequality, we obtain

$$2\sqrt{\frac{M_AM_B}{m_Am_B}}\sqrt{\Phi\left(A^{(2)}\right)\Phi\left(B^{(2)}\right)}\le\frac{M_B}{m_A}\Phi\left(A^{(2)}\right)+\frac{M_A}{m_B}\Phi\left(B^{(2)}\right)\le\left(\frac{M_AM_B+m_Am_B}{m_Am_B}\right)\Phi(A\circ B). \tag{65}$$

Through simple calculations, (65) implies the right side of (58).

Next, we check (59). It is easy to see that the left side of (58) implies the left side of (59). Therefore, we only need to prove the right side of (59). If $\frac{M_A}{m_B}=\frac{m_A}{M_B}$, then $M_A=m_A, M_B=m_B$. So, there exist positive numbers a,b such that $A=aE_{m,n}, B=bE_{m,n}$. In view of this, it is easy to verify (59). We therefore assume $\frac{M_A}{m_B}\ne\frac{m_A}{M_B}$. Let

$$\begin{aligned}x&=\Phi\left(A\circ B\circ\left(\frac{M_A}{m_B}E_{m,n}-A\circ B^{(-1)}\right)\right)=\frac{M_A}{m_B}\Phi\left(A\circ B\right)-\Phi\left(A^{(2)}\right),\\y&=\Phi\left(A\circ B\circ\left(A\circ B^{(-1)}-\frac{m_A}{M_B}E_{m,n}\right)\right)=\Phi\left(A^{(2)}\right)-\frac{m_A}{M_B}\Phi\left(A\circ B\right).\end{aligned} \tag{66}$$

By (62), we have ‘

$$x\ge 0, y\ge 0, x+y=\left(\frac{M_A}{m_B}-\frac{m_A}{M_B}\right)\Phi(A\circ B)>0. \tag{67}$$

By (66) and (67), we have

$$\frac{\Phi\left(A^{(2)}\right)}{\Phi(A\circ B)}=\frac{\frac{m_A}{M_B}x+\frac{M_A}{m_B}y}{x+y}. \tag{68}$$

By (64) and (68), we have

$$\frac{\Phi\left(B^{(2)}\right)}{\Phi(A\circ B)} \le \frac{M_A M_B x + m_A m_B y}{M_A m_A (x+y)}. \tag{69}$$

If $xy = 0$, by (68) and (69), we have

$$\frac{\Phi\left(A^{(2)}\right)}{\Phi(A\circ B)} - \frac{\Phi(A\circ B)}{\Phi\left(B^{(2)}\right)} = 0.$$

Therefore, the conclusion holds.

If $x > 0, y > 0$, by (68) and (69), we have

$$\begin{aligned}\frac{\Phi\left(A^{(2)}\right)}{\Phi(A\circ B)} - \frac{\Phi(A\circ B)}{\Phi\left(B^{(2)}\right)} &\le \frac{\frac{m_A}{M_B}x + \frac{M_A}{m_B}y}{x+y} - \frac{M_A m_A (x+y)}{M_A M_B x + m_A m_B y} \\ &= \frac{(M_A M_B - m_A m_B)^2 xy}{M_B m_B \left(M_A M_B x^2 + m_A m_B y^2 + (M_A M_B + m_A m_B)\, xy\right)}. \end{aligned} \tag{70}$$

From the arithmetic geometric mean inequality, we have

$$\begin{aligned} M_A M_B x^2 + m_A m_B y^2 + (M_A M_B + m_A m_B)\, xy &\ge \left(2\sqrt{M_A M_B m_A m_B} + M_A M_B + m_A m_B\right) xy \\ &= (\sqrt{M_A M_B} + \sqrt{m_A m_B})^2 xy. \end{aligned} \tag{71}$$

Finally, (59) follows from (70) and (71). □

Remark 4.2. *When $B = cA$ $(c > 0)$, it is easy to check the left equalities of (58) and (59) hold. When*

$$m = 2, n = 1, A = \begin{bmatrix} M_A \\ m_A \end{bmatrix}, B = A^{(-1)}, \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2,$$

the right equality in (58) holds.
When

$$m = 2, n = 1, A = \begin{bmatrix} M_A \\ m_A \end{bmatrix}, B = A^{(-1)}, \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = M_A x_1 + m_A x_2,$$

the right equality in (59) holds.

