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An existence result for three-component two-phase incompressible flow with dynamic capillary pressure in porous media

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Abstract. The aim of this work is to prove the existence of weak solutions of a two-phase (aqueous and oil) immiscible and incompressible flow model with dynamic capillary pressure in porous media and three components (water, polymer and oil). The mathematical model is obtained by writing down the mass conservation for wetting and non-wetting phase and the mass conservation for polymer component in the wetting phase. We obtain a nonlinear parabolic degenerate problem of equations in term of oil and water saturations, and polymer concentration in wetting phase.

1. Introduction

The primary and secondary enhanced oil recovery methods leave more than helf of hydrocarbon reserves in place, so tankers rely on tertiary assisted recovery methods. This mathematical model under study has applications in the tertiary oil recovery.

In [23], we prove the existence of weak solutions of a two-incompressible immiscible phase flow model in porous medium with dynamic capillary pressure, the authors to prove the existence of a solution of the problem after using the Galerkin approximation method.
In [25], we prove the existence of weak solutions of a two-incompressible immiscible phase flow model with dynamic capillary pressure in porous media with three components, the authors used the Leray-Schauder pixed point theorem, see [30], for to analyze mathematically a two-phase flow with three components modeling the enhanced oil recovery by polymer flooding. We prove existence of weak solutions of a two-incompressible immiscible "aqueous and oil" phase flow model with dynamic capillary pressure in porous media with three components (polymer, water and oil). This model is obtained by writing down the mass conservation for each phase and the mass conservation for polymer component in the wetting phase (water).

The aim of this paper is to analyze a two-phase flow with three-components modeling the Enhanced Oil Recovery (EOR) by polymer flooding with dynamic capillary pressure. The study of degenerate parabolic problems modeling the displacement of incompressible immiscible two-phase flows, for more details see

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[17, 18, 23] and [19] in case assumption that densities are increasing under a global pressure, also, we consider the dynamic capillary pressure in [23]. In [25], the authors demonstrated the existence of solutions of a two-phase with three components immiscible incompressible flow model in porous media, Also in [21], an existence result has been shown in the case where the density of wetting and non-wetting phase is fixed. There is only few mathematical study of two-phase three-components flow models. A partial existence according to [7, 24], in porous media the mixture of two phases water and oil are immiscible, the authors establish the existence of a weak solution; the authors derive a compositional model of multiphase incompressible flow in porous media.

The author, in [28], uses this model to prove another existence result, assuming that non degeneracy and of strictly liquid saturation. In [29], a nonlinear elliptic/parabolic problem describing the compositional of water oil incompressible flow modeling the process of infiltration of hydrocarbons in an aquifer is studied. The aurhors, in [8], present a technique consisting in the use of yield stress fluids as blocking agents in porous media presenting pore-scale heterogeneities concentrated polymer solutions developing a yield stress were used as microscopic blocking agents. The aurthors, in [2, 3], analyse and study of numerical simulation for nonisothermal multiphase immiscible compressible flow model in porous medium. The authors, in [1], introduce a model of the time evolution of a flow of compressible fluids and immiscible in porous medium, taking into account the thermal effects, the existence result of weak solutions of the more general model is obtained based on assumptions that are physically relevant to the problem data. This result is obtained in several steps involving an appropriate regularization and a time discretization. In [4, 5], the authors study the existence of weak solution of non-isothermal multiphase flows models in porous media. The model used in [29] combines two-phase incompressible which are dissolution of hydrocarbons in the wetting phase, also, the model is assumed that the transfer velocity of a dissolved chemical is finite.

In [13], the authors present a convergence analysis of the finite difference method for multicomponent transport in porous media. For the analysis, the authors consider a reduced system of equations in one spatial dimension involving only one component (polymer). This reduced system models Chemical Enhanced Oil Recovery (CEOR) by polymer flooding , see [14, 16], in one spatial dimension.

Now, we propose and analyze a mathematical model for incompressible, immiscible, three-component (water, polymer and oil) two-phase flow (wetting and non-wetting) with dynamic capillary pressure modeling the enhanced oil recovery by polymer flooding in porous media, it is using a global pressure introduced by [10, 12, 14, 16] and [11]. The mathematical model couples three non-linear degenerate parabolic equations arising from mass conservation of each component in wetting and non-wetting phases, we prove the existence of weak solutions of the problem with the assumption that the viscosity of water is increasing with respect to its own polymer concentration. Let us state the model used in this paper, we consider herein a porous medium saturated with a fluid composed of two phases (aqueous and oil). The water is supposed only in the liquid phase (no vapor of water due to evaporation).

In order to define the model, we write the governing equations for incompressible, immiscible, three-component two-phase flow , with dynamic capillary pressure, of fluids through porous media:

$$\phi\frac{\partial s_w}{\partial t} + \mathrm{div}(\mathbf{u}_w) = 0, \tag{1}$$

$$\phi\frac{\partial s_o}{\partial t} + \mathrm{div}(\mathbf{u}_o) = 0, \tag{2}$$

$$\phi\frac{\partial (c s_w)}{\partial t} + \mathrm{div}(c\mathbf{u}_w) - \mathrm{div}(\mathbf{D}(s_w)\nabla c) = 0, \tag{3}$$

here div $= \nabla\cdot$ denotes the divergence operator, where the subscripts w and o represent respectively the wetting phase and the nonwetting phase. Quantities $s_\alpha, \mathbf{u}_\alpha, c, \mathbf{D}$ and ϕ represent respectively the saturation of the α phase ($\alpha = w, o$), the velocity of the α phase, and the polymer concentration in wetting phase and the diffusion-dispersion tensor of the polymer in the aqueous phase and the porosity of the medium. We work under the hypotheses that

$$s_w + s_o = 1, \tag{4}$$

and the velocity of each phase $\mathbf{u}_\alpha$ is given by the Darcy's law

$$\mathbf{u}_w = -\lambda_w(s_w, c)\mathbf{K}(\mathbf{x})\nabla p_w, \qquad \mathbf{u}_o = -\lambda_o(s_o)\mathbf{K}(\mathbf{x})\nabla p_o, \tag{5}$$

with $\mathbf{K}$ is the intrinsic permeability tensor of the porous media and p_α is the pressure of α-phase. Where $\lambda_w(s_w, c)$ is the mobility of the wetting phase and $\lambda_o(s_o)$ is the mobility of the nonwetting one, they are defined by

$$\lambda_w(s_w, c) = \frac{k_{rw}(s_w)}{\mu_w(c)} \qquad \text{and} \qquad \lambda_o(s_o) = \frac{k_{ro}(s_o)}{\mu_o},$$

where $k_{r\alpha}$ the relative permeability of the α phase, μ_α the α-phase's viscosity, μ_o is supposed constant, also, the wetting phase (aqueous phase) viscosity "μ_w" is modeled by a linear function of the polymer concentration "c"

$$\mu_w(c) = (1 + \kappa c)\,\mu_p, \tag{6}$$

here μ_p is the viscosity of pure water and the coefficient κ characterizes the particular polymer, for more details, see [14, 16]. It is known that when two fluids immiscible are in contact with each other in porous space, a clear interface exists between them, also, the interface is a curved surface and the pressure on the concave side exceeds that in the convex side. In this work, the pressure difference is known as with dynamic capillary pressure:

$$p_o - p_w = p_c(s_w) + \zeta\partial_t s_w. \tag{7}$$

the function $s_w \longmapsto p_c(s_w)$ is increasing and stands for classical static capillary pressure ($\frac{dp_c}{ds_w} < 0$) for all $s_w \in [0, 1]$ and $p_c(1) = 0$, and $\zeta = \zeta(s_w)$ is the damping coefficient. Note that our problem (1)–(7) is closed. The unknowns are saturations s_o, s_w, pressures p_o, p_w, and concentration c.

It is known that equations originating from multiphase flow in porous medium are degenerated. The first type of degeneracy arises from the behavior of relative permeability of the phases which disappears when his saturation goes to 0. Also, the second type of degeneracy is due to the temporal term when the flow goes to a saturated state. In this paper, the strategy we base those difficulties is to introduce some regularizations as follows. Firstly, we regularize the problem (S) by adding diffusive terms on dynamic capillary pressure to obtain a non degenerate problem (S_ν) as described in the beginning of Section 3. Secondly, we make a time discretization of problem (S_ν) to treat the time degeneracy. This gives an elliptic problem $(S_{\nu,\delta t})$ which is helpful to prove the existence of a solution of (S_ν). This is the goal of Sections 3.1 and 3.2 Finally, Section 4 is devoted to establish uniform energy estimates independently of ν to pass to the limit (ν goes to zero) and demonstrate the existence of a weak solution of problem (S) in the sense of Theorem 2.3.

2. Main result and hypotheses

Let us begining by introducing the global pressure p, it is the function given by

$$p = p_o + \tilde{p}(s_w, c) = p_w - \bar{p}(s_w, c) - \zeta(s_w)\partial_t s_w, \tag{8}$$

such that $\tilde{p}(s_w, c)$ and $\bar{p}(s_w, c)$ are two functions where

$$\frac{d\tilde{p}}{ds_w} = \frac{\lambda_w(s_w, c)}{\lambda(s_w, c)}\frac{dp_c}{ds_w} \qquad \text{and} \qquad \frac{d\bar{p}}{ds_w} = -\frac{\lambda_o(s_o)}{\lambda(s_w, c)}\frac{dp_c}{ds_w}. \tag{9}$$

One can demonstrate that, see for instance the above references,

$$\lambda(s_w, c)\nabla p = \lambda_w(s_w, c)\nabla p_w + \lambda_o(s_o)\nabla p_o, \tag{10}$$

with the total mobility $\lambda(s_w, c)$ is defined by

$$\lambda(s_w, c) = \lambda_w(s_w, c) + \lambda_o(s_o). \tag{11}$$

We have the equality

$$\int_{\Omega_T} \lambda |\nabla p|^2 dxdt \quad + \quad \int_{\Omega_T} \frac{\lambda_w \lambda_o}{\lambda} |\nabla p_c|^2 dxdt$$
$$= \int_{\Omega_T} \lambda_w |\nabla p_w|^2 dxdt + \int_{\Omega_T} \lambda_o |\nabla p_o|^2 dxdt \leq C, \tag{12}$$

where the constant C is positive.

We introduce, see [7], the contribution of capillary terms by

$$\gamma(s_w, c) = -\frac{\lambda_w(s_w, c)\lambda_o(s_o)}{\lambda(s_w, c)} \frac{dp_c}{ds_w}(s_w) \geq 0 \quad \text{and} \quad \mathfrak{B}(s_w, c) = \int_0^{s_w} \gamma(y, c)\, dy. \tag{13}$$

We define the function $\mathfrak{B}_c$ by $\mathfrak{B}_c(s_w) = \mathfrak{B}(s_w, c)$ for $s_w \in [0, 1]$ and c fixed in $[0, c_{equ}]$.

Now, we complete the description of the our model (1)–(7) by introducing boundary and initial conditions. Let $T > 0$ be the final time fixed, and let be Ω a bounded open subset of $\mathbb{R}^d$ $(d \geq 1)$, whose boundary is denoted by Γ or $\partial\Omega$.

Now, we denote by Γ_w the part of the boundary of Ω where the wetting saturation is imposed to one; the imprevious part of the boundary is thus denoted $\Gamma_{\text{imp}} = \Gamma \setminus \Gamma_w$ and we impose the following boundary conditions on pressures, concentration and no fluxes through Γ_{imp}:

$$\left.\begin{array}{lll} c = c_{\Gamma_w}(t, x) & \text{on} & (0, T) \times \Gamma_w, \\ p_o(t, x) = p_w(t, x) = 0 & \text{on} & (0, T) \times \Gamma_w, \\ \mathbf{u}_w \cdot \mathbf{n} = \mathbf{u}_o \cdot \mathbf{n} = 0 & \text{on} & (0, T) \times \Gamma_{\text{imp}}, \\ (c\mathbf{u}_w - \mathbf{D}\nabla c) \cdot \mathbf{n} = 0 & \text{on} & (0, T) \times \Gamma_{\text{imp}}, \end{array}\right\} \tag{14}$$

with $\mathbf{n}$ is the outward unit normal to Γ.

The initial conditions are defined on saturations of each phase and polymer concentration

$$s_\alpha(t = 0, \cdot) = s_\alpha^0 \ \text{ for } \ \alpha = w, o \quad \text{and} \quad c(t = 0, \cdot) = c^0 \ \text{ in } \ \Omega. \tag{15}$$

In the sequel we note by (S) the problem consisting of partial differential equations (1)–(3), Darcy-Muskat's laws (5), the capillary relation (7), and boundary and intial conditions (14) and (15).

We note

$$\Omega_T = (0, T) \times \Omega \qquad \text{and} \qquad \Sigma_T = (0, T) \times \partial\Omega.$$

Now, we introduce some relevant hypotheses on the coefficients of this problem. We assume the following:

(A1) The function $\phi \in L^\infty(\Omega)$ and there exist two constants $\phi_\star > 0$, $\phi^\star > 0$ such that $\phi_\star \leq \phi(x) \leq \phi^\star$ a.e. $x \in \Omega$.

(A2) The permeability tensor $\mathbf{K} \in (L^\infty(\Omega))^{d\times d}$ and there exist two constants $k_0 > 0$ and $k_\infty > 0$ such that $||\mathbf{K}||_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty$ and

$$\langle \mathbf{K}(\mathbf{x})\xi, \xi \rangle \geq k_0 |\xi|^2 \quad \text{a.e.} \quad x \in \Omega, \quad \forall \xi \in \mathbb{R}^d.$$

(A3) The function $\lambda_o \in C^0([0, 1], \mathbb{R}_+)$ such that $\lambda_o(s_o = 0) = 0$ and the function $\lambda_w \in C^0([0, 1] \times [0, c_{equ}], \mathbb{R}_+)$ such that $\lambda_w(s_w = 0, \cdot) = 0$. And there is a positive constant $m_0 > 0$ with for all $s_w \in [0, 1]$ and $c \in [0, c_{equ}]$,

$$\lambda(s_w, c) = \lambda_w(s_w, c) + \lambda_o(s_o) \geq m_0.$$

(A4) The capillary pressure function $p_c(s_w, \partial_t s_w) \in C^1([0,1]\times]0,+\infty[;\mathbb{R}_+)$ and there exists $\underline{p_c} > 0$ such that $0 < \underline{p_c} \leq \left|\frac{dp_c}{ds_w}\right|$ and the damping coefficient $\zeta(s_w) \in C^1([0,1];\mathbb{R})$.

(A5) The equilibrium concentration $c_{equ} \geq 0$ is a constant.

(A6) The function $\gamma \in C^2([0,1]\times[0,c_{equ}];\mathbb{R}_+)$ and satisfies $\gamma(s_w,c) > 0$ for $s_w \in]0,1[$ and $c \in]0,c_{equ}[$ where $\gamma(0,\cdot) = \gamma(1,\cdot) = 0$. The function $\mathfrak{B}_c^{-1}$, inverse of the function $s_w \longmapsto \mathfrak{B}_c(s_w) = \mathfrak{B}(s_w,c) = \int_0^{s_w}\gamma(y,c)\,dy$, c fixed, is Hölderian of order θ, where $0 < \theta \leq 1$, on the interval $[0, \mathfrak{B}(1,c_{equ})]$.

(A7) There exists a function $\widehat{c_{\Gamma_w}} \in L^2(0,T;H^1(\Omega))$ where $\widehat{c_{\Gamma_w}} = c_{\Gamma_w}$ on $(0,T)\times\Gamma_w$, c_{Γ_w} being the prescribed concentration on $(0,T)\times\Gamma_w$.

