



On some F -spaces associated to the functions satisfying Mulholland inequality

Lav Kumar Singh^{a,*}, Aljoša Peperko^{b,a}

^a*Institute of Mathematics, Physics and Mechanics, Jadranska ulica 19, SI-1000 Ljubljana, Slovenia*
^b*Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia*

Abstract. In this article we explore a new growth condition on Young functions, which we call Mulholland condition, pertaining to the mathematician H.P. Mulholland, who studied these functions for the first time, albeit in a different context. We construct a non-trivial Young function Ω which satisfies Mulholland condition and Δ_2 -condition. We then associate F -norms to the vector space $X_1 \oplus X_2$, where X_1 and X_2 are Banach spaces, using the function Ω . This F -space contains the Banach space X_1 and X_2 as a maximal Banach subspace. Further, the Banach envelope $(X_1 \oplus X_2, \|\cdot\|_{\Omega_0})$ of this F -space corresponds to the Young function Ω_0 whose characteristic function is an asymptotic line to the characteristic function of the Young function Ω . Thus these F -spaces serve as "interpolation space" for Banach spaces X_1 and $(X_1 \oplus X_2, \|\cdot\|_{\Omega_0})$ in some sense. These F -space are well behaved in regards to Hahn-Banach extension property, which is lacking in classical F -spaces like L^p and H^p for $0 < p < 1$. Towards the end, some direct sums for Orlicz spaces are discussed.

1. Introduction

F -spaces often arise in various subdomains of operator theory. For an introductory literature on F -spaces one can refer the book "An F -space sampler" by N.J. Kalton [6], where he studies non-locally convex topological vector spaces (F -spaces in particular). Most of the text is dedicated to the study of F -space $L^p[0, 1]$ and H^p for $0 < p < 1$ (the p -norm being $\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$). One can construct Banach envelope of any F -space, which in some sense is the "smallest" Banach space containing the original F -space by giving a new norm to the underlying vector space. As mentioned by Kalton, modern functional analysis mostly focuses on locally convex space, and rightly so. But in some cases, there is no reason to restrict the study to locally convex space, for example there is no reason to restrict the study of H^p spaces only for $p \geq 1$. Several important results regarding F -spaces are explored in [11], [10], [3], [7] and [6].

In this article we consider a family of special type of Young functions $\Phi(x) = |x|e^{\chi(\ln|x|)}$, where χ is a continuous even convex function on \mathbb{R} which is also strictly increasing on positive axis. The function χ will

2020 Mathematics Subject Classification. 46A16, 46E30, 46B70

Keywords. F -spaces, Frechet Spaces, Banach spaces, Metric space, Banach envelope, Interpolation spaces, Orlicz spaces
Received: 18 September 2025; Accepted: 17 December 2025

Communicated by Dragan S. Djordjević

* Corresponding author: Lav Kumar Singh

Email addresses: lavksingh@hotmail.com (Lav Kumar Singh), aljosa.peperko@fs.uni-lj.si (Aljoša Peperko)

ORCID iDs: <https://orcid.org/0000-0001-9748-1493> (Lav Kumar Singh), <https://orcid.org/0000-0003-1080-956X> (Aljoša Peperko)

be called the characteristic function of Φ . Such Young functions are said to satisfy Mulholland condition. When χ is an affine function of the type $\chi(x) = ax + b$, then the associated Young function is of type $\Phi(x) = Cx^m$ for some $C > 0$ and $m \in \mathbb{N}$. But these Young functions are of less interest to us. In section 3, we construct a special kind of Young function Ω whose characteristic function χ has an ever increasing growth. But the growth of Ω is still restricted by Δ_2 -condition. These characteristic functions have a linear asymptote at infinity, which will serve as characteristic function for Young function Ω_0 . Now for any two Banach spaces X_1 and X_2 and any Young function Φ with Mulholland condition, we can assign an F -norm $\|\cdot\|_\Phi$ to the space $X_1 \oplus X_2$. These F -norms are actually norms if the characteristic function of Φ is an affine function. Otherwise these F -norms fails homogeneity condition in general. We will show that for our special type of Young function Ω constructed in example 3.1, the F -space $(X_1 \oplus X_2, \|\cdot\|_\Omega)$ is not a Banach space and contains both X_1 and X_2 as maximal Banach subspaces. Further they manifest an interesting geometric phenomenon. The Banach envelope of $(X_1 \oplus X_2, \|\cdot\|_\Omega)$ is $(X_1 \oplus X_2, \|\cdot\|_{\Omega_0})$, where Ω_0 is the Young function whose characteristic function is the asymptotic line to the characteristic function of Ω . Interestingly these spaces also enjoy the Hahn-Banach extension property (HBEP), which is not a common appearance in F -spaces.

2. Preliminaries to Young functions and associated Orlicz spaces.

2.1. Young functions

Definition 2.1. A **Young function** is a convex, left semicontinuous, even function $\Phi : \mathbb{R} \rightarrow [0, \infty]$ such that $\Phi(0) = 0$ and $\lim_{x \rightarrow \infty} \Phi(x) = \infty$.

A Young function is said to be *finite* if $\Phi(x) < \infty$ for each $x \in \mathbb{R}$. Finite Young functions are automatically continuous because finite convex functions are continuous. Further an Young function Φ is called *N-function* if it is continuous (and hence finite), $\lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty$ and $\lim_{x \rightarrow 0} \frac{\Phi(x)}{x} = 0$. A Young function Φ is said to satisfy *Mulholland conditions* if it is continuous and strictly increasing on $[0, \infty)$ and $\log \Phi(x)$ is a convex function of $\log x$. In fact it was shown in [1][5,Cor. 1] that a strictly increasing (on positive real axis) finite Young function Φ satisfying Mulholland condition is equivalent to it taking the form $\Phi(x) = |x|e^{\chi(\log|x|)}$ for some continuous increasing convex function χ . In such situation, we say that χ is the characteristic function of Φ .

Example 2.2. The functions such as $\sinh|x|$, $|x|^{1+a}e^{b|x|^c}$ ($a, b, c > 0$) are strict Young functions which satisfies Mulholland conditions.

A Young function Φ is said to satisfy Δ_2 -condition if there exists $M > 0$ such that $\Phi(2x) \leq M\Phi(x)$ for all $x \geq x_0 \geq 0$. For example, if we take χ to be an affine function on \mathbb{R} with positive slope, then $\Phi(x) = |x|e^{\chi(\log|x|)}$ gives us a trivial Young function which is strict and satisfies both Mulholland condition and Δ_2 -conditions. In-fact there is an ubundance of strict Young functions which satisfy both these conditions.

Associated to a Young function Φ , there exists another convex function Ψ given by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

The function Ψ is also a Young function and the pair (Φ, Ψ) is called complementary pair of Young functions. The complementary pair (Φ, Ψ) of Young functions is said to satisfy Δ_2 -condition if both Φ and Ψ satisfy Δ_2 -condition.

Remark 2.3. For $p \geq 1$, the maps $t \mapsto |t|^p$ are Young functions which satisfy both the Δ_2 -condition and Mulholland's condition. Further the associated complementary Young function also satisfies Δ_2 . In-fact taking χ to be any affine function of positive slope on \mathbb{R} makes $\Phi(t) = |t|e^{\chi(\log|t|)}$ a Young function which satisfy Mulholland condition and the complementary pair associated to it satisfies Δ_2 -condition.