Remark 4.3. *Replacing A, B with $A^{(\frac{1}{2})}, A^{(-\frac{1}{2})}$ respectively in (58) and (59), we can obtain*

$$1 \le \frac{\Phi(A)\Phi\left(A^{(-1)}\right)}{\left[\Phi\left(E_{m,n}\right)\right]^2} \le \frac{(R_A+1)^2}{4R_A}, \tag{72}$$

$$0 \le \frac{\Phi(A)}{\Phi\left(E_{m,n}\right)} - \frac{\Phi\left(E_{m,n}\right)}{\Phi(A^{(-1)})} \le \left(\sqrt{M_A} - \sqrt{m_A}\right)^2. \tag{73}$$

By Lemma 2.11, 2.12, 2.13 and (72), we have the following corollary.

Corollary 4.4. *Let $A \in P_n^+$, $m_A = \min_{1\le i\le n, 1\le j\le n} a_{ij}$, $M_A = \max_{1\le i\le n,1\le j\le n} a_{ij}$, $R_A = \frac{M_A}{m_A}$, and $c_A = \min_{1\le j\le n}\sum_{i=1}^n a_{ij}$. Then*

$$\frac{1}{\| A \|_1} n^2 \le \rho\left(A^{(-1)}\right) \le \frac{(R_A+1)^2}{4R_A c_A} n^2, \tag{74}$$

$$w(A) w(A^{(-1)}) \ge n^2. \tag{75}$$

Proof. Let y be the Perron vector of $A^{(-1)}$. Taking $\Phi(X) = E_{n,1}^T Xy$ ($X \in \mathbb{R}_n$) in (72), we have

$$1 \le \frac{E_{n,1}^T Ay E_{n,1}^T A^{(-1)} y}{[E_{n,1}^T E_n y]^2} \le \frac{(R_A + 1)^2}{4R_A}. \tag{76}$$

By Lemma 2.12, we have

$$c_A \le E_{n,1}^T Ay \le \| A \|_1 . \tag{77}$$

Through simple calculations, we obtain

$$E_{n,1}^T A^{(-1)} y = \rho(A^{(-1)}), E_{n,1}^T E_n y = n. \tag{78}$$

By simple substitution, (74) follows from (76), (77) and (78).

Similarly, for any non negative unit vector $x \in \mathbb{R}^n$ (i.e., $x > 0, x^T x = 1$), taking $\Phi(X) = x^T X x$ ($X \in \mathbb{R}_n$) in (72), we have

$$\frac{x^T A x x^T A^{(-1)} x}{[x^T E_n x]^2} \ge 1. \tag{79}$$

Through simple calculations, we know

$$n^2 = [w(E_n)]^2 = \sup\left\{\left[x^T E_n x\right]^2 : x > 0, x^T x = 1\right\} \le \sup\left\{x^T A x x^T A^{(-1)} x : x > 0, x^T x = 1\right\} \le w(A) w\left(A^{(-1)}\right).$$

□

If $A \in P_n^+$ is a doubly stochastic matrix, by (74), we know $n^2 \le \rho\left(A^{(-1)}\right) \le \frac{(R_A+1)^2}{4R_A} n^2$ and equalities hold when $A = \frac{1}{n} E_n$.

4.2. *Several Kantorovich type inequalities involving the permanent of positive matrices*

By (73), we have the following corollary.

Corollary 4.5. *If $\lambda_i > 0, i = 1, 2, \cdots, n$, $\rho_\lambda = \max_{1 \le i \le n}\{\lambda_i\}$, $\sigma_\lambda = \min_{1 \le i \le n}\{\lambda_i\}$, then*

$$0 \le \frac{\sum_{i=1}^n \lambda_i}{n} - \frac{n}{\sum_{i=1}^n \lambda_i^{-1}} \le \left(\sqrt{\rho_\lambda} - \sqrt{\sigma_\lambda}\right)^2. \tag{80}$$

Proof. Taking $A = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Phi(X) = X E_{n,1}, X \in \mathbb{R}_{1,n}$ in (73), the conclusion follows from simple substitution. □

By Theorem 4.1, we have the following corollary.

