Filomat 40:4 (2026), 1253–1269
https://doi.org/10.2298/FIL2604253Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Applications of weak metric structures to non-symmetrical gravitational theory

Milan Zlatanović[a,*], Vladimir Rovenski[b]

[a]*Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia*
[b]*Department of Mathematics, Faculty of Natural Science, University of Haifa, Mount Carmel, 3498838 Haifa, Israel*

Abstract. Linear connections satisfying the Einstein metricity condition are important in the study of generalized Riemannian manifolds $(M, G = g+F)$, where the symmetric part g of G is a non-degenerate $(0,2)$-tensor, and F is the skew-symmetric part. Such structures naturally arise in spacetime models in theoretical physics, where F can be defined as an almost complex or almost contact metric (a.c.m.) structure. In the paper, we first study more general models, where F has constant rank and is based on weak metric structures (introduced by the second author and R. Wolak), which generalize almost complex and a.c.m. structures. We consider linear connections with totally skew-symmetric torsion that satisfy both the Einstein metricity condition and the A-torsion condition, where A is a skew-symmetric (1,1)-tensor adjoint to F. In the almost Hermitian case, we prove that the manifold with such a connection is weak nearly Kähler, the torsion is completely determined by the exterior derivative of the fundamental 2-form and the Nijenhuis tensor, and the structure tensors are parallel, while in the weak a.c.m. case, the contact distribution is involutive, the Reeb vector field is Levi-Civita parallel, and the structure tensors are also parallel with respect to both connections. For $\operatorname{rank}(F) = \dim M$, we apply weak almost Hermitian structures to fundamental results (by the first author and S. Ivanov) on generalized Riemannian manifolds and prove that the manifold equipped with an Einstein's connection is a weighted product of several nearly Kähler manifolds. For $\operatorname{rank}(F) < \dim M$ we apply weak almost Hermitian and weak a.c.m. structures and obtain splitting results for generalized Riemannian manifolds equipped with Einstein's connections.

1. Introduction

1.1. *Motivation From General Relativity*

General relativity (GR) was developed by A. Einstein in 1916 [3], with contributions by many others after 1916. In GR the equation $ds^2 = g_{ij}dx^i dx^j$ is valid, where $g_{ij} = g_{ji}$ are functions of the point in the space. In GR which is the four dimensional space-time continuum metric properties depend on the mass distribution. The magnitudes g_{ij} are known as *gravitational potential*. The Christoffel symbols, usually denoted by Γ^k_{ij},

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25, 53B05.
Keywords. weak almost Hermitian structure, weak almost contact structure, weak nearly Kähler structure, weak nearly cosymplectic structure, Einstein metricity condition, totally skew-symmetric torsion, A-torsion condition, generalized Riemannian manifold.
Received: 09 September 2025; Accepted: 25 November 2025
Communicated by Mića Stanković
M.Z. was partially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, project no. 451-03-137/2025-03/200124 and by the Bulgarian Ministry of Education and Science, Scientific Programme ”Enhancing the Research Capacity in Mathematical Sciences (PIKOM)”, No. DO1-67/05.05.2022.
* Corresponding author: Milan Zlatanović
Email addresses: zlatmilan@yahoo.com (Milan Zlatanović), vrovenski@univ.haifa.ac.il (Vladimir Rovenski)

play the role of magnitudes defining the gravitational force field. General relativity explains gravity as a curvature of spacetime. In the GR a metric tensor is related by the Einstein equations

$$R_{ij} - \frac{1}{2} R\, g_{ij} = \mathcal{T}_{ij}, \tag{1.1}$$

where R_{ij} is Ricci tensor of metric of space time, R is scalar curvature of metric, and $\mathcal{T}_{ij}$ is energy-momentum tensor of matter. However, in 1922, A. Einstein believed that the universe is apparently static, and since a static cosmology was not supported by the GR field equations, he added a cosmological constant Λ to the field equations (1.1), which became

$$R_{ij} - \frac{1}{2} R\, g_{ij} + \Lambda\, g_{ij} = \mathcal{T}_{ij}. \tag{1.2}$$

Since 1923 until the end of his life, A. Einstein worked on various versions of the Unified Field Theory (Non-symmetric Gravitational Theory – NGT) [5]. This theory was intended to unite the gravitation theory, to which GR is related, and the theory of electromagnetism. Introducing various versions of his NGT, A. Einstein used a complex basic tensor, with a symmetric real part and a skew-symmetric imaginary part. Beginning in 1950, A. Einstein used a real non-symmetric basic tensor G, sometimes called a *generalized Riemannian metric/manifold*). Note that in the NGT the symmetric part g_{ij} of the basic tensor G_{ij} ($G_{ij} = g_{ij} + F_{ij}$) is associated with gravity, and the skew-symmetric part F_{ij} is associated with electromagnetism. The same is true for the symmetric part of the connection and torsion tensor, respectively.

Later, ideas of non-symmetric metric tensor appeared in Moffat's non-symmetric theory of gravity [10]. Moffat's theory extends general relativity by introducing a non-symmetric metric and connection, allowing for richer gravitational dynamics. In his theory the antisymmetric part is a Proca field (massive Maxwell's field), which is part of the gravitational interaction that contributes to the rotation of galaxies. The connection in NGT exhibits regularity, allowing for smooth and physically meaningful solutions without singularities.

In NGT [15], two new classes of path equations are derived using Bazański's variational approach. These equations describe how test particles and charged particles move, and they also reflect certain quantum characteristics of NGT. An explicit formula of such a connection, satisfying the *Einstein metricity condition* (EMC), is obtained by localizing the global formula given recently by S. Ivanov and M. Zlatanović [8]. These equations not only reveal quantum features intrinsic to the NGT, but also highlight possible interactions between torsion and electromagnetic potential, even in the absence of electromagnetic force. In [16], the authors establish new identities for a connection with totally skew-symmetric torsion on NGT. These identities, derived via the Dolan-McCrea variational method, naturally split into symmetric and skew-symmetric parts, generalizing the second Bianchi identity. Recent approaches to modified gravity often rely on post-Riemannian geometry, incorporating torsion and non-metricity as natural extensions of Einstein's GR framework [7]. Such formulations resonate with the original ideas of Einstein's NGT.

While in a Riemannian space the connection coefficients are expressed through the metric, g_{ij}, in Einstein's works on NGT the connection between these magnitudes is determined by the EMC, i.e. the non-symmetric metric tensor G and the connection components Γ^k_{ij} are related by the equations

$$\frac{\partial G_{ij}}{\partial x^m} - \Gamma^p_{im} G_{pj} - \Gamma^p_{mj} G_{ip} = 0. \tag{1.3}$$

A generalized Riemannian manifold satisfying the EMC (1.3) is also called an NGT-space [5, 10]. The choice of a connection in NGT is not uniquely determined. In particular, in NGT there exist two types of covariant derivatives, for example for the tensor a^i_j:

$$\overset{+}{a}{}^i_{\underset{+}{j|m}} = \frac{\partial a^i_j}{\partial x^m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_p; \quad \overset{-}{a}{}^i_{\underset{-}{j|m}} = \frac{\partial a^i_j}{\partial x^m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{mj} a^i_p,$$

where the lowering and the rising of indices is defined by equations

$$G_{pi} G^{pj} = G_{ip} G^{jp} = \delta^j_i. \tag{1.4}$$

A. Einstein considered in NGT only one curvature tensor:

$$R^i_{klm} = \Gamma^i_{kl,m} - \Gamma^i_{km,l} - \Gamma^i_{sl}\Gamma^s_{km} + \Gamma^i_{sm}\Gamma^s_{kl}, \tag{1.5}$$

and proved a Bianchi type identity for the covariant curvature tensor $R_{iklm} = G_{si}R^s{}_{klm}$ (see [4]):

$$R_{\underset{-+-+}{iklm}|n} + R_{\underset{-+++}{ikmn}|l} + R_{\underset{-+--}{iknl}|m} = 0.$$

1.2. Main Objectives and Structure of the Paper

The main goal of the paper is to study a generalized Riemannian manifold $(M, G = g + F)$ equipped with a linear connection satisfying EMC (1.3) and admitting a totally skew-symmetric torsion. Our key results show that the assumption of the A-torsion condition, see (2.4), or equivalently, the preservation of g, leads to a simplified form of the Nijenhuis tensor and characterizes weak nearly Kähler or weak nearly cosymplectic structures. In Section 2, we represent the geometric model with the EMC, define the A-torsion condition and prove Proposition 2.6. In Section 3, we study application of weak metric structures on generalized Riemannian manifolds, focusing on the interplay between a skew-symmetric endomorphism A and a self-adjoint endomorphism $Q > 0$. Under the key assumption that A commutes with Q, $[A, Q] = 0$, we establish conditions under which these tensors admit block-diagonalization, and analyze the existence of linear connections with totally skew-symmetric torsion compatible with the given structures. In particular, we prove that weak almost Hermitian manifolds admitting Einstein's connections are weak nearly Kähler, and describe their decomposition into weighted products of nearly Kähler manifolds. Theorems 3.5 and 3.9 show that if an Einstein's connection ∇ satisfies the A-torsion condition on a weak almost Hermitian manifold, then Q is parallel with respect to both connections, ∇ and the Levi-Civita connection, and the manifold is weak nearly Kähler.