Example 2.4. The Young function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$ defined as $t \mapsto |t|e^{\frac{t^2-|t|}{2|t|}}$ is strict and satisfy Mulholland's condition. Further the complementary pair (Φ, Ψ) satisfies Δ_2 -condition. Clearly, $\Phi(t) = |t|e^{\sinh(\log|t|)}$ and hence it satisfies Mulholland condition since \sinh is strictly increasing convex function on $[0, \infty)$. Now notice that

$$\frac{\Phi(2t)}{\Phi(t)} = 2e^{\frac{2t^2+1}{4t}} \rightarrow \infty$$

And hence, Φ does not satisfy Δ_2 -condition.

2.2. Orlicz spaces

We define Orlicz space associated to a measure space (X, μ) and Young function Φ as

$$L^\Phi(X) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable, } \int_X \Phi(\beta|f|)d\mu < \infty \text{ for some } \beta > 0\}$$

$L^\Phi(X)$ becomes a Banach space with respect to the Gauge norm

$$N_\Phi(f) = \inf \left\{ k > 0 : \int_X \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1 \right\}$$

There is an another equivalent norm on $L^\Phi(X)$, known as Orlicz norm

$$\|f\|_\Phi = \sup \left\{ \int_X |fg| d\mu : g \in L^\Psi(X) \text{ and } \int_X \Psi(|g|) d\mu \leq 1 \right\}.$$

Further, if (Φ, Ψ) are a pair of complementary Young functions, both satisfying Δ_2 -condition, then $(L^\Phi(X), N_\Phi)$ is a reflexive Banach space with $(L^\Psi(X), \|\cdot\|_\Psi)$ as its dual space.

2.3. F-spaces

Definition 2.5. An F-space is a vector space X over a field of real or complex numbers together with a metric $d : X \times X \rightarrow [0, \infty)$ such that

- Scalar multiplication in X is continuous with respect to the metric d on X and standard metric on \mathbb{C} (or \mathbb{R}).
- Addition in X is continuous with respect to d .
- d is translation invariant i.e, $d(x+a, y+a) = d(x, y)$ for all $x, y, a \in X$.
- Metric space (X, d) is complete.

The map $x \mapsto \|x\|_F = d(x, 0)$ is called F-norm. An F-space is called Fréchet space if the underlying topology is locally convex.

Definition 2.6. A topological vector space X is called locally bounded if there exists an open neighborhood A of 0 such that for each open neighborhood U of 0 there exists $t > 0$ such that $A \subset sU$ for all $s > t$.

F-spaces are not locally convex in general and the closed unit ball B_X is not convex. Hence, it is natural to consider the closed convex hull $co(B_X)$. The Gauge seminorm on X

$$\|x\|_C = \inf \left\{ \lambda > 0 : \frac{f}{\lambda} \in co(B_X) \right\}$$

is actually a norm on X if its dual X^* separate points (see [6, Ch.2, sec. 4]) and the identity mapping $i : (X, \|\cdot\|_F) \rightarrow (X, \|\cdot\|_C)$ is continuous. In-fact $(\overline{X}, \|\cdot\|_C)$ is the “smallest” Banach space containing $(X, \|\cdot\|_F)$.

Definition 2.7. Let (X, d) be an F -space with a separating dual. The Banach space $\overline{(X, \|\cdot\|_C)}$ generated by the Gauge norm on X is called the Banach envelope of the F -space (X, d) .

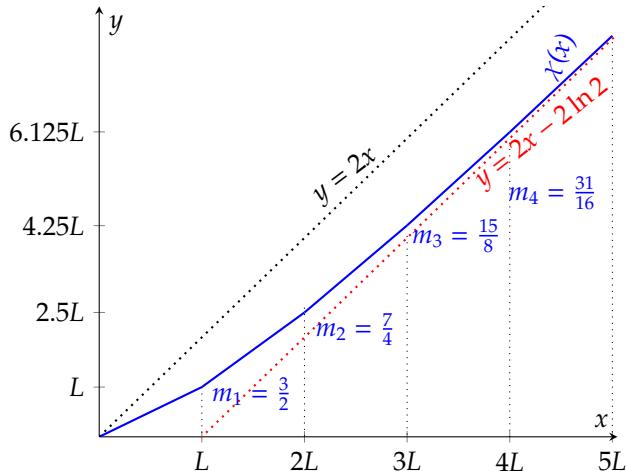
Example 2.8. Consider the F -space $(\mathbb{R}^2, \|\cdot\|_p)$ for any $0 < p < 1$. Then the closed convex hull of the unit ball of \mathbb{R}^2 with respect to $\|\cdot\|_p$ is same as the unit ball of $(\mathbb{R}^2, \|\cdot\|_1)$ and hence $(\mathbb{R}^2, \|\cdot\|_1)$ is the Banach envelope of $(\mathbb{R}^2, \|\cdot\|_p)$.

Theorem 2.9. [6, Ch.2 Sec. 4] If (X, d) is an F -space then the dual space (space of continuous linear functionals on X with respect to F -norm) of (X, d) is the same (isometrically isomorphic) as the dual space of its Banach envelope.

3. Interpolation F -spaces of Banach spaces

We notice that there is an abundance of non-trivial Young functions which satisfy both Δ_2 -condition and Mulholland condition. In the previous remark, we saw few trivial examples of Young functions which satisfy Mulholland condition and the complementary pair (Φ, Ψ) satisfies Δ_2 -condition. We now construct a non-trivial example of such a Young function Φ .

Example 3.1. Let $M > 0$ be a fixed number and $\{m_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of positive real numbers converging to a fixed $M > 0$. Consider the convex continuous even function $\chi : \mathbb{R} \rightarrow \mathbb{R}^+$ defined as piecewise straight lines on the intervals $[0, \ln 2], [\ln 2, 2 \ln 2], \dots, [r \ln 2, (r+1) \ln 2], \dots$. The slope of the line in interval $[r \ln 2, (r+1) \ln 2]$ is m_r . The growth of χ is depicted in figure below.



Now if we define $\Omega(t) = |t|e^{\chi(\log|t|)}$, then Ω is a Young function which satisfies Mulholland condition. It is very desirable for Young functions and their complementary functions to have Δ_2 -condition because it makes the associated Orlicz spaces reflexive and the simple functions becomes dense in $L^\Phi(X)$. Thus we show that the above constructed Young function is actually well behaved with respect to the growth and possess Δ_2 -condition. Notice that for $t > 0$ such that $\log t \in [r \log 2, (r+1) \log 2]$, we have

$$\begin{aligned} \frac{\Omega(2t)}{\Omega(t)} &= 2e^{\chi(\log 2 + \log t) - \chi(\log t)} \\ &\leq 2e^{m_{r+1} \ln 2} \\ &\leq 2e^{M \ln 2} = 2^{M+1}. \end{aligned}$$

Hence, Ω satisfies Δ_2 -condition. Although it is not necessary for our further results in this section, we will see that the complementary function Θ also satisfies Δ_2 -condition. To ease the computation, we fix $m_r = 2 - \frac{1}{2^r}$. Let

