Filomat 40:4 (2026), 1479–1490
https://doi.org/10.2298/FIL2604479B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation of certain operators on time scales via deferred Cesàro mean

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Abstract. The paper first studies the basic notions of statistical convergence for a sequence of time scale functions via deferred Cesàro summability mean. Some useful limit related and inclusion results among these newly defined notions have been examined. Also based upon these notions, two Korovkin-type approximation theorems with certain algebraic test functions have been established. An example is provided considering a positive linear operator in association with Bernstein polynomial to clarify the relevant results. Moreover, the rate of statistical deferred Cesàro summability is estimated.

1. Introduction and Motivation

The notion of statistical convergence was introduced by Zygmund (see [27]) in 1935. Fast (see [6]) and Steinhaus (see [24]) independently introduced Statistical Convergence in sequence space theory in the year 1951. Statistical convergence is closely related with the study of measure Theory, probability Theory, Fibonacci sequence, etc. Statistical convergence has been an active area of research in the current time. Some of the recent works in approximation theory using statistical convergence are [5, 11, 12, 15, 16, 18–23].

A time scale is an arbitrary non-empty closed subset of the real numbers. It is denoted by the symbol $\mathbb{T}$. Throughout the study we assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. The calculus of time scale was introduced by Stefan Hilger in his Ph.D. thesis supervised by Aulbach in 1988 (see [9, 10]). This allowed the unification of discrete and continuous analysis. Here one can replace the range of definition of the functions under consideration by an arbitrary time scale $\mathbb{T}$. There are many applications of time scales in dynamic equations (see [4]). The continuous version of statistical convergence was studied by Móricz (see [14]). Guseinov (see [8]) introduced and investigated the concept of Riemann's Δ- and ∇- integrals on time scale. Several other studies on time scale calculus have been presented in [1, 9, 17, 25] and there are many more.

We first discuss some important terms and notions on time scale (see [8]):

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) = \inf \{s \in \mathbb{T} \ : s > t\}$$

2020 *Mathematics Subject Classification*. Primary 40A30; Secondary 26E70, 40A35, 40C05.
Keywords. Sequence of functions; Time scale; Statistical convergence; Summability; Korovkin-type approximation theorems.
Received: 03 October 2025; Accepted: 30 November 2025
Communicated by Dragan S. Djordjević
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and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given by

$$\mu(t) = \sigma(t) - t$$

Here we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ is the empty set.

A closed interval, open interval and semi-closed (or semi-open) interval on a time scale $\mathbb{T}$ are given by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$, $(a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a < t < b\}$ and $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t < b\}$ respectively.

Next, let S be the collection of all left closed and right open intervals of the form $[a, b)_{\mathbb{T}}$. Then the set function $m : S \to [0, \infty)$ defined by $m([a, b)) = b - a$ is a countably additive measure. The Carathéodory extension of the set function m associated with the family S is called the Lebesgue Δ-measure on $\mathbb{T}$ and it is denoted by μ_Δ (see [8, 17]).

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is Δ-measurable if the set $f^{-1}(A)$ is Δ-measurable for every open subset A of $\mathbb{R}$.

Theorem 1.1 ([8]). *For each $a \in \mathbb{T} - \{\max \mathbb{T}\}$, the singleton point set $\{a\}$ is Δ-measurable, and its Δ-measure is given by*

$$\mu_\Delta(a) = \sigma(a) - a.$$

Theorem 1.2 ([8]). *If $a, b \in \mathbb{T}$ and $a \le b$, then*

$$\mu_\Delta([a, b)) = b - a, \text{ and } \mu_\Delta((a, b)) = b - \sigma(a).$$

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ anda $\le b$, then,

$$\mu_\Delta((a, b]) = \sigma(b) - \sigma(a), \text{ and } \mu_\Delta([a, b]) = \sigma(b) - a.$$

Definition 1.3 ([25]). *Let Ω be a Δ-measurable subset of $\mathbb{T}$. Then for $t \in \mathbb{T}$, we define the set $\Omega(t)$ by*

$$\Omega(t) = \{s \in [t_o, t]_{\mathbb{T}} : s \in \Omega\}$$

The density of the set Ω on $\mathbb{T}$, denoted by $\delta_{\mathbb{T}}(\Omega)$, is defined as

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0, t]_{\mathbb{T}})}$$

provided the above limit exists.

Here if $\mathbb{T} = \mathbb{N}$, then the concept reduces to asymptotic density (or natural density) and if $\mathbb{T} = [0, \infty)$, then the concept implies approximate density. In this paper, we shall mainly use the Lebesgue Δ-measure μ_Δ introduced by Guseinov in [8]. Here $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$.

In this paper, we have studied certain notions of a sequence of Δ-measurable functions on time scales and provided Korovkin-type approximation theorems in the time scale framework.

