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Mappings preserving the ascent or descent of triple skew products of operators

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Abstract. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $asc(A)$ and $desc(A)$ be, respectively, the ascent and descent of an operator A in $\mathcal{B}(\mathcal{H})$. In this paper, we determine the explicit form of all maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself that preserve the ascent or descent of triple skew product of operators.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ with unit I.

In the sequel, for any operator $A \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^*, respectively, the range, the kernel and the adjoint of A.

We denote, as usual, by $\langle ., . \rangle$ the inner product on $\mathcal{H}$. For any nonzero vectors $x, u \in \mathcal{H} \setminus \{0\}$, the rank-one operator $x \otimes u$ is defined by $(x \otimes u)y = \langle y, u \rangle x$ for all $y \in \mathcal{H}$. Note that every rank-one operator on $\mathcal{H}$ can be expressed in this form. The operator $x \otimes u$ is nilpotent if and only if $\langle x, u \rangle = 0$, and it is idempotent if and only if $\langle x, u \rangle = 1$. We denote by $\mathcal{F}_1(\mathcal{H})$ and $\mathcal{N}_1(\mathcal{H})$, respectively, the set of all rank-one operators and the set of all rank-one nilpotent operators on $\mathcal{B}(\mathcal{H})$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is algebraic if there exists a nonzero complex polynomial P such that $P(A) = 0$. Clearly, an operator A is algebraic if and only if its adjoint A^* is algebraic.

For any operator $A \in \mathcal{B}(\mathcal{H})$, the ascent and the descent of A are defined by

$$asc(A) = inf\{n \in \mathbb{N} \cup \{0\} : \mathcal{N}(A^n) = \mathcal{N}(A^{n+1})\},$$

$$desc(A) = inf\{n \in \mathbb{N} \cup \{0\} : \mathcal{R}(A^n) = \mathcal{R}(A^{n+1})\};$$

if no such numbers exists the ascent of A (resp. the descent of A) is defined to be infinite. Note that $asc(0) = desc(0) = 1$ and A is injective (resp. surjective) if and only if $asc(A) = 0$ (resp. $desc(A) = 0$). These notions where introduce by Riesz [16]. They play a central role in the development of one of the most important branches of spectral theory, namely the theory of Fredholm operators. For more information on these quantities one can see the books [1, 3, 10].

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Over the past few years, one of the classical problems that has attracted the attention of many mathematicians is the study of nonlinear preserver problems. These problems involve characterizing maps between algebras that preserve a given set, property, or relation, without assuming in advance any algebraic conditions such as linearity, additivity, or multiplicativity. see for instance [4, 8, 9, 11, 13, 15] and the references cited therein.

Recently, in [8], Hosseinzadeh and Petek characterized surjective maps on $\mathcal{B}(\mathcal{X})$, the algebra of all bounded linear operators on a complex or real Banach space $\mathcal{X}$, that preserve the ascent (respectively, the descent) of the product of operators. Furthermore, in [9], the same authors described the structure of all nonlinear maps that preserve either the ascent or, alternatively and simultaneously, the descent of the triple product of operators.

In line with this direction, the present paper determines the explicit form of all surjective maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself that preserve the ascent or the descent of the skew triple product of operators. Precisely, our main result is stated as follows.

Theorem 1.1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a surjective map satisfying*

$$asc(AB^*A) = asc(\phi(A)\phi(B)^*\phi(A)), \quad (A, B \in \mathcal{B}(\mathcal{H})), \tag{1}$$

or

$$desc(AB^*A) = desc(\phi(A)\phi(B)^*\phi(A)), \quad (A, B \in \mathcal{B}(\mathcal{H})). \tag{2}$$

Then there exist a unitary or conjugate-unitary operators $U : \mathcal{H} \to \mathcal{H}$ and a map $\mu : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \setminus \{0\}$ such that

$$\phi(A) = \mu(A)UAU^* \ \ (A \in \mathcal{B}(\mathcal{H})),$$

or,

$$\phi(A) = \mu(A)UA^*U^* \ \ (A \in \mathcal{B}(\mathcal{H})).$$

Note that any map ϕ of the above forms, with the corresponding properties for μ and U, is surjective and satisfies (1) and (2).

2. Preliminaries

In this section, we introduce some notation and preliminary results that will be needed in the sequel. We begin with the following lemma, which discribes the relationship between the ascent and descent of an operator and those of its adjoint.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Assume that $asc(A) = desc(A) = p < \infty$, then its adjoint A^* has finite ascent and finite descent, moreover*

$$asc(A^*) = desc(A^*) = p. \tag{3}$$

Proof. See [1, Problem 2.2.5]. □

The following lemma presents some known properties of the ascent and descent of nilpotent and idempotent operators.

Lemma 2.2. *For every $A \in \mathcal{B}(H)$, the following statements hold.*

1. *If A is nilpotent of nilindex k, then $asc(A) = desc(A) = k$.*
2. *If A is idempotent and $A \neq I$, then $asc(A) = desc(A) = 1$.*

The next lemma is quoted from [9], it characterizes nilpotent and non-nilpotent rank-at-most-one operators in terms of their ascent and descent.

