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Existence results for a coupled anisotropic ϕ-Laplacian system with variable exponents

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Abstract. We study a coupled system of anisotropic ϕ-Laplacian equations with variable exponents, extending earlier work on single equations. Under generalized Orlicz-Sobolev settings, we study nontrivial weak solutions via variational methods. Key tools include Ekeland's principle and Mountain Pass geometry, with compact embeddings linking anisotropic growth to variable exponent Lebesgue spaces.

1. Introduction

Recent advances in the study of anisotropic elliptic partial differential equations with nonstandard growth conditions have revealed profound connections between functional analysis and materials science. The anisotropic ϕ-Laplacian operator, which generalizes both the p-Laplacian and the variable exponent Laplacian, has emerged as a powerful tool for modeling phenomena where the energy density exhibits direction-dependent growth behavior. Such operators naturally arise in the study of electrorheological fluids, where the electrical conductivity depends on both the electric field strength and direction, as well as in image processing applications where anisotropic diffusion preserves edges while smoothing textures (see [1, 6–15]).

The mathematical analysis of these problems requires sophisticated tools from the theory of Orlicz-Sobolev spaces, particularly when dealing with anisotropic growth conditions. While single-equation cases have been extensively studied in recent years, systems of coupled equations present additional challenges due to the intricate interplay between different growth rates and coupling effects. The present work bridges this gap by developing a comprehensive framework for analyzing coupled systems in anisotropic Orlicz-Sobolev spaces with variable exponents.

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This paper investigates a coupled system of degenerate/singular elliptic equations with variable exponents, generalizing the framework of [9, 10] to a coupled system framework. We consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i(\phi_i(\partial_i u)) = \lambda \left(|u|^{q_1(x)-2}u + |v|^{q_2(x)}\right) & \text{in } \Omega, \\ -\sum_{i=1}^{N} \partial_i(\psi_i(\partial_i v)) = \lambda \left(|v|^{q_2(x)-2}v + |u|^{q_1(x)}\right) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

(H_1) $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary $\partial\Omega$.

(H_2) $\lambda \in \mathbb{R}^+$ and $q_i : \overline{\Omega} \to (1, +\infty)$ are continuous functions.

(H_3) For each $1 \leq i \leq N$, the functions ϕ_i and ψ_i are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$.

(H_4) For all $t \in \mathbb{R}$ and $1 \leq i \leq N$, define $\Theta_i(t) = \int_0^t \theta_i(s)ds$ for Θ_i and θ_i represents either (Φ_i, ϕ_i) or (Ψ_i, ψ_i).

(H_5) The Banach spaces $L_{\Theta_i}(\Omega)$ with the norm

$$\|u\|_{\Theta_i} := \inf\left\{k > 0, \int_\Omega \Theta_i(\frac{u(x)}{k})dx \leq 1\right\} < \infty,$$

for $\Theta_i = \Phi_i, \Psi_i$ where $1 \leq i \leq N$.

(H_6) Define

$$\begin{cases} (p_i^1)_0 := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \text{ and } (p_i^1)^0 := \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)},\ 1 \leq i \leq N, \\ (p_i^2)_0 := \inf_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} \text{ and } (p_i^2)^0 := \sup_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)},\ 1 \leq i \leq N. \end{cases}$$

(H_7) Assume

$$1 < (p_i^1)_0 \leq \frac{t\phi_i(t)}{\Phi_i(t)} \leq (p_i^1)^0 < \infty,\ 1 < (p_i^2)_0 \leq \frac{t\psi_i(t)}{\Psi_i(t)} \leq (p_i^2)^0 < \infty,\ \text{for all } t \geq 0.$$

(H_8) Assume for each $1 \leq i \leq N$ the functions $t \to \Phi_i(\sqrt{t})$ and $t \to \Psi_i(\sqrt{t})$ for $t \in [0, +\infty)$ are convex.

The primary objective of this work is to establish the existence of nontrivial weak solutions to problem (1) under various conditions on the eigenvalue parameter λ. Our approach combines variational methods with sophisticated techniques from the theory of Orlicz-Sobolev spaces, allowing us to handle the intricate interplay between anisotropic growth conditions and variable exponents. The results presented here significantly extend previous work on single equations to the more challenging case of coupled systems.