2. Statistical Convergence of a Sequence Functions on Time scales

Definition 2.1. *A sequence $(f_n)_{n=1}^{\infty}$ of Δ-measurable functions on $\mathbb{T}$ is said to be convergent to a Δ-measurable function f on $\mathbb{T}$ if, for each $\epsilon > 0$, there exists $m \in \mathbb{N}$ and $N_\epsilon \subset \mathbb{T}$ such that $\delta_{\mathbb{T}}(N_\epsilon) = 1$ and*

$$\left|f_n(t) - f(t)\right| < \epsilon \text{ for all } t \in N_\epsilon \text{ and } n \ge m.$$

We denote it by

$$\Delta\text{M} \lim_{n\to\infty} f_n = f.$$

Next, we present the notion of statistical convergence of a sequence of Δ - measurable functions on $\mathbb{T}$.

Definition 2.2. *A sequence (f_n) of Δ-measurable functions on $\mathbb{T}$ is said to be statistically convergent to a Δ-measurable function f on $\mathbb{T}$ if, for each $\epsilon > 0$, the set*

$$E(\epsilon) = \left\{ s \ : s \in [t_0, t_k]_{\mathbb{T}} \textit{ and } \left|f_k(s) - f(s)\right| \geq \epsilon \right\}$$

has zero density, where $t_k \in \mathbb{T}$, for $k = 1, 2, 3, \ldots$, and $t_k \to \infty$ as $k \to \infty$. Thus, for each $\epsilon > 0$,

$$\delta_{\mathbb{T}}(E(\epsilon)) = \lim_{k\to\infty} \frac{\mu_\Delta(E(\epsilon))}{\mu_\Delta([t_0, t_k]_{\mathbb{T}})} = 0.$$

We denote this by

$$\text{stat}_{\Delta\text{M}} \lim_{n\to\infty} f_n = f.$$

Next, we present some basic fundamental limit theorems on sequence of Δ-measurable functions on $\mathbb{T}$.

Theorem 2.3. *Let the sequence (f_n) of Δ-measurable functions is statistically convergent on $\mathbb{T}$, then it has unique limit.*

Proof: Let f and g be Δ-measurable functions on $\mathbb{T}$ such that

$$\text{stat}_{\Delta\text{M}} \lim_{n\to\infty} f_n = f \text{ and } \text{stat}_{\Delta\text{M}} \lim_{n\to\infty} f_n = g.$$

Since, $\left|f - g\right| = \left|f_k - f_k + f - g\right| \leq \left|f_k - f\right| + \left|f_k - g\right|$, for each $k = 1, 2, 3, ...$

Then, for statistical convergence of sequence of Δ-measurable functions on $\mathbb{T}$, for each $\epsilon > 0$, we have

$$\text{stat}_{\Delta\text{M}} \left|f - g\right| \leq \text{stat}_{\Delta\text{M}} \left|f_k - f\right| + \text{stat}_{\Delta\text{M}} \left|f_k - g\right| < \epsilon.$$

This implies $f = g$.

Theorem 2.4. *If a sequence (f_n) of Δ-measurable functions is convergent on $\mathbb{T}$, then it is statistically convergent on $\mathbb{T}$. But the converse may not be true.*

Proof: The sequence $f_n : \mathbb{T} \to \mathbb{R}$ being convergent, for each $\epsilon > 0$, there exists $m \in \mathbb{N}$ and $N_\epsilon \subset \mathbb{T}$ such that $\delta_{\mathbb{T}}(N_\epsilon) = 1$ and $\left|f_n(t) - f(t)\right| < \epsilon$ for each $t \in K_\epsilon$ and $n \geq m$.
Therefore, for each $\epsilon > 0$, $n \geq m$ and the set

$$E(\epsilon) = \left\{s : s \in [t_0,\ t_k]_{\mathbb{T}} \text{ and } \left|f_k(s) - f(s)\right| \geq \epsilon\right\},$$

the following inequality holds

$$0 \leq \lim_{k\to\infty} \frac{\mu_\Delta\,(E(\epsilon))}{\mu_\Delta\,([t_0, t_k]_{\mathbb{T}})} \leq \lim_{k\to\infty} \left|f_k(t) - f(t)\right| < \epsilon.$$

Hence, by definition 2.2, we get $\text{stat}_{\Delta\text{M}} \lim_{n\to\infty} f_n = f$ on $\mathbb{T}$.

The following example shows that the converse is not always true.

Example 2.5. *Let $\mathbb{T} = [0, 1]$. We consider the sequence $f_n : \mathbb{T} \to \mathbb{R}$ of functions defined by*

$$f_n(s) = \begin{cases} \frac{1}{2}, & \textit{for } 0 \leq s < \frac{1}{n},\ n = j^2\ (j \in \mathbb{N}) \\ 0, & \textit{otherwise} \end{cases}$$

Here the sequence (f_n) of Δ-measurable functions is statistically convergent to the function $f = 0$ over $\mathbb{T} = [0, \infty)$, but it is not usual convergent over $\mathbb{T} = [0, \infty)$.

