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A study of the properties of the gDMP inverse of an operator

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Abstract. This paper investigates the gDMP inverse of generalized Drazin invertible operators with closed range. Several characterizations and properties of the gDMP inverse are established. Furthermore, key properties are derived, and applications to the solution of certain linear operator equations are presented.

1. Introduction and Notation

Let $\mathcal{H}$ and $\mathcal{K}$ be arbitrary Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H},\mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and $I_{\mathcal{H}}$ to denote the identity operator on $\mathcal{H}$. Especially, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we denote by T^*, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\sigma(T)$, respectively, the adjoint, the null space, the range and the spectrum of T. An operator $R \in \mathcal{B}(\mathcal{H})$ is quasinilpotent if $\sigma(R) = \{0\}$.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is *generalized Drazin invertible* (or Koliha Drazin invertible), if there exists some $X \in \mathcal{B}(\mathcal{H})$ such that

$$XTX = X, \quad TX = XT, \quad T - T^2X \text{ is quasinilpotent.}$$

The generalized Drazin inverse X of T is unique (if exists) and it is denoted by T^d [1, 4]. The Drazin inverse is a special case of the generalized Drazin inverse for which $T - T^2X$ is nilpotent, and it is denoted by T^D. The condition $T - T^2X$ is nilpotent is equivalent to $T^{k+1}X = T^k$, for some nonnegative integer k. The smallest nonnegative integer k such that $T^{k+1}X = T^k$ holds, is called the index of T and it is denoted by $ind(T)$. If an operator $T \in \mathcal{B}(\mathcal{H})$ has an index of at most 1, it is called group invertible and T^D is called group inverse of T denoted by $T^\#$. Recent results about expressions for the Drazin inverse can be found in [13, 16–19].

An operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is the *Moore–Penrose* inverse of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ if satisfied the following four operator equations

$$TXT = T, \quad XTX = X, \quad (TX)^* = TX, \quad (XT)^* = XT.$$

The Moore–Penrose inverse of T is unique (if it exists) and denoted by $T^\dagger$ [1, 5]. As we know the operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has a Moore–Penrose inverse if and only if $\mathcal{R}(T)$ is closed in $\mathcal{K}$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called EP operator, if $\mathcal{R}(T)$ is closed and $T^\dagger T = TT^\dagger$.

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Let us recall that an operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is called an *outer inverse* of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ if $XTX = X \neq 0$. On the other hand, if X satisfies only the condition $TXT = T$, then X is called an *inner generalized inverse* of T.

In [6], Malik and Thome introduced a new generalized inverse, called the *DMP inverse*, by combining the Drazin inverse and the Moore–Penrose inverse of square matrices of arbitrary index. This generalized inverse was later extended in [15] to the setting of bounded linear operators that are Drazin invertible with closed range: let $T \in \mathcal{B}(\mathcal{H})$ with index k and has closed range, then the *DMP inverse* of T, denoted by $X = T^{D,\dagger} = T^D TT^\dagger$, is the unique operator $X \in \mathcal{B}(\mathcal{H})$ satisfying

$$XTX = X, \quad T^k X = T^k T^\dagger, \quad XT = TT^D.$$

In 2018, Dijana Mosić et al. introduced a new generalized inverse, called the *gDMP inverse*, for a generalized Drazin invertible operator with closed range. This inverse was defined using both the generalized Drazin inverse and the Moore–Penrose inverse, as a generalization of the DMP inverse to bounded operators [11]. Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator such that $\mathcal{R}(T)$ is closed. The *gDMP inverse* of T, denoted by $T^{d,\dagger}$, is defined as

$$T^{d,\dagger} = T^d TT^\dagger.$$

The authors also introduced another inverse, called the *MPgD inverse*, defined for a generalized Drazin invertible operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, as $T^{\dagger,d} = T^\dagger TT^d$. In the case where the operator $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible with closed range, $T^{d,\dagger}$ is reduced to $T^{D,\dagger}$.

These generalized inverses play a significant role in various applications: the Drazin and DMP inverses are used in solving singular linear control systems, the Moore–Penrose inverse is applied to least-squares problems and the group inverse finds applications in Markov chain[1, 3].