The paper is organized as follows: Section 2 provides the necessary background on anisotropic Orlicz-Sobolev spaces and establishes key embedding results. In Section 3, we present our main existence theorems, employing variational methods and critical point theory to prove the existence of nontrivial solutions to (1) under different conditions on the parameter λ.

2. Orlicz-Sobolev framewrok

The study of anisotropic elliptic problems requires a careful analysis of the underlying function spaces. In this section, we present the fundamental aspects of Orlicz-Sobolev spaces, which provide the natural setting for problems with nonstandard growth conditions. For comprehensive treatments of this subject, we refer to the seminal works [3–5, 9, 10].

The Orlicz-Sobolev space $W^1L_{\Theta_i}(\Omega)$ comprises functions that are weakly differentiable over Ω, with their weak derivatives belonging to $L_{\Theta_i}(\Omega)$. These spaces are Banach spaces, equipped with the norm $\|u\|_{1,\Theta_i} = \|u\|_{\Theta_i} + \|\nabla u\|_{\Theta_i}$ for $1 \leq i \leq N$.

Moreover, the Orlicz-Sobolev spaces $W^1_0L_{\Theta_i}(\Omega)$ are defined as the closures of $C^1_0(\Omega)$ within $W^1L_{\Theta_i}(\Omega)$, using the norm $\|u\|_{0,\Theta_i} = \|\nabla u\|_{\Theta_i}$. These spaces are reflexive Banach spaces for $1 \leq i \leq N$, where Θ_i can take the forms Φ_i or Ψ_i.

We introduce the vectorial function $\overline{\Theta} : \overline{\Omega} \to \mathbb{R}$, defined as $\overline{\Theta} = (\Theta_1, \ldots, \Theta_N)$. The anisotropic Orlicz-Sobolev space, noted as $W^1_0L_{\overline{\Theta}}(\Omega)$, is then established as the closure of $C^1_0(\Omega)$ with respect to the norm $\|u\|_{\overline{\Theta}} := \sum_{i=1}^N |\partial_i u|_{\Theta_i}$, where $\Theta = \Phi, \Psi$.

It is recalled that when $\overline{\Theta}$ is a constant vector then $W_0^{1,\overline{\Theta}}(\Omega)$ is a reflexive Banach space for any $\overline{\Theta} \in \mathbb{R}^N$ with $\Theta_i > 1$ for all $1 \leq i \leq N$, this shows that in general $W^1_0L_{\overline{\Theta}}(\Omega)$ is a reflexive Banach space for $\Theta = \Phi, \Psi$.

We introduce $\overline{P^{1^0}}, \overline{P^1{}_0}, \overline{P^{2^0}}, \overline{P^2{}_0} \in \mathbb{R}^N$ as

$$\overline{P^{1^0}} = ((p^1_1)^0, \cdots, (p^1_N)^0),\ \overline{P^1{}_0} = ((p^1_1)_0, \cdots, (p^1_N)_0),$$
$$\overline{P^{2^0}} = ((p^2_1)^0, \cdots, (p^2_N)^0),\ \overline{P^2{}_0} = ((p^2_1)_0, \cdots, (p^2_N)_0)$$

and $(P^{1^0})_+, (P^1{}_0)_+, (P^{1^0})_-, (P^{2^0})_+, (P^2{}_0)_+, (P^{2^0})_- \in \mathbb{R}^+$ by

$$(P^{1^0})_+ := \max\{(p^1_1)^0, \cdots, (p^1_N)^0\},\ (P^1{}_0)_+ := \max\{(p^1_1)_0, \cdots, (p^1_N)_0\},$$
$$(P^1{}_0)_- := \min\{(p^1_1)_0, \cdots, (p^1_N)_0\},$$
$$(P^{2^0})_+ := \max\{(p^2_1)^0, \cdots, (p^2_N)^0\},\ (P^2{}_0)_+ := \max\{(p^2_1)_0, \cdots, (p^2_N)_0\},$$
$$\text{and } (P^2{}_0)_- := \min\{(p^2_1)_0, \cdots, (p^2_N)_0\}.$$