(A8) The diffusion-dispersion symmetric tensor $\mathbf{D}$ is a non-linear continuous function of the saturation s_w and is bounded for $s_w \in [0,1]$. There exists a positive constant δ where

$$\forall v \in \mathbb{R}^d, \quad \forall s_w \in [0,1], \quad \langle \mathbf{D}(s_w)v, v\rangle \geq \delta\|v\|^2.$$

Now, we put

$$F(s_w) = \int_0^{s_w} p_c(z)dz \quad \text{so that} \quad \partial_t F(s_w) = F'(s_w)\partial_t s_w = p_c(s_w)\partial_t s_w. \tag{16}$$

Remark 2.1. *Our problem is degenerate because, as stated in assumption (A3),* $\lambda_o(s_o = 0) = 0$ *and* $\lambda_w(s_w = 0, c = 0) = 0$.

Remark 2.2. *In assumption (A6), the function* $\mathfrak{B}_c^{-1}$ *is supposed Hölderian, it means there exists a constant* $b \geq 0$ *such that*

$$|\mathfrak{B}_c^{-1}(\sigma_1) - \mathfrak{B}_c^{-1}(\sigma_2)| \leq b|\sigma_1 - \sigma_2|^\theta, \quad \forall \sigma_1, \sigma_2 \in [0, \mathfrak{B}(1,c_{equ})].$$

Also, for σ_1 *and* σ_2 *fixed in* $[0,\mathfrak{B}(1,c_{equ})]$*, there exists two numbers* $s_w^1, s_w^2 \in [0,1]$ *with* $\mathfrak{B}_c(s_w^1) = \sigma_1$ *and* $\mathfrak{B}_c(s_w^2) = \sigma_2$. *We have*

$$\begin{aligned} |\sigma_1 - \sigma_2| &= |\mathfrak{B}_c(s_w^1) - \mathfrak{B}_c(s_w^2)| \\ &= \left|\int_{s_w^2}^{s_w^1}\gamma(y,c)dy\right| \leq cM|s_w^1 - s_w^2|, \quad M = \sup_{\substack{y\in[0,1]\\ z\in[0,c_{equ}]}}\gamma(y,z). \end{aligned}$$

we get

$$|\mathfrak{B}_c^{-1}(\sigma_1) - \mathfrak{B}_c^{-1}(\sigma_2)| \leq bc^\theta M^\theta|s_w^1 - s_w^2|^\theta \quad \textit{with} \quad \mathfrak{B}_c(s_w^1) = \sigma_1, \quad \mathfrak{B}_c(s_w^2) = \sigma_2. \tag{17}$$

Now, we denote $\mathfrak{B}_c^{-1}$ simply by $\mathfrak{B}^{-1}$.

In all what follows, we define the following Sobolev space

$$V \doteq H^1_{\Gamma_w}(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \quad \text{on} \quad \Gamma_w\}$$

which is a Hilbert space such that the norm $\|u\|_V = \|\nabla u\|_{(L^2(\Omega))^d}$.

Now, let us state the main result of this work

Theorem 2.3. Let $(A1)-(A8)$ hold and let the initial conditions s_o^0, s_w^0, and $c^0 \in L^2(\Omega)$, with $0 \leq c^0(x) \leq c_{equ}$, and $0 \leq s_w^0 \leq 1$. Then, there exists a solution (s_o, s_w, p_o, p_w, c) satisfying

$$p_o,\ p_w,\ \sqrt{\lambda_w(s_w,c)}\,\nabla p_w,\ \sqrt{\lambda_o(s_o)}\,\nabla p_o \in L^2(\Omega_T), \quad \phi\partial_t s_\alpha \in L^2(0,T;V'), \quad \alpha = w,o, \tag{18}$$

$$0 \leq s_\alpha \leq 1,\ \alpha = w,o, \quad 0 \leq c \leq c_{equ} \quad \text{a.e. in} \quad \Omega_T \quad \text{and} \quad \mathfrak{B}(s_w,c) \in L^2(0,T;V), \tag{19}$$

$$c - \widehat{c_{\Gamma_w}} \in L^2(0,T;V), \quad \phi\partial_t(s_w c) \in L^2(0,T;V'), \tag{20}$$

and satisfying for all φ, ψ, $\chi \in V$, and a.e. $t \in (0,T)$ the following integral identities

$$\langle \phi \partial_t s_w, \varphi \rangle + \int_\Omega \lambda_w(s_w, c)\mathbf{K}\nabla p_w \cdot \nabla \varphi \, dx = 0, \tag{21}$$

$$\langle \phi \partial_t s_o, \psi \rangle + \int_\Omega \lambda_o(s_o)\mathbf{K}\nabla p_o \cdot \nabla \psi \, dx = 0, \tag{22}$$

$$\langle \phi \partial_t (c s_w), \chi \rangle + \int_\Omega c\lambda_w(s_w, c)\mathbf{K}\nabla p_w \cdot \nabla \chi \, dx + \int_\Omega \mathbf{D}(s_w)\nabla c \cdot \nabla \chi \, dx = 0. \tag{23}$$

and the initial conditions, in the sense that for all $\xi \in V$, the functions

$$t \longmapsto \int_\Omega \phi(x) s_\alpha(t,x)\xi(x)dx \quad \text{for} \quad \alpha = o, w \quad \text{and} \quad t \longmapsto \int_\Omega \phi(x)(c s_w)(t,x)\xi(x)dx$$

are in $C^0([0,T])$, and we have

$$\Big(\int_\Omega \phi s_\alpha \xi dx\Big)(0) = \int_\Omega \phi s_\alpha^0 \xi dx, \quad \alpha = o, w, \tag{24}$$

$$\Big(\int_\Omega \phi c s_w \xi dx\Big)(0) = \int_\Omega \phi c^0 s_w^0 \xi dx. \tag{25}$$

Let us explain the origin of the requirements (18)-(20). The main point is to handle a priori estimates on pressure, saturation and concentration. The studied problem represents two kinds of degeneracy: the degeneracy for evolution terms ($\partial_t(s_\alpha)$ and $\partial_t(c\, s_w)$), and the degeneracy for dissipative terms ($\mathrm{div}(\lambda_\alpha \nabla p_\alpha)$). We will see that even if we control the quantities $\lambda_\alpha \nabla p_\alpha$ in the L^2−norm, this does not permit the control of the gradient of pressure of wetting and non-wetting phase since the mobility of two phase vanishes in the region where the phase is absent. We will get estimation on the gradient of the global pressure and the gradient of the capillary term $\mathfrak{B}$ to treat the degeneracy of our problem. Also, we can give the estimates on the gradient of the global pressure p and on the gradient of the capillary term $\mathfrak{B}$, after using the assumptions $(A1) - (A8)$, we indicate that $p \in L^2(0,T;V)$ and $\mathfrak{B}(s_w, c) \in L^2(0,T;H^1(\Omega))$, see [25].

3. Construction of a regularized problem

We treat the degeneracy of pressures due to disappearance of mobilities, for $(s_w, c) = (0,0)$ and $s_o = 0$, we regularize the problem (S) by adding a dissipative capillary term.

We consider the non degenerate problem:

$$\phi \frac{\partial s_w^\nu}{\partial t} - \mathrm{div}\,(\lambda_w(s_w^\nu, c^\nu)\mathbf{K}\nabla p_w^\nu) + \nu\delta(p_o^\nu - p_w^\nu) = 0, \tag{26}$$

$$\phi \frac{\partial s_o^\nu}{\partial t} - \mathrm{div}\,(\lambda_o(s_o^\nu)\mathbf{K}\nabla p_o^\nu) + \nu\delta(p_w^\nu - p_o^\nu) = 0, \tag{27}$$

$$\phi \frac{\partial (c^\nu s_w^\nu)}{\partial t} - \mathrm{div}\,(c^\nu\lambda_w(s_w^\nu, c^\nu)\mathbf{K}\nabla p_w^\nu) + \nu\,\mathrm{div}\,(c^\nu\nabla(p_o^\nu - p_w^\nu)) - \mathrm{div}(\mathbf{D}\nabla c^\nu) = 0, \tag{28}$$

where $\nu > 0$ is a positive parameter intended to tend towards zero, here δ denotes the Laplacian operator, with the boundary conditions

$$\left.\begin{array}{lll}
c^\nu(t,x) = c_{\Gamma_w}(t,x), & \text{on} & (0,T)\times\Gamma_w, \\
p_o^\nu(t,x) = p_w^\nu(t,x) = 0, & \text{on} & (0,T)\times\Gamma_w, \\
(\mathbf{u}_w^\nu + \nu\nabla(p_o^\nu - p_w^\nu))\cdot\mathbf{n} = 0, & \text{on} & (0,T)\times\Gamma_{\mathrm{imp}}, \\
(\mathbf{u}_o^\nu - \nu\nabla(p_o^\nu - p_w^\nu))\cdot\mathbf{n} = 0, & \text{on} & (0,T)\times\Gamma_{\mathrm{imp}}, \\
\Big(\mathbf{u}_w^\nu c^\nu + \nu\nabla(p_o^\nu - p_w^\nu) - \mathbf{D}\nabla c^\nu\Big)\cdot\mathbf{n} = 0, & \text{on} & (0,T)\times\Gamma_{\mathrm{imp}},
\end{array}\right\} \tag{29}$$

and initial conditions

$$s_\alpha^\nu(t=0,\cdot) = s_\alpha^0 \ \text{ for } \ \alpha = w, o \quad \text{and} \quad c^\nu(t=0,\cdot) = c^0 \ \text{ in } \ \Omega, \tag{30}$$

where, $\mathbf{n}$ is the outward unit normal to Γ_{imp}, the impervious part of the boundary, and

$$\mathbf{u}_w^\nu = -\lambda_w(s_w^\nu, c^\nu)\mathbf{K}\nabla p_w^\nu, \qquad \mathbf{u}_o^\nu = -\lambda_o(s_o^\nu)\mathbf{K}\nabla p_o^\nu.$$

We use (S_ν) to denote the problem (26)–(28) with initial and boundary conditions (29)–(30). Let us state the following result about existence of solutions to the above non degenerated the problem (S_ν).

Theorem 3.1. *Let (A1)-(A8) hold. Assume that $p_o^0, p_w^0, c^0 \in L^2(\Omega)$ with $0 \le c^0(x) \le c_{equ}$ and $0 \le s_w^0(x) \le 1$ a.e. in Ω. For all $\nu > 0$ fixed, there exists $(p_o^\nu, p_w^\nu, c^\nu)$ satisfying $p_\alpha^\nu \in L^2(0,T;V)$, $\phi\partial_t(s_\alpha^\nu) \in L^2(0,T;V')$, $s_\alpha^\nu \in C^0([0,T];L^2(\Omega))$, $s_w^\nu \in L^2(0,T;H^1(\Omega))$, $s_o^\nu \in L^2(0,T;V)$, $c - \widehat{c_{\Gamma_w}} \in L^2(0,T;V)$, $\nabla c^\nu \in L^2(0,T;L^2(\Omega))$, $\phi\partial_t(s_w^\nu c^\nu) \in L^2(0,T;V')$, $s_w^\nu c^\nu \in C^0(0,T;L^2(\Omega))$, and for all φ, ψ, $\chi \in V$ the following hold true a.e. in $]0,T[$:*

$$\langle \phi\partial_t s_w^\nu, \varphi\rangle + \int_\Omega \lambda_w(s_w^\nu, c^\nu)\mathbf{K}\nabla p_w^\nu \cdot \nabla\varphi \, dx - \nu \int_\Omega \nabla(p_o^\nu - p_w^\nu)\cdot\nabla\varphi\, dx = 0, \tag{31}$$

$$\langle \phi\partial_t s_o^\nu, \psi\rangle + \int_\Omega \lambda_o(s_o^\nu)\mathbf{K}\nabla p_o^\nu \cdot \nabla\psi \, dx - \nu \int_\Omega \nabla(p_w^\nu - p_o^\nu)\cdot\nabla\psi\, dx = 0, \tag{32}$$

$$\begin{aligned}\langle \phi\partial_t (c^\nu s_w^\nu), \chi\rangle + \int_\Omega c^\nu\lambda_w(s_w^\nu, c^\nu)\mathbf{K}\nabla p_w^\nu \cdot \nabla\chi \, dx + \int_\Omega \mathbf{D}\nabla c^\nu\cdot\nabla\chi\, dx \\ -\nu\int_\Omega c^\nu\nabla(p_o^\nu - p_w^\nu)\cdot\nabla\chi\, dx = 0,\end{aligned} \tag{33}$$

here the bracket $\langle\cdot,\cdot\rangle$ is the duality product between V' and V. The water saturation s_w^ν and the concentration c^ν of dissolved polymer verify the maximum principle in the sense that $0 \le s_w^\nu \le 1$ and $0 \le c^\nu \le c_{equ}$ a.e. in Ω_T.

The proof of the above Theorem 3.1 needs several steps. We begin by approximating the nondegenerate parabolic problem (S_ν) by a family of elliptic problems parametrized by a time step for which we prove the existence of solution and making this time step goes to zero, we obtain a solution for (S_ν). The maximum principles hold for saturations solution of these elliptic problems. In substance, let $M \in >^*$ be an integer. We subdivide the time interval $[0,T]$ in M subintervals. We get the time step $\delta t = T/M$. Starting from the initial conditions s_o^0, s_w^0, and c^0, we construct recursively sequences of functions solutions of approximating elliptic problems as follows. If at the time level $t_n = n\delta t$, the quintuplet of approximating solutions $(s_w^{\nu,n}, s_o^{\nu,n}, p_w^{\nu,n}, p_o^{\nu,n}, c^{\nu,n}) \in (L^2(\Omega))^5$ with $s_\alpha^{\nu,n} \ge 0$ and $c^{\nu,n}s_w^{\nu,n} \ge 0$ is known, using $s_o^{\nu,n}$, $s_w^{\nu,n}$, and $c^{\nu,n}$ as initial conditions, we define the quintuplet $(s_w^{\nu,n+1}, s_o^{\nu,n+1}, p_w^{\nu,n+1}, p_o^{\nu,n+1}, c^{\nu,n+1})$ as a solution of the problem

$$(S_{\nu,\delta t}) \qquad \begin{aligned}
\phi\frac{s_w^{\nu,n+1} - s_w^{\nu,n}}{\delta t} &- \operatorname{div}(\lambda_w(s_w^{\nu,n+1}, c^{\nu,n+1})\mathbf{K}\nabla p_w^{\nu,n+1}) \\
&+ \nu\operatorname{div}(\nabla(p_o^{\nu,n+1} - p_w^{\nu,n+1})) = 0, \qquad (34)\\
\phi\frac{s_o^{\nu,n+1} - s_o^{\nu,n}}{\delta t} &- \operatorname{div}(\lambda_o(s_o^{\nu,n+1})\mathbf{K}\nabla p_o^{\nu,n+1}) \\
&+ \nu\operatorname{div}(\nabla(p_w^{\nu,n+1} - p_o^{\nu,n+1})) = 0, \qquad (35)\\
\phi\frac{c^{\nu,n+1}s_w^{\nu,n+1} - c^{\nu,n}s_w^{\nu,n}}{\delta t} &- \operatorname{div}(c^{\nu,n+1}\lambda_w(s_w^{\nu,n+1}, c^{\nu,n+1})\mathbf{K}\nabla p_w^{\nu,n+1}) \\
-\operatorname{div}(\mathbf{D}\nabla c^{\nu,n+1}) &+ \nu\operatorname{div}(c^{\nu,n+1}\nabla(p_o^{\nu,n+1} - p_w^{\nu,n+1})) = 0, \qquad (36)
\end{aligned}$$

satisfying the boundary conditions (29), such that the upper index ν is replaced by ν, n. In this problem the main unknowns are p_o, p_w, and c. The remaining unknowns, the saturations s_α are obtained from inverting the function of capillarity p_c. So, to keep the property of inversion, we extend the capillary pressure function p_c by continuity and strict monotony outside $[0,1]$ to get $\bar{p}_c$. This is possible in the case when the capillary function p_c is bounded, in another sense when $|p_c(0)| < \infty$. Therefore, we put $s_w = \bar{p}_c^{-1}(p_o - p_w)$ and $s_o = 1 - \bar{p}_c^{-1}(p_o - p_w)$.

In the next section, we are looking for the existence of solutions of the elliptic problem $(S_{\nu,\delta t})$ at fixed ν and δt.