For a weak almost contact metric (a.c.m.) manifold equipped with an Einstein's connection ∇ satisfying the A-torsion condition, we have proved that the Reeb vector field ξ is parallel with respect to the Levi-Civita connection, the Nijenhuis tensor N^{wac}_A is totally skew-symmetric, and the contact distribution $\mathcal{D} = \ker\eta$ is involutive, the tensor Q is parallel with respect to ∇ and the Levi-Civita and connection. Based on Proposition 3.15 on almost-nearly cosymplectic manifolds, we present Theorem 3.16, which complements [8, Theorem 3.8]. Theorems 3.18 and 3.19 show that if Q is conformal when restricted to $\mathcal{D}$, then the manifold locally splits as a weighted product of $\mathbb{R}$ and a nearly Kähler manifold; otherwise, $\mathcal{D}$ decomposes into mutually orthogonal eigen-distributions of Q with constant eigenvalues, locally giving a weighted product structure of $\mathbb{R}$ and several nearly Kähler manifolds.

2. Geometric Model

The fundamental (0,2)-tensor G in a non-symmetric (generalized) Riemannian manifold (M, G) is, in general, non-symmetric. It decomposes in two parts, $G = g + F$, the symmetric part g (called Riemannian metric) and the skew-symmetric part F (called fundamental 2-form), where

$$g(X, Y) = \frac{1}{2}\Big[G(X, Y) + G(Y, X)\Big], \qquad F(X, Y) = \frac{1}{2}\Big[G(X, Y) - G(Y, X)\Big]. \tag{2.1}$$

We assume that the symmetric part, g, is non-degenerate of arbitrary signature, and the skew-symmetric part, $F \neq 0$, has a constant rank, e.g., is non-degenerate. Therefore, we obtain a well-defined (1,1)-tensor $A \neq 0$ of constant rank determined by the following condition:

$$g(AX, Y) = F(X, Y) \quad \text{for all } X, Y \in \mathfrak{X}_M. \tag{2.2}$$

According to the above, since F is skew-symmetric, the tensor A is also skew-symmetric:

$$g(AX, Y) = -g(X, AY) \quad \text{for all } X, Y \in \mathfrak{X}_M.$$

Convention 2.1. In the whole paper we shall use the capital Latin letters $X, Y, \ldots$ to denote smooth vector fields on a smooth manifold M, which commute, $[X, Y] = 0$.

Using the vector fields defined in Convention 2.1, the Levi-Civita connection ∇^g corresponding to the symmetric non-degenerate (0,2)-tensor g reduces to the following:

$$g(\nabla^g_X Y, Z) = \frac{1}{2}\Big[Xg(Y, Z) + Yg(X, Z) - Zg(Y, X)\Big]. \tag{2.3}$$

2.1. Linear Connections on Generalized Riemannian Manifolds

We consider linear connections ∇ on a smooth manifold M with a torsion (1,2)-tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

We denote the torsion (0,3)-tensor with respect to g by the same letter,

$$T(X, Y, Z) := g(T(X, Y), Z).$$

A linear connection on a generalized Riemannian manifold (M, G) is completely determined by the torsion tensor and the covariant derivative ∇g of the symmetric part g of G, see [8].

Definition 2.2 (see [14]). A linear connection ∇ on a generalized Riemannian manifold $(M, G = g + F)$ is said to have *A-torsion condition*, where A is given by (2.2), if its torsion tensor T satisfies

$$T(AX, Y) = T(X, AY) \quad \text{for all } X, Y \in \mathfrak{X}_M. \tag{2.4}$$

A linear connection ∇ is said to have *Q-torsion condition* if its torsion tensor T satisfies

$$T(QX, Y) = T(X, QY) \quad \text{for all } X, Y \in \mathfrak{X}_M, \tag{2.5}$$

where $Q : TM \to TM$ is an endomorphism that is self-adjoint with respect to the symmetric part g of metric G, i.e. $g(QX, Y) = g(X, QY)$ for all $X, Y \in \mathfrak{X}_M$.

Note that the Q-torsion condition is trivial when $Q = \mathrm{Id}$ (or, Q is conformal: $Q = \lambda\,\mathrm{Id}$).

The Nijenhuis tensor N_P of a (1,1)-tensor P on a smooth manifold M is defined by (e.g. [9]),

$$N_P(X, Y) = [PX, PY] + P^2[X, Y] - P[PX, Y] - P[X, PY]. \tag{2.6}$$

The Nijenhuis tensor is skew-symmetric by definition. We denote the Nijenhuis (0,3)-tensor with respect to a Riemannian metric g with the same letter,

$$N_P(X, Y, Z) := g(N_P(X, Y), Z).$$

The Nijenhuis tensor N_A plays a fundamental role in almost complex (resp. almost para-complex) geometry. If $A^2 = -\mathrm{Id}$ (resp. $A^2 = \mathrm{Id}$) then the celebrated Nulander-Nirenberg theorem (see, e.g. [9]) shows that an almost complex structure is integrable if and only if N_A vanishes.

Using the definition of the torsion tensor T of a linear connection ∇ and the covariant derivative ∇A, we can express the Nijenhuis tensor N_A in terms of T and ∇A as follows:

$$\begin{aligned} N_A(X, Y) &= (\nabla_{AX} A)Y - (\nabla_{AY} A)X - A(\nabla_X A)Y + A(\nabla_Y A)X \\ &\quad - T(AX, AY) - A^2 T(X, Y) + AT(AX, Y) + AT(X, AY). \end{aligned} \tag{2.7}$$

For a self-adjoint endomorphism $Q : TM \to TM$ we have

$$\begin{aligned} N_Q(X, Y, Z) &= g((\nabla_{QX} Q)Y, Z) - g((\nabla_{QY} Q)X, Z) - g((\nabla_X Q)Y, QZ) + g((\nabla_Y Q)X, QZ) \\ &\quad - T(QX, QY, Z) - T(X, Y, Q^2 Z) + T(QX, Y, QZ) + T(X, QY, QZ). \end{aligned} \tag{2.8}$$

2.2. Einstein Metricity Condition

In his attempt to construct an unified field theory, briefly NGT, A. Einstein [5] considered a generalized Riemannian manifold $(M, G = g + F)$ with a linear connection ∇ satisfying the EMC (1.3), which has the following coordinate-free form, see [8]:

$$XG(Y,Z) - G(\nabla_Y X, Z) - G(Y, \nabla_X Z) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{X}_M. \tag{2.9}$$

In the paper we will call such linear connections *Einstein's connections*. Using the definition of the torsion (0,3)-tensor, (2.1) and (2.2), the EMC (2.9) can be presented in the following form, see [8]:

$$(\nabla_X G)(Y,Z) = -G(T(X,Y),Z) \;\Leftrightarrow\; (\nabla_X(g+F))(Y,Z) = -T(X,Y,Z) + T(X,Y,AZ). \tag{2.10}$$

Separating symmetric and skew-symmetric parts of (2.10) (w.r.t. Y and Z), we express the covariant derivatives ∇g and ∇F in terms of the exterior derivative dF and torsion:

$$(\nabla_X\, g)(Y,Z) = -\frac{1}{2}\Big[T(X,Y,Z) + T(X,Z,Y) - T(X,Y,AZ) - T(X,Z,AY)\Big], \tag{2.11}$$

$$\begin{aligned}(\nabla_Z\, F)(X,Y) &= \frac{1}{2}\Big[T(X,Z,Y) - T(X,Y,Z) + T(X,Y,AZ) - T(X,Z,AY)\Big]\\ &= \frac{1}{2}\Big[dF(X,Y,Z) + T(X,Y,Z) - T(Z,Y,AX) + T(Z,X,AY)\Big].\end{aligned} \tag{2.12}$$

Using the vector fields defined in Convention 2.1, the co-boundary formula for exterior derivative of a 2-form F reduces to the following formula (without the coefficient 3, unlike [1]):

$$dF(X,Y,Z) = X(F(Y,Z)) + Y(F(Z,X)) + Z(F(X,Y)). \tag{2.13}$$

The connection ∇ of (2.9) is represented in [8] as

$$\begin{aligned}g(\nabla_X Y, Z) &= g(\nabla^g_X Y, Z) + \frac{1}{2}\Big[T(X,Y,Z) - T(X,Z,AY) - T(Y,Z,AX)\Big]\\ &= g(\nabla^g_X Y, Z) - \frac{1}{2}\Big[dF(X,Y,Z) + T(Z,X,Y) + T(Y,Z,X)\Big] + \frac{1}{2}\Big[T(Z,X,AY) + T(Z,Y,AX)\Big].\end{aligned} \tag{2.14}$$

In local coordinates, the equations (2.11)-(2.14) have the following form, see [8]:

$$\begin{aligned}\nabla_k F_{ij} &= \frac{1}{2}\Big[dF_{ijk} + T_{ijk} - T_{kjs}A^s_i - T_{kis}A^s_j\Big],\\ \nabla_i g_{jk} &= -\frac{1}{2}\Big[T_{ijk} - T_{ijs}A^s_k + T_{ikj} - T_{iks}A^s_j\Big],\\ \Gamma_{ijk} &= \Gamma^g_{ijk} + \frac{1}{2}\Big[T_{ijk} - T_{ikp}A^p_j - T_{jkp}A^p_i\Big].\end{aligned} \tag{2.15}$$

The *contorsion (or, difference)* (1,2)-tensor K of a linear connection ∇ is defined by

$$K(X,Y) = \nabla_X Y - \nabla^g_X Y.$$

The contorsion $(0,3)$-tensor $K(X,Y,Z)$ is defined by $K(X,Y,Z) := g(K(X,Y),Z)$.