$\theta(x, t) = xt - \Omega(x)$ be a function defined for $x, t \geq 0$. On interval $I_r = [2^r, 2^{r+1}]$ the function $\Omega(x)$ takes the form $\Omega(x) = C_r x^{m_r+1}$, where C_r is a constant. Let t be fixed. The derivative of θ on the interval I_r is

$$\frac{d}{dx} \theta(x, t) = t - C_r(m_r + 1)x^{m_r}.$$

Thus, $\theta(x, t)$ attains a maximum value in interval I_r at point $x_r(t) = \left(\frac{t}{C_r(m_r+1)}\right)^{\frac{1}{m_r}}$ if $x_r(t) \in I_r$ or on the end points of the interval. The maximum possible value of $\theta(x, t)$ on interval I_r is

$$\Theta_r(t) = tx_r(t) - C_r(x_r(t))^{1+m_r} = \frac{m_r}{(1+m_r)^{\frac{1+m_r}{m_r}}} C_r^{-\frac{1}{m_r}} t^{\frac{1+m_r}{m_r}} \quad (1)$$

Now $\Theta(t) = \sup_{r \geq 0} \Theta_r(t)$. For a fixed t , the supremum is attained at some $r = r(t)$, i.e., $\Theta(t) = \Theta_{r(t)}(t)$. Now, consider the ratio $R(t) = \frac{\Theta(2t)}{\Theta(t)} = \frac{\Theta_{r(t)}(2t)}{\Theta_{r(t)}(t)}$. Since, $\Theta_r(t)$ grows like $t^{\frac{m_r+1}{m_r}}$, the maximizing $r(t)$ increase with an increase in t . Hence,

$$R(t) \leq \max\left\{\frac{\Theta_r(2t)}{\Theta_r(t)} : r \in \mathbb{N}\right\} \leq 2^{1+\frac{1}{m_r}} \quad (\text{due to 1})$$

As $m_r \rightarrow 2$ for large values of r , we see that $R(t) \leq 2^{3/2}$ eventually. Thus the complementary function Θ satisfies Δ_2 -condition.

Note: We will also need the Mulholland Young function Ω_0 associated to the asymptotic red line $y = 2x - 2 \ln 2$ in the above figure. In-fact $\Omega_0(x) = |x|e^{2 \ln |x| - 2 \ln 2} = \frac{1}{4}|x|^3$

Example 3.2. Let χ denote a function which is a combination of rotation of the graph of $f(x) = e^{-x}$ about origin by angle θ and appropriate translation such that the resulting curve has the $y = x \tan \theta - c$ as an asymptote. Then $\Omega(x) = |x|e^{\chi(\ln|x|)}$ is an Young function which satisfies Mulholland condition and Δ_2 -condition.

Suppose $(X_i, \|\cdot\|_i)$ are F -spaces for $i = 1, 2, \dots, n$ and Φ is a Young function which satisfies Mulholland condition. We consider the vector space direct sum $\bigoplus_{i=1}^n X_i$. Further we define a metric $d_\Phi : \bigoplus_{i=1}^n X_i \times \bigoplus_{i=1}^n X_i \rightarrow [0, \infty)$ as

$$d_\Phi((x_1, \dots, x_n), (y_1, \dots, y_n)) = \Phi^{-1} \left(\sum_{i=1}^n \Phi(\|x_i - y_i\|_i) \right).$$

The associated F -norm is $\|(x_1, \dots, x_n)\|_F = \Phi^{-1}(\sum_{i=1}^n \Phi(\|x_i\|_i))$. One can easily check that this is a well defined metric. $d_\Phi((x_1, \dots, x_n), (y_1, \dots, y_n)) = 0$ if and only $x_i = y_i$ for each $i = 1, \dots, n$. Due to Mulholland's condition on Φ , the following Minkowsky type inequality (also known as *Mulholland's inequality*) holds true (see [1][5], Th. 1]).

$$\Phi^{-1} \left(\sum_{i=1}^n \Phi(a_i + b_i) \right) \leq \Phi^{-1} \left(\sum_{i=1}^n \Phi(a_i) \right) + \Phi^{-1} \left(\sum_{i=1}^n \Phi(b_i) \right)$$

for all $a_i, b_i \geq 0$. And hence the triangle inequality follows for the F -norm on $\bigoplus_{i=1}^n X_i$ and hence for the metric d_Φ . We denote this metric space by $(\bigoplus_{i=1}^n X_i, d_\Phi)$.

Proposition 3.3. Metric space $(\bigoplus_{i=1}^n X_i, d_\Phi)$ is an F -space if Φ satisfies Mulholland condition.

Proof. We start by showing that the scalar multiplication continuous with respect to the metric d_Φ . Suppose $\{K_\alpha, (x_1^{(\alpha)}, \dots, x_n^{(\alpha)})\}_{\alpha \in \Gamma}$ be a net in $\mathbb{C} \times \bigoplus_{i=1}^n X_i$ converging to $\{K, (x_1, \dots, x_n)\}$. Then $\|K_\alpha x_i^{(\alpha)} - Kx_i\|_i \rightarrow 0$ for each i due the continuity of scalar multiplications in each space $(X_i, \|\cdot\|_i)$. Combining this with the fact that Φ is strictly increasing and continuous tells us that

$$d_\Phi((K_\alpha x_1^{(\alpha)}, \dots, K_\alpha x_n^{(\alpha)}), (Kx_1, \dots, Kx_n)) = \Phi^{-1} \left(\sum_{i=1}^n \Phi(\|K_\alpha x_i^{(\alpha)} - Kx_i\|_i) \right) \rightarrow 0.$$

Thus the scalar multiplication is continuous with respect to the metric d_Φ .

Now we turn to prove that the addition in $\bigoplus_{i=1}^n X_i$ is continuous with respect to the metric d_Φ . But this is evident from the fact that addition is a short map under the metric d_Φ and hence continuous. The fact that d_Φ is translation invariant follows from the fact that the F -norm, which is translation invariant, generates d_Φ . Only thing remains to be verified is whether $(\bigoplus_{i=1}^n X_i, d_\Phi)$ is a complete metric space. To see this, let $\{(x_1^{(r)}, \dots, x_n^{(r)})\}_{r=1}^\infty$ be a Cauchy sequence in $(\bigoplus_{i=1}^n X_i, d_\Phi)$. Then for each $\epsilon > 0$, there exist a N_ϵ such that $d_\Phi((x_i^{(r)}), (x_i^{(s)})) < \epsilon$ for all $r, s > N_\epsilon$. Then

$$d_\Phi((x_i^{(r)}), (x_i^{(s)})) = \Phi^{-1} \left(\sum_{i=1}^n \Phi(\|x_i^{(r)} - x_i^{(s)}\|_i) \right) < \epsilon$$

for all $r, s > N_\epsilon$. Hence, due to Φ being strictly increasing, we have $\sum_{i=1}^n \Phi(\|(x_i^{(r)}) - x_i^{(s)}\|_i) < \Phi(\epsilon)$ for all $r, s > N_\epsilon$. Hence for each i , again due to strictly increasing nature of Φ , we have $\|x_i^{(r)} - x_i^{(s)}\|_i < \Phi^{-1}(\Phi(\epsilon)) = \epsilon$ for all $r, s > N_\epsilon$. Thus, for each i , the sequence $\{x_i^{(r)}\}_{r=1}^\infty$ is a Cauchy sequence in the F -space X_i . Thus there exists $x_i \in X_i$ for each $i = 1, 2, \dots, n$ such that $\{x_i^{(r)}\} \xrightarrow{\|\cdot\|_i} x_i$ for each $i = 1, 2, \dots, n$. Now, we claim that $\{(x_1^{(r)}, \dots, x_n^{(r)})\} \xrightarrow{d_\Phi} (x_1, \dots, x_n)$. But this is the easy consequence of the continuity of Φ and Φ^{-1} . Hence, $(\bigoplus_{i=1}^n X_i, d_\Phi)$ is an F -space. \square