We assume

$$\sum_{i=1}^N \frac{1}{(p^1_i)_0} > 1 \text{ and } \sum_{i=1}^N \frac{1}{(p^2_i)_0} > 1. \tag{2}$$

We define $(P^1_0)^* \in \mathbb{R}^+$, $P^1{}_{0,\infty} \in \mathbb{R}^+$, $(P^2_0)^* \in \mathbb{R}^+$ and $P^2{}_{0,\infty} \in \mathbb{R}^+$ by

$$(P^1{}_0)^* = \frac{N}{\sum_{i=1}^N \frac{1}{(p^1)_0} - 1},\ (P^2{}_0)^* = \frac{N}{\sum_{i=1}^N \frac{1}{(p^2)_0} - 1},$$
$$P^1{}_{0,\infty} = \max\{(P^1{}_0)_+, (P^1{}_0)^*\} \text{ and } P^2{}_{0,\infty} = \max\{(P^2{}_0)_+, (P^2{}_0)^*\}.$$

Let $C_+(\overline{\Omega}) := \{\zeta : \zeta \in C(\overline{\Omega}),\ \zeta(x) > 1 \text{ for all } x \in \overline{\Omega}\}$ we define $\zeta^+ := \sup_{x\in\Omega} \zeta(x)$ and $\zeta^- := \inf_{x\in\Omega} \zeta(x)$.

For any $\zeta \in C_+(\overline{\Omega})$, $L^{\zeta(x)}(\overline{\Omega})$ is defined as the collection of all measurable functions $u : \overline{\Omega} \to \mathbb{R}$ that satisfy the integrability condition $\int_\Omega |u(x)|^{\zeta(x)}\,dx < \infty$ and is called the variable exponent Lebesgue space where

$$|u|_{\zeta(x)} := \inf\{\nu > 0, \int_\Omega \left|\frac{u(x)}{\nu}\right|^{\zeta(x)} dx \leq 1\}.$$

This framework generalizes classical Lebesgue spaces by allowing the exponent $\zeta(x)$ to vary spatially, offering flexibility to model systems with non-uniform growth or decay rates. Further properties of these spaces, such as modular functionals and norm equivalences, depend critically on the behavior of the variable exponent $\zeta(x)$.

We define $\tau_{\zeta(x)}(u) = \int_\Omega |u|^{\zeta(x)}dx$, for and $u \in L^{\zeta(x)}(\Omega)$. If $u_n, u \in L^{\zeta(x)}(\Omega)$ then

$$\begin{cases} \text{If } |u|_{\zeta(x)} > 1 \text{ then } |u|^{\zeta^-}_{\zeta(x)} \le \int_\Omega |u|^{\zeta(x)}dx \le |u|^{\zeta^+}_{\zeta(x)} \\ \text{If } |u|_{\zeta(x)} < 1 \text{ then } |u|^{\zeta^+}_{\zeta(x)} \le \int_\Omega |u|^{\zeta(x)}dx \le |u|^{\zeta^-}_{\zeta(x)}. \end{cases} \tag{3}$$

From now, on we set for $1 \le i \le N$

$$\begin{cases} a_i : \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ a_i(t) = \frac{\phi_i(t)}{t},\ \text{for } t > 0 \text{ and } a_i(0) = 0, \\ b_i : \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ b_i(t) = \frac{\psi_i(t)}{t},\ \text{for } t > 0 \text{ and } b_i(0) = 0. \end{cases}$$

Notice that $\phi(t) = a_i(|t|)t$ and $\psi(t) = b_i(|t|)t$, since ϕ, ψ are odd functions.

The following embedding result is the extension of [9] or [10, Proposition 2.1].

Proposition 2.1. *Suppose* (H_1) *and* (2) *hold, and* $q_1, q_2 \in C(\overline{\Omega})$ *satisfy* $1 < q_1(x) < P^1{}_{0,\infty}$ *and* $1 < q_2(x) < P^2{}_{0,\infty}$ *for all* $x \in \overline{\Omega}$. *The embedding* $X \hookrightarrow L^{q_1(x)}(\Omega) \times L^{q_2(x)}(\Omega)$ *is compact.*

3. Eigenvalue problem

Having established the necessary framework in Section 2, we now turn to the main objective of this work: proving existence results for the coupled system (1) The variational structure of the problem naturally leads us to employ critical point theory in anisotropic Orlicz-Sobolev spaces. This section presents our key existence theorems, each addressing different scenarios for the eigenvalue parameter λ.