Lemma 2.3. *Let an Einstein's connection ∇ on a generalized Riemannian manifold $(M, G = g + F)$ satisfy EMC (2.9). Then the contorsion and torsion $(0,3)$-tensors of ∇ are related as*

$$2\,K(X,Y,Z) = T(X,Y,Z) - T(X,Z,AY) - T(Y,Z,AX).$$

Proof. It follows directly from (2.14). □

2.3. NGT with Totally Skew-Symmetric Torsion

Here, we consider a linear connection ∇ with totally skew-symmetric torsion (0,3)-tensor, $T(X,Y,Z) = -T(X,Z,Y)$. In this case,

i) the A-torsion condition (2.4) implies

$$T(AX,Y,Z) = T(X,AY,Z) = T(X,Y,AZ). \tag{2.16}$$

ii) the Q-torsion condition (2.5) implies

$$T(QX,Y,Z) = T(X,QY,Z) = T(X,Y,QZ). \tag{2.17}$$

The following result, see [8, Theorem 3.1], presents conditions for the existence and uniqueness of the Einstein's connection on a generalized Riemannian manifold and gives its explicit expression.

Theorem 2.4. *A generalized Riemannian manifold* $(M, G = g + F)$ *admits an Einstein's connection with totally skew-symmetric torsion* T *if and only if the Nijenhuis tensor* N_A*, the tensor* A *and the exterior derivative of* F *satisfy the following relation:*

$$\begin{aligned} N_A(X,Y,Z) &= \frac{2}{3}dF(X,Y,AZ) + \frac{1}{3}dF(AX,Y,Z) + \frac{1}{3}dF(X,AY,Z) + \frac{1}{3}dF(AX,AY,AZ) \\ &\quad - \frac{1}{6}\Big[dF(A^2X,Y,AZ) + dF(A^2X,AY,Z) + dF(X,A^2Y,AZ) - dF(X,AY,A^2Z)\Big] \\ &\quad - \frac{1}{6}\Big[dF(AX,A^2Y,Z) - dF(AX,Y,A^2Z)\Big]; \end{aligned} \tag{2.18}$$

moreover, the covariant derivatives of the tensors F *and* A *with respect to* ∇^g *are related by*

$$(\nabla^g_X F)(Y,Z) = g((\nabla^g_X A)Y,Z) \tag{2.19}$$

$$= \frac{1}{3}dF(X,Y,Z) + \frac{1}{3}dF(X,AY,AZ) - \frac{1}{6}dF(AX,Y,AZ) - \frac{1}{6}dF(AX,AY,Z). \tag{2.20}$$

In this case, the totally skew-symmetric torsion $(0,3)$*-tensor is completely determined by* dF*:*

$$T(X,Y,Z) = -\frac{1}{3}dF(X,Y,Z), \tag{2.21}$$

the EMC (2.9) *is equivalent to the following two conditions, see* (2.11)-(2.12)*:*

$$\begin{aligned} (\nabla_X g)(Y,Z) &= -\frac{1}{6}\Big[dF(X,Y,AZ) - dF(X,AY,Z)\Big], \\ (\nabla_X F)(Y,Z) &= \frac{1}{6}\Big[2dF(X,Y,Z) - dF(X,Y,AZ) - dF(X,AY,Z)\Big], \end{aligned} \tag{2.22}$$

and the linear connection ∇ *is uniquely determined by the following formula, see* (2.14)*:*

$$g(\nabla_X Y,Z) = g(\nabla^g_X Y,Z) + \frac{1}{6}\Big[dF(AX,Y,Z) - dF(X,Y,Z) - dF(X,AY,Z)\Big]. \tag{2.23}$$

Remark 2.5. By (2.23), the contorsion (0,3)-tensor $K(X,Y,Z)$ of a connection ∇ with a totally skew-symmetric torsion $(0,3)$-tensor is given by the following formula:

$$K(X,Y,Z) = \frac{1}{6}\Big[dF(AX,Y,Z) - dF(X,AY,Z) - dF(X,Y,Z)\Big]. \tag{2.24}$$

The formulas in Theorem 2.4 are especially meaningful under the assumption that the torsion tensor satisfies the A-torsion condition (2.4) with totally skew-symmetric property.

Proposition 2.6. *Let $(M, G = g + F)$ be a generalized Riemannian manifold with a fundamental 2-form F. Then an Einstein's connection ∇, having a totally skew-symmetric torsion $(0, 3)$-tensor T, satisfies the A-torsion condition* (2.4) *if and only if it preserves the symmetric part of the metric G, i.e., $\nabla g = 0$. In this case,*

$$(i)\ (\nabla^g_X A)Y = -T(X, Y) \ \Leftrightarrow\ \nabla^g A = -T,$$

$$(ii)\ N_A(X, Y, Z) = \frac{4}{3}\, dF(X, Y, AZ). \tag{2.25}$$

Proof. Using the first equation in (2.22), the equivalence (with $\nabla g = 0$) follows.

(*i*) Using (2.16), from (2.19) and (2.21) we obtain

$$g\Big((\nabla^g_X A)Y, Z\Big) = \frac{1}{3} dF(X, Y, Z) = -T(X, Y, Z), \tag{2.26}$$

which completes the proof of (*i*).

(*ii*) Using (2.16) in (2.18), we get the required equation for the Nijenhuis (0,3)-tensor N_A. □

Remark 2.7. By (2.25) (i), we have $(\nabla^g_X A)X = 0$, which corresponds to weak nearly (para) Kähler or weak nearly (para) cosymplectic structures considered in Section 3.

3. Applications of Weak Metric Structures

In this section, we supply a number of examples (given below using weak metric structures, see [12, 13]) with a (1,1)-tensor A of constant rank. First, we prove the following.

Lemma 3.1. *Let a generalized Riemannian manifold $(M, G = g+F)$ be equipped with a self-adjoint (with respect to g) endomorphism $Q > 0$ such that $[Q, A] = 0$. Then at each point $x \in M$ there is a basis $\{e_1, Ae_1, \ldots, e_m, Ae_m, \xi_1, \ldots, \xi_s\}$ (called an A-Q-basis), consisting of mutually orthogonal nonzero vectors of T_xM such that A and Q have block-diagonal structures: $Q = [\lambda_1 \mathrm{Id}_{n_1}, \ldots, \lambda_k \mathrm{Id}_{n_k}, \nu_1, \ldots, \nu_s]$ and $A = [\sqrt{\lambda_1} J_{n_1}, \ldots, \sqrt{\lambda_k} J_{n_k}, 0_s]$, where $\lambda_i > 0$, $\nu_i \neq 0$ and J_{n_i} is a complex structure ($J^2_{n_i} = -\mathrm{Id}_{n_i}$) on a n_i-dimensional subspace of T_xM.*

Proof. Suppose that A is non-degenerate at $x \in M$. Let $e_1 \in T_xM$ be a unit eigenvector of the self-adjoint operator $Q > 0$ with the minimal eigenvalue $\lambda_1 \neq 0$. Then, $Ae_1 \in T_xM$ is orthogonal to e_1 and $Q(Ae_1) = A(Qe_1) = \lambda_1 Ae_1$. Thus, the subspace of T_xM orthogonal to the plane $\mathrm{Span}\{e_1, Ae_1\}$ is Q-invariant (and A-invariant). Continuing in the same manner, we find a basis $\{e_1, Ae_1, \ldots, e_m, Ae_m\}$ of T_xM consisting of mutually orthogonal vectors. Hence, Q has k different nonzero eigenvalues $\lambda_1 < \ldots < \lambda_k$ of even multiplicities $n_1, \ldots, n_k$, and $\sum_{i=1}^k n_i = 2m = \dim M$. In this basis, A and Q have the required block-diagonal structures. If A is degenerate at $x \in M$ and $(\ker A)_x$ is s-dimensional, then the proof is similar. □

Remark 3.2. The condition $[A, Q] = 0$, see Lemma 3.1, is satisfied by structural tensors of all weak metric structures considered in Section 3.

3.1. Weak Almost Hermitian Structure

Let us consider a weak almost Hermitian manifold $M(A, Q, g)$, i.e. a Riemannian manifold (M, g) of dimension $n\,(= 2m \geq 4)$ endowed with non-singular endomorphisms: A (skew-symmetric) and Q (self-adjoint), and the fundamental 2-form F such that the following conditions are valid, see [12]:

$$A^2 = -Q, \quad g(AX, AY) = g(QX, Y), \quad F(X, Y) = g(AX, Y). \tag{3.1}$$

From $A^2 = -Q$ we conclude that A commutes with Q: $[A, Q] = 0$; hence $F(X, QY) = F(QX, Y)$.