As one might have noticed, for Banach spaces X_1, \dots, X_n , the only thing preventing $(\bigoplus_{i=1}^n X_i, d_\Phi)$ from being a Banach space is the homogeneity of F -norm $x \mapsto d_\Phi(0, x)$ with respect to scalar multiplication. In-fact the F -norm is homogeneous if $\Phi(x) = |x|e^{\chi(\ln|x|)}$, where $\chi(x) = ax + b$ is any affine function. On the contrary if Φ is of the type constructed in example 3.1, then it is easy to see that the F -norm on $\bigoplus_{i=1}^n X_i$ is not homogeneous with respect to scalar multiplication (the ever changing growth of Φ would not permit homogeneity, and it also follows from the next result). Consider the set $\mathcal{W} = \{V : V \prec \bigoplus_{i=1}^n X_i, \|\alpha(x_i)\|_F = |\alpha| \|(x_i)\|_F \forall \alpha \in \mathbb{C}, (x_i) \in \bigoplus_{i=1}^n X_i\}$. Define the natural ordering on \mathcal{W} induced by inclusion i.e $V_1 \prec V_2$ if V_1 is a subspace of V_2 . Then (\mathcal{W}, \prec) becomes a partially ordered set. Further, \mathcal{W} is non-empty because $V = \{(x, 0, \dots, 0) : x \in X_1\} \in \mathcal{W}$. If $\{W_\alpha\}$ is a chain in \mathcal{W} , then $\cup W_\alpha$ is its upper bound. Hence, by Zorn's lemma the family \mathcal{W} has at-least one maximal member. Clearly, the maximal member of \mathcal{W} will be closed with respect to the metric d_Φ . Hence, the maximal member of \mathcal{W} will be a Banach space with respect to the norm $\|\cdot\|_F$. We will see that for Φ constructed in example 3.1, each space X_i can be identified isometrically (w.r.t d_Φ) with a maximal member of \mathcal{W} .

Definition 3.4. A locally bounded F -space (X, d) is called a p -interpolation F -space for Banach spaces X_1, X_2 and a $p \in [1, \infty)$ if

1. there exists a distance preserving linear maps $i_1 : X_1 \rightarrow X$ and $i_2 : X_2 \rightarrow X$ such that $i_1(X_1)$ and $i_2(X_2)$ are maximal Banach subspaces in X .
2. the Banach envelope of (X, d) is the p -direct sum $X_1 \oplus_p X_2$.

Theorem 3.5. If Ω is the Young function from example 3.1 and $(X_i, \|\cdot\|_i)$ be Banach spaces for $i = 1, 2, \dots, n$. Then each Banach space $(X_i, \|\cdot\|_i)$ is isometrically isomorphic to a maximal Banach subspace of the F -space $(\bigoplus_{i=1}^n X_i, d_\Omega)$.

Proof. To simplify the computations, we shall prove it for $n = 2$. The general case follows in similar fashion. Consider the natural embedding $\theta : (X_1, \|\cdot\|_1) \rightarrow (X_1 \oplus X_2, d_\Omega)$ given by $\theta(x) = (x, 0)$. This is clearly an isometry (w.r.t metric d_Ω). We claim that $\theta(X_1)$ is a maximal Banach subspace of the F -space $(X_1 \oplus X_2, d_\Omega)$. To prove this, suppose on the contrary, that $\theta(X_1)$ is not maximal. Then there exist $W \in \mathcal{W}$ such that $\theta(X_1) \prec W$. Hence, there exists a $(x, y) \in W \setminus \theta(X_1)$ such that

$$\Omega^{-1}(\Omega(\|\alpha x\|_1) + \Omega(\|\alpha y\|_2)) = |\alpha| \Omega^{-1}(\Omega(\|x\|_1) + \Phi(\|y\|_2)) \quad \forall \alpha \in \mathbb{C}. \quad (2)$$

Without loss of generality, we can choose (f, g) to be such that $u = \|x\|_1 = 1$ and $v = \|y\|_2 = 1$. Thus due to equation 2, we have

$$\frac{\Omega(2|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v)))}{\Omega(2|\alpha|u) + \Omega(2|\alpha|v)} = 1 \quad \forall \alpha \in \mathbb{C} \quad (3)$$

Hence,

$$\begin{aligned} \frac{\Omega(2|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v)))}{\Omega(2|\alpha|u) + \Omega(2|\alpha|v)} &= \frac{\Omega(2|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v)))}{|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v))} \frac{|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v))}{\Omega(2|\alpha|u) + \Omega(2|\alpha|v)} \\ &\leq M \frac{|\alpha|}{\Omega(2|\alpha|u) + \Omega(2|\alpha|v)} \quad \forall \alpha \in \mathbb{C} \text{ large enough} \quad (\because \Omega \text{ is } \Delta_2) \end{aligned} \quad (4)$$

Now notice that for large values of t the function $\Omega(t) \approx \frac{1}{4}t^3$. Hence, $\Omega(2|\alpha|u) \approx 2|\alpha|^3|u|^3$ and $\Omega(2|\alpha|v) \approx 2|\alpha|^3|v|^3$. For large values of α . Thus from the equation 4, we have

$$\lim_{\alpha \rightarrow \infty} \frac{\Omega(2|\alpha|\Omega^{-1}(\Omega(u) + \Omega(v)))}{\Omega(2|\alpha|u) + \Omega(2|\alpha|v)} \leq \lim_{\alpha \rightarrow \infty} M \frac{|\alpha|}{2|\alpha|^3(u^3 + v^3)} = 0,$$

which is a clear contradiction to the equation 3.

Hence, $\theta(X_1)$ is a maximal Banach subspace of the F -space $(X_1 \oplus X_2, d_\Omega)$. \square

Remark 3.6. *The above phenomenon is peculiar for the special Young function Ω constructed in example 3.1. For Young functions of the type $\Phi(x) = cx^m$, the subspace X_i is not a maximal Banach subspace in $(X_1 \oplus X_2, d_\Phi)$ because the latter is a Banach space itself in such case.*

In the preceding theorem, we have interpolated the F -space $(X_1 \oplus X_2, d_\Omega)$ from inside through maximal Banach space. Now in the next result we interpolate it from the outside i.e we compute its Banach envelope. Recall from example 3.1, the Young function Ω_o whose characteristic function was an asymptote to the characteristic function of Ω . We now demonstrate an interesting phenomena which outlines that a Banach envelope of an F -space associated to a Young function Ω with Mulholland condition is nothing but the space associated to the Young function whose characteristic function is an asymptote to the characteristic function of Ω . Since, $(X_1 \oplus X_2, d_\Omega)$ has a separating dual and is locally bounded (see theorem 3.13), we have the following result.