Here we set $X := W^1_0 L_{\overline{\Phi}}(\Omega) \times W^1_0 L_{\overline{\Psi}}(\Omega)$ and define the norm

$$\|(u, v)\|_X := |||\nabla u|||_\Phi + |||\nabla v|||_\Psi.$$

Definition 3.1. *The pair of functions* $(u, v) \in X$ *is defined as a weak solution to the problem* (1) *if it satisfies the integral identity*

$$\int_\Omega \sum_{i=1}^N (a_i(|\partial_i u|)\partial_i u \partial_i w_1 dx + b_i(|\partial_i v|)\partial_i v \partial_i w_2)\, dx - \lambda \int_\Omega \left(|u|^{q_1(x)-2}u(x)w_1(x) + |v|^{q_2(x)-2}v(x)w_2(x)\right) dx = 0$$

for all $(w_1, w_2) \in X$.

For any $\lambda > 0$ the energy functional $T_\lambda : X \to \mathbb{R}$ is defined by

$$T_\lambda(u, v) = \int_\Omega \sum_{i=1}^N (\Phi_i(|\partial_i u|) + \Psi_i(|\partial_i v|))\, dx - \lambda \left(\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)} + \frac{1}{q_2(x)}|v|^{q_2(x)} dx \right).$$

The Proposition 2.1 implies that $T_\lambda \in C^1(X, \mathbb{R})$ and

$$T'_\lambda(u, v) := \int_\Omega \sum_{i=1}^N (a_i(|\partial_i u|\partial_i u \partial w_1 + b_i(|\partial_i v|\partial_i v \partial w_2)\, dx - \lambda \int_\Omega |u|^{q_1(x)-2}uw_1 dx - \lambda \int_\Omega |v|^{q_2(x)-2}vw_2 dx.$$

The energy functional T_λ has the Mountain Pass geometry (see [10, Lemma 2.2 and 2.3]).

Lemma 3.2. *Suppose* $q_1(x), q_2(x) \in C^+(\overline{\Omega})$ *and*

$$\begin{aligned} &(P^{1^0})_+ < \min_{x\in\overline{\Omega}} q_1(x) \le \max_{x\in\overline{\Omega}} q_1(x) < (P^{1^0})^*, \\ &(P^{2^0})_+ < \min_{x\in\overline{\Omega}} q_2(x) \le \max_{x\in\overline{\Omega}} q_2(x) < (P^{2^0})^*. \end{aligned} \tag{4}$$

Then

(I) $\exists_{\eta>0,\alpha>0}$ *such that* $T_\lambda(u,v) \geq \alpha > 0$ *for any* $(u,v) \in X$ *with* $\|(u,v)\|_X = \eta$.

(II) $\exists_{(e_1,e_2)\in X}$ *with* $\|(e_1,e_2)\|_X > \eta$ *such that* $T_\lambda(e_1,e_2) < 0$.

The next result establishes the existence of a nontrivial solution via Mountain Pass geometry.

Theorem 3.3. *Assuming the conditions outlined in* (4) *of Lemma 3.2 are satisfied. Suppose* $({P^1}^0)_+ \leq ({P^2}^0)_+$ *and* $({P^1}_0)_- \leq ({P^2}_0)_-$. *Then for any* $\lambda > 0$ *the problem defined by equation* (1) *admits a nontrivial solution within X.*

Proof. Notice that the Mountain-Pass Theorem and Lemmas 3.2 imply there exist a sequence $(u_n, v_n) \subset E$ such that

$$T_\lambda(u_n,v_n) \to \bar{c} \text{ and } T'_\lambda(u_n,v_n) \to 0. \tag{5}$$

We proceed to demonstrate that the sequence $\{u_n, v_n\}$ is bounded in X. Assume not, thus there exists a subsequence (still denoted by $\{u_n, v_n\}$) such that $\|(u_n,v_n)\|_X \to \infty$. Consequently, this assumption implies that for sufficiently large n, the inequality $\|(u_n,v_n)\|_X > 1$ hold.