3.1. *Study of a nonlinear elliptic problem*

To simplify notations, we will omit the upper indexes ν and n (or $n+1$) in the Equations of problem $(S_{\nu,\delta t})$. We add two regularizations, we replace the mobilities λ_α, by two strictly positive functions,

$$\lambda_\alpha^\varepsilon = \lambda_\alpha + \varepsilon \geq \varepsilon, \quad \varepsilon > 0,$$

chosen to reinforce the passage to the limit in the regularization using ν. We denote that the above regularization of the mobilities can lead to the loss of maximum principle property of the saturations and concentration. For this reason, the functions λ_α are extended on $\mathbb{R}$ by continuity outside $[0,1]$ and, for the same reason, we introduce

$$Z(s_\alpha) = \begin{cases} 0 & \text{if} \quad s_\alpha \leq 0, \\ s_\alpha & \text{if} \quad s_\alpha \in [0,1], \\ 1 & \text{if} \quad s_\alpha \geq 1, \end{cases} \qquad \text{and} \qquad Y(c) = \begin{cases} 0 & \text{if} \quad c \leq 0, \\ c & \text{if} \quad c \in [0, c_{equ}], \\ c_{equ} & \text{if} \quad c \geq c_{equ}. \end{cases}$$

The aim of the following is to prove the existence of solution to (34)–(36). This needs three steps.

Firstly, we introduce the orthogonal projection $\mathcal{P}_m$ of $L^2(\Omega)$ on its subspace spanned by the first m eigenfunctions $\{p_1, ..., p_m\}$ of the eigenvalue problem

$$\left.\begin{array}{lll} \Delta p_i = \lambda_i p_i & \text{in} & \Omega, \\ p_i = 0 & \text{on} & \Gamma_w, \\ \nabla p_i \cdot \mathbf{n} = 0 & \text{on} & \Gamma_{\text{imp}}. \end{array}\right\} \tag{37}$$

The operator $\mathcal{P}_m$ is defined by

$$L^2(\Omega) \ni \bar{p} \quad \longmapsto \quad \mathcal{P}_m \bar{p} = \sum_{i=1}^{m} (\bar{p}, p_i)\, p_i \in \Pi_m = \text{Vec}\{p_1, ..., p_m\} \subset L^2(\Omega),$$

here Π_m is the subspace spanned by the eigenvectors p_1, ..., p_m; $(\cdot,\cdot)$ stands for the scalar product of $L^2(\Omega)$.

The first step consists therefore to study the following weak formulation of the non-degenerate problem $(S_{\nu,\delta t})$, for fixed parameters $\varepsilon > 0$ and $m > 0$:

$$\begin{aligned} \int_\Omega \phi \frac{Z(s_w^{\varepsilon,m}) - s_w^*}{\delta t} \varphi dx \quad &- \quad \nu \int_\Omega \nabla(\mathcal{P}_m p_o^{\varepsilon,m} - \mathcal{P}_m p_w^{\varepsilon,m}) \cdot \nabla \varphi dx \\ &+ \quad \int_\Omega \lambda_w^\varepsilon (s_w^{\varepsilon,m}, c^{\varepsilon,m}) \mathbf{K} \nabla p_w^{\varepsilon,m} \cdot \nabla \varphi dx = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} \int_\Omega \phi \frac{Z(s_o^{\varepsilon,m}) - s_o^*}{\delta t} \psi dx \quad &+ \quad \nu \int_\Omega \nabla(\mathcal{P}_m p_o^{\varepsilon,m} - \mathcal{P}_m p_w^{\varepsilon,m}) \cdot \nabla \psi dx \\ &+ \quad \int_\Omega \lambda_o^\varepsilon (s_o^{\varepsilon,m}) \mathbf{K} \nabla p_o^{\varepsilon,m} \cdot \nabla \psi dx = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} \int_\Omega \phi \frac{Z(s_w^{\varepsilon,m}) Y(c^{\varepsilon,m}) - c^* s_w^*}{\delta t} \chi dx \quad &- \quad \nu \int_\Omega Y(c^{\varepsilon,m}) \nabla(\mathcal{P}_m p_o^{\varepsilon,m} - \mathcal{P}_m p_w^{\varepsilon,m}) \cdot \nabla \chi dx \\ + \int_\Omega \mathbf{D} \nabla c^{\varepsilon,m} \cdot \nabla \chi dx \quad &+ \int_\Omega Y(c^{\varepsilon,m}) \lambda_w^\varepsilon (s_w^{\varepsilon,m}, c^{\varepsilon,m}) \mathbf{K} \nabla p_w^{\varepsilon,m} \cdot \nabla \chi dx = 0, \end{aligned} \tag{40}$$

$$\forall (\varphi, \psi, \chi) \in V^3.$$

We note the above problem by $(S^{\varepsilon,m}_{\nu,\delta t})$. This is an elliptical problem to be solved at each step $n = 1, \dots, M$. The work to be done at each time level is the same. Assuming known s^n, an approximation of the solution at the level $t_n = n\delta t$, we determine s^{n+1}, an approximation of the solution at the level t_{n+1} by solving the above problem where, to simplify, we set $s^{n+1} = s^m$ and $s^n = s^*$. We use recurrence, covering the whole time interval.

Step 1.

The first one consists in studying the problem $(S^{\varepsilon,m}_{\nu,\delta t})$ for fixed parameters $\varepsilon > 0$ and $m > 0$. We will show for fixed $m > 0$ and $\varepsilon > 0$ the existence of solutions $(p^{\varepsilon,m}_w, p^{\varepsilon,m}_o, c^{\varepsilon,m})$ of (38)–(40).

Proposition 3.2. *For fixed $m > 0$ and $\varepsilon > 0$, assume s^*_α and $s^*_w c^*$ belonging to $L^2(\Omega)$, $s^*_\alpha \geq 0$ and $s^*_w c^* \geq 0$. Then there exists $p^{\varepsilon,m}_w \in V$, $p^{\varepsilon,m}_o \in V$ and $c^{\varepsilon,m} \in H^1(\Omega)$, solution of the problem (38)-(40).*

Proof. We omit for the time being the dependence of solutions on parameters $m > 0$ and $\varepsilon > 0$. We shall use the Leray-Schauder Fixed Point Theorem [30]. Let $\mathcal{T}$ be the map from $(L^2(\Omega))^3$ to $(L^2(\Omega))^3$ defined by

$$\mathcal{T}(\bar{p}_w, \bar{p}_o, \bar{c}) = (p_w, p_o, c),$$

where (p_w, p_o, c) is the unique solution of the following problem

$$\begin{aligned}
\int_\Omega \phi\, \frac{Z(\bar{s}_w) - s^*_w}{\delta t}\, \varphi\, dx \quad &+ \quad \int_\Omega \lambda^\varepsilon_w(\bar{s}_w, \bar{c}) \mathbf{K} \nabla p_w \cdot \nabla\varphi\, dx \\
&- \quad \nu \int_\Omega \nabla(\mathcal{P}_m \bar{p}_o - \mathcal{P}_m \bar{p}_w) \cdot \nabla\varphi\, dx = 0, \qquad (41)\\
\int_\Omega \phi \frac{Z(\bar{s}_o) - s^*_o}{\delta t} \psi dx \quad &+ \quad \int_\Omega \lambda^\varepsilon_o(\bar{s}_o) \mathbf{K} \nabla p_o \cdot \nabla\psi\, dx \\
&+ \quad \nu \int_\Omega \nabla(\mathcal{P}_m \bar{p}_o - \mathcal{P}_m \bar{p}_w) \cdot \nabla\psi\, dx = 0, \qquad (42)\\
\int_\Omega \phi \frac{Z(\bar{s}_w) Y(\bar{c}) - s^*_w c^*}{\delta t} \chi dx \quad &= \quad -\int_\Omega Y(\bar{c}) \lambda^\varepsilon_w(\bar{s}_w, \bar{c}) \mathbf{K} \nabla p_w \cdot \nabla\chi\, dx \\
&- \quad \int_\Omega \mathbf{D} \nabla c \cdot \nabla\chi\, dx + \nu \int_\Omega Y(\bar{c}) \nabla(p_o - p_w) \cdot \nabla\chi\, dx, \qquad (43)\\
&\qquad\qquad \forall\, \varphi,\ \psi,\ \chi \in V.
\end{aligned}$$

The mapping $\mathcal{T}$ is well defined on $(L^2(\Omega))^3$. To see this, we use the Lax-Milgram Theorem (see for instance [26]). Let us begin by Equation (41). To prove the existence of an unique function $p_w \in V$ solution of this equation, we consider the bilinear form

$$a(p_w, \varphi) \;=\; \int_\Omega \lambda^\varepsilon_w(\bar{s}_w, \bar{c}) \mathbf{K} \nabla p_w \cdot \nabla\varphi dx,$$

and the linear functional

$$\ell(\varphi) = \nu \int_\Omega \nabla(\mathcal{P}_m \bar{p}_o - \mathcal{P}_m \bar{p}_w) \cdot \nabla\varphi\, dx - \int_\Omega \phi \frac{Z(\bar{s}_w) - s^*_w}{\delta t}\, \varphi\, dx.$$

It is easy to see that $a(\cdot,\cdot)$ is bilinear symmetric and ℓ is linear on V.

To see the continuity of $a(\cdot,\cdot)$, we use the fact that $\lambda^\varepsilon_w \in L^\infty(\Omega)$ and $\mathbf{K} \in (L^\infty(\Omega))^{d\times d}$ to write

$$\begin{aligned}
|a(p_w, \varphi)| \quad &= \quad \Big| \int_\Omega \lambda^\varepsilon_w(\bar{s}_w, \bar{c}) \mathbf{K} \nabla p_w \cdot \nabla\varphi\, dx \Big| \\
\text{(Cauchy-Schwarz Ineq.)} \quad &\leq \quad \sup |\lambda^\varepsilon_w| \|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \|\nabla p_w\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq \quad C \|p_w\|_V \|\varphi\|_V.
\end{aligned}$$

The bilinear form $a(\cdot,\cdot)$ is coercive, since $\lambda^{\varepsilon} \geq \varepsilon$, using the hypothesis $(H2)$, we can write:

$$\begin{aligned}|a(p_w, p_w)| &= \Big| \int_{\Omega} \lambda_w^{\varepsilon}(\bar{s}_w, \bar{c})\mathbf{K}\nabla p_w \cdot \nabla p_w \, dx \Big| \\ &\geq \varepsilon \int_{\Omega} \mathbf{K}\nabla p_w \cdot \nabla p_w \, dx \geq \varepsilon k_0 \|p_w\|_V^2.\end{aligned}$$

The functional $\ell(\cdot)$ is continuous on V:

$$\begin{aligned}|\ell(\varphi)| &= \Big| \nu \int_{\Omega} \nabla(\mathcal{P}_m\bar{p}_o - \mathcal{P}_m\bar{p}_w) \cdot \nabla\varphi \, dx - \int_{\Omega} \phi \frac{Z(\bar{s}_w) - s_w^*}{\delta t} \varphi \, dx \Big| \\ &\leq \nu \int_{\Omega} |\nabla(\mathcal{P}_m\bar{p}_o - \mathcal{P}_m\bar{p}_w) \cdot \nabla\varphi| \, dx + \int_{\Omega} \Big| \phi \frac{Z(\bar{s}_w) - s_w^*}{\delta t} \varphi \Big| \, dx \\ &\leq \nu \|\nabla(\mathcal{P}_m\bar{p}_o - \mathcal{P}_m\bar{p}_w)\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} + \phi^{\star} \Big\| \frac{Z(\bar{s}_w) - s_w^*}{\delta t} \Big\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\ &\leq C\|\varphi\|_V \quad \text{(by the Poincaré Inequality).}\end{aligned}$$

Thus, we can use Lax-Milgram Theorem to see that the equation (41) has an unique solution p_w in V. The same argument shows the existence of an unique solution $p_o \in V$ to the equation (42). Now plugging these found functions p_o and p_w into the equation (43).

We get a linear equation to determine the unknown function c. To do this we will use the following result, given for instance in [6], which asserts that if B is a reflexive Banach space and $\mathfrak{C} \subset B$ is a nonempty, closed, convex subset of B, each convex lower semi-continuous functional $F : \mathfrak{C} \longrightarrow (-\infty, +\infty]$ such that $F \not\equiv +\infty$ and $\lim\limits_{v \in \mathfrak{C}, \|v\| \to \infty} F(v) = +\infty$ (no assumption if $\mathfrak{C}$ is bounded), achieves its minimum on $\mathfrak{C}$. Let us therefore consider $\mathfrak{C}$, the convex closed subset of $H^1(\Omega)$ given by $\mathfrak{C} = V + \widehat{c_{\Gamma_w}}$, and define the functional F on $\mathfrak{C}$ by

$$\mathfrak{C} \ni v \longmapsto F(v) = \frac{1}{2}\sigma(v, v) - \tau(v) \in \mathbb{R},$$

with

$$\begin{aligned}\sigma(v, v) &= \int_{\Omega} \mathbf{D}\nabla v \cdot \nabla v \, dx, \\ \tau(v) &= \nu \int_{\Omega} Y(\bar{c})\nabla(p_o - p_w) \cdot \nabla v \, dx - \int_{\Omega} \phi \frac{Z(\bar{s}_w)Y(\bar{c}) - s_w^* c^*}{\delta t} v \, dx \\ &\qquad - \int_{\Omega} Y(\bar{c})\lambda_w^{\varepsilon}(\bar{s}_w, \bar{c})\mathbf{K}\nabla p_w \cdot \nabla v \, dx.\end{aligned}$$

As above, we can prove that $\sigma(\cdot,\cdot)$ is a continuous bilinear form and $\tau(\cdot)$ is a linear continuous form on $H^1(\Omega)$; this implies the continuity of the functional F. This functional is convex on $H^1(\Omega)$. In fact, for v, w two functions of $H^1(\Omega)$ and $t \in [0, 1]$ a real number, we can write

$$\sigma((1-t)v + tw, (1-t)v + tw) = (1-t)^2\sigma(v, v) + t(1-t)\{\sigma(v, w) + \sigma(w, v)\} + t^2\sigma(w, w).$$

Then

$$\begin{aligned}\sigma((1-t)v + tw, (1-t)v + tw) - (1-t)\sigma(v, v) - t\sigma(w, w) &= t(1-t)\{-\sigma(v, v) + \sigma(v, w) \\ &\qquad + \sigma(w, v) - \sigma(w, w)\} \\ &= -t(1-t)\sigma(w - v, w - v) \leq 0.\end{aligned}$$

Since τ is linear, the functional F is convex.

Now, using the continuity of the linear form τ and the Hypothesis ($H8$), we see that $F(v+\hat{c}_{\Gamma_w}) \geq \frac{\delta}{2}\|v\|^2 - C'\|v\| - C''$, $\forall v \in H^1(\Omega)$, with $C', C'' > 0$ the constants. This shows that $\lim_{\|v\|\to\infty} F(v) = +\infty$. We are now ready to use the above mentioned result to see the existence of a function c such that

$$c \in \mathfrak{C}, \quad F(c) = \frac{1}{2}\sigma(c,c) - \tau(c) = \min\Big\{\frac{1}{2}\sigma(v,v) - \tau(v) \mid v \in \mathfrak{C}\Big\}.$$

Note that $(1-t)v + tw \in \mathfrak{C}$ for all $v, w \in \mathfrak{C}$ and all $t \in \mathbb{R}$. This means that the function $\mathbb{R} \ni t \longmapsto \xi(t) = F((1-t)c + tw) \in \mathbb{R}$ reaches its minimum at $t = 0$. Since (we use the symmetry of $\mathbf{D}$)

$$\sigma(c + t(w-c), c + t(w-c)) = \sigma(c,c) + 2t\sigma(c, w-c) + t^2\sigma(w-c, w-c)$$

we get $\xi'(t) = \sigma(c, w-c) + t\sigma(w-c, w-c) - \tau(w-c)$, for all $t \in \mathbb{R}$. This gives

$$\sigma(c, w-c) = \tau(w-c), \quad \forall w \in \mathfrak{C}.$$

Put $\chi = w - c$, which is in V, we that the function c is a solution of Equation (43): $\sigma(c,\chi) = \tau(\chi)$, $\forall \chi \in V$. This solution is unique, by virtue of the coerciveness of the diffusion-dispersion tensor $\mathbf{D}$.