A. Gray defined in [6] a nearly Kähler structure (J, g), where J is an almost complex structure, using condition that the symmetric part of $\nabla^g J$ vanishes.

Definition 3.3. A weak almost Hermitian manifold is said to be *weak nearly Kähler* if the covariant derivative of A (or F) with respect to the Levi-Civita connection ∇^g is skew-symmetric:

$$(\nabla^g_X A)X = 0 \quad \Longleftrightarrow \quad (\nabla^g_X F)(X,\,\cdot\,) = 0.$$

If $\nabla^g A = 0$, then such $M(A, Q, g)$ is called a *weak Kähler manifold*; in this case, $\nabla^g Q = 0$.

At the same time, $M(A, Q, g)$ admits a generalized Riemannian structure $G = g + F$.

Example 3.4. Let $M(A, Q, g)$ be a weak nearly Kähler manifold with a fundamental 2-form F, considered as a generalized Riemannian manifold $(M, G = g + F)$. Suppose that ∇ is an Einstein's connection on $M(A, Q, g)$ with totally skew-symmetric torsion. Let us show that ∇ satisfies the A-torsion condition (2.16). Since the manifold is weak nearly Kähler, we have

$$g\big((\nabla^g_X A)Y, Z\big) + g\big((\nabla^g_Y A)X, Z\big) = 0.$$

On the other hand, using equation (2.19), we obtain the following:

$$g\big((\nabla^g_X A)Y, Z\big) + g\big((\nabla^g_Y A)X, Z\big) = \frac{1}{2}dF(X, AY, AZ) - \frac{1}{2}dF(AX, Y, AZ),$$

which leads to the equality $dF(AX, Y, AZ) = dF(X, AY, AZ)$. This equation, using (2.21) and the non-degeneracy of A, implies (2.4): $T(AX, Y, Z) = T(X, AY, Z)$, and by the totally skew-symmetry of torsion, we conclude the A-torsion condition (2.16) is true.

The following result complements Theorem 3.3 of [8].

Theorem 3.5. *Let $M(A, Q, g)$ be a weak almost Hermitian manifold with a fundamental 2-form F, considered as a generalized Riemannian manifold $(M, G = g + F)$. Suppose that ∇ is an Einstein's connection with totally skew-symmetric torsion $(0, 3)$-tensor T. If the A-torsion condition (2.4) is true, then we get the following:*

$$T(AX, Y, Z) = -\frac{1}{3}\, dF(AX, Y, Z) = -\frac{1}{4}\, N_A(X, Y, Z), \tag{3.2}$$

$$\nabla Q = \nabla^g Q = 0, \tag{3.3}$$

and $M(A, Q, g)$ is a weak nearly Kähler manifold.

Proof. Since the A-torsion condition (2.4) is true, the first equality of (3.2) follows from (2.21) and the second equality of (3.2) follows from Proposition 2.6 (ii). Using (3.1) and (2.19), we get

$$\begin{aligned} g\big((\nabla^g_X Q)Y, Z\big) &= -g\big((\nabla^g_X A)AY, Z\big) + g\big((\nabla^g_X A)Y, AZ\big) \\ &= -\frac{1}{3}dF(X, AY, Z) + \frac{1}{3}dF(X, QY, AZ) + \frac{1}{3}dF(X, Y, AZ) \\ &\quad - \frac{1}{3}dF(X, AY, QZ) - \frac{1}{6}dF(AX, QY, Z) + \frac{1}{6}dF(AX, Y, QZ). \end{aligned} \tag{3.4}$$

Using (2.21) and the A-torsion condition (which implies the Q-torsion condition) we obtain:

$$\begin{aligned} g\big((\nabla^g_X Q)Y, Z\big) &= T(X, AY, Z) - T(X, QY, AZ) - T(X, Y, AZ) \\ &\quad + T(X, AY, QZ) + \frac{1}{2}T(AX, QY, Z) - \frac{1}{2}T(AX, Y, QZ) = 0. \end{aligned}$$

Further, using (2.23) and $\nabla^g Q = 0$, we get

$$\begin{aligned} g\big((\nabla_X Q)Y, Z\big) &= g\big(\nabla_X QY, Z\big) - g\big(\nabla_X Y, QZ\big) \\ &= g\big((\nabla^g_X Q)Y, Z\big) + \frac{1}{6}dF(AX, QY, Z) - \frac{1}{6}dF(X, QY, Z) - \frac{1}{6}dF(X, AQY, Z) \\ &\quad - \frac{1}{6}dF(AX, Y, QZ) + \frac{1}{6}dF(X, Y, QZ) + \frac{1}{6}dF(X, AY, QZ) \\ &= \frac{1}{6}dF(AX, QY, Z) - \frac{1}{6}dF(X, Y, QZ) - \frac{1}{6}dF(AX, QY, Z) \\ &\quad - \frac{1}{6}dF(AX, Y, QZ) + \frac{1}{6}dF(X, Y, QZ) + \frac{1}{6}dF(AX, Y, QZ) = 0 \end{aligned} \tag{3.5}$$

Similarly to the case of $\nabla^g Q = 0$, the A-torsion condition yields $\nabla Q = 0$. By Proposition 2.6 (i), $(\nabla^g_X A)X = -T(X, X) = 0$ holds; hence $M(A, Q, g)$ is a weak nearly Kähler manifold. □

Remark 3.6. By [14, Equation (3.7)] we have $dF(AX, Y, Z) = 3\,T(QX, Y, Z)$. By (3.2), we have $dF(AX, Y, Z) = -3\,T(AX, Y, Z)$. The different expressions for $dF(AX, Y, Z)$ arise since we use different connections: we use a metric connection preserving G in [14], whereas in this paper we work with Einstein's connections. In both cases, we assume that the torsion is totally skew-symmetric.

Definition 3.7. Take two (or more) almost Hermitian manifolds $M_j(A_j, g_j)$, thus $A_j^2 = -\mathrm{Id}_j$. The product $\prod_{j=1}^k M_j(\sqrt{\lambda_j}\,A_j, g_j)$ of k weak almost Hermitian manifolds, where $\lambda_j > 0$ are different constants, is a weak almost Hermitian manifold with $Q = \bigoplus_j \lambda_j \,\mathrm{Id}_j$. We call $\prod_j M_j(\sqrt{\lambda_j}\,A_j, g_j)$ a $(\lambda_1, \ldots, \lambda_k)$-*weighed product of almost Hermitian manifolds* $M_j(A_j, g_j)$.

The following example represents a set of strictly weak nearly Kähler manifolds.

Example 3.8. Note that the $(\lambda_1, \ldots, \lambda_k)$-weighed product of nearly Kähler manifolds is a weak nearly Kähler manifold. A nearly Kähler manifold of dimension ≤ 4 is a Kähler manifold, see [6]. The 6-dimensional unit sphere in the set of purely imaginary Cayley numbers is an example of a strictly nearly Kähler manifold. The classification of weak nearly Kähler manifolds in dimensions ≥ 4 is an open problem. Some 4-dimensional weak nearly Kähler manifolds appear as (λ_1, λ_2)-weighed products of 2-dimensional Kähler manifolds. Some 6-dimensional weak nearly Kähler manifolds are $(\lambda_1, \lambda_2, \lambda_3)$-weighed products of 2-dimensional Kähler manifolds or (λ_1, λ_2)-weighed products of 2- and 4-dimensional Kähler manifolds. Some 8-dimensional weak nearly Kähler manifolds are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-weighed products of 2-dimensional Kähler manifolds or (λ_1, λ_2)-weighed products of 2-dimensional Kähler manifolds and 6-dimensional nearly Kähler manifolds, or (λ_1, λ_2)-weighed products of 4-dimensional nearly Kähler manifolds, and similarly for even dimensions > 8. The $(\lambda_1, \ldots, \lambda_k)$-weighed products of nearly Kähler manifolds serve as new models for NGT.

Theorem 3.9. *Let $M(A, Q, g)$ be a weak almost Hermitian manifold with a fundamental 2-form F, considered as a generalized Riemannian manifold $(M, G = g + F)$. Suppose that an Einstein's connection ∇ on M with totally skew-symmetric torsion satisfies the A-torsion condition* (2.4).

(i) If $Q = \lambda\,\mathrm{Id}$ for $\lambda \in C^\infty(M)$, then $\lambda = const > 0$ and $(\lambda^{-1/2}A, g)$ is a nearly Kähler structure.

(ii) If $Q \ne \lambda\,\mathrm{Id}$ for $\lambda \in C^\infty(M)$, then there exist $k > 1$ mutually orthogonal even-dimensional distributions $\mathcal{D}_i \subset TM$ $(1 \le i \le k)$ such that $\bigoplus_i \mathcal{D}_i = TM$ and $\mathcal{D}_i$ are the eigen-distributions of Q with constant eigenvalues $\lambda_i : 0 < \lambda_1 < \ldots < \lambda_k$; moreover, each $\mathcal{D}_i$ defines a ∇^g-totally geodesic foliation and $M(A, Q, g)$ is locally the $(\lambda_1, \ldots, \lambda_k)$-weighed product of nearly Kähler manifolds.