In recent years, the DMP inverse has become a subject of growing interest, with numerous studies addressing its extensions, characterizations, properties, and applications [2, 3, 7–10, 12, 20].

The purpose of this work is to present new properties and applications of the gDMP and MPgD inverses of a bounded operator that is generalized Drazin invertible with closed range.

2. Some properties of gDMP inverse

To establish certain properties of the gDMP inverse, we first present an auxiliary result regarding the expressions of its powers.

Lemma 2.1 ([3]). *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator such that $\mathcal{R}(T)$ is closed. Then*

(i) For all nonnegative integer $m \geq 1$,

$$(T^{d,\dagger})^m = (T^d)^m TT^\dagger.$$

(ii) For all nonnegative integer $m \geq 1$,

$$T^m (T^{d,\dagger})^m = TT^{d,\dagger} \quad \text{and} \quad (T^{d,\dagger})^m T^m = T^{d,\dagger} T.$$

The following corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator with closed range. Then the operator $(T^{d,\dagger})^m$ is an outer inverse of T^m, for all nonnegative integer $m \geq 1$.*

Proof. Since

$$\begin{aligned}(T^{d,\dagger})^m T^m (T^{d,\dagger})^m &= (T^{d,\dagger})^m T^m (T^d)^m TT^\dagger \\ &= (T^{d,\dagger})^m TT^d TT^\dagger \\ &= (T^d)^m TT^\dagger TT^d TT^\dagger \\ &= (T^d)^m TT^\dagger \\ &= (T^{d,\dagger})^m,\end{aligned}$$

for all $m \geq 1$, it follows that $(T^{d,\dagger})^m$ is an outer inverse of T^m. □

We now present a necessary and sufficient condition for $T^{d,\dagger}$ to be an inner inverse of T.

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator with closed range. Then $T^{d,\dagger}$ is an inner inverse of T if and only if T is group invertible.*

Proof. The equality $TT^{d,\dagger}T = T$ holds if and only if $TT^dT = T$, which is equivalent to the fact that T is group invertible. □

We now discuss the idempotent property of the $T^{d,\dagger}$ inverse.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator such that $\mathcal{R}(T)$ is closed. Then $T^{d,\dagger}$ is idempotent if and only if T^d is idempotent.*

Proof. Assume that $T^{d,\dagger}$ is idempotent. Since $(T^{d,\dagger})^2 = (T^d)^2TT^\dagger = T^dT^\dagger$, we obtain $T^dTT^\dagger = T^dT^\dagger$. Postmultiplying this equality by T, we get $T^dT = T^d$. Thus, T^d is idempotent. Conversely, postmultiplying the equality $(T^d)^2 = T^d$ by $TT^\dagger$, we have $(T^d)^2TT^\dagger = T^dTT^\dagger$, which implies $(T^{d,\dagger})^2 = T^{d,\dagger}$. Hence, $T^{d,\dagger}$ is idempotent. □

Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with closed range and let $U \in \mathcal{B}(\mathcal{H})$ be unitary. It is well known that $UTU^* \in \mathcal{B}(\mathcal{H})$ also has closed range and satisfies $(UTU^*)^\dagger = UT^\dagger U^*$. Based on this fact, we can state the following proposition.

Proposition 2.5. *Let $T, U \in \mathcal{B}(\mathcal{H})$ where T is a generalized Drazin invertible operator with closed range, and U is unitary. Then*

$$(UTU^*)^{d,\dagger} = UT^{d,\dagger}U^*.$$

Proof. Since $(UTU^*)^d = UT^dU^*$, we compute

$$\begin{aligned}(UTU^*)^{d,\dagger} &= (UTU^*)^d(UTU^*)(UTU^*)^\dagger\\ &= UT^dU^* \cdot UTU^* \cdot UT^\dagger U^*\\ &= UT^dTT^\dagger U^*\\ &= UT^{d,\dagger}U^*.\end{aligned}$$