3. Deferred Cesàro Statistical Convergence & Statistically Deferred Cesàro Summability

Following [2], we present notion of deferred Cesàro summability mean for a sequence of time scale functions as follows:

Definition 3.1. *Let (a_k) and (b_k) be two sequences of non-negative integers on $\mathbb{T}$ which satisfy the regularity conditions $a_k < b_k$ and $\lim_{k\to\infty} b_k = +\infty$. Then we define the deferred Cesàro summability mean for the sequence (f_n) of Δ-measurable functions on $\mathbb{T}$ as*

$$\phi_k = \frac{1}{b_k - a_k} \sum_{i=a_k+1}^{b_k} f_i.$$

Next, we present the definitions of statistical convergence and statistical summability of a sequence of Δ-measurable functions on $\mathbb{T}$ via deferred Cesàro summability mean as follows:

Definition 3.2. *Let (a_k) and (b_k) be two sequences of non-negative integers on $\mathbb{T}$ which satisfy the regularity conditions $a_k < b_k$ and $\lim_{k\to\infty} b_k = +\infty$. A sequence (f_n) of Δ-measurable functions is said to be deferred Cesàro statistically convergent to a Δ-measurable function f on $\mathbb{T}$ if, for each $\epsilon > 0$, the set*

$$E(\epsilon) = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{ and } \left|f_k(s) - f(s)\right| \geq \epsilon\right\}$$

has zero density. Thus, for each $\epsilon > 0$,

$$\delta_{\mathbb{T}}(E(\epsilon)) = \lim_{k\to\infty} \frac{\mu_\Delta(E(\epsilon))}{\mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} = 0.$$

We denote this by

$$\mathrm{D\Delta M_{stat}} \lim_{n\to\infty} f_n = f.$$

Definition 3.3. *Let (a_k) and (b_k) be two sequences of non-negative integers on $\mathbb{T}$ which satisfy the regularity conditions $a_k < b_k$ and $\lim_{k\to\infty} b_k = +\infty$. A sequence (f_n) of Δ-measurable functions is said to be statistically deferred Cesàro summable to a Δ-measurable function f on $\mathbb{T}$ if, for each $\epsilon > 0$, the set*

$$E(\epsilon) = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{ and } \left|\phi_k(s) - f(s)\right| \geq \epsilon\right\}$$

has zero density. Thus, for each $\epsilon > 0$,

$$\delta_{\mathbb{T}}(E(\epsilon)) = \lim_{k\to\infty} \frac{\mu_\Delta(E(\epsilon))}{\mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} = 0.$$

We denote this by

$$\mathrm{stat_{D\Delta M}} \lim_{n\to\infty} f_n = f.$$

Now, we give an inclusion theorem relating these two potentially useful notions.

Theorem 3.4. *Let (a_k) and (b_k) be two sequences of non-negative integers which satisfy the regularity conditions $a_k < b_k$ and $\lim_{k\to\infty} b_k = +\infty$. If a sequence (f_n) of Δ-measurable functions is deferred Cesàro Statistical convergent to a Δ-measurable function f on $\mathbb{T}$, then it is statistically deferred Cesàro summable to the same function f on $\mathbb{T}$, but the converse is not always true.*

Proof: Suppose the sequence $f_n : \mathbb{T} \to \mathbb{R}$ of Δ-measurable functions is deferred Cesàro statistical convergent to a Δ-measurable function f on $\mathbb{T}$, by Definition 3.2, we have

$$\delta_{\mathbb{T}}(E) = \lim_{k\to\infty} \frac{\mu_\Delta(E)}{\mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} = 0,$$

where $E = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{ and } \left|f_k(s) - f(s)\right| \geq \epsilon\right\}$. Now, we consider two sets as follows:

$$E_\epsilon = E = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{ and } \left|f_k(s) - f(s)\right| \geq \epsilon\right\},$$

$$\text{and, } (E_\epsilon)^c = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{ and } \left|f_k(s) - f(s)\right| < \epsilon\right\}.$$

Then we have

$$\begin{aligned}
\left|\phi_k - f\right| &= \left|\frac{1}{b_k - a_k}\sum_{i=a_k+1}^{b_k} f_i - f\right| \\
&\leq \left|\frac{1}{b_k - a_k}\sum_{i=a_k+1}^{b_k} f_i - f\right| + \left|\frac{1}{b_k - a_k}\sum_{i=a_k+1}^{b_k} f - f\right| \\
&\leq \frac{1}{b_k - a_k}\sum_{\substack{i = a_k+1 \\ (m \in E_\epsilon)}}^{b_k} \left|f_i - f\right| + \frac{1}{b_k - a_k}\sum_{\substack{i = a_k+1 \\ (m \in (E_\epsilon)^c)}}^{b_k} \left|f_i - f\right| + |f|\left|\frac{1}{b_k - a_k}\sum_{i=a_k+1}^{b_k} 1 - 1\right| \\
&\leq \frac{1}{P_k}\left|\mu_\Delta\left(E_\epsilon\right)\right| + \frac{1}{P_k}\left|\mu_\Delta(E_\epsilon)^c\right|
\end{aligned}$$

where $P_k = \mu_\Delta((a_k,\ b_k]_{\mathbb{T}})$. Then the above inequality implies $|\phi_k - f| < \epsilon$. This shows that the sequence (f_n) of Δ-measurable functions is statistically deferred Cesàro summable to the Δ-measurable function f on $\mathbb{T}$.