Now, we define for any $n \geq 0$ and $1 \leq i \leq N$

$$\alpha_{i,n} = \begin{cases} \max\{({P^1}^0)_+, ({P^2}^0)_+\} & \text{if } \|\partial_i u_n\|_{\Phi_i}, \|\partial_i v_n\|_{\Psi_i} < 1, \\ \max\{({P^1}_0)_+, ({P^2}_0)_-\} & \text{if } \|\partial_i u_n\|_{\Phi_i}, \|\partial_i v_n\|_{\Psi_i} > 1 \end{cases}$$

So, by the above considerations (see [1, 2, 9, 10]) we deduce that for n large enough we have

$$\begin{aligned}
1 + \bar{c} + \|(u_n,v_n)\|_X \quad &\geq \quad T_\lambda(u_n,v_n) - \frac{1}{q_1^-}\langle T'_\lambda(u_n,v_n), (u_n,v_n)\rangle \\
&= \quad \sum_{i=1}^N \int_\Omega \left(\Phi_i(|\partial_i u_n|) - \frac{1}{q_1^-}\phi_i(|\partial_i u_n|)|\partial_i u_n|\right)dx + \sum_{i=1}^N \int_\Omega \left(\Psi_i(|\partial_i v_n|) - \frac{1}{q_1^-}\psi_i(|\partial_i v_n|)|\partial_i v_n|\right)dx \\
&\quad + \lambda\left(\int_\Omega \left(\frac{1}{q_1^-} - \frac{1}{q_1(x)}\right)|u_n|^{q_1(x)} + \left(\frac{1}{q_1^-} - \frac{1}{q_2(x)}\right)|v_n|^{q_2(x)}dx\right) \\
&\geq \quad \sum_{i=1}^N \int_\Omega \left(\Phi_i(|\partial_i u_n|) - \frac{1}{q_1^-}\phi_i(|\partial_i u_n|)|\partial_i u_n|\right)dx \\
&\quad + \sum_{i=1}^N \int_\Omega \left(\Psi_i(|\partial_i v_n|) - \frac{1}{q_1^-}\psi_i(|\partial_i v_n|)|\partial_i v_n|\right)dx \\
&\geq \quad \left(1 - \frac{({P^1}^0)_+}{{q_1}^-}\right)\sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i u_n|)dx + \left(1 - \frac{({P^2}^0)_+}{{q_1}^-}\right)\sum_{i=1}^N \int_\Omega \Psi_i(|\partial_i v_n|)dx \\
&\geq \quad \left(1 - \frac{({P^1}^0)_+}{{q_1}^-}\right)\sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{\alpha_{i,n}} + \left(1 - \frac{({P^2}^0)_+}{{q_1}^-}\right)\sum_{i=1}^N \|\partial_i v_n\|_{\Psi_i}^{\alpha_{i,n}} \\
&\geq \quad \left(1 - \frac{({P^1}^0)_+}{{q_1}^-}\right)\left\{\frac{1}{N^{({P^1}_0)_- - 1}}\|u_n\|_{\Phi}^{({P^1}_0)_-} - N\right\} + \left(1 - \frac{({P^2}^0)_+}{{q_1}^-}\right)\left\{\frac{1}{N^{({P^2}_0)_- - 1}}\|v_n\|_{\Psi}^{({P^2}_0)_-} - N\right\} \\
&\geq \quad \left(1 - \frac{({P^1}^0)_+}{{q_1}^-}\right)\left\{\frac{1}{N^{({P^1}_0)_- - 1}}\|(u_n,v_n)\|_X^{({P^1}_0)_-} - N\right\}
\end{aligned} \tag{6}$$

where we assumed $({P^1}^0)_+ \leq ({P^2}^0)_+$ and $({P^1}_0)_- \leq ({P^2}_0)_-$. By dividing through by $\|(u_n,v_n)\|_X^{({P^2}_0)_-}$ in equation (6) and subsequently taking the limit as $n \to \infty$, a contradiction arises. This establishes that the sequence $\{(u_n,v_n)\}$ is bounded in X. Utilizing the reflexivity of X, there exists a subsequence denoted again as $\{(u_n,v_n)\}$ and as an element $(u_0,v_0) \in X$ such that $\{(u_n,v_n)\}$ converges weakly to (u_0,v_0) in X. Consequently, invoking Proposition 2.1, it follows that $\{(u_n,v_n)\}$ converges strongly to (u_0,v_0) in $L^{q_1(x)}(\Omega) \times L^{q_1(x)}(\Omega)$.

