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Higher-order commutators of the parameterized Marcinkiewicz integrals on central Morrey spaces with variable exponent

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Abstract. By using the method of function decomposition on central Morrey spaces with variable exponent and with the help of the boundedness of parameterized Marcinkiewicz integrals with rough kernels on $L^{p(\cdot)}(\mathbb{R}^n)$, the boundedness of higher-order Lipschitz commutator and higher-order BMO commutator of parameterized Marcinkiewicz integrals is estimated and obtained on central Morrey spaces with variable exponent.

1. Introduction

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^{\mathrm{n}-1})$ for $0 < \beta \leq 1$ be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\delta(x') = 0, \tag{1.1}$$

where, $x' = \frac{x}{|x|}$, $x \neq 0$.

Parameterized Marcinkiewicz integral operator μ_Ω^ρ is defined by

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(x)|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}^\rho(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\mathrm{d}y.$$

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When $\rho = 1$, It is Stein [13] introduced and studied the following Marcinkiewicz integral operator related to the Littlewood-Paley g function on $\mathbb{R}^n$

$$\mu_\Omega(f)(x) = \Big[\int_0^\infty \Big| \int\limits_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^3}\Big]^{1/2}.$$

It was shown that in [13] μ is of weak type $(1,1)$ and when $1 < p \leq 2$, μ is type (p,p), where the Lipschitz continuous function Ω is homogeneous of degree zero and vanishes on the unit sphere.

The boundedness of Marcinkiewicz integral operator on function spaces has aroused great interest among mathematicians. Currently, there have been numerous research results regarding the Marcinkiewicz integrals, we refer the readers to references [1,2,8,14-18,25].

In 2019, Fu et al. [4] introduced the central Morrey spaces with variable exponent and proved the boundedness of the fractional singular integrals and its commutator on above space. Since then, the central Morrey spaces with variable exponent have been widely studied by a significant number of authors, see [9,10,19,20] and the references therein.

Motivated by [1,2,4,8-10,13-20,25], the aim of this paper is to prove the boundedness of higher-order Lipschitz commutator and higher-order BMO commutator of the parameterized Marcinkiewicz integrals on central Morrey spaces with variable exponent.

We end this section by introducing some conventional notations which will be used later. Throughout this paper, the letter C represents a constant independent of parameters, and its value may vary in different contexts. We will denote the Lebesgue measure and the characteristic function of a measure set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively. The notation $f \approx g$ means that there exist constants $C_1,\ C_2 > 0$, such that $C_1 g \leq f \leq C_2 g$. If $f \leq Cg$, we then write $f \lesssim g$.

2. Preliminaries

The variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ become one of the important class function spaces due to the seminal paper [7] by Kováčik and Rákosník. Now, we give some notations and basic definitions on $L^{p(\cdot)}(\mathbb{R}^n)$. Given a measure function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$. Let $p'(\cdot)$ be the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$. The $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \lambda > 0 \right\},$$

then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \leqslant 1 \right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n\}$.

The set $\mathscr{P}(\mathbb{R})$ consists of all $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfying

$$p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1, \quad p^+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

The set $\mathscr{P}^0(\mathbb{R})$ consists of all $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ satisfying

$$p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0, \quad p^+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x) := \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y\,, \tag{2.1}$$

here, B is a circle with zero as the center and r as the radius. The set $\mathscr{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.1[4] Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The central Morrey space with variable exponent is defined by

$$\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Definition 2.2[12] For $0 < \beta \le 1$, the norm definition of Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is as follows

$$\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{|f(x+h)-f(x)|}{|h|^{\beta}} < \infty.$$

Definition 2.3[11] Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$.

(i) For any $x, y \in \mathbb{R}^n$, $|x-y| \le 1/2$, if

$$|\alpha(x)-\alpha(y)| \lesssim \frac{1}{-\log(|x-y|)}, \tag{2.2}$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$.

(ii) For all $x \in \mathbb{R}^n$, if

$$|\alpha(x)-\alpha(0)| \lesssim \frac{1}{\log(e+1/|x|)}, \tag{2.3}$$

then $\alpha(\cdot)$ is said log-Hölder continuous functions at origin, denote by $\mathscr{P}_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous at origin.

(iii) If there exist $\alpha_\infty \in \mathbb{R}$, for $x \in \mathbb{R}^n$, if

$$|\alpha(x)-\alpha_\infty| \lesssim \frac{1}{\log(e+|x|)}, \tag{2.4}$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity.

(iv) The set $LH(\mathbb{R}^n)$ consists of all exponents $p(\cdot)$ defined on $\mathbb{R}^n$ which are locally log-Hölder continuous and log-Hölder continuous at infinity. The set $\mathscr{P}_0^{\log}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ being log-Hölder continuous at the origin and the set $\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ which are log-Hölder continuous at infinity.