Example 3.5. *Let $a_k = 2k$ and $b_k = 4k$ and $\mathbb{T} = [0, \infty)$. Let (f_n) be a sequence of Δ-measurable functions on $\mathbb{T}$ of the form given by*

$$f_k(t) = \begin{cases} 1, & if\ t \in [0,\ k+1)_{\mathbb{T}}; k \text{ is even} \\ 0, & if\ t \in [k+1,\ \infty)_{\mathbb{T}};\ k \text{ is odd}. \end{cases}$$

By the definition of deferred Cesàro summability mean, we get

$$\begin{aligned}
\phi(t) &= \frac{1}{b_k - a_k}\sum_{i=a_k+1}^{b_k} f_i(t) \\
&= \frac{1}{2k}\sum_{i=2k+1}^{4k} f_i(t) = \frac{1}{2}.
\end{aligned}$$

Thus, the sequence (f_n) of Δ-measurable functions on $\mathbb{T}$ has deferred Cesàro summability mean $\frac{1}{2}$ over $\mathbb{T}$. Therefore, the sequence is statistically deferred Cesàro summable to $\frac{1}{2}$ over $\mathbb{T}$. But it is not deferred Cesàro statistically convergent.

4. Korovkin-Type Approximation Theorems on Time Scales

Let $[a, b]_{\mathbb{T}} \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale. And let $C([a, b]_{\mathbb{T}})$ denote the space of all continuous real valued functions defined on $[a, b]_{\mathbb{T}}$, then $C([a, b]_{\mathbb{T}})$ is a complete normed linear space (Banach space) with the sup norm $\|.\|_\infty$. Then for $f \in C([a, b]_{\mathbb{T}})$, the norm of f is given by

$$\left\|f\right\|_\infty = \sup\ \left\{\left|f(\eta)\right|\ : \eta \in [a, b]_{\mathbb{T}}\right\}.$$

We say that a sequence of linear operators $\Re_j : C([a,b]_{\mathbb{T}}) \to C([a,b]_{\mathbb{T}})$ is positive if

$$\Re_j(f,\ \eta) \geq 0 \text{ as } f \geq 0.$$

Now, with the help of our proposed mean, we use the notions of deferred Cesàro statistical convergence ($D\Delta M_{stat}$) and statistically deferred Cesàro summability ($stat_{D\Delta M}$) for the sequences of Δ-measurable functions on $\mathbb{T}$ to state and prove the following Korovkin-type approximation theorems. Jena *et al.* (see [11]) introduced statistical deferred Cesàro summability for a sequence of real numbers in approximation results of Korovkin-type.

Theorem 4.1. *Let $\Re_j : C([a,b]_{\mathbb{T}}) \to C([a,b]_{\mathbb{T}})$ be a sequence of positive linear operators. Then for all $f \in C(\mathbb{T})$,*

$$D\Delta M_{stat} \lim_{j\to\infty} \left\| \Re_j(f;\eta) - f(\eta)) \right\|_\infty = 0 \tag{1}$$

if and only if

$$D\Delta M_{stat} \lim_{j\to\infty} \left\| \Re_j(1;\eta) - f(\eta) \right\|_\infty = 0, \tag{2}$$

$$D\Delta M_{stat} \lim_{j\to\infty} \left\| \Re_j(\eta;\eta) - f(\eta) \right\|_\infty = 0, \tag{3}$$

$$\text{and } D\Delta M_{stat} \lim_{j\to\infty} \left\| \Re_j(\eta^2;\eta) - f(\eta) \right\|_\infty = 0. \tag{4}$$

Proof: Since, each of the following functions

$$f_0(\eta) = 1,\ f_0(\eta) = \eta, \text{ and } f_0(\eta) = \eta^2$$

are members of $C([a,b]_{\mathbb{T}})$, so the condition (1) obviously implies the conditions (2), (3) and (4).

For the converse part, we assume that the conditions (2), (3) and (4) hold. If $f \in C([a,b]_{\mathbb{T}})$, then there exists a constant $\tau > 0$ such that $\left|f(\eta)\right| \leq \tau$ for each $\eta \in [a,b]_{\mathbb{T}}$. We thus find that

$$\left|f(\xi) - f(\eta)\right| \leq 2\tau \text{ for each } \xi, \eta \in [a,b]_{\mathbb{T}} \tag{5}$$

This implies, for each $\epsilon > 0$, there exists $\theta > 0$ such that

$$\left|f(\xi) - f(\eta)\right| < \epsilon \tag{6}$$

whenever $|\xi - \eta| < \theta$ for each $\xi,\ \eta \in [a,b]_{\mathbb{T}}$.

We choose $\lambda_1 = \lambda_1(\xi,\eta) = (2\xi - 2\eta)^2$.