Lemma 3.3. *The map $\mathcal{T}$ is a continuous operator which maps every bounded subset of $L^2(\Omega)$ into a relatively compact set.*

Proof. Let us consider a sequence $\{(\bar{p}_{wk}, \bar{p}_{ok}, \bar{c}_k)\}$ bounded in $(L^2(\Omega))^3$ and converging to $(\bar{p}_w, \bar{p}_o, \bar{c}) \in (L^2(\Omega))^3$. We have prove that the image sequence $\{(p_{wk}, p_{ok}, c_k)\} = \{\mathcal{T}(\bar{p}_{wk}, \bar{p}_{ok}, \bar{c}_k)\}$ is bounded in V^3 and converges to $(p_w, p_o, c) = \mathcal{T}(\bar{p}_w, \bar{p}_o, \bar{c})$. In fact, the sequences $\{p_{wk}\}$, $\{p_{ok}\}$, $\{c_k\}$ verify respectively, for all $(\varphi, \psi, \chi) \in V^3$:

$$\begin{aligned}\int_\Omega \phi\frac{Z(\bar{s}_{wk}) - s_w^*}{\delta t}\varphi dx &+ \int_\Omega \lambda_w^\varepsilon(\bar{s}_{wk}, \bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla\varphi\, dx \\ &- \nu\int_\Omega \nabla(\mathcal{P}_m\bar{p}_{ok} - \mathcal{P}_m\bar{p}_{wk})\cdot\nabla\varphi\, dx = 0,\end{aligned} \tag{44}$$

$$\begin{aligned}\int_\Omega \phi\frac{Z(\bar{s}_{ok}) - s_o^*}{\delta t}\psi\, dx &+ \int_\Omega \lambda_o^\varepsilon(\bar{s}_{ok})\mathbf{K}\nabla p_{ok}\cdot\nabla\psi\, dx \\ &+ \nu\int_\Omega \nabla(\mathcal{P}_m\bar{p}_{ok} - \mathcal{P}_m\bar{p}_{wk})\cdot\nabla\psi\, dx = 0,\end{aligned} \tag{45}$$

and

$$\begin{aligned}\int_\Omega \phi\frac{Z(\bar{s}_{wk})Y(\bar{c}_k) - s_w^*c^*}{\delta t}\chi dx + \int_\Omega \mathbf{D}\nabla c_k\cdot\nabla\chi dx = &-\int_\Omega Y(\bar{c}_k)\lambda_w^\varepsilon(\bar{s}_{wk}, \bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla\chi dx \\ &+ \nu\int_\Omega Y(\bar{c}_k)\nabla(p_{ok} - p_{wk})\cdot\nabla\chi dx.\end{aligned} \tag{46}$$

Let us take $\varphi = p_{wk}$ in (44), we get

$$\begin{aligned}\int_\Omega \phi\frac{Z(\bar{s}_{wk}) - s_w^*}{\delta t}p_{wk}\, dx &+ \int_\Omega \lambda_w^\varepsilon(\bar{s}_{wk}, \bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla p_{wk}\, dx \\ &- \nu\int_\Omega \nabla(\mathcal{P}_m\bar{p}_{ok} - \mathcal{P}_m\bar{p}_{wk})\cdot\nabla p_{wk}\, dx = 0.\end{aligned}$$

Leading to

$$\begin{aligned}\int_\Omega \lambda_w^\varepsilon(\bar{s}_{wk}, \bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla p_{wk}\, dx \leq &\int_\Omega \Big|\phi\frac{Z(\bar{s}_{wk}) - s_w^*}{\delta t}p_{wk}\Big|\, dx \\ &+ \nu\int_\Omega |\nabla(\mathcal{P}_m\bar{p}_{ok} - \mathcal{P}_m\bar{p}_{wk})\cdot\nabla p_{wk}|\, dx.\end{aligned}$$

Now, using the Cauchy-Schwarz Inequality, one gets

$$\int_\Omega (\lambda_w(\bar{s}_{wk},\bar{c}_k)+\varepsilon)\mathbf{K}\nabla p_{wk}\cdot\nabla p_{wk}\,dx \;\leq\; \Big(\int_\Omega \Big|\phi\frac{Z(\bar{s}_{wk})-s_w^*}{\delta t}\Big|^2 dx\Big)^{\frac12}\Big(\int_\Omega |p_{wk}|^2\,dx\Big)^{\frac12} + \nu\Big(\int_\Omega |\nabla(\mathcal{P}_m\bar{p}_{ok}-\mathcal{P}_m\bar{p}_{wk})|^2\,dx\Big)^{\frac12}\|p_{wk}\|_V.$$

Now, according to hypothesis ($H2$) and Poincaré Inequality, we obtain

$$\varepsilon k_0\|p_{wk}\|_V^2 \;\leq\; C\Big(1+\sum_{\alpha=o,w}\|\nabla(\mathcal{P}_m\bar{p}_{\alpha k})\|_{L^2(\Omega)}\Big)\|p_{wk}\|_V, \tag{47}$$

where C is a positive constant depending on Ω, ν, δt, $\phi^\star$, k_∞, and $\|s_w^*\|_{L^2(\Omega)}$.

Since $\nabla\mathcal{P}_m\bar{p}_{\alpha k}=\sum_{i=1}^m(\bar{p}_{\alpha k},p_i)\nabla p_i$, there exists a constant C_m, depending on the norms of eigenfunctions such that

$$\|\nabla\mathcal{P}_m\bar{p}_{\alpha k}\|_{L^2(\Omega)}\leq C_m\|\bar{p}_{\alpha k}\|_{L^2(\Omega)} \qquad \alpha=o,w.$$

As the sequence $\{(\bar{p}_{wk},\bar{p}_{ok},\bar{c}_k)\}$ is taken bounded in $(L^2(\Omega))^3$, the estimate (47) ensures that the sequence $\{p_{wk}\}_k$ is uniformly bounded in V.

Using the same argument, we can prove that the sequence $\{p_{ok}\}_k$ is uniformly bounded in V.

To estimate the sequence $\{c_k\}_k$, we take $\chi=c_k-\widehat{c}_{\Gamma_w}\doteq z_k$ as a test function in (46). We have

$$\begin{aligned}\int_\Omega \mathbf{D}\nabla c_k\cdot\nabla(c_k-\widehat{c}_{\Gamma_w})\,dx \;=\; &-\int_\Omega \phi\frac{Z(\bar{s}_{wk})Y(\bar{c}_k)-s_w^*c^*}{\delta t}(c_k-\widehat{c}_{\Gamma_w})\,dx\\ &-\int_\Omega Y(\bar{c}_k)\lambda_w^\varepsilon(\bar{s}_{wk},\bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla(c_k-\widehat{c}_{\Gamma_w})\,dx\\ &+\nu\int_\Omega Y(\bar{c}_k)\nabla(\mathcal{P}_m\bar{p}_{ok}-\mathcal{P}_m\bar{p}_{wk})\cdot\nabla(c_k-\widehat{c}_{\Gamma_w})\,dx.\end{aligned}$$

Using hypothesis ($H1$), ($H2$), ($H3$), ($H8$) and using Cauchy-Schwartz inequality and Poincaré Inequality, we obtain

$$\begin{aligned}\delta\int_\Omega|\nabla c_k|^2\,dx \;\leq\; &\int_\Omega\phi\Big|\frac{Z(\bar{s}_{wk})Y(\bar{c}_k)-s_w^*c^*}{\delta t}(c_k-\widehat{c}_{\Gamma_w})\Big|dx\\ &+\int_\Omega|Y(\bar{c}_k)\lambda_w^\varepsilon(\bar{s}_{wk},\bar{c}_k)\mathbf{K}\nabla p_{wk}\cdot\nabla(c_k-\widehat{c}_{\Gamma_w})|dx\\ &+\nu\int_\Omega|Y(\bar{c}_k)\nabla(\mathcal{P}_m\bar{p}_{ok}-\mathcal{P}_m\bar{p}_{wk})\cdot\nabla(c_k-\widehat{c}_{\Gamma_w})|dx\\ &+\frac12\parallel\mathbf{D}\parallel_{(L^\infty(\Omega))^{d\times d}}\Big[\int_\Omega|\nabla c_k|^2dx+\int_\Omega|\nabla\widehat{c}_{\Gamma_w}|^2dx\Big],\end{aligned}$$

we deduce similarly that

$$\int_\Omega|\nabla z_k|^2dx\leq C(1+\|\nabla p_{wk}\|_{L^2(\Omega)}+\|\nabla p_{ok}\|_{L^2(\Omega)}) \tag{48}$$

where C depends on $\Omega,\delta t,\phi^\star,k_\infty,\delta$ and $\|s_w^*c^*\|_{L^2(\Omega)}$.
This establishes the relative compactness property of the map $\mathcal{T}$ in $(L^2(\Omega))^3$.
Furthermore, up to a subsequence, we have the convergences

$$p_{\alpha k}\longrightarrow p_\alpha \quad \text{weakly in} \quad L^2(0,T;V),\ \alpha=o,w \tag{49}$$

$$z_k\longrightarrow z \quad \text{weakly in} \quad L^2(0,T;V), \tag{50}$$

$$p_{\alpha k}\longrightarrow p_\alpha \quad \text{strongly in} \quad L^2(\Omega)\text{ and a.e. in }\Omega \tag{51}$$

$$z_k\longrightarrow z \quad \text{strongly in} \quad L^2(\Omega)\text{ and a.e. in }\Omega. \tag{52}$$

We have $\mathfrak{C} = V + \widehat{c_{\Gamma_w}}$, the convex closed subset of $H^1(\Omega)$, then

$$c_k \longrightarrow c \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)), \tag{53}$$

$$c_k \longrightarrow c \quad \text{strongly in} \quad L^2(\Omega) \text{ and a.e. in } \Omega. \tag{54}$$

Using the above convergence and the theorem of Lebesgue concerning dominated convergence, we show classically that $\mathcal{T}$ is continuous operator. □

Lemma 3.4. *(A priori estimate)*
There exists $r > 0$ such that, if $(p_w, p_o, c) = \beta\mathcal{T}(p_w, p_o, c)$ with $\beta \in (0,1)$, then

$$\|(p_w, p_o, c)\|_{(L^2(\Omega))^3} \leq r.$$

Proof. Assume that $(p_w, p_o, c) = \beta\mathcal{T}(p_w, p_o, c)$ exists, then (p_w, p_o, c) satisfies

$$\begin{aligned}\int_\Omega \mathbf{K}\lambda_w^\varepsilon(s_w, c)\nabla p_w \cdot \nabla\varphi dx = & - \beta\int_\Omega \phi\frac{Z(s_w) - s_w^*}{\delta t}\varphi dx \\ & + \beta\nu\int_\Omega \nabla(\mathcal{P}_m p_o - \mathcal{P}_m p_w)\cdot\nabla\varphi dx,\end{aligned} \tag{55}$$

$$\begin{aligned}\int_\Omega \mathbf{K}\lambda_o^\varepsilon(s_o)\nabla p_o \cdot \nabla\psi dx = & - \beta\int_\Omega \phi\frac{Z(s_o) - s_o^*}{\delta t}\psi dx \\ & - \beta\nu\int_\Omega \nabla(\mathcal{P}_m p_o - \mathcal{P}_m p_w)\cdot\nabla\psi dx,\end{aligned} \tag{56}$$

and

$$\begin{aligned}\int_\Omega \mathbf{D}\nabla c \cdot \nabla\chi dx = & - \beta\int_\Omega \phi\frac{Z(s_w)Y(c) - s_w^* c^*}{\delta t}\chi dx - \beta\int_\Omega Y(c)\mathbf{K}\lambda_w^\varepsilon(s_w, c)\nabla p_w\cdot\nabla\chi dx \\ & + \beta\nu\int_\Omega Y(c)\nabla(p_o - p_w)\cdot\nabla\chi dx,\end{aligned} \tag{57}$$

for all $(\varphi, \psi, \chi) \in (V)^3$.
Taking $\varphi = p_w \in V$ in (55), $\psi = p_o \in V$ in (56), adding them and using again the Cauchy-Schwarz and Poincaré inequalities, we deduce

$$\varepsilon\int_\Omega |\nabla p_w|^2 dx + \varepsilon\int_\Omega |\nabla p_o|^2 dx + \beta\nu\int_\Omega |\nabla(\mathcal{P}_m p_o - \mathcal{P}_m p_w)|^2 dx \leq C\Big(\|s_w^*\|_{L^2(\Omega)} + \|s_o^*\|_{L^2(\Omega)}\Big), \tag{58}$$

where C depends on ε and not on β.
In the same manner, we obtain by taking $\chi = c - \widehat{c_{\Gamma_w}}$ in (57), that

$$\begin{aligned}\frac{1}{2}\delta\int_\Omega |\nabla c|^2 dx \quad \leq \quad & C(\|\nabla p_o\|_{L^2(\Omega)} + \|\nabla p_w\|_{L^2(\Omega)} \\ & + \|\nabla c\|_{L^2(\Omega)} + \|s_w^* c^*\|_{L^2(\Omega)} + \|\nabla\widehat{c_{\Gamma_w}}\|^2_{L^2(\Omega)}),\end{aligned} \tag{59}$$

where C depends on ε and not on β. □

Lemma 3.3 and Lemma 3.4 allow to apply the Leray-Schauder fixed point theorem [30], thus the proof of Proposition 3.2 is completed. □

Step 2.

This step consists to pass to the limit when m goes to infinity, and the last step concerns to pass to the limit when ε goes to 0. This step is devoted go by the limit when m goes to ∞.
The estimates (58) and (59) are uniform with respect to m and β, then taking these estimates with $\beta = 1$, we get

$$\varepsilon \int_\Omega |\nabla p_w^{\varepsilon,m}|^2 dx + \varepsilon \int_\Omega |\nabla p_o^{\varepsilon,m}|^2 dx + \nu \int_\Omega |(\nabla(\mathcal{P}_m p_o^{\varepsilon,m} - \mathcal{P}_m p_w^{\varepsilon,m}))|^2 dx \le C, \tag{60}$$

and

$$\delta \int_\Omega |\nabla c^{\varepsilon,m}|^2 dx \le C, \tag{61}$$

where C depends on ε and not on m.
Then up to a subsequence, when m goes to infinity, we have the convergences

$$\begin{array}{lll} p_\alpha^{\varepsilon,m} & \longrightarrow & p_\alpha^\varepsilon \quad \text{weakly in } V, \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega \text{ with } \alpha = w, o \\ c^{\varepsilon,m} & \longrightarrow & c^\varepsilon \quad \text{weakly in } H^1(\Omega), \text{ strongly in } L^2(\Omega). \end{array}$$

We pass to the limit when m goes to infinity in (38)-(40) to obtain, for all $\varepsilon > 0$, the existence of $(p_w^\varepsilon, p_o^\varepsilon) \in (V)^2$ and $c^\varepsilon \in H^1(\Omega)$, satisfying

$$\int_\Omega \phi \frac{Z(s_w^\varepsilon) - s_w^*}{\delta t} \varphi dx + \int_\Omega \mathbf{K} \lambda_w^\varepsilon (s_w^\varepsilon, c^\varepsilon) \nabla p_w^\varepsilon \cdot \nabla \varphi dx - \nu \int_\Omega \nabla(p_o^\varepsilon - p_w^\varepsilon) \nabla \varphi dx = 0, \tag{62}$$

$$\int_\Omega \phi \frac{Z(s_o^\varepsilon) - s_o^*}{\delta t} \psi dx + \int_\Omega \mathbf{K} \lambda_o^\varepsilon (s_o^\varepsilon,) \nabla p_o^\varepsilon \cdot \nabla \psi dx + \nu \int_\Omega \nabla(p_o^\varepsilon - p_w^\varepsilon) \nabla \psi dx = 0, \tag{63}$$

and

$$\begin{aligned} \int_\Omega \phi \frac{Z(s_w^\varepsilon) Y(c^\varepsilon) - s_w^* c^*}{\delta t} \chi dx \quad &+ \quad \int_\Omega Y(c^\varepsilon) \mathbf{K} \lambda_w^\varepsilon (s_w^\varepsilon, c^\varepsilon) \nabla p_w^\varepsilon \cdot \nabla \chi dx \\ &- \quad \nu \int_\Omega Y(c^\varepsilon) \nabla(p_o^\varepsilon - p_w^\varepsilon) \nabla \chi dx + \int_\Omega \mathbf{D} \nabla c^\varepsilon \cdot \nabla \chi dx = 0, \end{aligned} \tag{64}$$

for all $(\varphi, \psi, \chi) \in (V)^3$.