Lemma 2.3. *Let $A \in \mathcal{F}_1(\mathcal{H})$ be a rank one operator. Then*

1. $A \in \mathcal{N}_1(\mathcal{H}) \iff asc(A) = 2 \iff desc(A) = 2$
2. $A \notin \mathcal{N}_1(\mathcal{H}) \iff asc(A) = 1 \iff desc(A) = 1.$

Proof. See [9, Lemma 2.2]. □

Definition 2.4. *We say that an operator $A \in \mathcal{B}(\mathcal{H})$ is s-idempotent if $A = \alpha P$ for some nonzero scalar $\alpha \in \mathbb{C}$ and an idempotent operator $P \in \mathcal{B}(\mathcal{H})$.*

Note that an operator $A \in \mathcal{B}(\mathcal{H})$ is s-idempotent if and only if $A^2 = \alpha A$ for some nonzero scalar $\alpha \in \mathbb{C}$. In what follows, we denote the set of all rank one s-idempotent operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{I}_{1_s}(\mathcal{H})$.

The following lemma determines the ascent and descent of s-idempotent operator.

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a non-scalar operator. If A is s-idempotent, then*

$$asc(A) = desc(A) = 1.$$

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ be a non-scalar operator such that $A = \alpha P$ for some nonzero scalar $\alpha \in \mathbb{C}$ and some idempotent operator $P \in \mathcal{B}(\mathcal{H})$. Then $asc(A) = desc(A) = asc(P) = desc(P) = 1$.
□

Note that if $A \in \mathcal{F}_1(\mathcal{H})$ a rank-one operator. Then, we have

$$A \in \mathcal{I}_{1_s}(\mathcal{H}) \iff asc(A) = 1 \iff desc(A) = 1 \tag{4}$$

In the next Lemma, we identify when a rank-one operator is s-idempotent in terms of ascent and descent.

Lemma 2.6. *Let $A \in \mathcal{F}_1(\mathcal{H})$ be a rank-one operator. The following statements are equivalent.*

1. $A \in \mathcal{I}_{1_s}(\mathcal{H})$.
2. $asc(ATA) = 1$ *for all operator* $T \in \mathcal{B}(\mathcal{H})$.
3. $desc(ATA) = 1$ *for all operator* $T \in \mathcal{B}(\mathcal{H})$.

Proof. Assume that $A = \alpha x \otimes u$ where $x, u \in \mathcal{H}$ and α is a nonzero scalar such that $\langle x, u\rangle = 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary operator. We have $ATA = \alpha^2 \langle Tx, u\rangle x \otimes u$. If $\langle Tx, u\rangle = 0$, then $asc(ATA) = desc(ATA) = asc(0) = desc(0) = 1$. If $\langle Tx, u\rangle \neq 0$, then $ATA \in \mathcal{I}_{1_s}(\mathcal{H})$. Thus, by (4), that $asc(ATA) = desc(ATA) = 1$.

For the implication (2) $\implies$ (1) and (3) $\implies$ (1), assume that $A \notin \mathcal{I}_{1_s}(\mathcal{H})$. Then $A \in \mathcal{N}_1(\mathcal{H})$. Set $A = y \otimes v$ where $y, v \in \mathcal{H} \setminus \{0\}$ such that $\langle y, v\rangle = 0$. There exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $\langle Ty, v\rangle \neq 0$. Then $ATA = \langle Ty, v\rangle y \otimes v \in \mathcal{N}_1(\mathcal{H})$. This implies, by Lemma 2.3, that $asc(ATA) = desc(ATA) = 2$, as desired.
□

Lemma 2.7. *Let $A \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent.*

1. $A \in \mathbb{C}^* I$.
2. $PAP \in \mathcal{I}_{1_s}(\mathcal{H})$ *for every* $P \in \mathcal{I}_{1_s}(\mathcal{H})$.

Proof. Assume that $A \notin \mathbb{C}^* I$. If $A = 0$, then $PAP = 0 \notin \mathcal{I}_{1_s}(\mathcal{H})$ for every $P \in \mathcal{I}_{1_s}(\mathcal{H})$. If $A \neq 0$, there exists a nonzero vector $x \in \mathcal{H}$ such that x and Ax are linearly independent. Let $u \in \mathcal{H}$ be a nonzero vector such that $\langle x, u\rangle = 1$ and $\langle Ax, u\rangle = 0$. For $P = x \otimes u$, we have $P \in \mathcal{I}_{1_s}(\mathcal{H})$ and $PAP = 0 \notin \mathcal{I}_{1_s}\mathcal{H})$. This proves that (2) $\implies$ (1) is true. The other implication is obvious. □

In [11], Li, Šemrl and Sze proved the following lemma.