If $|\xi - \eta| \geq \theta$, then we obtain

$$\left|f(\xi) - f(\eta)\right| < 2\tau = \frac{2\tau\lambda_1}{\lambda_1} = \frac{2\tau\lambda_1}{4(\xi-\eta)^2} \leq \frac{2\tau}{4\theta^2}\lambda_1 \leq \frac{2\tau}{\theta^2}\lambda_1(\xi,\ \eta) \tag{7}$$

From inequalities (6) and (7), we get

$$\left|f(\xi) - f(\eta)\right| < \epsilon + \frac{2\tau}{\theta^2}\lambda_1(\xi,\eta)$$

which implies

$$-\epsilon - \frac{2\tau}{\theta^2}\lambda_1(\xi,\eta) < f(\xi) - f(\eta) < \epsilon + \frac{2\tau}{\theta^2}\lambda_1(\xi,\eta) \tag{8}$$

Now, $\Re_m(1;\eta)$ being monotone and linear, we apply the operator $\Re_m(1;\eta)$ to the above inequality (8) as

$$\Re_m(1;\eta)\left[-\epsilon - \frac{2\tau}{\theta^2}\lambda_1(\xi,\eta)\right] \leq \Re_m(1;\eta)\left(f(\xi) - f(\eta)\right) \leq \Re_m(1;\eta)\left[\epsilon + \frac{2\tau}{\theta^2}\lambda_1(\xi,\eta)\right]$$

Here η is fixed, so $f(\eta)$ is constant. So, we have

$$-\epsilon\,\Re_m(1;\eta)-\frac{2\tau}{\theta^2}\Re_m(\lambda_1\,;\eta)\le\Re_m(f\,;\eta)-f(\eta)\Re_m(1;\eta)\le\epsilon\,\Re_m(1;\eta)+\frac{2\tau}{\theta^2}\Re_m(\lambda_1;\eta)\tag{9}$$

Also, we have

$$\Re_m(f;\eta)-f(\eta)=[\Re_m(f;\eta)-f(\eta)\Re_m(1;\eta)]+f(\eta)[\Re_m(1;\eta)-1]\tag{10}$$

Using the conditions (9) and (10), we have

$$\Re_m(f;\eta)-f(\eta)\le\epsilon\Re_m(1;\eta)+\frac{2\tau}{\theta^2}\Re_m(\lambda_1;\eta)+f(\eta)\,[\Re_m(1;\eta)-1]\tag{11}$$

Now, we estimate $\Re_m(\lambda_1;\eta)$ as follows

$$\begin{aligned}\Re_m(\lambda_1\,;\eta)&=\Re_m\left((2\xi-\;2\eta)^2\,;\eta\right)\\&=\Re_m\left(4\xi^2-8\xi\eta+4\eta^2\,;\eta\right)\\&=\Re_m(4\xi^2\,;\eta)-8\eta\Re_m(\xi;\eta)+4\eta^2\Re_m(1;\eta)\\&=4[\Re_m(\xi^2;\eta)-\eta^2]-8\eta[\Re_m(\xi;\eta)-\eta]+4\eta^2[\Re_m(1\,;\eta)-1]\end{aligned}$$

Using this estimation, the inequality (11) becomes

$$\begin{aligned}\Re_m(f;\eta)-f(\eta)&\le\epsilon\Re_m(1\,;\eta)+\frac{2\tau}{\theta^2}\Big\{4[\Re_m(\xi^2\,;\eta)-\eta^2]-8\eta[\Re_m(\xi;\eta)-\eta]\\&\quad+4\eta^2[\Re_m\,(1\,;\eta)-1]\Big\}+f(\eta)\,[\Re_m\,(1\,;\eta)-1]\\&=\epsilon\,[\Re_m\,(1\,;\eta)-1]+\epsilon+\frac{2\tau}{\theta^2}\{4[\Re_m(\xi^2\,;\eta)-\eta^2]-8\eta[\Re_m(\xi;\eta)-\eta]\\&\quad+4\eta^2[\Re_m(1\,;\eta)-1]+f(\eta)\,[\Re_m(1\,;\eta)-1]\end{aligned}$$

Since, $\epsilon>0$, we can write this as

$$\begin{aligned}\left|\Re_m(f;\eta)-f(\eta)\right|&\le\epsilon+\left(\epsilon+\frac{8\tau b^2}{\theta^2}+\tau\right)\left|\Re_m(1;\eta)-1\right|\frac{16\tau b}{\theta^2}\left|\Re_m(\xi\,;\;\eta)-\eta\right|+\frac{8\tau}{\theta^2}\left|\Re_m(\xi^2;\eta)-\eta^2\right|\\&\le\vartheta\;\left\{\left|\Re_m\,(1\,;\;\eta)-1\right|+\left|\Re_m\,(\xi\,;\;\eta)-\eta\right|+\left|\Re_m\left(\xi^2\,;\;\eta\right)-\eta^2\right|\right\}\end{aligned}\tag{12}$$

where

$$\vartheta=\max\left(\epsilon+\frac{8\tau b^2}{\theta^2}+\tau,\frac{16\tau b}{\theta^2},\frac{8\tau}{\theta^2}\right)$$

and $\epsilon>0$ being arbitrary. Now, for a given $\chi>0$, there exists $\epsilon>0$ $(\epsilon<\chi)$ such that

$$\mathcal{L}_m\,(\chi)=\left\{\eta\;:\eta\in(a_k,\;b_k]_{\mathbb{T}}\text{ and }\left|\Re_m\,(f\;;\;\eta)-f\,(\eta)\right|\ge\chi\right\}.$$