The aforementioned arguments, alongside relations (5) and assumption (H_8), further imply that the convergence of $\{(u_n, v_n)\}$ to (u_0, v_0) is, in fact, strong within X. Therefore, by virtue of relation (5), one concludes that $T_\lambda(u_0, v_0) = \bar{c}$ and $T'_\lambda(u_0, v_0) = 0$. This demonstrates that (u_0, v_0) constitutes a nontrivial weak solution to the equation given by (1). □

For the proof of the following lemma see [9] and [10, Lemma 2.4].

Lemma 3.4. *Suppose* $q_1(x), q_2(x) \in C^+(\overline{\Omega})$ *and*

$$\begin{aligned} &1 < \min_{x\in\overline{\Omega}} q_1(x) < (P_0^1)_- \text{ and } \max_{x\in\overline{\Omega}} q_1(x) < P^1_{0,\infty}, \\ &1 < \min_{x\in\overline{\Omega}} q_2(x) < (P_0^2)_- \text{ and } \max_{x\in\overline{\Omega}} q_2(x) < P^2_{0,\infty}. \end{aligned} \tag{7}$$

Thus, there exists a parameter $\lambda^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*)$*, one can identify constants* $\rho > 0$ *and* $a > 0$ *satisfying the condition* $T_\lambda(u, v) \geq a > 0$ *for all* $(u, v) \in X$ *with* $\|(u, v)\|_X = \rho$

The following result complements Theorem 3.3 by establishing existence for small values of the parameter λ.

Theorem 3.5. *Assuming the conditions outlined in* (7) *of Lemma 3.4. Then there exists* λ^* *such that problem* (1) *has a nontrivial solution in* X *for any* $\lambda \in (0, \lambda^*)$.

Proof. Let λ^* be defined as above and fix $\lambda \in (0, \lambda^*)$. By Lemma 3.4, we have the boundary estimate

$$\inf_{\partial B_\rho(0)\times B_\rho(0)} J_\lambda > 0.$$

Step 1: Constructing the sublevel set. Standard variational arguments yield the existence of nonnegative functions $(\wp, \varpi)$ such that for sufficiently small $t > 0$:

$$T_\lambda(t\wp, t\varpi) < 0.$$

Moreover, the functional satisfies the growth estimate for all $(u, v) \in B_\rho(0) \times B_\rho(0)$:

$$T_\lambda(u, v) \geq C_1 \|(u, v)\|_X^{(P^{1^0})_+} - C_2 \|(u, v)\|_X^{q_1^-},$$

where $C_1, C_2 > 0$ are constants. This implies the existence of a negative infimum:

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)\times B_\rho(0)}} T_\lambda < 0.$$

Step 2: Applying Ekeland's principle. Fix

$$0 < \epsilon < \inf_{\partial B_\rho(0)\times B_\rho(0)} T_\lambda - \inf_{B_\rho(0)\times B_\rho(0)} T_\lambda.$$

Ekeland's variational principle yields $(u_\epsilon, v_\epsilon) \in B_\rho(0) \times B_\rho(0)$ satisfying:

- $T_\lambda(u_\epsilon, v_\epsilon) < \inf T_\lambda + \epsilon$
- $T_\lambda(u_\epsilon, v_\epsilon) < T_\lambda(u, v) + \epsilon\|(u, v) - (u_\epsilon, v_\epsilon)\|_X$ for $(u, v) \neq (u_\epsilon, v_\epsilon)$

Step 3: Variational analysis. Consider the perturbed functional

$$I_\lambda(u, v) = T_\lambda(u, v) + \epsilon\|(u, v) - (u_\epsilon, v_\epsilon)\|_X.$$

At the minimum point (u_ϵ, v_ϵ), we derive for any $(w_1, w_2) \in B_\rho(0) \times B_\rho(0)$ and small $t > 0$:

$$\frac{I_\lambda(u_\epsilon + tw_1, v_\epsilon + tw_2) - I_\lambda(u_\epsilon, v_\epsilon)}{t} \geq 0.$$

Passing to the limit $t \to 0^+$ yields

$$\langle T'_\lambda(u_\epsilon, v_\epsilon), (w_1, w_2)\rangle + \epsilon\|(w_1, w_2)\|_X \geq 0,$$

from which we conclude $\|T'_\lambda(u_\epsilon, v_\epsilon)\| \leq \epsilon$.