Lemma 2.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Assume that $Bx \in span\{x, Ax\}$ for every $x \in H$. Then $B = \alpha I + \mu A$ for some $\alpha,\ \mu \in \mathbb{C}$.*

Proof. See [11, Lemma 2.6]. □

Lemma 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If for every $N \in \mathcal{N}_1(\mathcal{H})$ we have*

$$NAN \in \mathcal{N}_1(\mathcal{H}) \iff NBN \in \mathcal{N}_1(\mathcal{H}).$$

Then, there exist $\alpha, \mu \in \mathbb{C}$ such that $B = \alpha I + \mu A$.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $NAN \in \mathcal{N}_1(\mathcal{H}) \iff NBN \in \mathcal{N}_1(\mathcal{H})$, for every $N \in \mathcal{N}_1(\mathcal{H})$. Suppose, by the way of contradiction, that there exists a nonzero vector x such that x, Ax and Bx are linearly independent. We can find a nonzero vector $u \in \mathcal{H}$ such that $\langle x, u\rangle = 0$, $\langle Ax, u\rangle = 1$ and $\langle Bx, u\rangle = 0$. Consider the rank-one nilpotent opertaor $N = x \otimes u$, note that $NAN = N \in \mathcal{N}_1(\mathcal{H})$ but $NBN = 0 \notin \mathcal{N}_1(\mathcal{H})$. This contradiction ensures that x, Ax and Bx are linearly dependent for every $x \in \mathcal{H}$. Thus, by Lemma 2.8, we conclude that $B = \alpha I + \mu A$ for some $\alpha,\ \mu \in \mathbb{C}$. □

From the above Lemma we obtain immediately the following result.

Lemma 2.10. *Let $A, B \in \mathcal{F}_1(\mathcal{H})$ be rank-one operators. The following statements are equivalent.*

1. *A and B are linearly dependent.*
2. *For every $N \in \mathcal{N}_1(\mathcal{H})$ we have*

$$NAN \in \mathcal{N}_1(\mathcal{H}) \iff NBN \in \mathcal{N}_1(\mathcal{H}). \tag{5}$$

Proof. (1) $\implies$ (2) is obvious.

(2) $\implies$ (1) Suppose that the condition 2 holds true. Then by Lemma 2.9, we have $B = \alpha I + \mu A$ for some $\alpha,\ \mu \in \mathbb{C}$. Since A and B are of rank-one, then $\alpha = 0$. Hence A and B are linearly dependent.
□

For two rank one operators $A, B \in \mathcal{F}_1(\mathcal{H})$, we write $A \sim B$ if $\mathcal{N}(A) = \mathcal{N}(B)$ or $\mathcal{R}(A) = \mathcal{R}(B)$. In other words for $x, y, u, v \in \mathcal{H}$, we have $x \otimes u \sim y \otimes v$ if and only if x and y are linearly dependent or u and v are linearly dependent.

Lemma 2.11. *Let M, N be linearly independent operators in $\mathcal{N}_1(\mathcal{H})$. The following statements are equivalent.*

1. *$M \sim N$.*
2. *There exists an operator $R \in \mathcal{N}_1(\mathcal{H})$ such that $R \notin \mathbb{C}^*M$ and $R \notin \mathbb{C}^*N$, and satisfies the property that for every $T \in \mathcal{B}(\mathcal{H})$*

$$MTM \notin \mathcal{N}_1(\mathcal{H}) \quad \text{and} \quad NTN \notin \mathcal{N}_1(\mathcal{H}) \Longrightarrow RTR \notin \mathcal{N}_1(\mathcal{H}).$$

Proof. See [8, Proposition 2.12]. □

We finish this section with the following lemma, established in [8, Proposition 2.13], which will be useful in the proof of our main result.

Lemma 2.12. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Assume that $\phi : \mathcal{N}_1(\mathcal{H}) \cup \{0\} \to \mathcal{N}_1(\mathcal{H}) \cup \{0\}$ is a surjective mapping satisfying $\phi(0) = 0$, preserving linear dependency in both directions and*

$$N \sim M \iff \phi(N) \sim \phi(M)$$

for all $M, N \in \mathcal{N}_1(\mathcal{H})$.

Then there exist an inverible bounded operator or conjugate-linear operator $T : \mathcal{H} \to \mathcal{H}$ and a map $\mu : \mathcal{N}_1(\mathcal{H}) \to \mathbb{C} \setminus \{0\}$ such that

$$\phi(N) = \mu(N) T N T^{-1}, \quad (N \in \mathcal{N}_1(\mathcal{H})), \tag{6}$$

or,

$$\phi(N) = \mu(N) T N^* T^{-1}, \quad (N \in \mathcal{N}_1(\mathcal{H})). \tag{7}$$

3. Proof of the main result

Before proving the main theorem, we establish the following lemmas. The first gives an important property of locally linearly dependent operators.

Definition 3.1. *Let U and V be vector spaces over a field $\mathbb{F}$. Linear operators $T_1, \ldots, T_n : U \to V$ are locally linearly dependent if $T_1u, \ldots, T_nu$ are linearly dependent for every $u \in U$.*

Lemma 3.2. *Let A and B be linear operators. If A, B and I are locally linearly dependent, then there exist scalars λ and μ such that $(A-\lambda)(B-\mu) = 0$ and either $(A-\lambda)^2 = 0$ or $(B-\mu)^2 = 0$.*

Proof. See [14] and [7]. □

The second lemma gives a necessary and sufficient condition for an operator to be zero.