Also, for $j=0,\;1,\;2$, we have

$$\mathcal{L}_{j,\;m}\,(\chi)=\left\{\eta\;:\eta\in(a_k,\;b_k]_{\mathbb{T}}\text{ and }\left|\Re_m\left(f_j\;;\;\eta\right)-f_j(\eta)\right|\ge\frac{\chi-\epsilon}{3\vartheta}\right\}$$

such that

$$\mathcal{L}_m(\chi)\le\sum_{j=0}^{2}\mathcal{L}_{j,\;m}\,(\chi).$$

This clearly gives

$$\frac{\mu_\Delta(\mathcal{L}_m(\chi))}{\mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} \leq \sum_{j=0}^{2} \frac{\mu_\Delta\left(\mathcal{L}_{j,\ m}(\chi)\right)}{\mu_\Delta\left((a_k,\ b_k]_{\mathbb{T}}\right)} \tag{13}$$

From the inequality (12), the RHS of this inequality (13) vanishes as $k \to \infty$. Hence

$$\lim_{k\to\infty} \frac{\mu_\Delta(\mathcal{L}_m(\chi))}{\mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} = 0.$$

This completes the proof.

Theorem 4.2. *Let $\Re_j : C([a,b]_{\mathbb{T}}) \to C([a,b]_{\mathbb{T}})$ be a sequence of positive linear operators. Then for all $f \in C([a,b]_{\mathbb{T}})$,*

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{j\to\infty} \left\|\Re_j(f\ ;\ \eta) - f(\eta)\right\|_\infty = 0 \tag{14}$$

if and only if

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{j\to\infty} \left\|\Re_j(1\ ;\ \eta) - f(\eta)\right\|_\infty = 0\ , \tag{15}$$

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{j\to\infty} \left\|\Re_j(\eta\ ;\eta) - f(\eta)\right\|_\infty = 0\ , \tag{16}$$

$$\text{and}, \text{stat}_{\text{D}\Delta\text{M}} \lim_{j\to\infty} \left\|\Re_j(\eta^2\ ;\eta) - f(\eta)\right\|_\infty = 0. \tag{17}$$

Proof: The proof of Theorem 4.2 is similar to that of Theorem 4.1.

Next, we consider an example of a sequence of positive linear operators that does not work via the deferred Cesàro statistical convergence of a sequence of Δ-measurable functions on $\mathbb{T}$ (Theorem 4.1), but it fairly works on Theorem 4.2. In view of this example, we can say that Theorem 4.2 is a non-trivial extension of deferred Cesàro statistical convergence of a sequence of Δ-measurable functions on $\mathbb{T}$.

We now recall the operator

$$\rho(1+\rho D) \text{ where } (D = \tfrac{d}{d\rho}) \tag{18}$$

This operator was used by Al-Salam (see [3]), Viskov and Srivastava (see [26]) and many more.

Example 4.3. *Consider the Bernstein polynomial $\beta_n(f;t)$ on $C(\mathbb{T})$ given by*

$$\beta_k(f;t) = \sum_{i=0}^{k} f(\frac{i}{k}) \binom{k}{i} t^i (1-t)^{k-i} \tag{19}$$

where $t \in \mathbb{T} = [0,\infty)$ and $k = 0,\ 1,\ 2, \ldots.$

We now introduce the positive linear operators on $C(\mathbb{T})$ under the composition of Bernstein polynomial $\beta_n(f;t)$ and the operator mentioned above in (18) *as given below:*

$$\Re_i(f;t) = [1+f_i]t(1+tD)\beta_i(f;t), \text{for each} f \in\ C(\mathbb{T}) \tag{20}$$

where (f_i) is the same sequence of Δ-measurable functions on $\mathbb{T}$ mentioned in Example 3.5.

Next, we will estimate the values of each of the testing functions $1, t$ and t^2 by using the operator mentioned in (20) *in the following way*

$$\Re_i(1;t) = [1+f_i]\,t\,(1+tD)\,1 = [1+f_i]t$$

$$\Re_i(x;t) = [1+f_i]\,t(1+tD)t = [1+f_i]\,t(1+t)$$

$$\begin{aligned}\Re_i\left(x^2\ t\right) &= [1+f_i]\,t(1+tD)\left\{t^2 + \frac{t(1-t)}{i}\right\} \\ &= [1+f_i]\left\{t^2\left(2 - \frac{3t}{i}\right)\right\}\end{aligned}$$

This gives

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{i\to\infty} \left\|\Re_i(1\,;\eta) - 1\right\|_\infty = 0\,, \tag{21}$$

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{i\to\infty} \left\|\Re_i\,(\eta\,;\eta) - \eta\right\|_\infty = 0\,, \tag{22}$$

$$\text{and, stat}_{\text{D}\Delta\text{M}} \lim_{i\to\infty} \left\|\Re_i(\eta^2\,;\eta) - \eta^2\right\|_\infty = 0. \tag{23}$$

This shows that the sequence $\Re_i(f;t)$ *satisfies the conditions* (15), (16) *and* (17). *Hence, we can conclude by Theorem 4.2 that*

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{i\to\infty} \left\|\Re_i(f;\eta) - f(\eta)\right\|_\infty = 0.$$

We have already seen that the sequence (f_k) *of* Δ*-measurable functions mentioned in Example 3.5 is statistically deferred Cesàro summable, but not deferred Cesàro statistically convergence. Therefore, the operator defined above in* (20) *satisfies Theorem 4.2, but not the Theorem 4.1.*

5. Rate of Deferred Cesàro Summability

In this section, we study the rate of statistical deferred Cesàro summability of sequences of positive linear operators from $C(\mathbb{T})$ to $C(\mathbb{T})$.