Step 4: Constructing the solution. There exists a Palais-Smale sequence $\{(w_{1n}, w_{2n})\} \subset B_\rho(0) \times B_\rho(0)$ satisfying:

- $T_\lambda(w_{1n}, w_{2n}) \to \underline{c}$
- $T'_\lambda(w_{1n}, w_{2n}) \to 0$

The boundedness of this sequence in X implies weak convergence to some (w_1, w_2). Following arguments similar to Theorem 3.3, we establish strong convergence. Consequently, (w_1, w_2) satisfies:

$$T_\lambda(w_1, w_2) = \underline{c} < 0 \quad \text{and} \quad T'_\lambda(w_1, w_2) = 0,$$

proving it is a nontrivial weak solution of (1). □

We now present our final existence result, which establishes multiple solutions under critical growth conditions.

Theorem 3.6. *Assume the variable exponents $q_1, q_2 \in C^+(\overline{\Omega})$ satisfy the critical growth conditions:*

$$\begin{cases} 1 < \inf_{x\in\Omega} q_1(x) \leq \sup_{x\in\Omega} q_1(x) < (P_0^1)_- \\ 1 < \inf_{x\in\Omega} q_2(x) \leq \sup_{x\in\Omega} q_2(x) < (P_0^2)_- \end{cases} \tag{8}$$

Then there exist critical parameters $0 < \lambda_* < \lambda_{**}$ such that:

(i) For all $\lambda \in (0, \lambda_*)$, problem (1) *admits a nontrivial solution*

(ii) For all $\lambda > \lambda_{**}$, problem (1) *admits another distinct nontrivial solution*

Both solutions belong to the space $X = W_0^{1,\Phi}(\Omega) \times W_0^{1,\Psi}(\Omega)$.

Proof. The proof proceeds in two main steps:

Part 1: Existence for small λ

By Theorem 3.5, there exists $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$, the energy functional T_λ possesses a mountain pass critical point, yielding the first nontrivial solution.

Part 2: Existence for large λ

We now establish the existence of a second solution for $\lambda > \lambda_{**}$ via global minimization:

(a) *Coercivity and weak lower semicontinuity*: The functional T_λ is:

- Coercive due to the strict subcritical growth condition (8)
- Weakly lower semicontinuous as a sum of convex N-functions and compact perturbations

(b) *Global minimizer*: By the fundamental result of Struwe [16, Theorem 1.2], there exists $(u_\lambda, v_\lambda) \in X$ realizing the global minimum of T_λ.

(c) *Non-triviality for large λ*: Fix $t_0 > 1$ and let $\Omega_1 \subset \Omega$ be an open subset with $|\Omega_1| > 0$. Construct test functions $(w_{10}, w_{20}) \in C_0^\infty(\Omega) \subset X$ satisfying:

$$(w_{10}(x), w_{20}(x)) = \begin{cases} (t_0, t_0) & x \in \overline{\Omega_1} \\ \in [0, t_0] & x \in \Omega \setminus \Omega_1 \end{cases}$$

A direct computation shows:

$$
\begin{aligned}
T_\lambda(w_{10}, w_{20}) =& \int_\Omega \sum_{i=1}^{N} (\Phi_i(|\partial_i w_{10}|) + \Psi_i(|\partial_i w_{20}|))\, dx - \lambda \left(\int_\Omega \frac{1}{q_1(x)} |w_{10}|^{q_1(x)} + \frac{1}{q_2(x)} |w_{20}|^{q_2(x)} dx \right) \\
\le& L - \frac{\lambda}{q_2^+} \int_{\Omega_1} |w_{10}|^{q_1(x)} + w_{20}^{q_2(x)} dx \\
\le& L - \frac{2\lambda}{q_2^+} (t_0^{q_1^-}) |\Omega_1|.
\end{aligned}
$$

where $L > 0$ depends only on Ω and t_0. Choosing $\lambda_{**} > \frac{Lq_2^+}{2t_0^{q_1^-}|\Omega_1|}$ ensures $T_\lambda(w_{10}, w_{20}) < 0$ for all $\lambda > \lambda_{**}$. Consequently, the global minimizer satisfies $T_\lambda(u_\lambda, v_\lambda) < 0$ and is therefore nontrivial.