Proof. (i) By (3.3), we get $\lambda = const > 0$, hence $(\lambda^{-1/2}A, g)$ is an almost Hermitian structure. Since the A-torsion condition (2.4) is true, by Theorem 3.5, $(\lambda^{-1/2}A, g)$ is a nearly Kähler structure.

(ii) Since $Q = -A^2$ is not conformal, it has $k > 1$ different eigenvalues $0 < \lambda_1 < \ldots < \lambda_k$ of even multiplicities $n_1, \ldots, n_k$. By Lemma 3.1, there exists an A-Q-basis at a point $x \in M$, in which A and Q have block-diagonal structures: $Q = [\lambda_1 \mathrm{Id}_{n_1}, \ldots, \lambda_k \mathrm{Id}_{n_k}]$ and $A = [\sqrt{\lambda_1} J_{n_1}, \ldots, \sqrt{\lambda_k} J_{n_k}]$, where J_{n_i} is a complex structure on a n_i-dimensional subspace of $T_x M$.

Since Q is ∇^g-parallel, see (3.3), we get the same structure at every point of M, that is, k and all λ_i are constant on M, and there exist mutually orthogonal ∇^g-parallel (and A-invariant) eigen-distributions $\mathcal{D}_i$ of Q with constant different eigenvalues λ_i. Since the Q-torsion condition is true, using (2.5) for any vector fields $X, Y \in \mathcal{D}_i$, i.e., $QX = \lambda_i X$ and $QY = \lambda_i Y$, we have

$$
\begin{aligned}
Q[X, Y] &= Q\{\nabla^g_X Y - \nabla^g_Y X\} = \nabla^g_X (QY) - \nabla^g_Y (QX) \\
&= \lambda_i \{\nabla^g_X Y - \nabla^g_Y X\} = \lambda_i [X, Y].
\end{aligned}
$$

Hence each $\mathcal{D}_i$ is involutive and defines a foliation $\mathcal{F}_i$. Similarly we can show that $Q(\nabla^g_X Y) = \nabla^g_X (QY) = \lambda_i \nabla^g_X Y$, hence $\mathcal{F}_i$ is a ∇^g-totally geodesic foliation, and by de Rham Decomposition Theorem (see [9]), our manifold splits and is the $(\lambda_1, \ldots, \lambda_k)$-weighed product of almost Hermitian manifolds. By Theorem 3.5, the factors are nearly Kähler manifolds. □

Example 3.10. Let a generalized Riemannian manifold $(M, G = g + F)$ be represented as the $(\lambda_1, \ldots, \lambda_k)$-weighed product of nearly Kähler manifolds $M_j(A, g_j)$ $(1 \le j \le k)$. We get a weak nearly Kähler structure on M with $Q = \bigoplus_j \lambda_j \,\mathrm{Id}_j$ for some constants $\lambda_j > 0$, hence $\nabla^g Q = 0$.

By [8, Theorem 3.3], for any j there exists a unique Einstein's connection $\nabla^{(j)}$ on a nearly Kähler manifold $M_j(A_j, g_j)$. Its torsion is determined by (3.2). It was shown in Example 3.4 that this $\nabla^{(j)}$ is an Einstein's connection on the weak nearly Kähler manifold $M_j(\sqrt{\lambda_j} A_j,\ \lambda_j \,\mathrm{Id}_j,\ g_j)$ satisfying the A-torsion condition (2.4). A unique linear connection ∇ on $(M, G = g + F)$ with a totally skew-symmetric torsion satisfying EMC (2.9) and the A-torsion condition (2.4) is the metric connection, i,e. $\nabla g = 0$, its torsion is $T = -\nabla^g A$, see Proposition 2.6 (i), and $\nabla Q = 0$, see (3.3). By the above, this Einstein metric connection has the following form: $\nabla = \bigoplus_j \nabla^{(j)}$.

3.2. *Weak Almost Contact Metric Structure*

Contact Riemannian geometry is of growing interest due to its important role in both theoretical physics and pure mathematics. Weak a.c.m. structures, i.e., the complex structure on the contact distribution is approximated by a non-singular skew-symmetric tensor, allowed us to take a new look at the theory of contact manifolds and find new applications.

Definition 3.11. A *weak a.c.m. manifold* $M(A, Q, \xi, \eta, g)$ is a $(2m + 1)$-dimensional Riemannian manifold equipped with a skew-symmetric (1,1)-tensor A of rank $2m$, a unit vector field ξ, a 1-form η dual to ξ with respect to the metric g, $\eta(\xi) = 1$, $\eta(X) = g(X, \xi)$, and a self-adjoint (1,1)-tensor $Q > 0$, satisfying the following compatibility conditions:

$$
A^2 = -Q + \eta \otimes \xi, \quad g(AX, AY) = g(QX, Y) - \eta(X)\eta(Y), \quad A\xi = 0, \quad Q\xi = \xi. \tag{3.6}
$$

Put $F(X, Y) := g(AX, Y)$. A weak a.c.m. manifold $M(A, Q, \xi, \eta, g)$ is said to be *weak almost-nearly cosymplectic* if it satisfies the following condition (see also [8] for $Q = \mathrm{Id}$):

$$
g((\nabla^g_X A)Y, Z) = -\frac{1}{3} dF(AX, AY, Z) + \frac{1}{6}\eta(Z) d\eta(Y, AX) - \frac{1}{2}\eta(Y) d\eta(AZ, X). \tag{3.7}
$$

From (3.6) we conclude that A commutes with Q: $[A, Q] = 0$; hence $F(X, QY) = F(QX, Y)$.

If we assume $d\eta = 0$, then (3.7) reduces to

$$
g((\nabla^g_X A)Y, Z) = -\frac{1}{3} dF(AX, AY, Z), \tag{3.8}
$$

and a weak almost-nearly cosymplectic manifold becomes weak nearly cosymplectic: $(\nabla^g_X A)X = 0$.

The following lemma generalizes [8, Corollary 3.9].

Lemma 3.12. *The Reeb field ξ of a weak almost-nearly cosymplectic manifold $M(A,Q,\xi,\eta,g)$ is a geodesic vector field, i.e., $\nabla^g_\xi\,\xi=0$, and a Killing vector field, i.e., $g(\nabla^g_X\,\xi,\ Y)+g(\nabla^g_Y\,\xi,\ X)=0$.*

Proof. Replacing Y by ξ in (3.7), we obtain

$$g(\nabla^g_X\,\xi,AZ)=\frac{1}{6}\,\eta(Z)\,d\eta(\xi,AX)+\frac{1}{2}\,d\eta(X,AZ). \tag{3.9}$$

Replacing X by ξ in the above equation and using $g(\nabla^g_X\,\xi,\xi)=0$, we obtain

$$g(\nabla^g_\xi\,\xi,Y)=\frac{1}{2}\,d\eta(\xi,Y). \tag{3.10}$$

Using the identity

$$d\eta(X,Y)=g(\nabla^g_X\,\xi,Y)-g(\nabla^g_Y\,\xi,X) \tag{3.11}$$

with $X=\xi$ (without the coefficient 2, unlike [1]) in (3.10), we get $d\eta(\xi,Y)=g(\nabla^g_\xi\,\xi,Y)$. Comparing with (3.10), we conclude that ξ is a geodesic vector field:

$$g(\nabla^g_\xi\,\xi,Y)=0,\quad d\eta(\xi,Y)=0. \tag{3.12}$$

Representing any vector Y as $Y-\eta(Y)\,\xi=AZ$ and using (3.10) and (3.12), gives

$$g(\nabla^g_X\,\xi,Y)=g(\nabla^g_X\,\xi,AZ)=\frac{1}{2}\,d\eta(X,AZ)=\frac{1}{2}\,d\eta(X,Y). \tag{3.13}$$

Using (3.13), we have $g(\nabla^g_X\,\xi,\ Y)+g(\nabla^g_Y\,\xi,\ X)=0$. Therefore, ξ is a Killing vector field. □

Proposition 3.13. *Let $M(A,Q,\xi,\eta,g)$ be a weak a.c.m. manifold considered as a generalized Riemannian manifold $(M,G=g+F)$, and ∇ an Einstein's connection with totally skew-symmetric torsion. Then $d\eta(X,\xi)=0$ and $\nabla^g_\xi\,\xi=0$ hold, i.e., the Reeb vector field ξ is a geodesic vector field.*

Proof. Using the fact that $g(\xi,\xi)=1$, $g(\nabla^g_X\,\xi,\xi)=0$, (3.6) and (2.19), we have

$$(\nabla^g_X\eta)(QZ)=g(\nabla^g_X\,\xi,QZ)=-g((\nabla^g_XA)\xi,AZ)=\frac{1}{3}dF(X,AZ,\xi)+\frac{1}{6}dF(AX,QZ,\xi). \tag{3.14}$$

Taking $X=\xi$ in (3.14), yields $(\nabla^g_\xi\,\eta)(QZ)=g(\nabla^g_\xi\,\xi,QZ)=0$ for all $Z\in\mathfrak{X}_M$. By this, since Q is non-degenerate, $\nabla^g_\xi\,\eta=\nabla^g_\xi\,\xi=0$ and $d\eta(X,\xi)=0$ hold, hence, ξ is a ∇^g-geodesic vector field. □

Let us introduce the (0,3)-tensor, see [14],

$$N^{\rm wac}_A=N_A+d\eta\otimes\eta, \tag{3.15}$$

called the *Nijenhuis tensor in the weak a.c.m. geometry*.