Theorem 3.7. *If X_1 and X_2 are Banach spaces, then the Banach envelope of the F -space $(X_1 \oplus X_2, d_\Omega)$ is the Banach space $(X_1 \oplus X_2, d_{\Omega_o})$.*

Proof. Recall that $\|(x_1, x_2)\|_{\Omega_o} = d_{\Omega_o}((0, 0), (x_1, x_2)) = (\|x_1\|^3 + \|x_2\|^3)^{1/3}$. We just have to prove that the gauge norm is $p(x_1, x_2) = \inf \left\{ \lambda > 0 : \frac{(x_1, x_2)}{\lambda} \in co(B) \right\} = (\|x_1\|^3 + \|x_2\|^3)^{1/3}$, where $co(B)$ denotes the closed convex hull

of the unit ball of $(X_1 \oplus X_2, d_\Omega)$. Let $k > 0$ be very large. Further let $\lambda = (\|kx_1\|^3 + \|kx_2\|^3)^{1/3}$. Then

$$\begin{aligned}
 \left\| \frac{(kx_1, ky_2)}{\lambda} \right\|_\Omega &= \Omega^{-1} \left(\Omega \left(\frac{\|kx_1\|}{\lambda} \right) + \Omega \left(\frac{\|kx_2\|}{\lambda} \right) \right) \\
 &\leq \Omega_o^{-1} \left(\Omega \left(\frac{\|kx_1\|}{\lambda} \right) + \Omega \left(\frac{\|kx_2\|}{\lambda} \right) \right) && \because \Omega^{-1} \leq \Omega_o^{-1} \text{ eventually} \\
 &\leq \Omega_o^{-1} \left(\frac{1}{\lambda} \Omega(\|kx_1\|) + \frac{1}{\lambda} \Omega(\|kx_2\|) \right) && \because \Omega \text{ is convex and } \lambda \text{ is very large} \\
 &\approx \Omega_o^{-1} \left(\frac{1}{4\lambda} \|kx_1\|^3 + \frac{1}{4\lambda} \|kx_2\|^3 \right) && \because \Omega(t) = \frac{1}{4}t^3 \text{ for large } t \\
 \therefore \left\| \frac{(kx_1, ky_2)}{\lambda} \right\|_\Omega &= \frac{1}{\lambda} (\|kx_1\|^3 + \|kx_2\|^3)^{1/3} + \epsilon_k \\
 &= 1 + \epsilon_k
 \end{aligned}$$

where, $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, $p(kx_1, kx_2) \leq (1 + \epsilon_k)\lambda$ for large k . But p is a norm and hence $p(kx_1, kx_2) = |k|p(x_1, x_2)$ for all k . Hence, $p(x_1, x_2) \leq (\|x_1\|^3 + \|x_2\|^3)^{1/3} = \|(x_1, x_2)\|_{\Omega_o}$. To prove the reverse inequality, fix $a \in X_1$ and $b \in X_2$. Then, define a function $p_o : \mathbb{R}^2 \rightarrow [0, \infty)$ as $p_o(s, t) = p(sa, tb)$. This is a well defined function and it can be easily verified that this is convex and homogeneous with respect to scalar multiplication on \mathbb{R}^2 . If p is a norm other than $\|\cdot\|_{\Omega_o}$, then the associated convex functions p_o should be sandwiched between p_{Ω_o} and p_Ω , where $p_{\Omega_o}(s, t) = \Omega_o(\sqrt{s^2 + t^2})$ and $p_\Omega(s, t) = \Omega(\sqrt{s^2 + t^2})$. But the only homogeneous convex functions sandwiched between Ω_o and Ω is Ω_o itself (since its characteristic function is asymptote to Ω). Hence, $(X_1 \oplus X_2, d_{\Omega_o})$ is the Banach envelope of the F -space $(X_1 \oplus X_2, d_\Omega)$. \square

Summary: We have so far worked with Ω , where the sequence of slopes for characteristic function is assumed to be $m_r = 2 - \frac{1}{2^r}$ for the ease of computation. But all of the above results holds in a general case as well. The following result captures the essence of if in general setup.

Theorem 3.8. *Let X_1 and X_2 be two Banach spaces and $(X_1 \oplus X_2, \|\cdot\|_p)$ be their p -direct sum for some $1 \leq p < \infty$. Then there exists a p -interpolation F -space $(X_1 \oplus X_2, \|\cdot\|_\Omega)$ which contains X_1 and X_2 as maximal Banach subspaces and the Banach envelope of $(X_1 \oplus X_2, \|\cdot\|_\Omega)$ is the p -direct sum $(X_1 \oplus X_2, \|\cdot\|_p)$.*

Proof. Consider the Young function $\Omega_p = \frac{1}{p}|x|^p$. Then its characteristic function is $\chi_p(x) = (p-1)x - \ln p$. Now choose a increasing sequence $\{m_r\}_{r=1}^\infty$ of positive real numbers such that $m_r \rightarrow p-1$. Construct a continuous even function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ of piece-wise straight lines of slope m_r and such that χ has χ_p as an asymptote (this is always possible). Then the F -space $(X_1 \oplus X_2, d_\Omega)$ associated to the Young function $\Omega(x) = |x|e^{\chi(\ln|x|)}$ is the required interpolation space. \square

Corollary 3.9. *Let Ω be an Young function with Mulholland condition such that the line $\chi_p(x) = (p-1)x - \ln p$ is asymptote to Ω (as constructed in preceding theorem for $1 \leq p < \infty$). Then the dual of the F -space $(X_1 \oplus X_2, d_\Omega)$ is the Banach space $(X_1^* \oplus X_2^*, \|\cdot\|_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Follows easily from the preceding theorem and the fact that the continuous dual of an F -space is equal to the continuous dual of its Banach envelope. \square

Remark 3.10. *One might be tempted to ask- what about the F -space $X_1 \oplus X_2$ equipped with $\|(x, y)\|_p = \|x\|^p + \|y\|^p$ for some $p < 1$? Well, these are non-locally convex F -space and they do not carry a copy of X_1 or X_2 , let alone contain them as maximal subspaces. Also, its Banach envelope is always $(X_1 \oplus X_2, \|\cdot\|_1)$. Further they do not possess Hahn-Banach extension property and hence deemed as object of less interest.*

Definition 3.11. *An F -space (X, d) is said to have the Hahn-Banach Extension property (HBEP) if for any closed subspace M of X and any continuous linear functional $\varphi : M \rightarrow \mathbb{C}$ has a continuous extension $\varphi' : X \rightarrow \mathbb{C}$ such that $\varphi'(x) = \varphi(x)$ for all $x \in M$.*

It is proved in [6, Ch. 2,3] that the spaces ℓ^p and H^p does not have HBEP for $0 < p < 1$. Motivated by these, Duren, Romberg and Shields formulated the problem in 1969—“**Is every F -space X with HBEP locally convex?**”. Shapiro answered this question in affirmative if X has a basis [2]. The answer to this question was proved to be affirmative in general by N.J Kalton (see [6, Theorem 4.8]). For two Banach spaces X_1, X_2 and the Young function Ω from example 3.1, the interpolation F -space $(X_1 \oplus_{\Omega} X_2, d_{\Omega})$ turns out to be locally convex and hence a Fréchet space, since it posses HBEP, as proved in the next result.