Step 3.

The third step is devoted go by the limit when ε goes to 0. We resort to make use of use the feature of global pressure to get uniform estimates on the solutions independent of the regularization ε. A maximum principle on saturation is possible after going to the limit in ε.

We state the following two lemmas in order to pass to the limit in ε.

Lemma 3.5. *(Uniform estimates with respect to ε)*
The sequences $(s_w^\varepsilon)_\varepsilon$, $(c^\varepsilon)_\varepsilon$, and $(p^\varepsilon = p_w^\varepsilon + \bar{p}(s_w^\varepsilon, c^\varepsilon))_\varepsilon$ defined by (62)-(64) satisfy

$$(p^\varepsilon)_\varepsilon \quad \text{is uniformly bounded in} \quad V \tag{65}$$

$$(\mathfrak{B}(s_w^\varepsilon, c^\varepsilon))_\varepsilon \quad \text{is uniformly bounded in} \quad H^1(\Omega) \tag{66}$$

$$(\nabla p_c(s_w^\varepsilon))_\varepsilon \quad \text{is uniformly bounded in} \quad L^2(\Omega) \tag{67}$$

$$(p_\alpha^\varepsilon)_\varepsilon \quad \text{is uniformly bounded in} \quad V, \ \alpha = o, w \tag{68}$$

$$(c^\varepsilon)_\varepsilon \quad \text{is uniformly bounded in} \quad H^1(\Omega) \tag{69}$$

Proof. Consider the test function $\varphi = p_w^\varepsilon \in V$ in (62) and $\psi = p_o^\varepsilon \in V$ in (63) and add them. We deduce with the help of Cauchy-Schwarz inequality and the relationship (8) between the global pressure and the pressure of each phase that

$$\int_\Omega \mathbf{K}\lambda_w^\varepsilon(s_w^\varepsilon, c^\varepsilon)\nabla p_w^\varepsilon \nabla p_w^\varepsilon dx + \int_\Omega \mathbf{K}\lambda_o^\varepsilon(s_o^\varepsilon)\nabla p_o^\varepsilon \nabla p_o^\varepsilon dx + \nu \int_\Omega \nabla p_c(s_w^\varepsilon)\cdot \nabla p_c(s_w^\varepsilon) dx \leq C(1 + \|p^\varepsilon\|^2_{L^2(\Omega)} + \|\tilde{p}(s_w^\varepsilon)\|^2_{L^2(\Omega)} + \|\tilde{p}(s_o^\varepsilon)\|^2_{L^2(\Omega)}),$$

here $C = C(\delta t, \lambda_\alpha, \phi, k_\infty, s_w^*, s_o^*)$ independent of ε.
Using the identity (12) which links the gradient of the pressure of each phase to the global pressure, we obtain

$$k_o \int_\Omega \lambda(s_w^\varepsilon, c^\varepsilon)|\nabla p^\varepsilon|^2 dx + k_o \int_\Omega \frac{\lambda_w(s_w^\varepsilon, c^\varepsilon)\lambda_o(s_o^\varepsilon)}{\lambda(s_w^\varepsilon, c^\varepsilon)}|\nabla p_c(s_w^\varepsilon)|^2 dx + \nu \int_\Omega \nabla p_c(s_w^\varepsilon)\cdot \nabla p_c(s_w^\varepsilon) dx + \varepsilon k_0 \int_\Omega \nabla p_w^\varepsilon \cdot \nabla p_w^\varepsilon dx + \varepsilon k_0 \int_\Omega \nabla p_o^\varepsilon \cdot \nabla p_o^\varepsilon dx \leq C_1, \tag{70}$$

here k_0 is the constant of coercivity of the tensor $\mathbf{K}$ and C_1 is a constant independent of ε.
Then, the assumption (A3) ensures the estimate (65).
For the estimate (66), we have

$$\int_\Omega |\mathfrak{B}(s_w^\varepsilon, c^\varepsilon)|^2 dx \leq \sup_{s\in[0,1]; c\in[0,c_{equ}]} |\lambda(s,c)| \int_\Omega \frac{\lambda_w(s_w^\varepsilon, c^\varepsilon)\lambda_o(s_o^\varepsilon)}{\lambda(s_w^\varepsilon, c^\varepsilon)}|\nabla p_c(s_w^\varepsilon)|^2 dx \leq C_2,$$

such that C_2 is independent of ε.
The estimate (67) is a consequence of (70). The estimate (68) is a direct consequence of (8) and the estimates (65) and (67).
The last estimate (69) is obtained by taking $\chi = c^\varepsilon - \widehat{c_{\Gamma_w}}$ in the formulation (64), and from the Cauchy-Schwarz inequality, we deduce

$$\|\nabla c^\varepsilon\|_{L^2(\Omega)} \leq C(\|c^\varepsilon\|_{L^2(\Omega)} + \|\widehat{c_{\Gamma_w}}\|_{L^2(\Omega)} + \|\nabla\widehat{c_{\Gamma_w}}\|^2_{L^2(\Omega)} + \|\nabla p_w^\varepsilon\|^2_{L^2(\Omega)}) + \|\nabla p_o^\varepsilon\|^2_{L^2(\Omega)}).$$

We use the Poincaré inequality and the estimate (68) to obtain (69). □

From Lemma 3.5 (up to a subsequence), the sequences $(s_\alpha^\varepsilon)_\varepsilon, (p^\varepsilon)_\varepsilon, (p_\alpha^\varepsilon)_\varepsilon$, verify the following convergences

$$p^\varepsilon \longrightarrow p \text{ weakly in } V \text{ and a.e. in } \Omega \tag{71}$$
$$\mathfrak{B}(s_w^\varepsilon, c^\varepsilon) \longrightarrow \mathfrak{B}(s_w, c) \text{ weakly in } H^1(\Omega) \text{ and a. e. in } \Omega \tag{72}$$
$$Z(s_w^\varepsilon) \longrightarrow Z(s_w) \text{ weakly in } L^2(\Omega) \text{ and almost evreywhere in } \Omega \tag{73}$$
$$p_\alpha^\varepsilon \longrightarrow p_\alpha \text{ weakly in } L^2(\Omega) \text{ and almost evreywhere in } \Omega \tag{74}$$
$$c^\varepsilon \longrightarrow c \text{ weakly in } H^1(\Omega) \text{ and almost evreywhere in } \Omega. \tag{75}$$

Then, we pass to the limit as ε goes to 0 in formulations (62)-(64) to get $(p_w, p_o) \in (V)^2$ and $c \in H^1(\Omega)$ solution of

$$\int_\Omega \phi \frac{Z(s_w) - s_w^*}{\delta t}\varphi dx + \int_\Omega \mathbf{K}\lambda_w(s_w, c)\nabla p_w \cdot \nabla\varphi dx - \nu \int_\Omega \nabla(p_o - p_w)\cdot\nabla\varphi dx = 0, \tag{76}$$

$$\int_\Omega \phi \frac{Z(s_o) - s_o^*}{\delta t}\psi dx + \int_\Omega \mathbf{K}\lambda_o(s_o)\nabla p_o \cdot \nabla\psi dx + \nu \int_\Omega \nabla(p_o - p_w)\cdot\nabla\psi dx = 0, \tag{77}$$

and

$$\int_\Omega \phi \frac{Z(s_w)Y(c) - s_w^* c^*}{\delta t}\chi dx + \int_\Omega Y(c)\mathbf{K}\lambda_w(s_w, c)\nabla p_w \cdot \nabla\chi dx - \nu \int_\Omega Y(c)\nabla(p_o - p_w)\cdot\nabla\chi dx + \int_\Omega \mathbf{D}\nabla c \cdot \nabla\chi dx = 0, \tag{78}$$

for all $\varphi, \psi, \chi \in V$.

Let us show the following the maximum principle.

Proposition 3.6. *Assume $s_\alpha^* \geq 0$, $c^* \geq 0$, s_α^* and $s_w^* c^*$ belong to $L^2(\Omega)$. Then, for all $\delta t > 0$, the solutions (p_w, p_o, c) of (76)-(78) satisfy*

$$0 \leq s_\alpha \leq 1 \quad \textit{and} \quad 0 \leq c \leq c_{equ} \quad \textit{a.e. in } \Omega.$$

Proof. Let us show $s_\alpha \geq 0$ and $c \geq 0$ a.e. in Ω. For that consider $\varphi = -s_w^-$, $\psi = -s_o^-$, and $\chi = -(c - \widehat{c}_{\Gamma_w})^-$ respectively in (76)-(78) with the notation $u = u^+ - u^-$, $u^+ = \max(0, u)$ and $u^- = -\min(0, u)$. Note that, according to the extension of the mobility of two phase we have $\lambda_w(s_w, c)s_w^- = \lambda_o(s_o)s_o^- = 0$ and from the definition of the function Z we have also $Z(s_\alpha)s_\alpha^- = Y(c)(c - \widehat{c}_{\Gamma_w})^- = 0$. We obtain

$$\int_\Omega \phi \frac{s_w^*}{\delta t} s_w^- dx - \nu \int_\Omega \bar{p}_c'(s_w) \nabla s_w^- \cdot \nabla s_w^- dx = 0,$$

$$\int_\Omega \phi \frac{s_o^*}{\delta t} s_o^- dx - \nu \int_\Omega \bar{p}_c'(s_w) \nabla s_o^- \cdot \nabla s_o^- dx = 0,$$

$$\int_\Omega \frac{s_w^*(c^* - \widehat{c}_{\Gamma_w})}{\delta t}(c - \widehat{c}_{\Gamma_w})^- dx + \int_\Omega \mathbf{D} \nabla (c - \widehat{c}_{\Gamma_w})^- \cdot \nabla (c - \widehat{c}_{\Gamma_w})^- dx = 0.$$

Since it is possible to choose an extension $\bar{p}_c$ of p_c outside $[0, 1]$ in a way that ensures $\bar{p}_c'(s_w)$ different from zero outside $[0, 1]$, we obtain

$$\int_\Omega |\nabla s_\alpha^-|^2 dx \leq 0 \text{ and } \delta \int_\Omega |\nabla c^-|^2 dx \leq 0,$$

which indicate that $s_\alpha^- = 0$ and $c^- = 0$ a. e. in Ω since s_α^- and c^- vanish on Γ_w.
To prove $c \leq c_{equ}$, we multiply (76) by $-c_{equ}$, and consider $\varphi = \chi = (c - c_{equ})^+$ respectively in (76) and (78), and adding them, we get

$$\int_\Omega \phi \frac{-s_w^*(c^* - c_{equ})}{\delta t}(c - c_{equ})^+ dx \quad + \quad \delta \int_\Omega |\nabla (c - c_{equ})^+|^2 dx \leq 0,$$

which indicates that $(c - c_{equ})^+ = 0$ a. e. in Ω since $(c - c_{equ})^+$ vanishes on Γ_w . □

3.2. Study of the regularized problem (S_ν)

We have shown the existence of a solution $(s_w^{\nu,n+1}, s_o^{\nu,n+1}, p_w^{\nu,n+1}, p_o^{\nu,n+1}, c^{\nu,n+1})$ of (76)-(78) in Section 3.1. So that, the sequence $(s_w^{\nu,n+1}, s_o^{\nu,n+1}, p_w^{\nu,n+1}, p_o^{\nu,n+1}, c^{\nu,n+1})$ defined in (34)-(36) is well defined. Moreover, for given $s_\alpha^{\nu,n} \geq 0$, $s_w^{\nu,n} c^{\nu,n} \geq 0$ and $s_\alpha^{\nu,n} \in L^2(\Omega)$, $s_w^{\nu,n} c^{\nu,n} \in L^2(\Omega)$, $(\alpha = w, o)$, we construct $(s_w^{\nu,n+1}, s_o^{\nu,n+1}, p_w^{\nu,n+1}, p_o^{\nu,n+1}, c^{\nu,n+1})$ so $s_\alpha^{\nu,n+1} \in [0, 1]$ and $c^{\nu,n} \in [0, c_{equ}]$.

We now omit the index ν (for the sake of clarity).

This section is devoted go by the limit as δt goes to 0 to prove a existing solution of the problem (S_ν). We will show some uniform estimates with respect to δt to obtain uniformly bounded on some quantities. Next, these estimations allow us go by the limit as δt goes to zero in the problem (34)-(36).

The next lemma gives some uniform estimates with respect to δt.

Lemma 3.7. *(Uniform estimates with respect to δt)*
The solution of (34)-(36) satisfies

$$\frac{1}{\delta t} \int_\Omega \phi(x)(s_w^{n+1} p_w^{n+1} - s_w^n p_w^n) dx + \frac{1}{\delta t} \int_\Omega \phi(x)(s_o^{n+1} p_o^{n+1} - s_o^n p_o^n) dx + \nu \int_\Omega |\nabla (p_o^{n+1} - p_w^n)|^2 dx$$
$$- \frac{1}{\delta t} \int_\Omega \phi(x)(\mathcal{F}(s_w^{n+1}) - \mathcal{F}(s_w^n)) dx + k_0 \int_\Omega \lambda_w(s_w^{n+1}, c^{n+1}) \nabla p_w^{n+1} \cdot \nabla p_w^{n+1} dx$$
$$+ k_0 \int_\Omega \lambda_o(s_o^{n+1}) \nabla p_o^{n+1} \cdot \nabla p_o^{n+1} dx \leq C, \tag{79}$$

and

$$\frac{1}{\delta t}\int_\Omega \phi(x)(s_w^{n+1}((c-\widehat{c}_{\Gamma_w})^{n+1})^2 - s_w^n((c-\widehat{c}_{\Gamma_w})^n)^2dx + \delta\int_\Omega \nabla((c-\widehat{c}_{\Gamma_w})^{n+1})^2dx$$
$$\leq C(\|\sqrt{\lambda_w(s_w^{n+1},c^{n+1})}\nabla p_w^{n+1})\|_{L^2(\Omega)} + \nu\|\nabla(p_o^{n+1}-p_w^{n+1})\|^2_{L^2(\Omega)}), \tag{80}$$

such that C does not depend on δt. *The function* $\mathcal{F}$ *is defined by*

$$\mathcal{F}(s_w) = \int_0^{s_w} p_c(z)dz. \tag{81}$$

Proof. Let forget the exponent $n+1$ in the proof and let note with the exponent $(*)$ the physical quantities at time t_n.

$$\begin{array}{rcl}(s_w - s_w^*)p_w + (s_o - s_o^*)p_o & = & s_wp_w - s_w^*p_w + s_op_o - s_o^*p_o \\ & \geq & s_wp_w - s_w^*p_w + s_op_o - s_o^*p_o - (s_w - s_w^*)p_c(s_w).\end{array} \tag{82}$$

Using the concavity of $\mathcal{F}$ we have the inequality: $(s_w - s_w^*)p_c(s_w) \leq \mathcal{F}(s_w) - \mathcal{F}(s_w^*)$, and the above inequality (82), we obtain that for all $s_\alpha \geq 0$ and $s_\alpha^* \geq 0$ where $s_w + s_o = s_w^* + s_o^* = 1$ the following inequality

$$(s_w - s_w^*)p_w + (s_o - s_o^*)p_o \geq s_wp_w - s_w^*p_w + s_op_o - s_o^*p_o - \mathcal{F}(s_w) + \mathcal{F}(s_w^*). \tag{83}$$