Proposition 3.14 (see [12]). *For a weak a.c.m. structure (A,Q,ξ,η,g), we get*

$$\begin{aligned}2\,g\Big((\nabla^g_XA)Y,Z\Big)&=N^{(5)}(X,Y,Z)+dF(X,Y,Z)-dF(X,AY,AZ)+N^{\rm wac}_A(Y,Z,AX)\\&\quad+\Big[\,d\eta(AY,Z)-d\eta(AZ,Y)\,\Big]\eta(X)-d\eta(X,AY)\eta(Z)+d\eta(X,AZ)\eta(Y)\end{aligned} \tag{3.16}$$

where the skew-symmetric with respect to Y and Z tensor $N^{(5)}(X,Y,Z)$ is defined by

$$\begin{aligned}N^{(5)}(X,Y,Z)\;&=(AZ)\,(g(X,\widetilde{Q}Y))-(AY)\,(g(X,\widetilde{Q}Z))+g([X,AZ],\widetilde{Q}Y)\\&\quad-g([X,AY],\widetilde{Q}Z)+g([Y,AZ]-[Z,AY]-A[Y,Z],\ \widetilde{Q}X),\end{aligned} \tag{3.17}$$

using the tensor $\widetilde{Q} = Q - \mathrm{Id}$. *For particular values of the tensor* $N^{(5)}$ *we get*

$$N^{(5)}(X,\xi,Z) = g([\xi, AZ] - A[\xi, Z],\, \widetilde{Q}X) = \frac{1}{2}\, g((\pounds_\xi A)Z,\, \widetilde{Q}X),$$
$$N^{(5)}(\xi,Y,Z) = g([\xi, AZ], \widetilde{Q}Y) - g([\xi, AY], \widetilde{Q}Z), \quad N^{(5)}(\xi,\xi,Z) = 0.$$

Proposition 3.15. *Let* $M(A, Q, \xi, \eta, g)$ *be a weak almost-nearly cosymplectic manifold with a fundamental 2-form* F*, considered as a generalized Riemannian manifold* $(M, G = g + F)$*. Suppose that* ∇ *is an Einstein's connection with totally skew-symmetric torsion. Then the manifold is weak nearly cosymplectic, and locally is the metric product of a real line and a weak nearly Kähler manifold.*

Proof. From (3.13) and (2.19), we have

$$\begin{aligned} d\eta(X, AZ) &= 2\, g(\nabla^g_X\, \xi, AZ) = -2\, g(A\nabla^g_X\, \xi, Z) = 2\, g\big((\nabla^g_X A)\xi, Z\big) \\ &= -\frac{2}{3} dF(X, Z, \xi) + \frac{1}{3} dF(AX, AZ, \xi). \end{aligned} \tag{3.18}$$

So, for any vector fields X and Z, we have

$$d\eta(X, AZ) = d\eta(AX, Z). \tag{3.19}$$

Comparing (2.19) and (3.7) with X or Y or Z equal to ξ, and using (3.12), we get, respsctively,

$$dF(\xi, AY, AZ) = -dF(\xi, Y, Z), \tag{3.20}$$
$$2\, dF(X, \xi, Z) = dF(AX, \xi, AZ) + 3\, d\eta(X, AZ), \tag{3.21}$$
$$2\, dF(X, Y, \xi) = -dF(AX, AY, \xi) - d\eta(AX, Y). \tag{3.22}$$

From (3.21) and (3.22) we find

$$dF(AX, \xi, AY) + 3\, d\eta(X, AY) = 2\, dF(X, \xi, Y) = dF(AX, AY, \xi) + d\eta(AX, Y),$$

hence, using (3.19), we get

$$dF(AX, AY, \xi) = d\eta(X, AY). \tag{3.23}$$

Applying this in (3.22) and keeping in mind (3.19), yields

$$2\, dF(X, Y, \xi) = -d\eta(X, AY) - d\eta(AX, Y) = -2\, d\eta(AX, Y). \tag{3.24}$$

Next, we calculate

$$d\eta(QY, AZ) = dF(AY, AZ, \xi) = dF(Y, Z, \xi) = -d\eta(AY, Z) = -d\eta(Y, AZ),$$

hence

$$d\eta(Y + QY, AZ) = 0. \tag{3.25}$$

Since Q is self-adjoint and positive definite, from (3.25), using $d\eta(\xi, \cdot) = 0$ of (3.12), we get $d\eta = 0$. Hence the distribution $\mathcal{D}$ is involutive, i.e., tangent to a codimension-one foliation $\mathcal{F}$. In view of (2.21), $T(X, Y, \xi) = 0$ and (3.8) are true. Since ξ is a Killing vector field (see Lemma 3.12), using (3.11) we find $g(\nabla^g_X\, \xi, Y) = 0$ for all $X, Y \in \mathfrak{X}_M$, hence ξ is a ∇^g-parallel vector field: $\nabla^g\, \xi = 0$. Thus, the foliation $\mathcal{F}$ is totally geodesic. By de Rham Decomposition Theorem, the manifold is weak nearly cosymplectic and locally is the metric product $\mathbb{R} \times \bar{M}^{2m}$ of a real line and a weak nearly Kähler manifold. □

Therefore we generalize [8, Theorem 3.8] as follows.

Theorem 3.16. *Let $M(A,\xi,\eta,g)$ be an almost-nearly cosymplectic manifold with a fundamental 2-form F, considered as a generalized Riemannian manifold $(M, G=g+F)$. Then an Einstein's connection ∇ has a totally skew-symmetric torsion if and only if the manifold is nearly cosymplectic and locally is the metric product of a real line and a nearly Kähler manifold. The torsion is determined by the condition*

$$T(X,Y,Z) = -\frac{1}{3}\,dF(X,Y,Z) = -\frac{1}{4}\,N_A(AX,AY,AZ),$$

the connection ∇ is uniquely determined by the formula

$$g(\nabla_X Y,Z) = g(\nabla^g_X Y,Z) - \frac{1}{6}\,dF(X,Y,Z),$$

the covariant derivative of g vanishes: $\nabla g=0$, and the covariant derivative of F is

$$(\nabla_X F)(Y,Z) = \frac{1}{3}\Big\{dF(X,Y,Z) - dF(X,Y,AZ)\Big\}.$$

Proof. This follows from [8, Theorem 3.8] and our Proposition 3.15. □

Example 3.17. Let $M(A,Q,\xi,\eta,g)$ be a weak nearly cosymplectic manifold with a fundamental 2-form F, considered as a generalized Riemannian manifold $(M,G=g+F)$. Suppose that ∇ is an Einstein's connection on M with totally skew-symmetric torsion. Let us show (similarly, to Example 3.4) that ∇ satisfies the A-torsion condition (2.16).

Using equation (2.19), we obtain the following:

$$0 = g\Big((\nabla^g_X A)Y,Z\Big) + g\Big((\nabla^g_Y A)X,Z\Big) = \frac{1}{2}dF(X,AY,AZ) - \frac{1}{2}dF(AX,Y,AZ),$$

which leads to the equality

$$dF(AX,Y,AZ) = dF(X,AY,AZ).$$

This equation, using (2.21), $dF(X,Y,\xi)=0$ (see the proof of Proposition 3.15) and the non-degeneracy of A on $\mathcal{D}$, implies (2.4):

$$T(AX,Y,Z) = T(X,AY,Z).$$

By the totally skew-symmetry of torsion, we conclude that the A-torsion condition (2.16) is true.

Theorem 3.18. *Let $M(A,Q,\xi,\eta,g)$ be a weak a.c.m. manifold, considered as a generalized Riemannian manifold $(M,G=g+F)$, and ∇ is an Einstein's connection with totally skew-symmetric torsion satisfying the A-torsion condition* (2.16). *Then the following properties hold:*

(i) The contact distribution $\mathcal{D}=\ker\eta$ is involutive and the Reeb vector field ξ is parallel with respect to the Levi-Civita connection.

(ii) The tensor N^{wac}_A, defined by (3.15), *is totally skew-symmetric and $N^{wac}_A(\cdot\,,\cdot\,,\xi)=0$.*

(iii) The tensor $N^{(5)}$, defined by (3.17), *is totally skew-symmetric and is given by*

$$N^{(5)}(X,Y,Z) = -\frac{1}{3}dF(X,Y,Z) - \frac{1}{3}dF(X,AY,AZ). \tag{3.26}$$

(iv) The tensor Q satisfies the following equalities: $\nabla^g Q = \nabla Q = 0$.