Theorem 3.12. *Let X_1, X_2 be two Banach spaces and Φ be a Young function with Mulholland condition. Then, the F -space $(X_1 \oplus X_2, d_{\Phi})$ has the HBEP.*

Proof. Let M be a closed subspace of $(X_1 \oplus_{\Phi} X_2, d_{\Phi})$ and $\varphi : M \rightarrow \mathbb{C}$ be a continuous linear functional. Consider the projection subspace $\pi_1(M)$ and $\pi_2(M)$ in X_1 and X_2 respectively. Let $\varphi_{\pi_1} : \pi_1(M) \rightarrow \mathbb{C}$ and $\varphi_{\pi_2} : \pi_2(M) \rightarrow \mathbb{C}$ be the restriction maps i.e $\varphi_{\pi_1}(x) = \varphi(x, 0)$ and $\varphi_{\pi_2}(y) = \varphi(0, y)$. Then φ_{π_1} and φ_{π_2} are continuous linear maps. By Hahn-Banach theorem, there exists a continuous extensions $\varphi_1 : X_1 \rightarrow \mathbb{C}$ and $\varphi_2 : X_2 \rightarrow \mathbb{C}$ of φ_{π_1} and φ_{π_2} respectively. Let $\varphi' : X_1 \oplus X_2 \rightarrow \mathbb{C}$ be defined as $\varphi'(x, y) = \varphi_1(x) + \varphi_2(y)$. Clearly, φ' is an extension of $\varphi : M \rightarrow \mathbb{C}$. We claim that it is continuous with respect to the metric d_{Φ} . Suppose (x_n, y_n) is a sequence in $X_1 \oplus X_2$ converging to (x, y) with respect to the metric d_{Φ} . Then

$$\lim_{n \rightarrow \infty} \Phi^{-1}(\Phi(\|x_n - x\|) + \Phi(\|y_n - y\|)) = 0.$$

Since Φ^{-1} and Φ are strictly increasing continuous functions, we can deduce that $\|x_n - x\| \rightarrow 0$ and $\|y_n - y\| \rightarrow 0$. Hence, $x_n \rightarrow x$ and $y_n \rightarrow y$. Using the continuity of φ_1 and φ_2 , we know that $\varphi_1(x_n) \rightarrow \varphi_1(x)$ and $\varphi_2(y_n) \rightarrow \varphi_2(y)$. Hence, $\varphi'(x_n, y_n) = \varphi_1(x_n) + \varphi_2(y_n) \rightarrow \varphi_1(x) + \varphi_2(y) = \varphi'(x, y)$. Thus, φ' is an continuous extension of φ . \square

Theorem 3.13. *For a family of locally bounded F -spaces $\{X_i\}_{i=1}^n$ and an Young function Φ which satisfies Mulholland condition, the F -space $(\oplus_i X_i, d_{\Phi})$ is locally bounded.*

Proof. It would be sufficient to prove this for the case $n = 2$. Let $B_r^{(i)}$ denote the open ball of the F -space X_i of radius r and centered at 0. The open ball of $(X \oplus X_2, d_{\Phi})$ centered at $(0, 0)$ and of radius r will be denoted by B_r^{Φ} . Notice that for $(x, y) \in B_1$, we have $\Phi^{-1}(\Phi(\|x\|) + \Phi(\|y\|)) < 1$. Thus $\Phi(\|x\|) < \Phi(1)$ and $\Phi(\|y\|) < 1$. Hence, $\|x\| < 1$ and similarly $\|y\| < 1$. Thus, $(x, y) \in B_1^{(1)} \times B_1^{(2)}$ and

$$B_1^{\Phi} \subset B_1^{(1)} \times B_1^{(2)}. \tag{5}$$

Further, if $c, r > 0$ then for any $(cx, cy) \in cB_r^{(1)} \times cB_r^{(2)}$, we have

$$\begin{aligned} \left\| \left(\frac{cx}{2c}, \frac{cy}{2c} \right) \right\|_{\Phi} &= \Phi^{-1} \left(\Phi \left(\frac{\|cx\|}{2c} \right) + \Phi \left(\frac{\|cy\|}{2c} \right) \right) \\ &\leq \Phi^{-1} \left(\frac{1}{2} \Phi(\|x\|) + \frac{1}{2} \Phi(\|y\|) \right) && \text{due to convexity of } \Phi \\ &\leq \Phi^{-1} \left(\frac{1}{2} \Phi(r) + \frac{1}{2} \Phi(r) \right) && \text{due to } \Phi \text{ being increasing} \\ &= r. \end{aligned} \tag{6}$$

Thus,

$$cB_r^{(1)} \times cB_r^{(2)} \subset 2cB_r^{\Phi} \quad \text{for each } c, r > 0 \tag{7}$$

Now fix a $r_o < 1$. Since X_1 and X_2 are locally, we can find a t such that $B_1^{(1)} \subset sB_{r_o}^{(1)}$ and $B_1^{(2)} \subset sB_{r_o}^{(2)}$ for all $s > t$. Hence,

$$B_1^{(1)} \times B_1^{(2)} \subset sB_{r_o}^{(1)} \times sB_{r_o}^{(2)} \quad \forall s > t. \tag{8}$$

Combining the above inclusion with the inclusion in equation-5 and 7, we get

$$B_1^\Phi \subset B_1^{(1)} \times B_1^{(2)} \subset sB_{r_o}^{(1)} \times sB_{r_o}^{(2)} \subset 2sB_{r_o}^\Phi \quad \forall s > t. \quad (9)$$

Thus, the collection of open balls of radius less than one and centered at origin forms a neighborhood base at origin. The set B_1 is bounded and hence $(X_1 \oplus X_2, d_\Phi)$ is a locally bounded F -space. \square

3.1. Coefficient of non-homogeneity of F -spaces

Since F -norms are non-homogeneous, the best property an F -space $(X, \|\cdot\|_F)$ they could exhibit is that there exists $M > 0$ such that $\|kx\|_F \leq M|k| \cdot \|x\|_F$ holds for all $x \in X$ and $k \in \mathbb{C}$. We define the coefficient of non homogeneity for F -spaces in the obvious way as follows.

Definition 3.14. Let $(X, \|\cdot\|_F)$ be an F -space. Then the coefficient of non-homogeneity of X is

$$\nu_{(X, \|\cdot\|_F)} = \inf\{M > 0 : \|kx\| \leq M|k|\|x\|, \forall k \in \mathbb{C}, x \in X\}.$$

Obviously not all spaces have a finite coefficient of non-homogeneity. For example, if $0 < p < 1$, then $(\mathbb{R}^2, \|\cdot\|_p)$ does not have finite coefficient of non-homogeneity because $\|(kx, ky)\|_p = k^p\|(x, y)\|_p$. We will show that F -norms associated to the Young functions with Mulholland condition, whose characteristic function has linear asymptote, have finite coefficient of non-homogeneity.

Lemma 3.15. If Φ is a Young function with Mulholland condition such that its characteristic function has a line as its asymptote and X_1, X_2 are F -spaces with finite coefficient of non-homogeneity, then the coefficient of non-homogeneity of the F -space $(X_1 \oplus X_2, \|\cdot\|_\Phi)$ is $\max\{\nu_{X_1}, \nu_{X_2}\}$. i.e, $\nu_{(X_1 \oplus X_2, \|\cdot\|_\Phi)} = \max\{\nu_{X_1}, \nu_{X_2}\}$.