Lemma 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$. The following assertions are equivalent.*

1. $A = 0$.
2. $asc(ATA) = 1$ *and* $asc(TA^*T) = 1$ *for every operator* $T \in \mathcal{B}(\mathcal{H})$.
3. $desc(ATA) = 1$ *and* $desc(TA^*T) = 1$ *for every operator* $T \in \mathcal{B}(\mathcal{H})$.

Proof. The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are obvious.

Conversely, let $A \in \mathcal{B}(\mathcal{H})$ be a nonzero operator. Assume first that $A^2 = 0$, then, by Lemma 2.2, $asc(A) = desc(A) = 2$. Thus, from Lemma 2.1, $asc(A^*) = desc(A^*) = 2$. Taking $T = I$, we obtain

$$asc(TA^*T) = desc(TA^*T) = asc(A^*) = desc(A^*) = 2.$$

Next, assume that $A^2 \neq 0$. Consider the following two cases:

Case 1. If x, Ax and A^2x are linearly independent for certain $x \in \mathcal{H}$. Then, there is a vector $u \in \mathcal{H}$ such that $\langle x, u\rangle = \langle Ax, u\rangle = 1$ and $\langle A^2x, u\rangle = 0$. Setting $T = x \otimes u$, we get ATA is a rank-one nilpotent operator. Thus, by Lemma 2.3 ,

$$asc(ATA) = desc(ATA) = 2.$$

Case 2. If x, Ax and A^2x are linearly dependent for all $x \in \mathcal{H}$. Then, by lemma 3.2, there exist scalars λ and μ such that $(A-\lambda)(A^2-\mu) = 0$ and either $(A-\lambda)^2 = 0$ or $(A^2-\mu)^2 = 0$.

Subcase 1. Suppose that $(A-\lambda)(A^2-\mu) = 0$ and $(A-\lambda)^2 = 0$. Then,

$$A^3 - \lambda A^2 - \mu A + \lambda\mu = 0 \tag{8}$$

and

$$A^2 = 2\lambda A - \lambda^2. \tag{9}$$

Hence, by (9), we obtain that $A^3 = 2\lambda A^2 - \lambda^2 A$. Substituting into (8), we get $\lambda A^2 - \lambda^2 A - \mu A + \lambda\mu = 0$, which simplifies to $\lambda(A^2 - 2\lambda A) + (\lambda^2 - \mu)A + \lambda\mu = 0$. This together with (9) gives that $-\lambda^3 + (\lambda^2-\mu)A + \lambda\mu = 0$, implying that

$$(\mu - \lambda^2)A = \lambda(\mu - \lambda^2).$$

Since $A^2 \neq 0$, from (9), we see that $\lambda \neq 0$. If $\mu - \lambda^2 \neq 0$, then A is a nonzero scalar operator. Thus, A^* is also a nonzero scalar operator. By setting $T = I$, we have

$$asc(TA^*T) = desc(TA^*T) = asc(A^*) = desc(A^*) = 0.$$

If $\mu - \lambda^2 = 0$, by (9), we have $\mu A = \frac{\lambda}{2}A^2 + \frac{\lambda^3}{2}$. Substituting into (8), we get $\frac{3}{\lambda^2}A^2 - \frac{2}{\lambda^3}A^3 = I$. This implies that $A(\frac{3}{\lambda^2}I - \frac{2}{\lambda^3}A)A = I$. By Setting $T = \frac{2}{\lambda^3}A - \frac{3}{\lambda^2}I$, we have

$$asc(ATA) = desc(ATA) = asc(I) = desc(I) = 0.$$

Subcase 2. Assume now that $(A-\lambda)(A^2-\mu)=0$ and $(A^2-\mu)^2=0$. Then,

$$A^4 - 2\mu A^2 + \mu^2 = 0 \tag{10}$$

From (8), we have $A^4 = \lambda A^3 + \mu A^2 - \lambda\mu A$. This together with (10) leads to $\lambda A^3 - \mu A^2 - \lambda\mu A + \mu^2 = 0$, which implies that

$$\lambda(A^3 - \lambda A^2 - \mu A) + (\lambda^2 - \mu)A^2 + \mu^2 = 0.$$

Hence, by (8), we have $-\lambda^2\mu + (\lambda^2-\mu)A^2 + \mu^2 = 0$. Thus,

$$(\mu - \lambda^2)A^2 = \mu(\mu - \lambda^2).$$

If $\mu - \lambda^2 \neq 0$. Then, $\mu \neq 0$ because $A^2 \neq 0$. Consequently, A^2 is a nonzero scalar operator. Let $T = I$, we have