Proof. From (3.14), assuming the A-torsion condition, we obtain

$$\begin{aligned}(\nabla^g_X\eta)(QZ) &= \frac{1}{3}dF(X,AZ,\xi)+\frac{1}{6}dF(AX,QZ,\xi)\\ &= \frac{1}{3}dF(X,Z,A\xi)+\frac{1}{6}dF(X,QZ,A\xi)=0 \quad \text{for all } X,Z\in\mathfrak{X}_M.\end{aligned}\tag{3.27}$$

Restricting to $Z\in\mathcal{D}=\ker\eta$, we obtain $A^2Z=-QZ$; hence, setting $U=AZ+\xi$ yields

$$AU=A(AZ+\xi)=A^2Z+A\xi=-QZ.$$

Therefore, $QZ=-AU$ lies in $\mathcal{D}$. By (3.27), $(\nabla^g_X\eta)(QZ)$ vanishes for every QZ with $Z\in\mathcal{D}$, and since Q is invertible on $\mathcal{D}$, it follows that

$$(\nabla^g_X\eta)(W)=0 \qquad \text{for all } W\in\mathcal{D}.$$

Finally, $\eta(\xi)=1$ implies $(\nabla^g_X\eta)(\xi)=0$, so the above equation extends to all $W\in\mathfrak{X}_M$. Thus,

$$g(\nabla^g_X\,\xi,W)=(\nabla^g_X\,\eta)(W)=0 \quad \text{for all } X,W\in\mathfrak{X}_M,\tag{3.28}$$

hence $\nabla^g_X\,\xi=0$ for every X, i.e. the Reeb vector field ξ is parallel with respect to the Levi-Civita connection. From (3.28), we obtain

$$d\eta(X,Y)=0 \quad \text{for all } X,Y\in\mathfrak{X}_M,\tag{3.29}$$

which implies that the contact distribution $\mathcal{D}=\ker\eta$ is involutive. This completes the proof of (i).

Taking into account (3.29), we observe that

$$N_A^{\rm wac}(X,Y,Z)=N_A(X,Y,Z).$$

Substituting the expression (2.19) into (3.16), we then obtain

$$\begin{aligned}N_A^{wac}(Y,Z,AX) = &-\frac{1}{3}dF(X,Y,Z)+dF(X,AY,AZ)-N^{(5)}(X,Y,Z)\\ &+\frac{2}{3}dF(X,AY,AZ)-\frac{1}{3}dF(AX,Y,AZ)-\frac{1}{3}dF(AX,AY,Z).\end{aligned}\tag{3.30}$$

In view of the A-torsion condition, the above equation becomes

$$N_A^{wac}(Y,Z,AX) = -\frac{1}{3}dF(X,Y,Z)+dF(X,AY,AZ)-N^{(5)}(X,Y,Z).\tag{3.31}$$

By setting $X=\xi$ in the last equation, we obtain

$$N^{(5)}(\xi,Y,Z)=-\frac{1}{3}dF(\xi,Y,Z)=T(\xi,Y,Z).$$

Applying (3.6) and (3.29), we simplify (2.18) to get

$$\begin{aligned}N_A(X,Y,Z) = &\,\frac{2}{3}dF(X,Y,AZ)+\frac{1}{3}dF(AX,Y,Z)+\frac{1}{3}dF(X,AY,Z)+\frac{1}{3}dF(AX,AY,AZ)\\ &+\frac{1}{6}\Big[dF(QX,Y,AZ)+dF(QX,AY,Z)+dF(X,QY,AZ)-dF(X,AY,QZ)\Big]\\ &+\frac{1}{6}\Big[dF(AX,QY,Z)-dF(AX,Y,QZ)\Big].\end{aligned}\tag{3.32}$$

From equation (3.32), and using the A-torsion condition together with its consequence

$$dF(AX, AY, AZ) = -dF(AX, QY, Z) = -dF(AX, Y, QZ) = -dF(QX, AY, Z),$$

we obtain

$$N_A^{wac}(X, Y, Z) = \frac{4}{3} dF(AX, Y, Z), \tag{3.33}$$

which shows that the Nijenhuis tensor N_A^{wac} is totally skew-symmetric and

$$N_A^{wac}(X, Y, \xi) = N_A^{wac}(\xi, X, Y) = 0,$$

that completes the proof of (ii).

Substituting (3.33) in (3.31), we obtain (3.26), that completes the proof of (iii).

Using the first equation of (3.6), $A^2 = -Q + \eta \otimes \xi$, we get

$$g\big((\nabla^g_X Q)Y, Z\big) = -g\big((\nabla^g_X A)AY, Z\big) + g\big((\nabla^g_X A)Y, AZ\big) + (\nabla^g_X \eta)(Y)\, \eta(Z) + \eta(Y)\, g\big(\nabla^g_X\, \xi, Z\big).$$

From the property (i) ($\nabla^g \xi = \nabla^g \eta = 0$) and (2.19), we obtain

$$\begin{aligned} g\big((\nabla^g_X Q)Y, Z\big) &= -\frac{1}{3} dF(X, AY, Z) + \frac{1}{3} dF(AX, QY, Z) - \frac{1}{3}\eta(Y)\, dF(X, \xi, AZ) \\ &\quad + \frac{1}{6} dF(AX, AY, AZ) - \frac{1}{6} dF(AX, QY, Z) + \frac{1}{6}\eta(Y) dF(AX, \xi, Z) \\ &\quad + \frac{1}{3} dF(X, Y, AZ) - \frac{1}{3} dF(X, AY, QZ) + \frac{1}{3}\eta(Z) dF(X, AY, \xi) \\ &\quad + \frac{1}{6} dF(AX, Y, QZ) - \frac{1}{6}\eta(Z) dF(AX, Y, \xi) - \frac{1}{6} dF(AX, AY, AZ) \\ &= -\frac{1}{3} dF(X, AY, Z) + \frac{1}{3} dF(X, Y, AZ) + \frac{1}{6} dF(AX, QY, Z) - \frac{1}{3} dF(X, AY, QZ) \\ &\quad + \frac{1}{6} dF(AX, Y, QZ) + \frac{1}{3}\Big[\eta(Y)\, dF(X, AZ, \xi) + \eta(Z) dF(X, AY, \xi)\Big] \\ &\quad - \frac{1}{6}\Big[\eta(Z) dF(AX, Y, \xi) + \eta(Y) dF(AX, Z, \xi)\Big]. \end{aligned}$$

Under the assumption of the A-torsion condition (2.16), and after rearranging the terms in the preceding equation, we obtain:

$$\begin{aligned} g\big((\nabla^g_X Q)Y, Z\big) &= -\frac{1}{3} dF(X, AY, Z) + \frac{1}{3} dF(X, AY, Z) + \frac{1}{6} dF(AX, QY, Z) - \frac{1}{3} dF(X, AY, QZ) \\ &\quad + \frac{1}{6} dF(X, AY, QZ) + \frac{1}{3}\Big[\eta(Y)\, dF(X, Z, A\xi) + \eta(Z) dF(X, Y, A\xi)\Big] \\ &\quad - \frac{1}{6}\eta(Z)\Big[\eta(Z) dF(X, Y, A\xi) + \eta(Y) dF(X, Z, A\xi)\Big]. \\ &= 0. \end{aligned} \tag{3.34}$$

Similarly we obtain $\nabla Q = 0$, that completes the proof of (iv). □

The following theorem is similar to Theorem 3.9.

Theorem 3.19. *Let conditions of Theorem 3.13 be satisfied. Then the following properties are true.*

(i) If $Q|_{\mathcal{D}} = \lambda\, \mathrm{Id}_{\mathcal{D}}$ for some $\lambda \in C^\infty(M)$, then $\lambda = const > 0$ and $M(\lambda^{-1/2} A, \xi, \eta, g)$ is locally the product of $\mathbb{R}$ and a nearly Kähler manifold.

(ii) If $Q|_{\mathcal{D}} \neq \lambda\, \mathrm{Id}_{\mathcal{D}}$ where $\lambda \in C^\infty(M)$, then there exist $k > 1$ mutually orthogonal even-dimensional distributions $\mathcal{D}_i \subset \mathcal{D}$ such that $\bigoplus_{i=1}^k \mathcal{D}_i = \mathcal{D}$ and $\mathcal{D}_i$ are eigen-distributions of Q with constant eigenvalues $0 < \lambda_1 < \ldots < \lambda_k$; moreover, the distributions $\mathcal{D}_i$ are involutive and define ∇^g-totally geodesic foliations and $M(A, Q, \xi, \eta, g)$ is locally a $(1, \lambda_1, \ldots, \lambda_k)$-weighed product of a real line and k nearly Kähler manifolds.