To obtain the inequality (79), we just have to multiply (34) by p_w and (35) by p_o, sum this two equations and use the inequality (83).
In the same way, to obtain the inequality (80), we just have to multiply (36) by $c - \widehat{c}_{\Gamma_w}$ and use Cauchy-Schwarz inequality.
To multiply (34) by p_w and (35) by p_o, after integration and sum this two equations and use the inequality (83), we obtain

$$\frac{1}{\delta t}\int_\Omega \phi(x)(s_w^{n+1}p_w^{n+1} - s_w^np_w^n)dx + \frac{1}{\delta t}\int_\Omega \phi(x)(s_o^{n+1}p_o^{n+1} - s_o^np_o^n)dx + \nu\int_\Omega |\nabla(p_o^{n+1} - p_w^n)|^2dx$$
$$-\frac{1}{\delta t}\int_\Omega \phi(x)(\mathcal{F}(s_w^{n+1}) - \mathcal{F}(s_w^n))dx + k_0\int_\Omega \lambda_w(s_w^{n+1},c^{n+1})\nabla p_w^{n+1}\cdot\nabla p_w^{n+1}dx$$
$$+k_0\int_\Omega \lambda_o(s_o^{n+1})\nabla p_o^{n+1}\cdot\nabla p_o^{n+1}dx \leq C.$$

For (80), we replace c by $\widehat{c}_{\Gamma_w}$ in (36), we get

$$\begin{array}{rcl}\phi\dfrac{\widehat{c}_{\Gamma_w}^{n+1}s_w^{n+1} - \widehat{c}_{\Gamma_w}^n s_w^n}{\delta t} & - & \mathrm{div}(\widehat{c}_{\Gamma_w}^{n+1}\lambda_w(s_w^{n+1},c^{n+1})\mathbf{K}\nabla p_w^{n+1}) \\ -\mathrm{div}(\mathbf{D}\nabla\widehat{c}_{\Gamma_w}^{n+1}) & + & \nu\,\mathrm{div}(\widehat{c}_{\Gamma_w}^{n+1}\nabla(p_o^{n+1} - p_w^{n+1})) = 0,\end{array} \tag{84}$$

we just have to multiply (36) by $c - \widehat{c}_{\Gamma_w}$ and (84) by $-(c - \widehat{c}_{\Gamma_w})$, and summation and integration the equations, we get

$$\frac{1}{\delta t}\int_\Omega \phi(x)(s_w^{n+1}((c-\widehat{c}_{\Gamma_w})^{n+1})^2 - s_w^n((c-\widehat{c}_{\Gamma_w})^n))^2dx$$
$$+\int_\Omega((c^{n+1}\lambda_w(s_w^{n+1},c^{n+1}) - \widehat{c}_{\Gamma_w}^{n+1}\lambda_w(s_w^{n+1},\widehat{c}_{\Gamma_w}^{n+1}))\mathbf{K}\nabla p_w^{n+1}\nabla(c-\widehat{c}_{\Gamma_w})^{n+1}dx$$
$$+\int_\Omega(\mathbf{D}(\nabla(c-\widehat{c}_{\Gamma_w})^{n+1})^2dx$$
$$-\frac{1}{2}\nu\int_\Omega(\nabla(p_o^{n+1} - p_w^{n+1}))\nabla((c-\widehat{c}_{\Gamma_w})^{n+1})^2dx = 0,$$

hence

$$\frac{1}{\delta t}\int_{\Omega}\phi(x)(s_w^{n+1}((c-\widehat{c}_{\Gamma_w})^{n+1})^2 - s_w^n((c-\widehat{c}_{\Gamma_w})^n))^2dx + \delta\int_{\Omega}(\nabla(c-\widehat{c}_{\Gamma_w})^{n+1})^2dx$$
$$\leq \frac{1}{2}k_\infty\int_{\Omega}(\lambda_w(s_w^{n+1},c^{n+1}))\nabla p_w^{n+1}(\nabla(c-\widehat{c}_{\Gamma_w})^{n+1})^2dx$$
$$+\frac{1}{2}\nu\int_{\Omega}(\nabla(p_o^{n+1}-p_w^{n+1}))\nabla((c-\widehat{c}_{\Gamma_w})^{n+1})^2dx,$$

after using Cauchy-Schwarz inequality,

$$\frac{1}{\delta t}\int_{\Omega}\phi(x)(s_w^{n+1}((c-\widehat{c}_{\Gamma_w})^{n+1})^2 - s_w^n((c-\widehat{c}_{\Gamma_w})^n)^2dx + \delta\int_{\Omega}\nabla((c-\widehat{c}_{\Gamma_w})^{n+1})^2dx$$
$$\leq C(\|\sqrt{\lambda_w(s_w^{n+1},c^{n+1})}\nabla p_w^{n+1})\|_{L^2(\Omega)} + \nu\|\nabla(p_o^{n+1}-p_w^{n+1})\|^2_{L^2(\Omega)}),$$

such that C does not depend on δt. □

The next goal is to get uniform estimates on the solutions reconstructed in time, namely the time piecewise constant function and the corresponding continuous linear function in time. To do that, by introducing some of notations. For a given sequence $(u_n)_{n=0,M}$, we define the time piecewise constant function as

$$u^{\delta t}(0)=u^0 \qquad \text{and} \qquad u^{\delta t}(t)=\sum_{n=0}^{M-1}u^{n+1}1_{]n\delta t,(n+1)\delta t]}(t), \qquad \forall\in]0,T], \tag{85}$$

here $1_{]n\delta t,(n+1)\delta t]}(t)=1$ for $t\in]n\delta t,(n+1)\delta t]$ and zero otherwise. We also define $\tilde{u}^{\delta t}$ by

$$\tilde{u}^{\delta t}(t)=\sum_{n=0}^{M-1}\left[\left(1+n-\frac{t}{\delta t}\right)u^n+\left(\frac{t}{\delta t}-n\right)u^{n+1}\right]1_{]n\delta t,(n+1)\delta t]}(t), \qquad t\in[0,T]. \tag{86}$$

so that, we can compute

$$\partial_t\tilde{u}^{\delta t}(t)=\frac{1}{\delta t}\sum_{n=0}^{M-1}(u^{n+1}-u^n)1_{]n\delta t,(n+1)\delta t[}(t), \qquad \forall t\in[0,T]\setminus\left\{\cup_{n=0}^{M}n\delta t\right\}.$$

For $\alpha=w,o$, we note by $s_\alpha^{\delta t}$ the function defined by (85), we note by $v^{\delta t}$ the function defined by (85) corresponding to $v^n=s_w^nc^n$. Finally, consider $\tilde{s}_\alpha^{\delta t}$, $\tilde{v}^{\delta t}$ the functions defined by (86) corresponding to $s_\alpha^{\delta t}$ and $v^{\delta t}$.

Proposition 3.8. *The sequence*

$$(s_\alpha^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;V), \tag{87}$$
$$(p_\alpha^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;V), \tag{88}$$
$$(c^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;H^1(\Omega)), \tag{89}$$
$$(v^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;H^1(\Omega)), \tag{90}$$
$$(\tilde{v}^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;H^1(\Omega)), \tag{91}$$
$$(\phi\partial_t\tilde{s}_\alpha^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;V'), \tag{92}$$
$$(\phi\partial_t\tilde{v}^{\delta t}) \quad \textit{is uniformly bounded in} \quad L^2(0,T;V'). \tag{93}$$

Proof. We remark that

$$\int_{\Omega_T} \lambda_w(s_w^{\delta t}, c^{\delta t})|\nabla p_w^{\delta t}|^2 dxdt = \delta t \sum_{n=0}^{M-1} \int_\Omega \lambda_w(s_w^{n+1}, c^{n+1})|\nabla p_w^{n+1}|^2 dx,$$

$$\int_{\Omega_T} \lambda_o(s_o^{\delta t})|\nabla p_o^{\delta t}|^2 dxdt = \delta t \sum_{n=0}^{M-1} \int_\Omega \lambda_o(s_o^{n+1})|\nabla p_o^{n+1}|^2 dx,$$

and

$$\int_{\Omega_T} |\nabla p_c(s_w^{\delta t})|^2 dxdt = \delta t \sum_{n=0}^{M-1} \int_\Omega |\nabla p_c(s_w^{n+1})|^2 dx.$$

We multiply (79) by δt and summing it from $n = 0$ to $M - 1$ to get the following estimation

$$\begin{aligned} \frac{1}{\delta t}\int_\Omega \phi(x)(s_w^{\delta t}(T) - s_w^{\delta t}(0))dx &+ \frac{1}{\delta t}\int_\Omega \phi(x)(s_o^{\delta t}(T) - s_o^{\delta t}(0))dx \\ &- \frac{1}{\delta t}\int_\Omega \phi(x)(\mathcal{F}(s_w^{\delta t}(T)) - \mathcal{F}(s_w^{\delta t}(0)))dx + \nu\int_{\Omega_T} |\nabla p_c(s_w^{\delta t})|^2 dx \\ &+ k_0\int_\Omega \lambda_w(s_w^{\delta t}, c^{\delta t})|\nabla p_w^{\delta t}|^2 dxdt + k_0\int_\Omega \lambda_o(s_o^{\delta t})|\nabla p_o^{\delta t}|^2 dxdt \le C_1. \end{aligned}$$

If we use the fact that $p_\alpha^{\delta t}(0) \in L^2(\Omega)$, $\quad 0 \le s_\alpha^{\delta t} \le 1$ for $\alpha = w, o$, we deduce that

$$k_0\Big(\int_{\Omega_T} \lambda_w(s_w^{\delta t}, c^{\delta t})|\nabla p_w^{\delta t}|^2 dxdt + \int_{\Omega_T} \lambda_o(s_o^{\delta t})|\nabla p_o^{\delta t}|^2 dxdt\Big) + \nu\Big(\int_{\Omega_T} |\nabla p_c(s_w^{\delta t})|^2 dxdt\Big) \le C_2,$$

then, one gets by the help of (12), the relationship between global pressure, capillary pressure and pressures, that

$$\int_{\Omega_T} \lambda(s_w^{\delta t}, c^{\delta t})|\nabla p^{\delta t}|^2 dxdt + \nu\int_{\Omega_T} |\nabla p_c(s_w^{\delta t})|^2 dxdt \le C_3, \tag{94}$$

here C_1, C_2, C_3 are constant independent of δt.
The assumption (A4) on the capillary function p_c with the second term of (94) achieves the estimate (87). Since we have the relationship (8) between the pressure of two phase, the capillary pressure and the global pressure, then the estimate (88) becomes a consequence of (94). The estimate (89) is a consequence of (80) and (87)-(88).
To get the uniform estimate (87), we compute the gradient of δt.

$$\nabla s_\alpha^{\delta t} = \sum_{n=0}^{M-1} (\nabla s_\alpha^{n+1}) 1_{]n\delta t,(n+1)\delta t]}(t).$$

In the same method, we get the uniform estimate for (90) and (91). From equations (34), we have for all $\varphi \in L^2(0, T; V)$,

$$\langle \phi\partial_t \tilde{s}_w^{\delta t}, \varphi\rangle = -\int_{\Omega_T} \mathbf{K}\lambda_w(s_w^{\delta t}, c^{\delta t})\nabla p_w^{\delta t} \cdot \nabla\varphi dxdt - \nu\int_{\Omega_T} \nabla(p_o^{\delta t} - p_w^{\delta t}) \cdot \nabla\varphi dxdt.$$

The above estimates (87)-(88) with (94) ensure that $(\phi\partial_t \tilde{s}_\alpha^{\delta t})_h$ is uniformly bounded in $L^2(0, T; V)$. In the same way, we get (92) for $\alpha = o, w$ and (93). □

The next step is devoted to pass to the limit as δt goes to 0 in order to study the problem (S_ν). This is the subject of the next proposition.

Proposition 3.9. *(Convergence as δt goes to 0)*
We have the following convergences as δt goes to 0,

$$\|s_\alpha^{\delta t} - \tilde{s}_\alpha^{\delta t}\|_{L^2(\Omega_T)} \longrightarrow 0, \quad \text{with} \quad \alpha = o, w \tag{95}$$

$$\|v^{\delta t} - \tilde{v}^{\delta t}\|_{L^2(\Omega_T)} \longrightarrow 0, \tag{96}$$

$$s_\alpha^{\delta t} \longrightarrow s_\alpha \quad \text{weakly in} \quad L^2(0,T;V), \tag{97}$$

$$p_\alpha^{\delta t} \longrightarrow p_\alpha \quad \text{weakly in} \quad L^2(0,T;V), \tag{98}$$

$$c^{\delta t} \longrightarrow c \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)), \tag{99}$$

$$v^{\delta t} \longrightarrow v \quad \text{strongly in} \quad L^2(\Omega_T). \tag{100}$$

Furthermore

$$s_\alpha^{\delta t} \longrightarrow s_\alpha \quad a.\ e.\ in \quad \Omega_T \tag{101}$$

$$0 \le s_\alpha \le 1 \quad a.\ e.\ in \quad \Omega_T \tag{102}$$

$$p_\alpha^{\delta t} \longrightarrow p_\alpha \quad a.\ e.\ in \quad \Omega_T \tag{103}$$

$$c^{\delta t} \longrightarrow c \quad a.\ e.\ in \quad \Omega_T \tag{104}$$

$$0 \le c \le c_{equ} \quad a.\ e.\ in \quad \Omega_T \tag{105}$$

and

$$v = s_w c \quad a.\ e.\ in \quad \Omega_T. \tag{106}$$

Finally, we have

$$\phi \partial_t \tilde{s}_\alpha \longrightarrow \phi \partial_t s_\alpha \quad \text{weakly in} \quad L^2(0,T;V'), \tag{107}$$

$$\phi \partial_t \tilde{v}_\alpha \longrightarrow \phi \partial_t (s_w c) \quad \text{weakly in} \quad L^2(0,T;V'). \tag{108}$$

Proof. Note that

$$\begin{aligned}
\|s_\alpha^{\delta t} - \tilde{s}_\alpha^{\delta t}\|^2_{L^2(\Omega_T)} &= \sum_{n=0}^{M-1} \int_{n\delta t}^{(n+1)\delta t} \|(1 + n - \frac{t}{\delta t})(s_\alpha^{n+1} - s_\alpha^n)\|^2_{L^2(\Omega)} dt \\
&= \frac{\delta t}{2} \sum_{n=1}^{M-1} \|s_\alpha^{n+1} - s_\alpha^n\|^2_{L^2(\Omega)}.
\end{aligned}$$

Now, we multiply (34) by $s_w^{n+1} - s_w^n$ and sum it from $n = 0$ to $M - 1$ to obtain the following estimation

$$\frac{\phi_\star}{\delta t} \sum_{n=0}^{M-1} \|s_w^{n+1} - s_w^n\|^2_{L^2(\Omega)} \le \sum_{n=0}^{M-1} \Big[\|\nabla s_w^n\|^2_{L^2(\Omega)} + \|\nabla s_w^{n+1}\|^2_{L^2(\Omega)} + \|\nabla p_w^{n+1}\|^2_{L^2(\Omega)}\Big].$$

This yields

$$\sum_{n=0}^{M-1} \|s_w^{n+1} - s_w^n\|^2_{L^2(\Omega_T)} \le C\Big(1 + \|\nabla s_w^{\delta t}\|^2_{L^2(\Omega_T)} + \|\nabla s_w^{\delta t}\|^2_{L^2(\Omega_T)} + \|\nabla p_w^{\delta t}\|^2_{L^2(\Omega_T)}\Big).$$

And from (87) and (88), we conclude that

$$\|s_w^{\delta t} - \tilde{s}_w^{\delta t}\|^2_{L^2(\Omega_T)} \longrightarrow 0.$$

We multiply scalary (35) with $s_o^{n+1} - s_o^n$ and this gives us (95). For (96), we multiply (36) with $(v^{n+1} - v^n)$, and from (88), (89) and (90), we deduce (96). Next, from (87), (88) and (89), the sequences $(s_o^{\delta t})$, $(p_\alpha^{\delta t})$ and $(c^{\delta t})$ are uniformly bounded in $L^2(0,T;V)$, then, we have up to a subsequence the convergence result (97), (98) and (99).