Corollary 4.6. *Let $\lambda_i, \mu_i > 0, i = 1, 2, \cdots, n$ and $w_i > 0$ with $\sum_{i=1}^n w_i = 1$. Let $\rho_\lambda = \max_{1 \le i \le n}\{\lambda_i\}$, $\sigma_\lambda = \min_{1 \le i \le n}\{\lambda_i\}$, $\rho_\mu = \max_{1 \le i \le n}\{\mu_i\}$, $\sigma_\mu = \min_{1 \le i \le n}\{\mu_i\}$, $\kappa_\lambda = \frac{\rho_\lambda}{\sigma_\lambda}$ and $\kappa_\mu = \frac{\rho_\mu}{\sigma_\mu}$. Then*

$$\left|\sum_{i=1}^n w_i \lambda_i \mu_i - \sum_{i=1}^n w_i \lambda_i \sum_{i=1}^n w_i \mu_i\right| \le \frac{1}{4}\left(\rho_\mu - \sigma_\mu\right)\left(\rho_\lambda - \sigma_\lambda\right), \tag{81}$$

$$\left(\frac{\sqrt{\kappa_\lambda} + \sqrt{\kappa_\mu}}{\sqrt{\kappa_\lambda \kappa_\mu} + 1}\right)^2 \le \frac{\sum_{i=1}^n w_i \lambda_i \mu_i}{\sum_{i=1}^n w_i \lambda_i \sum_{i=1}^n w_i \mu_i} \le \left(\frac{\sqrt{\kappa_\lambda \kappa_\mu} + 1}{\sqrt{\kappa_\lambda} + \sqrt{\kappa_\mu}}\right)^2. \tag{82}$$

Proof. Setting $A = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $B = diag\{\mu_1, \mu_2, \cdots, \mu_n\}$ $x = \left(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_n}\right)$, $\Phi(X) = x^{\dagger}Xx, X \in M_n$ in (16) and (17), it is easy to see that $\rho(A) = \rho_\lambda$, $\sigma(A) = \sigma_\lambda$, $\rho(B) = \rho_\mu$, $\sigma(B) = \sigma_\mu$, $\kappa(A) = \kappa_\lambda$ and $\kappa(B) = \kappa_\mu$. Through simple substitution, (81) and (82) follow from (16) and (17). □

From Corollary 4.5, and Corollary 4.6, we have the Kantorovich type inequality involving the permanent of positive matrices as follows:

Corollary 4.7. *Let* $A = [a_{ij}], B = [b_{ij}] \in P_n^+$, $P_A = \frac{M_A^n}{m_A^n}$, $P_B = \frac{M_B^n}{m_B^n}$. *Then*

$$0 \leq \frac{Per(A)}{n!} - \frac{n!}{Per(A^{(-1)})} \leq \left(M_A^{\frac{n}{2}} - m_A^{\frac{n}{2}}\right)^2, \tag{83}$$

$$\left|\frac{1}{n!}Per(A \circ B) - \frac{1}{(n!)^2}Per(A)Per(B)\right| \leq \frac{1}{4}\left(M_A^n - m_A^n\right)\left(M_B^n - m_B^n\right), \tag{84}$$

$$\frac{1}{n!}\left(\frac{\sqrt{P_A} + \sqrt{P_B}}{\sqrt{P_A P_B} + 1}\right)^2 \leq \frac{Per(A \circ B)}{Per(A)Per(B)} \leq \frac{1}{n!}\left(\frac{\sqrt{P_A P_B} + 1}{\sqrt{P_A} + \sqrt{P_B}}\right)^2. \tag{85}$$