□

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator with closed range. Then*

$$(T^{d,\dagger})^* = (T^*)^{\dagger,d}.$$

Proof. It is known that $(T^*)^d = (T^d)^*$ and $(T^*)^\dagger = (T^\dagger)^*$. Therefore,

$$\begin{aligned}(T^{d,\dagger})^* &= (T^dTT^\dagger)^*\\ &= (T^\dagger)^*T^*(T^d)^*\\ &= (T^*)^\dagger T^*(T^*)^d\\ &= (T^*)^{\dagger,d}.\end{aligned}$$

□

The following theorem shows that the gDMP inverse of a generalized Drazin invertible operator with closed range is group invertible.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then $T^{d,\dagger}$ and $T^2T^{d,\dagger}$ are group invertible. Furthermore,*

$$(T^{d,\dagger})^{\#} = T^2T^{d,\dagger}.$$

Proof. We show that $T^{d,\dagger}$ is group invertible. Define $X = T^2T^{d,\dagger}$. We then verify that

- $T^{d,\dagger}X = T^{d,\dagger}T^2T^{d,\dagger} = T(T^dTT^{d,\dagger}) = TT^{d,\dagger}$, and $XT^{d,\dagger} = T^2(T^{d,\dagger})^2 = TT^{d,\dagger}$.
- $XT^{d,\dagger}X = T^2(T^{d,\dagger})^2X = TT^{d,\dagger}T^2T^{d,\dagger} = T^2T^{d,\dagger} = X$.
- $T^{d,\dagger}XT^{d,\dagger} = T(T^{d,\dagger})^2 = T^{d,\dagger}$.

Hence, $T^{d,\dagger}$ is group invertible and $(T^{d,\dagger})^{\#} = T^2T^{d,\dagger}$. □

The following corollary is a direct consequence of Theorem 2.7.

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then*

$$T^{d,\dagger} = \left((T^{d,\dagger})^{\#}\right)^{\#}.$$

Zuo et al. [20, Theorem 3.8] proved an interesting result concerning the DMP inverse of a square matrix A:

$$A^{D,\dagger} = AA^{\dagger}\left(I_n - \overline{A}AA^{\dagger}\right)^D = \left(I_n - \overline{A}AA^{\dagger}\right)^D AA^{\dagger},$$

where $\overline{A} = I_n - A$, A^D denotes the Drazin inverse of A, and I_n is the $n \times n$ identity matrix. The next theorem provides a generalization of this for the gDMP inverse. We begin by presenting an important preliminary result.

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then $T^2T^{\dagger}$ is generalized Drazin invertible, and*

$$(T^2T^{\dagger})^d = T^{d,\dagger}.$$

Proof. Define $X = T^{d,\dagger}$. Then we have

- $T^2T^{\dagger}X = T^2T^{\dagger}TT^dT^{\dagger} = TX$, and $XT^2T^{\dagger} = T^dTT^{\dagger}T^2T^{\dagger} = TX$.
- $XT^2T^{\dagger}X = XTX = X$.
- The spectrum satisfies

$$\sigma\left(T^2T^{\dagger} - (T^2T^{\dagger})^2T^{d,\dagger}\right) \cup \{0\} = \sigma\left((TT^{\dagger} - T^2T^dT^{\dagger})T\right) \cup \{0\} = \sigma\left(T - T^2T^d\right) \cup \{0\} = \{0\}.$$

Therefore, $T^2T^{\dagger}$ is generalized Drazin invertible and $(T^2T^{\dagger})^d = T^{d,\dagger}$. □

It is well-known that for two generalized Drazin invertible operators $A, B \in \mathcal{B}(\mathcal{H})$ satisfying $AB = BA = 0$, we have $(A + B)^d = A^d + B^d$. From this fact, we can derive the following theorem.

Theorem 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then*

$$T^{d,\dagger} = TT^{\dagger}\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d = \left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d TT^{\dagger},$$

where $\overline{T} = I_{\mathcal{H}} - T$.