Definition 5.1. *Let* (a_k) *and* (b_k) *be two sequences of non-negative integers which satisfy the regularity conditions* $a_k < b_k$ *and* $\lim_{k\to\infty} b_k = +\infty$. *Let* (γ_k) *be a sequence of non-increasing positive real numbers. A sequence* $f_n : \mathbb{T} \to \mathbb{R}$ *of* Δ*-measurable functions is statistically deferred Cesàro summable to a* Δ*-measurable function* f *on* $\mathbb{T}$ *with rate* $o(\gamma_k)$, *if for each* $\epsilon > 0$ *and the set*

$$E(\epsilon) = \left\{ s \;: s \in (a_k,\; b_k]_{\mathbb{T}} \text{and} \;\; \left|\phi_k\,(s) - f(s)\right| \geq \epsilon \right\},$$

$$\lim_{k\to\infty} \frac{\mu_\Delta(E(\epsilon))}{\gamma_k \mu_\Delta((a_k,\; b_k]_{\mathbb{T}})} = 0$$

where ϕ_k *is the deferred Cesàro summability mean.*

We write this as

$$\text{stat}_{\text{RD}\Delta\text{M}}\; f_k - f = o(\gamma_k) \text{ on } \mathbb{T}.$$

Next, we prove the following lemma

Lemma 5.2. *Let* (a'_k) *and* (b'_k)*be two non-increasing positive sequences, and let* $(f_k), (g_k) \in C(\mathbb{T})$ *with the conditions*

$$\text{stat}_{\text{RD}\Delta\text{M}}\; f_k - f = o(a'_k) on \mathbb{T}$$
$$\text{and, stat}_{\text{RD}\Delta\text{M}}\; g_k - g = o(b'_k) on \mathbb{T}$$

Then the following assertions are true:

(i) $\text{stat}_{\text{RD}\Delta\text{M}}\; (f_k + g_k) - (f + g) = o(c'_k)$ *on* $\mathbb{T}$.
(ii) $\text{stat}_{\text{RD}\Delta\text{M}}\; (f_k - f)(g_k - g) = o(a'_k b'_k)$ *on* $\mathbb{T}$.
(iii) $\text{stat}_{\text{RD}\Delta\text{M}}\; K(f_k - f) = o(a'_k)$ *on* $\mathbb{T}$, *for any scalar* K.
(iv) $\text{stat}_{\text{RD}\Delta\text{M}}(f_k - f)^{\frac{1}{2}} = o(a'_k)$ *on* $\mathbb{T}$.

where, $c'_k = \max\{a'_k,\; b'_k\}$.

Proof: To prove the assertion (i) of the lemma, we consider the following sets for which $\epsilon > 0$ and $t \in \mathbb{T}$.

$$E(t) = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{and } \left|\phi_k + \psi_k)(s) - (f+g)(t)\right| \geq \epsilon\right\},$$
$$E_1(t) = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{and } \left|\phi_k(t) - f(t)\right| \geq \epsilon\right\},$$
$$E_2(t) = \left\{s : s \in (a_k,\ b_k]_{\mathbb{T}} \text{and } \left|\psi_k(t) - g(t)\right| \geq \epsilon\right\},$$

Then, clearly$E(t) \subseteq E_1(t) \cup E_2(t)$. Since, $c'_k = \max\{a'_k,\ b'_k\}$, we can obtain

$$\frac{\mu_\Delta(E(t))}{c'_k \mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} \leq \frac{\mu_\Delta(E_1(t))}{a'_k \mu_\Delta((a_k,\ b_k]_{\mathbb{T}})} + \frac{\mu_\Delta(E_2(t))}{b'_k \mu_\Delta((a_k,\ b_k]_{\mathbb{T}})}.$$

Using the given conditions of the lemma, the RHS of this inequality tends to zero as $k \to \infty$. So we obtain our desired result. Similar technique can be applied to prove the assertions (ii), (iii) and (iv) of Lemma 5.2.

Now, we define the modulus of continuity of a time scale function $f : \mathbb{T} \to \mathbb{R}$ as follows:

$$\omega(f, \delta) = \sup_{x,\ y \in \mathbb{T}} \left\{\left|f(x) - f(y)\right| : \left|x - y\right| \leq \delta\right\}, \text{ where } 0 < \delta \leq \max\mathbb{T}.$$

Now we establish a theorem on rates of statistically deferred Cesàro summable sequences of time scale positive linear operators with the help of modulus of continuity defined above.