Proof. Let S_n be a permutation group with order n. According to the definition of permanent, we have

$$\begin{aligned} per(A) &= \sum_{\pi_1\pi_2\cdots\pi_n \in S_n} a_{1\pi_1}a_{2\pi_2}\cdots a_{n\pi_n}, per(B) = \sum_{\pi_1\pi_2\cdots\pi_n \in S_n} b_{1\pi_1}b_{2\pi_2}\cdots b_{n\pi_n}, \\ per(A \circ B) &= \sum_{\pi_1\pi_2\cdots\pi_n \in S_n} a_{1\pi_1}a_{2\pi_2}\cdots a_{n\pi_n}b_{1\pi_1}b_{2\pi_2}\cdots b_{n\pi_n}. \end{aligned} \tag{86}$$

Let

$$\lambda_{\pi_1\pi_2\cdots\pi_n} = a_{1\pi_1}a_{2\pi_2}\cdots a_{n\pi_n}, \mu_{\pi_1\pi_2\cdots\pi_n} = b_{1\pi_1}b_{2\pi_2}\cdots b_{n\pi_n}, \omega_{\pi_1\pi_2\cdots\pi_n} = \frac{1}{n!}.$$

It is easy to see that

$$\rho_\lambda \leq M_A^n, \sigma_\lambda \geq m_A^n, \rho_\mu \leq M_B^n, \sigma_\mu \geq m_B^n, \frac{\sqrt{\kappa_\lambda \kappa_\lambda} + 1}{\sqrt{\kappa_\lambda} + \sqrt{\kappa_\mu}} \leq \frac{\sqrt{P_A P_B} + 1}{\sqrt{P_A} + \sqrt{P_B}}. \tag{87}$$

By simply substituting (79), (80), (81), (86) and (87), we obtain (83), (84) and (85). □

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] M. Adam, I. Oikonomou, A. Aretaki, *Sequences of lower and upper bounds for the spectral radius of a nonnegative matrix*, Linear Algebra Appl. **667** (2023), 165–191.

[2] H. Alzer, C. M. da Fonseca, A. Kovačec, *Young-type inequalities and their matrix analogues*, Linear Multilinear A. **63** (2015), 622–635.

[3] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. **26** (1979), 203–241.

[4] VI. Bogachev, AV. Shaposhnikov, F. Y. Wang, *Sobolev-Kantorovich inequalities under* $CD(0, \infty)$ *condition*, Commun. Contemp. Math. **24** (2022), 2150027.

[5] V. Bui, *On the joint spectral radius of nonnegative matrices*, Linear Algebra Appl. **654** (2022), 89–101.

[6] W. Greub, W. Rheinboldt, *On a generalization of an inequality of LV Kantorovich*, Proc. Amer. Math. Soc. **10** (1959), 407–415.

[7] RA. Horn, CR. Johnson, *Matrix Analysis*, Cambridge university press, Cambridge, 2012.

[8] LV. Kantorovich,*Functional analysis and applied mathematics (Russian), Uspekhi Mat Nauk (NS)*. **3** (1948),89-185.
[9] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), 262–269.
[10] F. Kittaneh, Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear A. **59** (2011), 1031–1037.
[11] B. Mond, JD. Pečarić, *A matrix version of the Ky Fan generalization of the Kantorovich inequality*, Linear Multilinear A. **36** (1994), 217–221.
[12] B. Mond, JD. Pečarić, *A matrix version of the Ky Fan generalization of the Kantorovich inequality II*, Linear Multilinear A. **38** (1995), 309–313.
[13] HR. Moradi, IH. Gümüş, Z. Heydarbeygi, *A glimpse at the operator Kantorovich inequality*, Linear Multilinear A. **67** (2019), 1031–1036.
[14] P. Sabancigil, M. Kara, NI. Mahmudov, *Higher order Kantorovich-type Szász-Mirakjan operators*, J. Inequal. Appl. **2022** (2022), 91.
[15] S. Steinerberger, *On a Kantorovich-Rubinstein inequality*, J. Math. Anal. Appl. **501** (2021), 125185.
[16] Z. Xie, J. Liu, M. Zhang, *A Kantorovich type inequality for positive matrices*, Linear Multilinear A. **73** (2025), 1777–1791.