Proof. Note that $I_{\mathcal{H}} - \overline{T}TT^{\dagger} = I_{\mathcal{H}} - TT^{\dagger} + T^2T^{\dagger}$. Since $(I_{\mathcal{H}} - TT^{\dagger})T^2T^{\dagger} = T^2T^{\dagger}(I_{\mathcal{H}} - TT^{\dagger}) = 0$, it follows that

$$\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d = \left(I_{\mathcal{H}} - TT^{\dagger} + T^2T^{\dagger}\right)^d = I_{\mathcal{H}} - TT^{\dagger} + (T^2T^{\dagger})^d.$$

By Theorem 2.9, we have $(T^2T^{\dagger})^d = T^{d,\dagger}$. Therefore, $\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d = I_{\mathcal{H}} - TT^{\dagger} + T^{d,\dagger}$. Multiplying by $TT^{\dagger}$ on the left, $TT^{\dagger}\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d = TT^{\dagger}T^{d,\dagger} = T^{d,\dagger}$. Similarly, multiplying on the right, $\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d TT^{\dagger} = T^{d,\dagger}TT^{\dagger} = T^{d,\dagger}$. Hence,

$$T^{d,\dagger} = TT^{\dagger}\left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d = \left(I_{\mathcal{H}} - \overline{T}TT^{\dagger}\right)^d TT^{\dagger}.$$

□

Theorem 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator with closed range. Then the operator $T - T^2T^{d,\dagger}$ is quasinilpotent.*

Proof. Since $T - T^2T^d$ is quasinilpotent, we have

$$\sigma(T - T^2T^{d,\dagger}) \cup \{0\} = \sigma\Big((I_{\mathcal{H}} - TT^{d,\dagger})T\Big) \cup \{0\} = \sigma\Big((I_{\mathcal{H}} - T^dT)T\Big) \cup \{0\} = \{0\}.$$

Hence, $T - T^2T^{d,\dagger}$ is quasinilpotent. □

The following proposition is a consequence of Theorems 2.7 and 2.11.

Proposition 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then there exist operators $X, Y \in \mathcal{B}(\mathcal{H})$ such that*

$$T = X + Y \quad \text{and} \quad YX = 0,$$

where X is group invertible and Y is quasinilpotent.

Proof. Let $T^{d,\dagger}$ denote the gDMP inverse of T. Define $X = T^2T^{d,\dagger}$ and $Y = T - T^2T^{d,\dagger}$. Clearly $T = X + Y$ and $YX = (T - T^2T^{d,\dagger})T^2T^{d,\dagger} = 0$. By Theorem 2.7, X is group invertible, and by Theorem 2.11, Y is quasinilpotent. □

Theorem 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$ be group invertible operator with closed range. Then the gDMP inverse $T^{\#,\dagger}$ is Moore–Penrose invertible, and*

$$(T^{\#,\dagger})^{\dagger} = T^2T^{\dagger}.$$

Proof. We will show that $T^2T^{\dagger}$ is the Moore–Penrose inverse of $T^{\#}TT^{\dagger}$. Let $X = T^{\#}TT^{\dagger}$. Then we verify that

- $T^2T^{\dagger}XT^2T^{\dagger} = TT^{\dagger}T^2T^{\dagger} = T^2T^{\dagger}$,
- $XT^2T^{\dagger}X = T^{\#}TT^{\dagger}T^2T^{\dagger} = T^{\#}TT^{\dagger} = X$,
- $(T^2T^{\dagger}X)^* = (TT^{\dagger})^* = TT^{\dagger} = T^2T^{\dagger}T^{\#}TT^{\dagger} = T^2T^{\dagger}X$,
- $(XT^2T^{\dagger})^* = (TT^{\dagger})^* = TT^{\dagger} = T^{\#}TT^{\dagger}T^2T^{\dagger} = XT^2T^{\dagger}$.

Thus, $(T^{\#,\dagger})^{\dagger} = T^2T^{\dagger}$. □

Corollary 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$ be group invertible with closed range. Then the gDMP inverse $T^{\#,\dagger}$ is an EP operator.*

In [4], it is shown that an operator $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible if and only if there exists an idempotent operator $P \in \mathcal{B}(\mathcal{H})$ commuting with T such that

$$TP \text{ is quasinilpotent} \quad \text{and} \quad T + P \text{ is invertible.}$$

In this case, the Drazin inverse of T is uniquely determined and is given by

$$T^D = (T + P)^{-1}(I_{\mathcal{H}} - P).$$

In the same spirit, we establish analogous results for the DMP inverse of Drazin invertible operators with closed range. We begin with the following proposition.