$$asc(ATA) = desc(ATA) = asc(A^2) = desc(A^2) = 0.$$

If $\mu - \lambda^2 = 0$ and $\mu = 0$, then $A^3 = 0$, that is A is nilpotent operator of nilindex 3. Then, by Lemma 2.1, $asc(A^*) = desc(A^*) = 3$ and by choosing $T = I$, we get

$$asc(TA^*T) = desc(TA^*T) = asc(A^*) = desc(A^*) = 3.$$

If $\mu - \lambda^2 = 0$ and $\mu \neq 0$. Then by (10), we easily get $A(\frac{2}{\mu}I - \frac{1}{\mu^2}A^2)A = I$. By taking $T = \frac{2}{\mu}I - \frac{1}{\mu^2}A^2$, we obtain

$$asc(ATA) = desc(ATA) = asc(I) = desc(I) = 0.$$

The proof is therefore complete. □

The following Lemma gives a characterization of rank one operators in terms of the ascent and the descent.

Lemma 3.4. *Let $A \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$. Then following statements are equivalent.*

1. *A is of rank one.*
2. *For any operator $T \in \mathcal{B}(\mathcal{H})$ we have $asc(ATA) \in \{1,2\}$ and $asc(TA^*T) \in \{1,2\}$.*
3. *For any operator $T \in \mathcal{B}(\mathcal{H})$ we have $desc(ATA) \in \{1,2\}$ and $desc(TA^*T) \in \{1,2\}$.*

Proof. The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are clear.

To prove (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1). Assume that A is not of rank one. If $A = \lambda I$, where λ is a nonzero scalar in $\mathbb{C}$, for $T = \frac{1}{\lambda^2}I$ we have

$$asc(ATA) = desc(ATA) = asc(I) = desc(I) = 0.$$

If A has rank at least two. Suppose that there exists a vector $x \in \mathcal{H}$ such that x, Ax and A^2x are linearly independent. Using a similar approach to the proof of [8, Lemma 2.8], we can find an operator $S \in B(\mathcal{H})$ such that $asc(SAS) = desc(SAS) = 3$. For $T = S^*$, by Lemma 2.1, we get

$$asc(TA^*T) = desc(TA^*T) = 3.$$

If x, Ax and A^2x are linearly dependent for all $x \in \mathcal{H}$. Similar to the previous discussion in Lemma 3.3, we show that there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $asc(ATA) = desc(ATA) \notin \{1,2\}$ or, $asc(TA^*T) = desc(TA^*T) \notin \{1,2\}$. □

Now we are able to prove our main theorem.

Proof of Theorem 1.1

We will only prove the theorem for the ascent, the proof for the descent is done in the same way. Assume that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective map satisfying (1). We break the proof into several steps.

Step 1. $\phi(A) = 0$ if and only if $A = 0$.

Let $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = 0$. For every $T \in \mathcal{B}(\mathcal{H})$ we have

$$asc(ATA) = asc(A(T^*)^*A) = asc(\phi(A)\phi(T^*)^*\phi(A)) = asc(0) = 1$$

and

$$asc(TA^*T) = asc(\phi(T)\phi(A)^*\phi(T)) = asc(0) = 1.$$

Then, by Lemma 3.3, we obtain that $A = 0$.

Conversely, let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary operator. Since ϕ is surjective, there exist $S, R \in \mathcal{B}(\mathcal{H})$ such that $T = \phi(S)$ and $T^* = \phi(R)$. By (1) we have

$$asc(\phi(0)T\phi(0)) = asc(\phi(0)\phi(R)^*\phi(0)) = asc(0R^*0) = 1,$$

and

$$asc(T\phi(0)^*T) = asc(\phi(S)\phi(0)^*\phi(S)) = asc(S0^*S) = 1.$$

This implies, by Lemma 3.3, that $\phi(0) = 0$, as desired.

Step 2. ϕ preserves rank-one operators in both directions.

Let $A \in \mathcal{F}_1(\mathcal{H})$ be a rank-one operator and $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary operator. Since ϕ is surjective, there exist $S, R \in \mathcal{B}(\mathcal{H})$ such that $T = \phi(S)$ and $T^* = \phi(R)$. Using (1) and Lemma 3.4, we obtain that

$$asc(\phi(A)T\phi(A)) = asc(\phi(A)\phi(R)^*\phi(A)) = asc(AR^*A) \in \{1; 2\},$$

and

$$asc(T\phi(A)^*T) = asc(\phi(S)\phi(A)^*\phi(S)) = asc(SA^*S) \in \{1; 2\}.$$

It follows, by Lemma 3.4 once again, that $\phi(A) \in \mathcal{F}_1(\mathcal{H})$. In the same way we can show the other direction.

Step 3. $A \in \mathcal{I}_{1_s}(\mathcal{H})$ if and only if $\phi(A) \in \mathcal{I}_{1_s}(\mathcal{H})$.