Theorem 5.3. *Let (a'_k) and (b'_k) be two non-increasing positive sequences in $\mathbb{T}$, and let $\Re_k : C([a,b]_{\mathbb{T}}) \to C([a,b]_{\mathbb{T}})$, $(k \in \mathbb{N}$N be a sequence of positive linear operators such that*

(i) $\text{stat}_{\text{RD}\Delta\text{M}}\ \Re_k(1,t) - 1 = o(a'_k)$ *on* $\mathbb{T}$.
(ii) $\text{stat}_{\text{RD}\Delta\text{M}}\ \omega(f, \delta_k) = o(b'_k)$ *on* $\mathbb{T}$.

where, $\delta_k(t) = \left\{\mathcal{L}_k(\theta^2; t)\right\}^{1/2}$ *and* $\theta(k) = (k - t)$,
then for each $f \in C([a,b]_{\mathbb{T}})$*, the below mentioned assertion holds:*

$$\text{stat}_{\text{RD}\Delta\text{M}} \left\|\mathcal{L}_k(f;t) - f(t)\right\| = o(c'_k) \text{ on } [a,b]_{\mathbb{T}}$$

where $c'_k = \max\{a'_k,\ b'_k\}$.

Proof: Suppose, $\mathbb{T} \subset \mathbb{R}$ be compact and let $f \in C([a,b]_{\mathbb{T}})$ and $t \in \mathbb{T}$. Then

$$\begin{aligned}
\left|\mathcal{L}_k(f;t) - f(t)\right| &\leq \mathcal{L}_k\left(\left|f(k) - f(t)\right|;\ t\right) + \left|f(t)\right| |\mathcal{L}_k(1;t) - 1| \\
&\leq \mathcal{L}_k\left(\frac{|k-t|}{\delta_k} + 1\ ;\ t\right)\omega(f,\delta_k) + \left|f(t)\right| |\mathcal{L}(1;t) - 1| \\
&\leq \mathcal{L}_k\left(1 + \frac{|k-t|^2}{{\delta_k}^2}\ ;\ t\right)\omega(f,\delta_k) + \left|f(t)\right| |\mathcal{L}_k(1;t) - 1| \\
&\leq \left[\mathcal{L}_k(1\ ;\ t) + \frac{1}{{\delta_k}^2}\mathcal{L}_k\left(\theta^2\ ;\ t\right)\right]\omega(f,\delta_k) + \left|f(t)\right| |\mathcal{L}_k(1;t) - 1| \\
&= [\mathcal{L}_k(1\ ;\ t) + 1]\,\omega(f,\delta_k) + \left|f(t)\right| |\mathcal{L}_k(1;t) - 1|
\end{aligned}$$

This gives

$$\left\|\mathcal{L}_k(f;t) - f(t)\right\|_\infty \leq 2\omega(f,\delta_k) + \omega(f,\delta_k)\|\mathcal{L}_k(1\ ;\ t) - 1\|_\infty + \left\|f(t)\right\|_\infty \|\mathcal{L}_k(1;t) - 1\|_\infty$$

which again gives

$$\left\|\Re_k(f;t) - f(t)\right\|_\infty \leq 2\omega(f,\delta_k) + \omega(f,\delta_k)\,\|\Re_k(1\ ;\ t) - 1\|_\infty + Q\|\Re_k(1;t) - 1\|_\infty$$

where $Q = \left\|f(t)\right\|_\infty$. Using the conditions (i) and (ii) of the Theorem 5.3 along with the Lemma 5.2, we get the desired result.

6. Concluding Remark

We further observe the following:

Remark 6.1. Let $(f_k) : \mathbb{T} \to \mathbb{R}$ be a sequence of Δ-measurable functions that is already mentioned in Example (3.5). The sequence (f_k) is statistically deferred Cesàro summable to $\frac{1}{2}$, i.e.

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{k\to\infty} f_k = \frac{1}{2} \text{ on } \mathbb{T}.$$

Then we have

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{k\to\infty} \left\| \Re_k(f_j\ ;\ \eta) - f_j(\eta) \right\|_\infty = 0 \text{ for j} = 0,\ 1,\ 2.$$

Thus, by Theorem 4.2, we immediately get

$$\text{stat}_{\text{D}\Delta\text{M}} \lim_{k\to\infty} \left\| \Re_k(f;\ \eta) - f(\eta) \right\|_\infty = 0\ .$$

where $f_0(\eta) = 1,\ f_1(\eta) = \eta$ and $f_2(\eta) = \eta^2$.

As the given sequence (f_k) of time scale functions is statistically deferred Cesàro summable on $\mathbb{T}$, but neither deferred Cesàro statistical convergence nor usual convergence on $\mathbb{T}$. Therefore, under the operator defined in equation (20), the Korovkin-type approximation Theorem 4.2 works properly, but the classical and statistical convergence via deferred Cesàro mean do not work for the same operators. This helps us to conclude that the Theorem 4.2 is a non-trivial extension of the Theorem 4.1 and the classical Korovkin-type approximation Theorem (see [13]).

Conflict of Interest Statement:

The authors do not have any conflict of interest to declare.

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