Proposition 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible with index k and has closed range. Then the operator $T + I_{\mathcal{H}} - TT^{D,\dagger}$ is invertible. Moreover, we have*

$$(T + I_{\mathcal{H}} - TT^{D,\dagger})^{-1} = T^{D,\dagger} + T^{\pi} \sum_{i=0}^{k-1} (-T)^i,$$

where $T^{\pi} = I_{\mathcal{H}} - TT^D$.

Proof. Let $T^{D,\dagger}$ be the DMP inverse of T. Define $P = I_{\mathcal{H}} - TT^{D,\dagger}$, $T^{\pi} = I_{\mathcal{H}} - TT^D$ and $S = \sum_{i=0}^{k-1}(-T)^i$. Note that $TS + S = I_{\mathcal{H}} + (-1)^{k-1}T^k$, $PT^{\pi} = P$ and $T^{\pi}P = T^{\pi}$. Then

$$\begin{aligned}
(T + P)(T^{D,\dagger} + T^{\pi}S) &= TT^{D,\dagger} + TT^{\pi}S + PT^{D,\dagger} + PS \\
&= TT^{D,\dagger} + TS + S - TT^{D,\dagger}TS - TT^{D,\dagger}S \\
&= TT^{D,\dagger}(I_{\mathcal{H}} - TS - S) + TS + S \\
&= TT^{D,\dagger}(I_{\mathcal{H}} - (I_{\mathcal{H}} + (-1)^{k-1}T^k)) + I_{\mathcal{H}} + (-1)^{k-1}T^k \\
&= -(-1)^{k-1}TT^{D,\dagger}T^k + I_{\mathcal{H}} + (-1)^{k-1}T^k \\
&= I_{\mathcal{H}}.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
(T^{D,\dagger} + T^{\pi}S)(T + P) &= T^{D,\dagger}T + T^{D,\dagger}P + T^{\pi}ST + T^{\pi}SP \\
&= T^DT + TS + S - TT^DST - T^dTS \\
&= TT^D(I_{\mathcal{H}} - TS - S) + TS + S \\
&= TT^D(I_{\mathcal{H}} - (I_{\mathcal{H}} + (-1)^{k-1}T^k)) + I_{\mathcal{H}} + (-1)^{k-1}T^k \\
&= -(-1)^{k-1}TT^DT^k + I_{\mathcal{H}} + (-1)^{k-1}T^k \\
&= I_{\mathcal{H}}.
\end{aligned}$$

□

Theorem 2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible with index k and has closed range. Then there exists an idempotent $P \in \mathcal{B}(\mathcal{H})$ such that both TP and PT are nilpotent and $T + P$ is invertible. Moreover,*

$$T^{D,\dagger} = (T + P)^{-1}(I_{\mathcal{H}} - P).$$

Proof. Assume that $T^{D,\dagger}$ is the DMP inverse of T. Let $P = I_{\mathcal{H}} - TT^{D,\dagger}$. We verify that P is idempotent

$$P^2 = (I_{\mathcal{H}} - TT^{D,\dagger})^2 = I_{\mathcal{H}} - 2TT^{D,\dagger} + (TT^{D,\dagger})^2 = I_{\mathcal{H}} - TT^{D,\dagger} = P.$$

Next, we compute $PT = (I_{\mathcal{H}} - TT^{D,\dagger})T = T - T^2T^D$, and $TP = T(I_{\mathcal{H}} - TT^{D,\dagger}) = T - T^2T^{D,\dagger}$. Since both TP and PT are nilpotent, from Proposition 2.15, we know that $T + P$ is invertible.