Let $A \in \mathcal{I}_{1_s}(\mathcal{H})$ be rank-one s-idempotent operator. So, by the previous step, $\phi(A)$ is of rank one. Let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary operator. Since ϕ is surjective, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $T^* = \phi(S)$. Using Lemma 2.6, we have

$$asc(\phi(A)T\phi(A)) = asc(\phi(A)\phi(S^*)^*\phi(A)) = asc(AS^*A) = 1.$$

By Lemma 2.6 once again, we conclude that $\phi(A) \in \mathcal{I}_{1_s}(\mathcal{H})$. The proof of the reverse direction is done in the same way. This proves that

$$A \in \mathcal{N}_1(\mathcal{H}) \iff \phi(A) \in \mathcal{N}_1(\mathcal{H}) \tag{11}$$

Step 4. $A \in \mathbb{C}I$ if and only if $\phi(A) \in \mathbb{C}I$.

Let $A = \lambda I$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then the equivalence is verfied by step 1. If $\lambda \neq 0$, let P be an arbitrary operator in $\mathcal{I}_{1_s}(\mathcal{H})$, by surjectivity of ϕ and step 3, there is an $R \in \mathcal{I}_{1_s}(\mathcal{H})$, such that $\phi(R) = P$. Applying (4) and Lemma 2.7, we get

$$asc(P\phi(A)^*P) = asc(\phi(R)\phi(A)^*\phi(R)) = asc(RA^*R) = 1.$$

Thus, $\phi(A) \in \mathbb{C}^*I$. By a similar way, we show the reverse direction.

Step 5. ϕ preserves linear dependency in both directions on $\mathcal{F}_1(\mathcal{H})$.

Let $M, N \in \mathcal{F}_1(\mathcal{H})$ be rank-one operators such that M and N are linearly dependent. Then for any $R \in \mathcal{N}_1(\mathcal{H})$, by surjectivity and (11), there is an $S \in \mathcal{N}_1(\mathcal{H})$ such that $\phi(S) = R$. By Lemma 2.3 and Lemma

2.10, we see that

$$\begin{aligned}
R\phi(M)^*R \in \mathcal{N}_1(\mathcal{H}) &\iff asc(R\phi(M)^*R) = 2\\
&\iff asc(\phi(S)\phi(M)^*\phi(S)) = 2\\
&\iff asc(SM^*S) = 2\\
&\iff SM^*S \in \mathcal{N}_1(H)\\
&\iff SN^*S \in \mathcal{N}_1(H)\\
&\iff asc(SN^*S) = 2\\
&\iff asc(\phi(S)\phi(N)^*\phi(S)) = 2\\
&\iff asc(R\phi(N)^*R) = 2\\
&\iff R\phi(N)^*R \in \mathcal{N}_1(\mathcal{H}).
\end{aligned}$$

Then, by Lemma 2.10, we conclude that $\phi(M)$ and $\phi(N)$ are linearly dependent. Similarly, we can check that $\phi(M)$ and $\phi(N)$ are linearly dependent implies that M and N are linearly dependent.

Step 6. For every $M, N \in \mathcal{N}_1(\mathcal{H})$ we have $M \sim N \iff \phi(M) \sim \phi(N)$.

Let $M, N \in \mathcal{N}_1(\mathcal{H})$ be rank-one nilpotent operators such that $M \sim N$. Firstly, suppose that M and N are linearly dependent. Then, by the previous step, $\phi(M)$ and $\phi(N)$ are also linearly dependent. Consequently, $\phi(M) \sim \phi(N)$.

Now, suppose that M and N are linearly independent. By Lemma 2.11, there exists an operator $R \in \mathcal{N}_1(\mathcal{H})$ such that $R \notin \mathbb{C}^*M$ and $R \notin \mathbb{C}^*N$, and for every $T \in \mathcal{B}(\mathcal{H})$, we have

$$MTM \notin \mathcal{N}_1(\mathcal{H}) \quad \text{and} \quad NTN \notin \mathcal{N}_1(\mathcal{H}) \Longrightarrow RTR \notin \mathcal{N}_1(\mathcal{H}). \tag{12}$$

Let $F = \phi(R)$. Then, by steps 4 and 6, $F \in \mathcal{N}_1(\mathcal{H})$, $F \notin \mathbb{C}^*\phi(M)$ and $F \notin \mathbb{C}^*\phi(N)$.

To prove that $\phi(M) \sim \phi(N)$, consider $S \in \mathcal{B}(\mathcal{H})$ such that $\phi(M)S\phi(M) \notin \mathcal{N}_1(\mathcal{H})$ and $\phi(N)S\phi(N) \notin \mathcal{N}_1(\mathcal{H})$, and let us show that $FSF \notin \mathcal{N}_1(\mathcal{H})$. Indeed, since ϕ is surjective there exists $T \in \mathcal{B}(\mathcal{H})$ such that $S = \phi(T)^*$. Note that

$$asc(\phi(M)S\phi(M)) = asc(\phi(N)S\phi(N))) = 1.$$

That is

$$asc(\phi(M)\phi(T)^*\phi(M)) = asc(\phi(N)\phi(T)^*\phi(N)) = 1.$$

Then, by (1), we get

$$asc(MT^*M) = asc(NT^*N) = 1.$$

This implies that

$$MT^*M \notin \mathcal{N}_1(\mathcal{H}) \ \textit{and} \ NT^*N \notin \mathcal{N}_1(\mathcal{H}).$$

Therefore, by (12), we have $RT^*R \notin \mathcal{N}_1(H)$. Thus $asc(RT^*R) = 1$. Using the condition (1) we see that $asc(\phi(R)\phi(T)^*\phi(R)) = 1$. Hence

$$FSF \notin \mathcal{N}_1(\mathcal{H}).$$

From Lemma 2.11, we see that $\phi(P) \sim \phi(Q)$ in this case too.