The two solutions are necessarily distinct since one is obtained via mountain pass geometry while the other is a global minimizer. □

Conclusion

In this paper, we prove existence results for a class of anisotropic ϕ-Laplacian systems with variable exponents. Under hypotheses (H_1)-(H_8), Theorems 3.3, 3.5, 3.6 demonstrate that nontrivial solutions exist for λ in various ranges, leveraging Mountain Pass theory and sublinear perturbations. The anisotropic Orlicz-Sobolev setting required careful analysis of embedding properties (Proposition 2.1) and energy functional geometry (Lemmas 3.2, 3.4).

Future directions include studying sign-changing solutions, bifurcation phenomena, and applications to non-Newtonian fluids. Extensions to quasilinear systems with convection terms or fractional anisotropy remain open challenges.

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References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Ph. Clément, B. de Pagter, G. Sweers and F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. **1** (2004), 241-267.

[3] A. Cianchi, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J. **45(1)** (1996), 39-65.

[4] A. Cianchi, *Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **23** (1996), no. 3, 575-608.

[5] T.K. Donaldson and N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. Anal. **8** (1971), 52-75.

[6] G. Ercole, G. Figueiredo and A. Razani, *Uniform convergence of global least energy solutions to Dirichlet systems in non-reflexive Orlicz-Sobolev spaces*, Results Math. **79** (2024), article number 237, 1-16. https://doi.org/10.1007/s00025-024-02270-9

[7] S. Heidari and A. Razani, *Infinitely many solutions for nonlocal elliptic systems in Orlicz–Sobolev spaces*, Georgian Math. J. **29** (2021) no. 1, 45–54. https://doi.org/10.1515/gmj-2021-2110

[8] S. Heidari and A. Razani, *Multiple solutions for a class of nonlocal quasilinear elliptic systems in Orlicz-Sobolev spaces*, Bound. Value Probl. **2021** (2021), No.22 15 pages. https://doi.org/10.1186/s13661-021-01496-8

[9] M. Mihăilescu, G. Moroşanu and V. Rădulescu, *Eigenvalue problems for anisotropic elliptic equations: An Orlicz-Sobolev space setting*, Nonlinear Anal. **73** (2010), 3239-3253.

[10] M. Mihăilescu, G. Moroşanu and V. Rădulescu, *Eigevalue problems in anisotropic Orlicz-Sobolev spaces*, C. R. Acad. Sci. Paris, Ser. I **347** (2009), 521-526.

[11] D. Motreanu, A. Razani and E. Tornatore, *An anisotropic Dirichlet system including unbounded coefficients*, Discrete Contin. Dyn. Syst. Ser. S **18** (2025), No. 12, 3799-3817. https://doi.org/10.3934/dcdss.2025215

[12] A. Razani, *Nonstandard competing anisotropic (p, q)-Laplacians with convolution*, Bound. Value Probl. **2022:87** (2022). https://doi.org/10.1186/s13661-022-01669-z

[13] A. Razani, *Entire weak solutions for an anisotropic equation in the Heisenberg group*, Proc. Amer. Math. Soc. **151** (2023), No. 11 4771-4779. https://doi.org/10.1090/proc/16488

[14] A. Razani and S. Baraket, *Non-variational solutions for anisotropic (p, q)-Laplacian systems with nonhomogeneous growth*, Bull. Iranian Math. Soc. **51:83** (2025). https://doi.org/10.1007/s41980-025-01007-w

[15] A. Razani and G. M. Figueiredo, *Degenerated and competing anisotropic (p, q)-Laplacians with weights*, Appl. Anal. **102** (2023), No. 16, 4471-4488. https://doi.org/10.1080/00036811.2022.2119137

[16] M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, Heidelberg, 1996.