The sequence $(\tilde{s}_\alpha^{\delta t})$ and $(\tilde{v}^{\delta t})$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$. Due to (92) and (93) we have the strong convergence

$$\tilde{s}_\alpha^{\delta t} \longrightarrow s_\alpha \quad \text{strongly in} \quad L^2(\Omega_T), \tag{109}$$
$$\tilde{v}^{\delta t} \longrightarrow v \quad \text{strongly in} \quad L^2(\Omega_T). \tag{110}$$

This compactness result is classical and can be found in [10] when the porosity is constant, and under the assumption $(A1)$ (the porosity belongs to $W^{1,\infty}(\Omega)$) but the proof can be adapted with minor modifications. The convergences (109) and (110) with (95) and (96) ensure the following strong convergences

$$s_w^{\delta t} \longrightarrow s_w \quad \text{strongly in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T \tag{111}$$
$$s_o^{\delta t} \longrightarrow s_o \quad \text{strongly in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T \tag{112}$$
$$s_w^{\delta t} c^{\delta t} \longrightarrow v \quad \text{strongly in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T \tag{113}$$

and this achieves (100). We are looking for the almost everywhere convergence on pressures $p_\alpha^{\delta t}$, saturations $s_\alpha^{\delta t}$ and concentration $c^{\delta t}$. For this, let define a map $\mathcal{G} : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R} \times [0,1]$, $\mathcal{G}(\xi,\zeta) = (p_w^{\delta t}, s_w^{\delta t})$ where ξ and ζ are solutions of the problem

$$\begin{cases} \xi(p_w^{\delta t}, s_w^{\delta t}) = p_w^{\delta t} s_w^{\delta t}, \\ \varrho(p_w^{\delta t}, s_w^{\delta t}) = (p_w^{\delta t} + p_c(s_w^{\delta t}))(1 - s_w^{\delta t}). \end{cases} \tag{114}$$

Note that $\mathcal{G}$ is well defined by computing the Jacobian of $\mathcal{G}$

$$\begin{vmatrix} \dfrac{\partial \xi}{\partial p_w^{\delta t}} & \dfrac{\partial \xi}{\partial s_w^{\delta t}} \\ \dfrac{\partial \varrho}{\partial p_w^{\delta t}} & \dfrac{\partial \varrho}{\partial s_w^{\delta t}} \end{vmatrix} = (1 - s_w^{\delta t}) s_w^{\delta t} p_c'(s_w^{\delta t}) - s_w^{\delta t} p_c(s_w^{\delta t}) - p_w^{\delta t} < 0.$$

As we have the almost everywhere convergences (111)-(112) and the map $\mathcal{G}$ defined in (114) is continuous, we deduce that

$$p_w^{\delta t} \longrightarrow p_w \quad \text{a. e. in} \quad \Omega_T,$$
$$s_w^{\delta t} \longrightarrow s_w \quad \text{a. e. in} \quad \Omega_T.$$

The identification of the limit is due to (87), (88). The continuity of the capillary pressure function ensures that

$$p_o^{\delta t} \longrightarrow p_o \quad \text{a. e. in} \quad \Omega_T,$$

and the saturation equation ensures also

$$s_o^{\delta t} \longrightarrow s_o \quad \text{a. e. in} \quad \Omega_T,$$

and this achieves (101)-(103), also, we have

$$p_w^{\delta t} \longrightarrow p_w \quad \text{a. e. in} \quad \Omega_T,$$

we deduce

$$p_w^{\delta t} \longrightarrow p_w \quad \text{strongly in} \quad L^1(\Omega_T),$$

and then we have the following convergence

$$c^{\delta t} \longrightarrow c \quad \text{strongly in} \quad L^1(\Omega_T).$$

This achieves (104). Finally the weak convergence (107)-(108) are a consequence of (92) and (93) and the identification of the limit is a due to (106). □

Let us consider the weak formulations obtained from the problem (34)-(36) written after summation from $n = 0$ to $M - 1$ on which we have to go by the limit as δt goes to 0,

$$\langle \phi \partial_t \tilde{s}_w^{\delta t}, \varphi \rangle + \int_{\Omega_T} \mathbf{K} \lambda_w(s_w^{\delta t}, c^{\delta t}) \nabla p_w^{\delta t} \cdot \nabla \varphi dx dt - \nu \int_{\Omega_T} \nabla (p_o^{\delta t} - p_w^{\delta t}) \cdot \nabla \varphi dx dt = 0, \quad (115)$$

$$\langle \phi \partial_t \tilde{s}_o^{\delta t}, \psi \rangle + \int_{\Omega_T} \mathbf{K} \lambda_o(s_o^{\delta t}) \nabla p_o^{\delta t} \cdot \nabla \psi dx dt - \nu \int_{\Omega_T} \nabla (p_w^{\delta t} - p_o^{\delta t}) \cdot \nabla \psi dx dt = 0, \quad (116)$$

and

$$\begin{aligned} \langle \phi \partial_t (\tilde{v}^{\delta t}), \chi \rangle \quad &+ \quad \int_{\Omega_T} \mathbf{K} c^{\delta t} \lambda_w(s_w^{\delta t}, c^{\delta t}) \nabla p_w^{\delta t} \cdot \nabla \chi dx dt \\ &+ \quad \int_{\Omega_T} \mathbf{D} \nabla c^{\delta t} \cdot \nabla \chi dx dt - \nu \int_{\Omega_T} c^{\delta t} \nabla (p_o^{\delta t} - p_w^{\delta t}) \cdot \nabla \chi dx dt = 0, \end{aligned} \quad (117)$$

where $\varphi, \psi, \chi \in L^2(0, T; V)$.
Then, the convergences obtained in Proposition 3.9 let us to access to the limit on each term of (115), (117).
Then, we have established the weak formulation (31)-(33) of Theorem 3.1.
Furthermore, we have obtained by Proposition 3.9 the following properties

$$\begin{array}{l} 0 \le s_\alpha(t, x) \le 1 \text{ a.e. in } \Omega_T, \qquad s_\alpha \in L^2(0, T; V), \\ p_\alpha \in L^2(0, T; V), \qquad \phi \partial_t s_\alpha \in L^2(0, T; V'), \quad \alpha = 0, w \\ 0 \le c \le c_{equ}, \qquad c \in L^2(0, T; V), \qquad \phi \partial_t (c s_w) \in L^2(0, T; V'). \end{array}$$

The compactness property on $s_\alpha^{\delta t}$ implies that $s_\alpha \in C^0(0, T; L^2(\Omega))$.
In the same way also the compactness property on $c^{\delta t} s_w^{\delta t}$ implies that $c s_w \in C^0([0, T]; L^2(\Omega))$ and Theorem 3.1 is proved.

4. Existence of solutions of the degenerate problem

In the Section 3.2, we have shown a existing solution $(s_w^\nu, s_o^\nu, p_w^\nu, p_o^\nu, c^\nu)$ of the problem (S_ν) given in Section 3. The goal of the section is to pass to the limit as ν goes to the 0 to demonstrate the main result of this work: the existence of solution for the problem (1)-(3) with some estimations in the sense of Theorem 2.3.

The first point to do this is to obtain regular estimates respect to ν to deduce strong convergences as ν goes to 0. Next we will be able to pass to the limit as ν goes to 0.

4.1. Uniform estimates with respect to ν

We state the following two lemmas in order to establish uniform estimates with respect to ν.

Lemma 4.1. *The sequence $(s_\alpha^\nu)_\nu$, $(c^\nu)_\nu$ and $(p^\nu = p_w^\nu + \bar{p}(s_w^\nu, c^\nu))_\nu$ and defined by Theorem 3.1 satisfy*

$$\begin{array}{rll} 0 \le s_\alpha^\nu \le 1 & \textit{a. e. in} & \Omega_T, \quad (118) \\ 0 \le c^\nu \le c_{equ} & \textit{a. e. in} & \Omega_T, \quad (119) \\ (p^\nu)_\nu & \textit{is uniformly bounded in} & L^2(0, T; V), \quad (120) \\ (\sqrt{\nu} \nabla p_c(s_w^\nu))_\nu & \textit{is uniformly bounded in} & L^2(\Omega_T), \quad (121) \\ (\sqrt{\lambda_\alpha(s_\alpha^\nu, c^\nu)} \nabla p_\alpha^\nu)_\nu & \textit{is uniformly bounded in} & L^2(\Omega_T), \quad (122) \\ (c^\nu)_\nu & \textit{is uniformly bounded in} & L^2(0, T; H^1(\Omega)), \quad (123) \\ (\mathfrak{B}(s_w^\nu, c^\nu))_\nu & \textit{is uniformly bounded in} & L^2(0, T; V), \quad (124) \\ (\phi \partial_t (s_\alpha^\nu))_\nu & \textit{is uniformly bounded in} & L^2(0, T; V'), \quad (125) \\ (\phi \partial_t (c^\nu s_w^\nu))_\nu & \textit{is uniformly bounded in} & L^2(0, T; V'). \quad (126) \end{array}$$

Proof. It is easy to verify that the maximum principle (118)-(119) is conserved through the limit process. For the next estimates, we have to multiply (26) by p_w^ν and (27) by p_o^ν and add these two equations. Using the assumptions (A1)-(A8), the Cauchy-Schwarz inequality and (12), we deduce the following estimation.

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi\sum_\alpha (s_\alpha^\nu p_\alpha^\nu)dx &+ \int_\Omega \mathbf{K}\sum_\alpha \lambda_\alpha(s_\alpha^\nu,c^\nu)|\nabla p_\alpha^\nu|^2dx + \nu\int_\Omega |\nabla p_c(s_w^\nu)|^2dx \\ &\leq C(\|\nabla p^\nu\|_{L^2(\Omega)} + \|\tilde{p}(s_w^\nu)\|_{L^2(\Omega)} + \|\bar{p}(s_w^\nu)\|_{L^2(\Omega)}).\end{aligned}$$

We use again (12) and the assumptions (A1)-(A8), Cauchy-Schwarz and Young inequalities, the fact that the function p_α is nonnegative to obtain after integration over $(0,T)$ the following estimation

$$\begin{aligned}&\int_{\Omega_T}\lambda(s_w^\nu,c^\nu)|\nabla p^\nu|^2dxdt + \int_{\Omega_T}\frac{\lambda_w(s_w^\nu,c^\nu)\lambda_o(s_o^\nu)}{\lambda(s_w^\nu,c^\nu)}|\nabla p_c(s_w^\nu)|^2dxdt \\ &\quad +\nu\int_{\Omega_T}|\nabla p_c(s_w^\nu)|^2dxdt + \frac{1}{2}k_0\sum_\alpha\Big[\int_{\Omega_T}\lambda_\alpha(s_\alpha^\nu,c^\nu)|\nabla p_\alpha^\nu|^2dxdt\Big] \leq C'.\end{aligned} \tag{127}$$

The previous estimation (127) and the assumption (A3) ensure the estimate (120). The estimations (121)-(122) come directly from (127).

For the estimate (123), we multiply (28) by c^ν and (26) by $-\frac{1}{2}(c^\nu)^2$, we add them and this gives us the equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi|s_w^\nu(c^\nu)^2|dx + \int_\Omega \mathbf{D}\nabla c^\nu\cdot\nabla c^\nu dx = 0. \tag{128}$$

After integration in time of this equation, assumption (A5) and the fact that c^ν is bounded, we get estimation (123).

The estimation (124) is a consequence of the assumption (A3) and the following estimation

$$\begin{aligned}\int_{\Omega_T}|\nabla\mathfrak{B}(s_w^\nu,c^\nu)|^2dxdt &= \int_{\Omega_T}\frac{\lambda_w^2(s_w^\nu,c^\nu)\lambda_o^2(s_o^\nu)}{\lambda^2(s_w^\nu,c^\nu)}|\nabla p_c(s_w^\nu)|^2dxdt \\ &\leq C,\end{aligned}$$

here C is a constant independent of ν. For all $\varphi,\psi,\chi\in L^2(0,T;V)$, we have

$$\langle\phi\partial_t s_w^\nu,\varphi\rangle + \int_{\Omega_T}\mathbf{K}\lambda_w(s_w^\nu,c^\nu)\nabla p_w^\nu\cdot\nabla\varphi dxdt - \int_{\Omega_T}\nabla(p_o^\nu - p_w^\nu)\cdot\nabla\varphi dxdt = 0, \tag{129}$$

$$\langle\phi\partial_t s_o^\nu,\psi\rangle + \int_{\Omega_T}\mathbf{K}\lambda_o(s_o^\nu)\nabla p_o^\nu\cdot\nabla\psi dxdt - \int_{\Omega_T}\nabla(p_w^\nu - p_o^\nu)\cdot\nabla\psi dxdt = 0, \tag{130}$$

and

$$\begin{aligned}\langle\phi\partial_t(c^\nu s_w^\nu),\chi\rangle &+ \int_{\Omega_T}c^\nu\mathbf{K}\lambda_w(s_w^\nu,c^\nu)\nabla p_w^\nu\cdot\nabla\chi dxdt \\ &- \int_{\Omega_T}\mathbf{D}\nabla c^\nu\cdot\nabla\chi dxdt - \int_{\Omega_T}c^\nu\nabla(p_o^\nu - p_w^\nu)\cdot\nabla\chi dxdt = 0,\end{aligned} \tag{131}$$

here the bracket $\langle\cdot,\cdot\rangle$ represents the duality product between $L^2(0,T;V')$ and $L^2(0,T;V)$. After using (8), one gets

$$\begin{aligned}|\langle\phi\partial_t s_\alpha^\nu,\varphi\rangle| &\leq \nu\Big|\int_{\Omega_T}\nabla p_c(s_w^\nu)\cdot\nabla\varphi dxdt\Big| \\ &+ \Big|\int_{\Omega_T}\mathbf{K}(\lambda_\alpha(s_\alpha^\nu,c^\nu)\nabla p^\nu + \nabla\mathfrak{B}(s_w^\nu,c^\nu))\cdot\nabla\varphi dxdt\Big|,\end{aligned} \tag{132}$$

and

$$\begin{aligned} |\langle \phi\partial_t(c^\nu s_w^\nu), \chi\rangle| &\leq \left|\int_{\Omega_T} \mathbf{D}\nabla c^\nu \cdot \nabla\chi dxdt\right| \\ &+ \left|\int_{\Omega_T} c^\nu \mathbf{K}(\lambda_w(s_w^\nu, c^\nu)\nabla p_w^\nu)\cdot\nabla\chi dxdt\right| \\ &+ \nu\left|\int_{\Omega_T} c^\nu\nabla p_c(s_w^\nu)\cdot\nabla\chi dxdt\right|. \end{aligned} \tag{133}$$

From estimations (119)-(124), we deduce

$$|\langle \phi\partial_t s_\alpha^\nu, \varphi\rangle| \leq C\|\varphi\|_{L^2(0,T;V)}, \text{ for } \alpha = w, o$$

and

$$|\langle \phi\partial_t(c^\nu s_w^\nu), \chi\rangle| \leq C\|\chi\|_{L^2(0,T;V)}.$$

This establishes (125)-(126) and demonstrates the lemma. □

Lemma 4.2. *(Compactness result for degenerate case)*
For every $M > 0$, the following implicit set

$$\begin{aligned} S_M = \{ & (s_w, s_o, c) \in L^2(\Omega_T)\times L^2(\Omega_T)\times L^2(\Omega_T), \quad \text{such that} \\ & \|\mathfrak{B}(s_w,c)\|_{L^2(0,T;H^1(\Omega))} \leq M, \quad \|\sqrt{\lambda_w(s_w,c)}\nabla p_w\|_{L^2(\Omega_T)} + \|\sqrt{\lambda_o(s_o)}\nabla p_o\|_{L^2(\Omega_T)} \leq M, \\ & \|\phi\partial_t s_w\|_{L^2(0,T;V')} \leq M, \quad \|\phi\partial_t s_o\|_{L^2(0,T;V')} \leq M, \\ & \|\phi\partial_t(cs_w)\|_{L^2(0,T;V')} \leq M\} \end{aligned}$$

is relatively compact in $L^2(\Omega_T)\times L^2(\Omega_T)\times L^2(\Omega_T)$ and $\gamma(S_M)$ is relatively compact in $L^2(\Sigma_T)\times L^2(\Sigma_T)\times L^2(\Sigma_T)$, ($\gamma$ denotes the trace on Σ_T operator).