Proof. Let $\chi(x)$ be the characteristic function of Φ and $\chi'(x) = mx - c$ be the asymptote to χ . Further, let Φ' be the Young function whose characteristic function is χ' . Then, $mx \leq \chi(x) \leq \chi'(x)$. And hence,

$$\Phi''(x) = x^{m+1} \leq \Phi(x) \leq \frac{1}{e^c}x^{m+1} = \Phi'(x).$$

Let $m_1 = \nu_{X_1}$ and $m_2 = \nu_{X_2}$ and $M = \max\{m_1, m_2\}$. Now notice that

$$\begin{aligned} \|k(x, y)\|_\Phi &= \Phi^{-1}(\Phi(\|kx\|) + \Phi(\|ky\|)) \\ &\leq \Phi'^{-1}(\Phi'(m_1|k|\|x\|) + \Phi'(m_2|k|\|y\|)) \\ &\leq \Phi'^{-1}\left(\frac{M|k|^{m+1}}{e^c}(\|x\|^{m+1} + \|y\|^{m+1})\right) \\ &= \frac{M|k|}{e^{\frac{c}{m+1}}}(\|x\|^{m+1} + \|y\|^{m+1})^{\frac{1}{m+1}} \\ &= \frac{M|k|}{e^{\frac{c}{m+1}}}\Phi'^{-1}(\Phi'(\|x\|) + \Phi'(\|y\|)) \\ &\leq \frac{M|k|}{e^{\frac{c}{m+1}}}\Phi'^{-1}(e^c(\Phi(\|x\|) + \Phi(\|y\|))) \\ &= \frac{M|k|}{e^{\frac{c}{m+1}}} \cdot e^{\frac{c}{m+1}}\Phi'^{-1}(\Phi(\|x\|) + \Phi(\|y\|)) \\ &\leq M|k| \cdot \Phi^{-1}(\Phi(\|x\|) + \Phi(\|y\|)) \end{aligned} \quad (10)$$

Thus, $\|(kx, ky)\|_\Phi \leq M|k| \cdot \|(x, y)\|_\Phi$ holds for all $k \in \mathbb{C}$ and $x \in X_1, y \in Y_1$. Hence, $\nu_{(X_1 \oplus X_2, \|\cdot\|_\Phi)} = \max\{\nu_{X_1}, \nu_{X_2}\}$. \square

4. Direct Sums of Orlicz spaces

This section is dedicated to establishing a proper notion of direct sums of Orlicz spaces. Recall that if (S, \mathcal{A}, μ) is a sigma-finite measure space, then we can define the p -direct sum $L^p(S) \oplus_p L^p(S)$ equipped with the p -norm $\|(f, g)\|_p = (\|f\|^p + \|g\|^p)^{1/p}$. Further the $L^p(S) \oplus_p L^p(S)$ is isometrically isomorphic to $L^p(\{1, 2\}, L^p(S))$ (Bochner space, see [9, 1.2b]) through the identification $\theta(f, g)(1) = f$, $\theta(f, g)(2) = g$. Further $L^p(\{1, 2\}, L^p(S))$ is isometrically isomorphic to $L^p(\{1, 2\} \times S)$ (see [9, Prop. 1.2.24]). Hence, the p -direct sum of $L^p(S)$ is again an L^p space, albeit over a different measure space $\{1, 2\} \times S$ (product measure of counting measure space $\{1, 2\}$ and measure space S).

Now suppose Φ is any Young function and (S, \mathcal{A}, μ) be a measure space. Consider the Orlicz space $L^\Phi(S)$ as defined in section 2.2. We aim to define an appropriate norm N_Γ on the vector space $L^\Phi(S) \oplus L^\Phi(S)$ such that it becomes a L^Φ space on some measure space.

Let $\Gamma : \mathbb{R}^2 \rightarrow [0, \infty)$ be any convex continuous function which is radially increasing (i.e, $\Gamma(rx, ry)$ is an increasing function of r for a fixed (x, y)) and the contours $U_c = \{(x, y) : \Gamma(x, y) = c\}$ for any $c > 0$ are all convex polygons with fixed number of sides and centered at origin. Further the extreme points of the polygon U_1 are no farther than one unit from origin and $\Gamma(0, 0) = 0$. Define N_Γ on $L^\Phi(S) \oplus L^\Phi(S)$ as

$$N_\Gamma(f, g) = \inf \left\{ \lambda > 0 : \Gamma \left(\int_S \Phi \left(\frac{|f(t)|}{\lambda} \right) dt, \int_S \Phi \left(\frac{|g(t)|}{\lambda} \right) dt \right) \leq 1 \right\}$$

We need to verify that this is a well defined norm. Clearly $N_\Gamma(f, g) \geq 0$. If $f = g = 0$ then $N_\Gamma(f, g) = 0$ follows easily. Further if $N_\Gamma(f, g) = 0$, then there exists a sequence $\{\lambda_n\}_{n=1}^\infty \rightarrow 0$ such that $\Gamma \left(\int_S \Phi \left(\frac{|f(t)|}{\lambda_n} \right) dt, \int_S \Phi \left(\frac{|g(t)|}{\lambda_n} \right) dt \right) \leq 1$ for each n . Hence,

$$\int_S \Phi \left(\frac{|f(t)|}{\lambda_n} \right) dt, \int_S \Phi \left(\frac{|g(t)|}{\lambda_n} \right) dt \in \text{Int}(U_1)$$

In particular $\int_S \Phi \left(\frac{|f(t)|}{\lambda_n} \right) dt \leq 1$ for each n . Now suppose that f is a non-zero function, then there exists a set A of positive measure and an $\epsilon > 0$ such that $|f(t)| \geq \epsilon$ for all $t \in A$. Thus,

$$\begin{aligned} \mu(A) \Phi \left(\frac{\epsilon}{\lambda_n} \right) &\leq \int_A \Phi \left(\frac{|f(t)|}{\lambda_n} \right) dt \\ &\leq 1 \quad \text{for each } n \end{aligned}$$

Which means $\Phi \left(\frac{\epsilon}{\lambda_n} \right) \leq \frac{1}{\mu(A)}$ for each n . But this is absurd because $\frac{\epsilon}{\lambda_n}$ increases indefinitely and Φ is an increasing function. Hence, contrary to our assumption, f must be a zero almost everywhere function. Similarly, g also vanishes almost everywhere. Now we verify the triangle inequality. Suppose $(f_1, g_1), (f_2, g_2) \in L^\Phi(S) \oplus L^\Phi(S)$ and $N_\Gamma(f_1, g_1) = k_1$ and $N_\Gamma(f_2, g_2) = k_2$. Now notice that

$$\begin{aligned} &\Gamma \left(\int_S \Phi \left(\frac{|f_1(t) + f_2(t)|}{k_1 + k_2} \right) dt, \int_S \Phi \left(\frac{|g_1(t) + g_2(t)|}{k_1 + k_2} \right) dt \right) \\ &\leq \Gamma \left(\frac{k_1}{k_1 + k_2} \left(\int_S \Phi \left(\frac{|f_1(t)|}{k_1} \right) dt, \int_S \Phi \left(\frac{|g_1(t)|}{k_1} \right) dt \right) + \frac{k_2}{k_1 + k_2} \left(\int_S \Phi \left(\frac{|f_2(t)|}{k_2} \right) dt, \int_S \Phi \left(\frac{|g_2(t)|}{k_2} \right) dt \right) \right) \\ &\leq \frac{k_1}{k_1 + k_2} \Gamma \left(\int_S \Phi \left(\frac{|f_1(t)|}{k_1} \right) dt, \int_S \Phi \left(\frac{|g_1(t)|}{k_1} \right) dt \right) + \frac{k_2}{k_1 + k_2} \Gamma \left(\int_S \Phi \left(\frac{|f_2(t)|}{k_2} \right) dt, \int_S \Phi \left(\frac{|g_2(t)|}{k_2} \right) dt \right) \\ &\leq \frac{k_1}{k_1 + k_2} + \frac{k_2}{k_1 + k_2} \\ &= 1 \end{aligned}$$

Hence, $N_\Gamma((f_1, g_1) + (f_2, g_2)) \leq N_\Gamma(f_1, g_1) + N_\Gamma(f_2, g_2)$.