Proof. (i) Since $\nabla^g Q = 0$, see Theorem 3.13 iii), we get $\lambda = const > 0$ (and $\lambda \neq 1$), hence $(\lambda^{-1/2}A, \xi, \eta, g)$ is an a.c.m. structure. Since ∇ satisfies the A-torsion condition, then using (2.4) for any vector fields $X, Y \in \mathcal{D}$, i.e., $QX = \lambda X$ and $QY = \lambda Y$ with $\lambda \neq 1$, we have

$$Q[X,Y] = Q\{\nabla^g_X Y - \nabla^g_Y X\} = \nabla^g_X(QY) - \nabla^g_Y(QX)$$
$$= \lambda\{\nabla^g_X Y - \nabla^g_Y X\} = \lambda[X,Y].$$

Hence the contact distribution $\mathcal{D}$ is involutive and defines a foliation $\mathcal{F}$. Similarly we show that $Q(\nabla^g_X Y) = \nabla^g_X(QY) = \lambda \nabla^g_X Y$, that is, $\mathcal{F}$ is ∇^g-totally geodesic. By Theorem 3.13, ξ is a ∇^g-geodesic vector field. By de Rham Decomposition Theorem (see [9]), $M(\lambda^{-1/2}A, \xi, \eta, g)$ splits and is locally the product of $\mathbb{R}$ and an almost Hermitian manifold. Since ∇ satisfies the A-torsion condition, the second factor is a nearly Kähler manifold.

(ii) Sinse $Q|_{\mathcal{D}}$ is not conformal, it has eigenvalues $0 < \lambda_1 < \ldots < \lambda_k$ of even multiplicities $n_1, \ldots, n_k$, and $\sum_{i=1}^k n_i = 2m$. By Lemma 3.1, there exists an A-Q-basis of $T_x M$, in which A and Q restricted on $\mathcal{D}_x$ have block-diagonal forms: $A|_{\mathcal{D}} = [\sqrt{\lambda_1} J_{n_1}, \ldots, \sqrt{\lambda_k} J_{n_k}]$ and $Q|_{\mathcal{D}} = [\lambda_1 \mathrm{Id}_{n_1}, \ldots, \lambda_k \mathrm{Id}_{n_k}]$, where J_{n_i} is a complex structure on a n_i-dimensional subspace of $\mathcal{D}_x$. Since Q is ∇^g-parallel, we get the same structure at each point of M, i.e., k and all λ_i are constant on M, and there exist mutually orthogonal ∇^g-parallel (and A-invariant) eigen-distributions $\mathcal{D}_i \subset \mathcal{D}$ of Q with constant different eigenvalues λ_i. The rest of the proof is similar to the proof of Theorem 3.9. □

Example 3.20. Let a generalized Riemannian manifold $(M, G = g + F)$ be represented as the $(1, \lambda_1, \ldots, \lambda_k)$-weighed product of $\mathbb{R}$ and k nearly Kähler manifolds $M_j(A, g_j)$ $(1 \leq j \leq k)$. We get a weak nearly Kähler structure on $\mathcal{D}$ with $Q|_{\mathcal{D}} = \bigoplus_j \lambda_j \,\mathrm{Id}_j$ for some $\lambda_j \in \mathbb{R}_+$, hence $\nabla^g Q = 0$.

By [8, Theorem 3.8], for any j there exists a unique Einstein's connection $\nabla^{(j)}$ on a nearly Kähler manifold $M_j(A, g_j)$. Its torsion is determined by the condition (3.2), which is invariant under the change $A \to \lambda A$. This $\nabla^{(j)}$ is also an Einstein's connection on the weak nearly Kähler manifold $M_j(\sqrt{\lambda_j}\, A,\ \lambda_j \,\mathrm{Id}_j,\ g_j)$ satisfying the A-torsion condition (2.4) with $A = \sqrt{\lambda_j}\, A$.

A unique linear connection ∇ on $(M, G = g + F)$ with a totally skew-symmetric torsion satisfying EMC (2.9) and the A-torsion condition (2.4) is the metric connection, i,e. $\nabla g = 0$, its torsion is $T = -\nabla^g A$, see Proposition 2.6 (i), and $\nabla Q = 0$, see (3.3). By the above, this Einstein metric connection on M has the following form: $\nabla = \bigoplus_j \nabla^{(j)}$.

Remark 3.21 (Weak para-Hermitian and weak almost para-contact structures). A *weak almost para-Hermitian manifold* $M(A, Q, g)$, is a (pseudo-) Riemannian manifold (M, g) of dimension n $(= 2m \geq 4)$ endowed with non-singular endomorphisms: A (skew-symmetric) and $Q > 0$ (self-adjoint) and the fundamental 2-form F, such that the following conditions hold:

$$A^2 = Q, \quad g(AX, AY) = -g(QX, Y), \quad F(X,Y) = g(AX, Y). \tag{3.35}$$

In this case, the skew-symmetric part F of $G = g + F$ is non-degenerate and rank$F = 2m$.

A *weak almost para-contact metric manifold* $M(A, Q, \xi, \eta, g)$ is a $(2m+1)$-dimensional pseudo-Riemannian manifold of signature $(m+1, m)$ equipped with a skew-symmetric (1,1)-tensor A of rank $2m$, a vector field ξ, a 1-form η dual to ξ with respect to the metric g, and a self-adjoint (1,1)-tensor $Q > 0$, satisfying the following conditions:

$$A^2 = Q - \eta \otimes \xi, \;\; g(AX, AY) = -g(QX, Y) + \eta(X)\eta(Y), \;\; A\xi = 0, \;\; Q\xi = \xi. \tag{3.36}$$

In this case, the skew-symmetric part $F(X,Y) := g(AX, Y)$ of $G = g + F$ is degenerate, $F(\xi, X) = 0$, and rank$F = 2m$. By applying analogous technique, one obtains results similar to Theorems 3.5, 3.16, and 3.18, in the cases of weak almost para-Hermitian and weak almost para-contact manifolds. Para cases indicate the sign changes type behaviour of the structure endomorphism Q. These sign changes do not significantly affect the essence of the above mentioned results. For conciseness, their statements and proofs are omitted.

Results analogous to Theorems 3.9 and 3.19 generally fail in the pseudo-Riemannian setting because the spectral theorem does not hold unless g is positive definite. In pseudo-Riemannian geometry, a self-adjoint operator may admit light-like eigenvectors, and this prevents diagonalizability; see, e.g., B. O'Neill [11, pp. 260–262]. Namely, in a pseudo-Euclidean space (V, g) of dimension ≥ 3, a self-adjoint operator Q is diagonalizable with respect to a g-orthonormal basis only if $g(QX, X) \neq 0$ for all light-like vectors $X \in V$, cf. [2]. Thus, principal directions and curvatures are not well defined unless one restricts to spacelike or timelike hypersurfaces.

4. Conclusion

The paper presents new applications of weak contact metric structures to NGT manifolds with totally skew-symmetric torsion. Our future research aims to extend these results by incorporating the Q-torsion condition instead of the A-torsion condition, which offers the potential for a richer geometric framework. We expect that analyzing the interaction involving the Q-torsion condition will lead to more general (specific) results and a deeper understanding of the Einstein's non-symmetric geometry with totally skew-symmetric torsion, including new classifications and possible applications in theoretical physics. In our future analysis of the Q-torsion condition, we examine the interaction between the metric g, the fundamental form F, and the self-adjoint tensor Q, paying special attention to the special cases, where the equalities $\nabla^g Q = 0$, $\nabla Q = 0$ and $[A, Q] = 0$ are satisfied.

References

[1] D.E. Blair, Riemannian geometry of contact and symplectic manifolds. Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.
[2] J. Bognár, Indefinite Inner Product Spaces, Springer, 1974.
[3] A. Einstein, Die Grundlagen der allgemeinen Relativitätstheorie, Ann. der Physik, 49, 769, 1916.
[4] A. Einstein, *The Bianchi Identities in the Generalized Theory of Gravitation,* Canadian Journal of Mathematics 2 (1950), 120-128.
[5] A. Einstein, The Meaning of Relativity, Princeton Univ. Press, 1953.
[6] A. Gray, *Nearly Kähler manifolds*, J. Diff. Geom. 4, 283-309 (1970).
[7] D. Iosifidis, L. Ravera, *The cosmology of quadratic torsionful gravity,* Eur. Phys. J. C 81, 736 (2021).
[8] S. Ivanov, M. Lj. Zlatanović, *Connections on Non-Symmetric (Generalized) Riemannian Manifold and Gravity,* Classical and Quantum Gravity, 33, No. 7, 075016, (2016).
[9] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry. vol I, 1963; vol. II, 1967.
[10] J.W. Moffat, *Regularity Theorems in the Nonsymmetric Gravitational Theory*, J. Math. Phys. 36, No. 10, 5897-5915 (1995).
[11] B. O'Neill, Semi-Riemannian Geometry With Applications to Relativity, Academic Press, 1983.
[12] V. Rovenski, *Weak almost contact structures: a survey*, Facta Universitatis, Ser. Math. Inform. Vol. 39, No. 5 (2024), 821-841.
[13] V. Rovenski and R. Wolak, *New metric structures on g-foliations*, Indagationes Mathematicae, 33 (2022), 518-532.
[14] V. Rovenski, M.Lj. Zlatanović, *Weak metric structures on generalized Riemannian manifolds*, J. Geom. Phys. 221, (2026), 105741.
[15] M.I. Wanas et al., *New path equations in Einstein non-symmetric geometry,* Gravitation and Cosmology, 30:4 (2024), 489-495.
[16] M.I. Wanas et al., *Differential Identities in Einstein Nonsymmetric Geometry,* Gravitation and Cosmology, 31 (2025), 185-194.