The inverse implication can be proved using the same method.

Step 7. ϕ takes the desired forms.

By the previous steps, the restriction

$$\phi_{|\mathcal{N}_1(\mathcal{H})\cup\{0\}} : \mathcal{N}_1(\mathcal{H}) \cup \{0\} \to \mathcal{N}_1(\mathcal{H}) \cup \{0\}$$

is a surjective mapping satisfying the conditions of Lemma 2.12, then there exist an inverible bounded operator linear or conjugate-linear operator $T : \mathcal{H} \to \mathcal{H}$ and a map $\mu : \mathcal{N}_1(\mathcal{H}) \to \mathbb{C} \setminus \{0\}$ such that ϕ takes either of form (6) or (7).

Assume that ϕ takes the form (6). First, we claim that for every $x, u \in \mathcal{H}$ such that $\langle x, u\rangle = 0, Tx \neq 0$ and $(T^{-1})^*u \neq 0$ we have

$$\langle (T^*T)^{-1}x, u\rangle = 0 \quad \text{or} \quad \langle T^*Tx, u\rangle = 0. \tag{13}$$

Indeed, let $x, u \in \mathcal{H}$ be such vectors. For $N = x \otimes u$ we have

$$\begin{aligned} (TNT^{-1})(TN^*T^{-1})^*(TNT^{-1}) &= (Tx \otimes uT^{-1})(Tu \otimes xT^{-1})^*(Tx \otimes uT^{-1}) \\ &= \langle (T^*T)^{-1}x, u\rangle\langle T^*Tx, u\rangle Tx \otimes (T^{-1})^*u. \end{aligned}$$

Moreover,

$$\begin{aligned} asc(\langle (T^*T)^{-1}x, u\rangle\langle T^*Tx, u\rangle Tx \otimes (T^{-1})^*u) &= asc((TNT^{-1})(TN^*T^{-1})^*(TNT^{-1})) \\ &= asc(\phi(N)\phi(N^*)^*\phi(N)) \\ &= asc(N^3) \\ &= asc(0) \\ &= 1. \end{aligned}$$

Since $Tx \otimes (T^{-1})^*u \in \mathcal{N}_1(\mathcal{H})$, then

$$\langle (T^*T)^{-1}x, u\rangle = 0 \quad \text{or} \quad \langle T^*Tx, u\rangle = 0.$$

Next, let us prove that T^*T is a scalar operator. To do that, suppose that there exists a nonzero vector $x \in \mathcal{H}$ such that T^*Tx and x are linearly independent.

If x, T^*Tx and $(T^*T)^{-1}x$ are linearly independent, then we can find $u \in \mathcal{H}$ such that $\langle x, u\rangle = 0$, $\langle T^*Tx, u\rangle \neq 0$ and $\langle (T^*T)^{-1}x, u\rangle \neq 0$, which contradicts (13).

If $(T^*T)^{-1}x = \alpha T^*Tx + \beta x$ for some scalars α, β not both equal to zero. Since T^*Tx and x are linearly independent, then there exists a vector $u \in \mathcal{H}$ such that $\langle x, u\rangle = 0$ and $\langle T^*Tx, u\rangle \neq 0$. Thus, $\langle (T^*T)^{-1}x, u\rangle = \alpha\langle T^*Tx, u\rangle$. If $\alpha \neq 0$, we obtain that $\langle (T^*T)^{-1}x, u\rangle \neq 0$, which is contradiction. If $\alpha = 0$, we infer that $(T^*T)^{-1}x = \beta x$. This gives that $\langle T^*Tx, u\rangle = 0$, again a contradiction. Therefore, T^*T is a scalar operator. Consequently, there exists a positive scalar λ such that $T^*T = \lambda I$. By setting $U = \frac{1}{\sqrt{\lambda}}T$, we conclude that

$$\phi(N) = \mu(N)UNU^* \text{ for every } N \in \mathcal{N}_1(\mathcal{H}),$$

where U is an unitary operator in $\mathcal{B}(\mathcal{H})$.

Define the map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\varphi(A) = U^*\phi(A)U$ for any $A \in \mathcal{B}(\mathcal{H})$. Note that φ has the same properties as ϕ and $\varphi(N) = \mu(N)N$ for every $N \in \mathcal{N}_1(\mathcal{H})$.