Finally, we verify that $L^\Phi(S) \oplus L^\Phi(S)$ is complete with respect to the norm N_Γ . Let $\{(f_i, g_i)\}_{i=1}^\infty$ be a Cauchy sequence in $L^\Phi(S) \oplus L^\Phi(S)$ with respect to norm N_Γ . Then for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $N_\Gamma((f_n, g_n) - (f_m, g_m)) < \epsilon$ for all $n, m \geq N_\epsilon$. Hence for each pair of positive integers (n, m) such that $n, m \geq N_\epsilon$, we can choose a $0 < \lambda_{n,m} < \epsilon$ such that

$$\Gamma\left(\int_X \Phi\left(\frac{|f_n(t) - f_m(t)|}{\lambda_{n,m}}\right) dt, \int_X \Phi\left(\frac{|g_n(t) - g_m(t)|}{\lambda_{n,m}}\right) dt\right) \leq 1 \quad \forall n, m \geq N_\epsilon.$$

Hence, $\int_X \Phi\left(\frac{|f_n(t) - f_m(t)|}{\lambda_{n,m}}\right) dt \leq 1$ for all $n, m \geq N_\epsilon$, which in turn means $N_\Phi(f_n - f_m) \leq \lambda_{n,m} \leq \epsilon$ for each $n, m \geq N_\epsilon$. Hence $\{f_n\}_{n=1}^\infty$ and similarly $\{g_n\}_{n=1}^\infty$ are Cauchy sequences in $L^\Phi(S)$. Suppose $f_n \rightarrow f \in L^\Phi(S)$ and $g_n \rightarrow g \in L^\Phi(S)$. Then with easy computations similar to above, one can conclude that (f_n, g_n) converges to (f, g) with respect to N_Γ norm. Hence $(L^\Phi(S)) \oplus L^\Phi(S)$ is complete with respect to N_Γ .

We now establish that the appropriate norm for the direct sum $L^\Phi(S) \oplus L^\Phi(S)$ corresponds to the function $\Gamma(x, y) = |x| + |y|$.

Theorem 4.1. *Let $\Gamma : \mathbb{R}^2 \rightarrow [0, \infty)$ be the function $\Gamma(x, y) = |x| + |y|$ and (S, \mathcal{A}, μ) be a measure space. Then the map $\eta : (L^\Phi(S) \oplus L^\Phi(S), N_\Gamma) \rightarrow L^\Phi(\{1, 2\} \times S)$ defined as*

$$\eta(f, g)(1, x) = f(x), \quad \eta(f, g)(2, x) = g(x)$$

is an isometric isomorphism of Banach spaces.

Proof. Clearly η is a linear bijection, as can be verified easily. Further for any $(f, g) \in L^\Phi(S) \oplus L^\Phi(S)$, we have

$$N_\Phi(\eta(f, g)) = \inf \left\{ \lambda > 0 : \int_{\{1,2\} \times S} \Phi\left(\frac{|\eta(f, g)(t, s)|}{\lambda}\right) d\mu(s, t) \leq 1 \right\}. \quad (11)$$

But

$$\begin{aligned} \int_{\{1,2\} \times S} \Phi\left(\frac{|\eta(f, g)(t, s)|}{\lambda}\right) d\mu(s, t) &= \int_S \Phi\left(\frac{|f(x)|}{\lambda}\right) dx + \int_S \Phi\left(\frac{|g(x)|}{\lambda}\right) dx \\ &= \Gamma\left(\int_S \Phi\left(\frac{|f(x)|}{\lambda}\right) dx, \int_S \Phi\left(\frac{|g(x)|}{\lambda}\right) dx\right) \end{aligned}$$

Hence, by equation 11, we have

$$\begin{aligned} N_\Phi(\eta(f, g)) &= \inf \left\{ \lambda > 0 : \Gamma\left(\int_S \Phi\left(\frac{|f(x)|}{\lambda}\right) dx, \int_S \Phi\left(\frac{|g(x)|}{\lambda}\right) dx\right) \leq 1 \right\} \\ &= N_\Gamma(f, g) \end{aligned}$$

Thus, the isometric isomorphism is established. \square

Data availability: There is no data associated to this article.

Statements and Declarations: Both the authors have equal contributions in this article.

Funding: This research was carried out with the financial support of Slovenian Agency for Research and Innovation (research program P1-0222-Algebra, Operator theory and Financial mathematics at Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia). The second author is also a project team member of the UL VIP project KTTK21 financed by ARIS (contract no. SN-ZRD/22-27/510).

References

- [1] Mulholland H.P. On Generalizations of Minkowski's Inequality in the Form of a Triangle Inequality, *Proc. Lond. Math. Soc.* Vol s2-51,4, 1949.
- [2] Shapiro J.J. Extension of linear functionals on F -spaces with bases, *Duke Math. J.* 37(1970), 639-645.
- [3] Kalton N.J. Basic sequences in F -spaces and their applications. *Proceedings of the Edinburgh Mathematical Society.* 1974;19(2):151-167.
- [4] Roberts J.W. A nonlocally convex F -space with the Hahn-Banach approximation property, *Banach spaces 9: analytic functions*, Springer Lecture Notes 604, Berlin- Heidelberg, New York, 1977, 76-81.
- [5] Kalton N.J. The three space problem for locally bounded F -spaces, *Compositio Mathematica*, tome 37, n- 3 (1978), p. 243-276.
- [6] Kalton N.J, N. T. Peck, James W. Roberts, *An F -space Sampler*, London Mathematical Society lecture notes series, Cambridge university press.
- [7] Mastylo M., Mackey completion of some interpolation F -spaces, *Studia Mathematica*, 98(2), 115-129, 1991.
- [8] Rao M.M, Ren Z.D, *Theory of Orlicz spaces*, Text book in pure and applied mathematics.
- [9] Hytönen, T., van Neerven, J., Veraar, M., Weis, L. (2016). *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory.* (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge; Vol. 63). Springer.
- [10] Drewnowski, L., Labuda, I. Ideals of subseries convergence and F -spaces. *Arch. Math.* 108, 55–64 (2017).
- [11] Cui, Y., Hudzik, H., Kaczmarek, R. et al. Geometric properties of F -normed Orlicz spaces. *Aequat. Math.* 93, 311–343 (2019).