We now verify $(T+P)T^{D,\dagger} = TT^{D,\dagger} + PT^{D,\dagger} = TT^{D,\dagger}$, since $PT^{D,\dagger} = 0$. Therefore, $(T+P)T^{D,\dagger} = TT^{D,\dagger} = I_{\mathcal{H}} - P$, and multiplying both sides on the left by $(T+P)^{-1}$ gives

$$T^{D,\dagger} = (T+P)^{-1}(I_{\mathcal{H}} - P).$$

□

Proposition 2.17. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then the operator $T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}$ is invertible. Moreover,*

$$\left(T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}\right)^{-1} = T^2T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}.$$

Proof. We verify that $(T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger})(T^2T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}) = I_{\mathcal{H}}$ and similarly, $(T^2T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger})(T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}) = I_{\mathcal{H}}$. Hence, the inverse of $T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}$ is given by

$$(T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger})^{-1} = T^2T^{d,\dagger} + I_{\mathcal{H}} - TT^{d,\dagger}.$$

□

Theorem 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then there exists an idempotent $P \in \mathcal{B}(\mathcal{H})$ such that both TP and PT are quasinilpotent, and the operator $T^{d,\dagger} + P$ is invertible. Moreover,*

$$T^{d,\dagger} = (T^{d,\dagger} + P)^{-1}T^dT^{\dagger}.$$

Proof. Assume that $T^{d,\dagger}$ is the gDMP inverse of T. Define $P = I_{\mathcal{H}} - TT^{d,\dagger}$ is idempotent By Theorem 2.11, both PT and TP are quasinilpotent. From Proposition 2.17, the operator $T^{d,\dagger} + P$ is invertible, and its inverse is $(T^{d,\dagger} + P)^{-1} = T^2T^{d,\dagger} + P$. Now, observe that $(T^{d,\dagger} + P)T^{d,\dagger} = (T^{d,\dagger})^2 + PT^{d,\dagger} = T^dT^{\dagger}$. Hence, $T^{d,\dagger} = (T^{d,\dagger} + P)^{-1}T^dT^{\dagger}$. □

Theorem 2.19. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible operator with index k and has closed range. Then there exists an idempotent $P \in \mathcal{B}(\mathcal{H})$ such that*

$$T^pT^{\dagger}P = 0, \quad PT^p = 0 \quad \text{and} \quad T^{D,\dagger} + P \text{ is invertible,}$$

for every nonnegative integer $p \geq k$.

Proof. Suppose $T^{D,\dagger}$ is the DMP inverse of T and $p \geq k$. Set $P = I_{\mathcal{H}} - TT^{D,\dagger}$.
It is easy to verify that P is idempotent, and

$$T^pT^{\dagger}P = T^pT^{\dagger}(I_{\mathcal{H}} - TT^{D,\dagger}) = T^pT^{\dagger} - T^{p+1}T^DT^{\dagger} = 0,$$

$$PT^p = (I_{\mathcal{H}} - TT^{d,\dagger})T^p = T^p - T^p = 0.$$

By Proposition 2.17, $T^{D,\dagger} + P$ is invertible. □

Solving certain types of operator equations, we present the gDMP inverse.

Theorem 2.20. *Assume that $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible with index k and has closed range. Then the operator system*

$$T^DTX = X \quad \text{and} \quad T^{p+1}X = T^{p+1}T^{\dagger} \tag{1}$$

has a unique solution $X = T^{D,\dagger}$ for every nonnegative integer $p \geq k$.

Proof. We can easily show that (1) holds for $X = T^{D,\dagger}$. Now suppose X and X_1 are two solutions of the system (1), then

$$T^{p+1}X = T^{p+1}X_1,$$

which further implies

$$X_1 = T^DTX_1 = (T^D)^{p+1}(T^{p+1}X_1) = (T^D)^{p+1}(T^{p+1}X) = T^DTX = X.$$

Thus, the solution $T^{D,\dagger}$ of the system (1) is unique. □

3. Applications of gDMP inverse

In [14], the authors applied the Inner-gMP and gMP-inner inverses to solve certain types of linear equations. Motivated by this, we apply the gDMP inverse to solve some systems of linear equations.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible with index k and has closed range. Then the linear system*

$$T^p(x) = T^p(T^\dagger(h)) \tag{2}$$

is consistent for each $h \in \mathcal{H}$ and every $p \geq k$, its general solution is given by

$$x = T^{D,\dagger}(h) + T^\pi(y), \tag{3}$$

where $y \in \mathcal{H}$ is arbitrary and $T^\pi = I_\mathcal{H} - TT^D$. Moreover,

$$x = T^{D,\dagger}(h)$$

is the unique solution to the system in $\mathcal{R}(T^p)$ with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T^p) \oplus \mathcal{N}(T^p)$.