Now, let us show that $\varphi(A) = \mu(A)A$, where $\mu(A) \in \mathbb{C} \setminus \{0\}$ for all $A \in \mathcal{B}(\mathcal{H})$. To do so, let A be an operator in $\mathcal{B}(\mathcal{H})$. If A is a scalar operator, then the result follows from Step 4. Otherwise, assume that $A \notin \mathbb{C}I$. For any $N \in \mathcal{N}_1(\mathcal{H})$, we have

$$\begin{aligned} NA^*N \in \mathcal{N}_1(H) &\iff asc(NA^*N) = 2 \\ &\iff asc(\varphi(N)\varphi(A)^*\varphi(N)) = 2 \\ &\iff \varphi(N)\varphi(A)^*\varphi(N) \in \mathcal{N}_1(H) \\ &\iff N\varphi(A)^*N \in \mathcal{N}_1(H). \end{aligned}$$

Thus, by Lemma 2.9, $\varphi(A)^*$ is a linear combination of I and A^*. As a result, there exist $\alpha, \mu \in \mathbb{C}$ such that

$$\varphi(A) = \alpha I + \mu A. \tag{14}$$

Since $A \notin \mathbb{C}I$, then $\mu \neq 0$. We claim that $\alpha = 0$. If A is of rank-one, then by Step 2, $\varphi(A)$ is of rank-one too. This together with (14) gives that $\alpha = 0$. Now, if A is of rank at least 2, we discuss two cases.

Case 1. If x, Ax and A^2x are linearly independent for certain $x \in \mathcal{H}$. We can take a vector $u \in \mathcal{H}$ such that $\langle x, u\rangle = 0$, $\langle Ax, u\rangle = 1$ and $\langle A^2x, u\rangle = 0$. Obviously, $Ax \neq 0$ and $A^*u \neq 0$. This implies that, $Ax \otimes uA \in \mathcal{N}_1(\mathcal{H})$. On the other hand, we have

$$\begin{aligned} Ax \otimes uA \in \mathcal{N}_1(\mathcal{H}) &\implies asc(Ax \otimes uA) = 2 \\ &\implies asc(A(u \otimes x)^*A) = 2 \\ &\implies asc(\varphi(A)\varphi(u \otimes x)^*\varphi(A)) = 2 \\ &\implies \varphi(A)x \otimes u\varphi(A) \in \mathcal{N}_1(\mathcal{H}) \\ &\implies \langle \varphi(A)^2x, u\rangle = 0. \end{aligned}$$

From the equation (14), we get

$$0 = \langle \varphi(A)^2x, u\rangle = 2\alpha\mu.$$

This together with $\mu \neq 0$ leads to $\alpha = 0$.

Case 2. If x, Ax and A^2x are linearly dependent for all $x \in \mathcal{H}$. Then, A is an algebraic operator of degree two and of rank greater than one, so A^* is. Moreover, by (14), we have, $\varphi(A)^* = \bar{\alpha}I + \bar{\mu}A^*$. Without loss of generality, assume that $\mu = 1$. Suppose, by the way of contradiction, that $\alpha \neq 0$. From Lemma [8, Lemma 2.6 (ii)], there exists an algebraic operator B of degree at least three such that

$$asc(BA^*B) \neq asc(B\varphi(A)^*B) \quad \text{and} \quad desc(BA^*B) \neq desc(B\varphi(A)^*B) \tag{15}$$

or,

$$asc(A^*BA^*) \neq asc(\varphi(A)^*B\varphi(A)^*) \quad \text{and} \quad desc(A^*BA^*) \neq desc(\varphi(A)^*B\varphi(A)^*). \tag{16}$$

Since B is algebraic of degree at least three, then there exists a nonzero vector $y \in \mathcal{H}$, such that y, By and B^2y are linearly independent. By case 1, we deduce that $\varphi(B) = \mu(B)B$, for certain $\mu(B) \in \mathbb{C} \setminus \{0\}$. Then,

$$asc(BA^*B) = asc(\varphi(B)\varphi(A)^*\varphi(B)) = asc(B\varphi(A)^*B),$$

this contradicts equation (15). On the other hand, the construction of the operator B, as presented in [8, Lemma 2.6 (ii)], ensures that $asc(A^*BA^*) = desc(A^*BA^*) < \infty$. Thus, by Lemma 2.1, we get

$$\begin{aligned} asc(A^*BA^*) &= asc(AB^*A) \\ &= asc(\varphi(A)\varphi(B)^*\varphi(A)) \\ &= asc(\varphi(A)B^*\varphi(A)) \\ &= asc(\varphi(A)^*B\varphi(A)^*). \end{aligned}$$

This contradicts equation (16). Therefore $\varphi(A) = \mu(A)A$ for all $A \in \mathcal{B}(\mathcal{H})$, as desired.

Finally, if ϕ takes the form (7), similarly to the above proof one can show that there exists a conjugate-unitary operator $U : \mathcal{H} \to \mathcal{H}$ and a map $\mu : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \setminus \{0\}$ such that

$$\phi(A) = \mu(A)UA^*U^* \;\; (A \in \mathcal{B}(\mathcal{H})).$$

The proof is therefore complete.

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