Proof. The demonstration is inspired by the compactness lemma in the reference [18] which is introduced for compressible degenerate model, we use the compactness result of Jacques Simon, see [27]. □

We deduce the following convergences.

Lemma 4.3. *(Weak and strong convergences) Up to a subsequence the sequence $(s_\alpha^\nu)_\nu$, $(p^\nu)_\nu$, and $(p_\alpha^\nu)_\nu$ verify the following convergence*

$$\begin{aligned} p^\nu &\longrightarrow p && \text{weakly in} && L^2(0,T;V) && (134)\\ \mathfrak{B}(s_w^\nu, c^\nu) &\longrightarrow \mathfrak{B}(s_w, c) && \text{weakly in} && L^2(0,T;V) && (135)\\ c^\nu &\longrightarrow c && \text{weakly in} && L^2(0,T;V) && (136)\\ p^\nu &\longrightarrow p && \text{a. e. in} && \Omega_T && (137)\\ s_\alpha^\nu &\longrightarrow s_\alpha && \text{a. e. in} && \Omega_T \text{ with } \alpha = w,o && (138)\\ 0 \leq s_\alpha(t,x) &\leq 1 && \text{a. e. in} && \Omega_T && (139)\\ 0 \leq c(t,x) &\leq c_{equ} && \text{a. e. in} && \Omega_T && (140)\\ p_\alpha^\nu &\longrightarrow p_\alpha && \text{a. e. in} && \Omega_T \text{ with } \alpha = w,o && (141)\\ c^\nu &\longrightarrow c && \text{a. e. in} && \Omega_T && (142)\\ \phi\partial_t(s_\alpha^\nu) &\longrightarrow \phi\partial_t(s_\alpha) && \text{weakly in} && L^2(0,T;V') \text{ with } \alpha = w,o && (143)\\ \phi\partial_t(c^\nu s_w^\nu) &\longrightarrow \phi\partial_t(cs_w) && \text{weakly in} && L^2(0,T;V') && (144) \end{aligned}$$

Proof. Firstly, the weak convergences (134)-(136) follows from the uniform estimates (120), (123) and (124) of Lemma 4.1.
Secondly, the Lemma 4.2 ensures the following strong convergences

$$s_w^\nu \longrightarrow s_w \quad \text{in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T,$$
$$s_w^\nu \longrightarrow s_w \quad \text{in } L^2(\Sigma_T) \quad \text{and a.e. in } \Sigma_T.$$

As the map $\mathcal{G}$ defined in (114) is continuous, we can deduce

$$p_w^\nu \longrightarrow p_w \quad \text{in } \Omega_T \quad \text{and a.e. in } \Sigma_T,$$

and consequently the following convergences hold

$$p^\nu, p_o^\nu \longrightarrow p, p_o \quad \text{in } \Omega_T \quad \text{and a.e. in } \Sigma_T.$$

From (118), the estimate (139) is holds.
By the help of (136), we get

$$c^\nu \longrightarrow c \quad \text{strongly in} \quad L^1(0,T;L^1(\Omega)),$$

this achieve the convergence (142), the limits have been identified by (136), and then the maximum principle (140) has been establish. The following convergences hold

$$\mathfrak{B}(s_w^\nu, c^\nu) \longrightarrow \mathfrak{B}(s_w, c) \quad \text{in } \Omega_T \quad \text{and a.e. in } \Sigma_T.$$

At last, the weak convergence (143) and (144) is a consequence of the estimate (125) and (126), and the identification of the limit follows from the previous convergence. □

4.2. *Proof of the main result*

In order to achieve the demonstration of Theorem 2.3, it remains to pass to the limit as ν goes to 0 in the formulations (31)-(33), for all smooth test functions φ, ψ and χ belongs in $C^1([0,T];V) \cap L^2(0,T;H^2(\Omega))$ such that $\varphi(T,\cdot) = \psi(T,\cdot) = \chi(T,\cdot) = 0$

$$\begin{aligned} -\int_{\Omega_T} \phi s_w^\nu \partial_t \varphi dxdt \quad &+ \quad \int_{\Omega_T} \mathbf{K}\lambda_w(s_w^\nu, c^\nu)\nabla p_w^\nu \cdot \nabla\varphi dxdt \\ &- \quad \nu \int_{\Omega_T} \nabla(p_o^\nu - p_w^\nu)\cdot\nabla\varphi dxdt = \int_\Omega \phi s_w^0 \varphi(0,x)dx, \end{aligned} \tag{145}$$

$$\begin{aligned} -\int_{\Omega_T} \phi s_o^\nu \partial_t \psi dxdt \quad &+ \quad \int_{\Omega_T} \mathbf{K}\lambda_o(s_o^\nu)\nabla p_w^\nu \cdot \nabla\psi dxdt \\ &- \quad \nu \int_{\Omega_T} \nabla(p_w^\nu - p_o^\nu)\cdot\nabla\psi dxdt = \int_\Omega \phi s_w^0 \psi(0,x)dx, \end{aligned} \tag{146}$$

and

$$\begin{aligned} -\int_{\Omega_T} \phi c^\nu s_w^\nu \partial_t \chi dxdt \quad &+ \quad \int_{\Omega_T} c^\nu \mathbf{K}\lambda_w(s_w^\nu, c^\nu)\nabla p_w^\nu \cdot \nabla\chi dxdt + \int_{\Omega_T} \mathbf{D}\nabla c^\nu \cdot \nabla\chi dxdt \\ &- \quad \nu \int_{\Omega_T} c^\nu \nabla(p_o^\nu - p_w^\nu)\cdot\nabla\chi dxdt = \int_\Omega \phi c^0 s_w^0 \chi(0,x)dx. \end{aligned} \tag{147}$$

The first terms converges due to the strong convergence of s_α^ν to s_α in $L^2(\Omega_T)$ and the strong convergence of $s_w^\nu c^\nu$ to $s_w c$ in $L^2(\Omega_T)$.
Now, the two seconds terms of (145) and (146) can be written as,

$$\int_{\Omega_T} \mathbf{K}\lambda_o(s_o^\nu)\nabla p_o^\nu \cdot \nabla\psi dxdt = \int_{\Omega_T} \mathbf{K}\lambda_o(s_o^\nu)\nabla p^\nu \cdot \nabla\psi dxdt - \int_{\Omega_T} \mathbf{K}\nabla\mathfrak{B}(s_w^\nu, c^\nu)\cdot\nabla\psi dxdt, \tag{148}$$

and

$$\int_{\Omega_T} \mathbf{K}\lambda_w(s_w^\nu, c^\nu)\nabla p_w^\nu \cdot \nabla\varphi dxdt = \int_{\Omega_T} \mathbf{K}\lambda_w(s_w^\nu, c^\nu)\nabla p^\nu \cdot \nabla\varphi dxdt + \int_{\Omega_T} \mathbf{K}\nabla\mathfrak{B}(s_w^\nu, c^\nu) \cdot \nabla\varphi dxdt. \tag{149}$$

The two terms on the right hand side of (147) converge arguing in 2 steps. Firstly, the convergences (138),(141) and (142) and the Lebesgue dominated convergence theorem, establish

$$\begin{array}{lll} \lambda_o(s_o^\nu)\nabla\psi \longrightarrow \lambda_o(s_o)\nabla\psi & \text{strongly in} & (L^2(\Omega_T))^d, \\ \lambda_w(s_w^\nu, c^\nu)\nabla\varphi \longrightarrow \lambda_w(s_w, c)\nabla\varphi & \text{strongly in} & (L^2(\Omega_T))^d. \end{array}$$

Secondly, the convergence (135) combined to the above strong convergence validate the convergence for the second term of the right hand side of (148) and (149), and the weak convergence on global pressure (134) combined to the above strong convergence validate the convergence for the first term of the right hand side of (148) and (149), and this achieves the passage to the limit on the second terms of (145) and (146).

For the third term of (147), we can deduce by the Lebesgue dominated convergence theorem and the convergences (138) that

$$\mathbf{D}(s_w^\nu)\nabla\chi \longrightarrow \mathbf{D}(s_w)\nabla\chi \quad \text{strongly in} \quad (L^2(\Omega_T))^d,$$

the convergence on concentration (136) combined to the above strong convergence validate the convergence for the second term of (147).
After that, the fourth terms of (145) and (146) can be written as,

$$\nu \int_{\Omega_T} \nabla(p_o^\nu - p_w^\nu) \cdot \nabla\varphi \, dxdt = \sqrt{\nu} \int_{\Omega_T} (\sqrt{\nu}\nabla p_c(s_w^\nu))\nabla\varphi \, dxdt, \tag{150}$$

the uniform estimate (121) and the inequality of Cauchy-Schwarz ensures the convergence of this term to 0.

The other terms converge classically by to use the theorem of Lebesgue concerning dominated convergence and the convergences (138), (141) and (142).
The formulations (21)-(23) are then established, and the Theorem 2.3 is then established.

5. Conclusion

The nonlinear mathematical model under study has applications in the enhanced oil recovery EOR by polymer flooding.

The goal of this manuscript is to proof the existence of weak solutions for a model of incompressible and two-phase (aqueous and oil) immiscible flow with dynamic capillary pressure for three components (polymer, water, and oil) in porous medium. We obtained the mathematical model by using the mass conservation equation for the two phases, and the mass conservation equation for the polymer component in the wetting phase (water).

In the future vision, we will generalized our results in case non-isothermal, also, we can study the same problem in the presence of other components and providing numerical simulations.

References

[1] B. Amaziane, M. Jurak, L. Pankratov, A. Piatnitski, *Existence of weak solutions for nonisothermal immiscible compressible two-phase flow in porous media,* Nonlinear Analysis: Real World Applications 85 (2025), 104364.

[2] B. Amaziane, M. El Ossmani, Y. Zahraoui, *Convergence of a TPFA finite volume scheme for nonisothermal immiscible compressible two-phase flow in porous media,* Comput. Math. Appl. 165 (2024) 118–149, http://dx.doi.org/10.1016/j.camwa.2024.04.010.

[3] E. Ahusborde, B. Amaziane, F. Croccolo, N. Pillardou, *Numerical simulation of a thermal-hydraulic-chemical multiphase flow model for CO2 sequestration in saline aquifers,* Math. Geosci. 56 (3) (2024) 541–572, http://dx.doi.org/10.1007/s11004-023-10093-7.

[4] M. Beneš, *Weak solutions of coupled variable-density flows and heat transfer in porous media,* Nonlinear Anal. 221 (2022) 112973, http://dx.doi.org/10.1016/j.na.2022.112973.

[5] M. Beneš, *Analysis of non-isothermal multiphase flows in porous media,* Math. Meth. Appl. Sci. 45 (16) (2022) 9653-9677, http://dx.doi.org/10.1002/mma.8328.
[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science+Business Media, LLC 2011.
[7] F. Caro, B. Saad, and M. Saad, *Two-component two-compressible flow in a porous medium*, Acta Applicandae Mathematicae, (2011).
[8] A. R. Castro, A. B. Abdelwahed, & H. Bertin, *Enhancing pollutant removal from contaminated soils using yield stress fluids as selective blocking agents*. Journal of Contaminant Hydrology, (2023), 255, 104142.
[9] G. Chavent, *A new formulation of diphasic incompressible flows in porous media*, in Applications of Methods of Functional Analysis to Problems in Mechanics, Lecture Notes in Mathematics, Volume 503, Edited by Paul Germain & Bernard Nayroles, 1976.
[10] G. Chavent and J. Jaffré, *Mathematical Models and Finite Elements for Reservoir Simulation: Single Phase, Multiphase and Multicomponent Flows Through Porous Media*, Studies in Mathematics and its Applications, Vol. 17, North-Holland, Amsterdam (1986).
[11] G. Chavent and J. Jaffré, *Mathematical models and finite elements for reservoir simulation*, North Holland, 1986.
[12] Z. Chen, G. Hwan, Y. Ma, *Computational Methods for Multiphase Flows in Porous Media*, Siam, Philadelphia, (2006).
[13] P. Daripa and S. Dutta *On the convergence analysis of a hybrid numerical method for multicomponent transport in porous media*. Applied Numerical Mathematics, (2019), doi:10.1016/j.apnum.2019.07.009
[14] P. Daripa and S. Dutta *Modeling and simulation of surfactant-polymer flooding using a new hybrid method*. J. Comput. Phys., 335:249–282, (2017).
[15] P. Daripa, J. Glimm, B. Lindquist, M. Maesumi and O. McBryan *On the simulation of heterogeneous petroleum reservoirs*. In Numerical Simulation in Oil Recovery, IMA Vol. Math. Appl. 11, pages 89–103, New York, NY, (1988). Springer.
[16] P. Daripa, J. Glimm, B. Lindquist and O. McBryan, *Polymer floods: A case study of nonlinear wave analysis and of instability control in tertiary oil recovery*. SIAM J. Appl. Math., 48:353–373, (1988).
[17] C. Galusinski and M. Saad, *On a degenerate parabolic problem for compressible immiscible two-phase flows in porous media.* Adv. Differ. Equ. 9(11-12), 1235-1278 (2004).
[18] C. Galusinski, M. Saad, *Two compressible immiscible fluids in porous media*. J. Differ. Equ. 244, 1741-1783 (2008).
[19] C. Galusinski, M. Saad, *A nonlinear degenerate problem modeling water-gas in reservoir flow*. Discrete Contin. Dyn. Syst., Ser. B 9(2), 281-308 (2008).
[20] R. Helmig, *Multiphase Flow and Transport Processes in the Subsurface,* Springer–Verlag, 1997.
[21] Z. Khalil, M. Saad, *Solutions to a model for compressible immiscible two phase flow in porous media*. J. Differ. Equ. 2010(122), 1-33 (2010).
[22] J.L. Lions & E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968.
[23] M.L. Mostefai, A. Choucha, S. Boulaaras, M. Alrawashdeh. *Two-Phase Incompressible Flow with Dynamic Capillary Pressure in a Heterogeneous Porous Media.* Mathematics 2024, 12, 3038. https:// doi.org/10.3390/math12193038.
[24] M.L. Mostefai, A. Choucha and B. Bahri. *On convergence of explicit finite volume scheme for one dimensional three-component two-phase flow model in porous media.* De Gruyter, Demonstr. Math. (2021), 1–17.
[25] M.L. Mostefai and A. Choucha, *Three-component two-phase flow model in porous media,* Applicable Analysis, (2025). https://doi.org/10.1080/00036811.2025.2544290
[26] S. Salsa, *Partial Differential equations in Action, From Modeling to Theory*, Springer, Italy, 2008.
[27] J. Simon, *Compact Sets in the Space* $L^p(0,T;B)$, Annali di Mathematica pura ed applicata, (IV), Vol. CXLVI. 65-96, (1987).
[28] F. Smai, *A model of multiphase flow and transport in porous media applied to gas migration in under-ground nuclear waste repository*. C.R. Acad. Sci. Paris, Ser. I 347 (2009).
[29] P. Fabrie, P. Le Thiez & P. Tardy, *On a problem of nonlinear elliptic and degenerate parabolic equations describing compositional water-oil flows in porous media*. Nonlinear Anal. 28(9), 1565-1600 (1997).
[30] E. Zeidler, *Nonlinear Analysis and Fixed-Point Theorems*. Springer, Berlin 1993.