Proof. For $x = T^{D,\dagger}(h) + T^\pi(y)$, we have

$$T^p(x) = T^p(T^{D,\dagger}(h) + T^\pi(y)) = T^pT^{D,\dagger}(h) + T^p(y) - T^{p+1}T^D(y) = T^pT^{D,\dagger}(h) = T^pT^\dagger(h).$$

Therefore, x is a solution of (2).
Conversly, suppose that x satisfies (2). Then

$$T^DT(x) = (T^DT)^p(x) = (T^D)^pT^pT^\dagger(h) = T^DTT^\dagger(h) = T^{D,\dagger}(h),$$

implies

$$x = T^{D,\dagger}(h) + x - T^DT(x) = T^{D,\dagger}(h) + T^\pi(h),$$

Hence, x has the form given in (3).

To prove that $x = T^{D,\dagger}(h)$ is the unique solution to the system in $\mathcal{R}(T^p)$, suppose that there is another solution $x_1 \in \mathcal{R}(T^p)$ of (2). Since, $x - x_1 \in \mathcal{N}(T^p)$, then we have $x - x_1 \in \mathcal{R}(T^p) \cap \mathcal{N}(T^p) = \{0\}$. Therefore,

$$x = x_1.$$

□

The following corollary follows from Theorem 3.1.

Corollary 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible with index k and has closed range. Then*

$$T^{D,\dagger}(h) - T^\dagger(h) \in \mathcal{N}(T^p)$$

for all $h \in \mathcal{H}$ and every nonnegative integer $p \geq k$.

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then the system*

$$T(x) = TT^d(h) \tag{4}$$

is consistent and its general solution is given by

$$x = T^{\dagger,d}(h) + (I_\mathcal{H} - T^\dagger T)(y), \tag{5}$$

where $y \in \mathcal{H}$ is arbitrary.

Proof. For $x = T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger}T)(y)$, we have

$$T(x) = T(T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger}T)(y)) = TT^{d}(h).$$

Hence, x is a solution of (4).
On the other hand, assume that x is a solution to (7). Then

$$T^{\dagger}T(x) = T^{\dagger}TT^{d}(h) = T^{\dagger,d}(h).$$

Therefore,

$$x = T^{\dagger,d}(h) + x - T^{\dagger}T(x) = T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger}T)(y),$$

which is of the form given. □

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be generalized Drazin invertible with closed range. Then the system*

$$TT^{d}(x) = T^{d}(h), \tag{6}$$

is consistent and its general solution is given by

$$x = T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger,d}T)(y), \tag{7}$$

where $y \in \mathcal{H}$ is arbitrary.

Proof. Let $x = T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger,d}T)(y)$, where $y \in \mathcal{H}$. Then

$$\begin{aligned} TT^{d}(x) &= TT^{d}\Big(T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger,d}T)(y)\Big) \\ &= TT^{d}T^{\dagger,d}(h) \\ &= T^{d}(h), \end{aligned}$$

which shows that x is a solution of (6).

Conversely, suppose that equation (6) has a solution x. Then

$$T^{\dagger,d}T(x) = T^{\dagger}TT^{d}(h) = T^{\dagger,d}(h).$$

Hence,

$$\begin{aligned} x &= T^{\dagger,d}(h) + x - T^{\dagger,d}T(x) \\ &= T^{\dagger,d}(h) + (I_{\mathcal{H}} - T^{\dagger,d}T)(x), \end{aligned}$$

which implies that x is of the form (7). □

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