Lemma 2.4[7] (Generalized Hölder's inequality) Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R})$ and $g \in L^{p'(\cdot)}(\mathbb{R})$, then fg is integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

Lemma 2.5[5] Suppose $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.6[5] Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $\delta_1, \delta_2 \in (0,1)$ are constants.

Lemma 2.7[21] Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH$. Then

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)}, & |Q| \leq 2^n, x \in Q; \\ |Q|^{1/p(\infty)}, & |Q| \geq 1, \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x\to\infty} p(x)$.

3. Boundedness of higher-order Lipschitz commutator of Parameterized Marcinkiewicz integrals

In this section, we will give the boundedness of higher-order Lipschitz commutator of the Parameterized Marcinkiewicz integrals. Now, we recall two Lemmas.

Lemma 3.1[3] Let $p(\cdot), q(\cdot), s(\cdot) \in (\mathbb{R}^n)$ be such that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},$$

for almost every $x \in \mathbb{R}^n$. Then

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 3.2[22] Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$, $B_j \in B_k$, we have

(1) $C^{-1}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \leq \sup_B |B|^{-m\beta/n}\|\chi_B\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}\|(b-b_B)^m\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}$;

(2) $\|(b-b_{B_j})^m\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C|B_k|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$,

where

$$B_k = B(0,2^k) = \{x \in \mathbb{R} : |x| \leq 2^k\}, \;\; B_j = B(0,2^j) = \{x \in \mathbb{R} : |x| \leq 2^j\}.$$

Theorem 3.3 Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0 < \beta \leq 1$, m is a positive integer, $\Omega \in L^s(S^{n-1})(1 \leq s < \infty)$ is a homogeneous function of degree zero and satisfies (1.1) . Suppose that

$$1/p_2(\cdot) + 1/s = 1/p_1'(\cdot), \quad p_1(\cdot),\ p_2(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH,$$

$$\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}, \quad \lambda_1 < -m\beta/n - 1,\;\; 0 < \rho < \frac{n}{3s},\;\; \lambda = \lambda_1 + m\beta/n,$$

then $[b^m, \mu_\Omega^\rho]$ is bounded from $\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)$ into $\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and the following inequality holds:

$$\left\|[b^m, \mu_\Omega^\rho]f\right\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Proof of Theorem 3.3

Let $f \in \dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)$, We have decomposition $f(x) = f_1 + f_2$, $f_1 = f\chi_{2B}$, $f_2 = f\chi_{(2B)^c}$. Let $R = 2^i, i \in \mathbb{Z}$, for fixed $R > 0$, denote $B(0, R)$ by B. Let b_B be the mean value of b on the ball B. Write

$$\begin{aligned} \|[b^m, \mu_\Omega^\rho]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq \|(b-b_B)^m(\mu_\Omega^\rho(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|(b-b_B)^m(\mu_\Omega^\rho(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\mu_\Omega^\rho((b-b_B)^m(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\mu_\Omega^\rho((b-b_B)^m(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &= U_1 + U_2 + U_3 + U_4. \end{aligned}$$

Firstly, we estimate U_1. In 2018, Wang H. B. et al published an article. In this article, they proved μ_Ω^ρ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. For detailed process, we refer the readers to reference [23]. At the same time, by Definition 2.1, Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} U_1 &\leq C\left\|\mu_\Omega^\rho(f\chi_{2B})\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|(b-b_B)^m\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C\left\|f\chi_{2B}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} |B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1}\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Where

$$\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}} \approx |B|^{\frac{1}{q(\cdot)}} \approx \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{3.1}$$

Next we estimate U_3, by Definition 2.1, Lemma 3.1, Lemma 3.2 and the boundedness of Parameterized Marcinkiewicz integral operator on Lebesgue spaces with variable exponents.

$$\begin{aligned} U_3 &\leq C\|(b-b_B)^m f\chi_{2B}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|(b-b_B)^m\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \left\|f\chi_{2B}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1}\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

To estimate U_2, we first need to estimate $|\mu_\Omega^\rho\left(f\chi_{(2B)^c}\right)(x)|$, we have

$$\begin{aligned}
&|\mu_\Omega^\rho\left(f\chi_{(2B)^c}\right)(x)|\\
&=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac12}\\
&\le\left(\int_0^{|x|+|y|}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac12}\\
&+\left(\int_{|x|+|y|}^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac12}\\
&=J_1+J_2.
\end{aligned}$$

Firstly, we estimate J_1. According to Minkowski's inequality, we can obtain

$$\begin{aligned}
J_1 &\le \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x-y|\le t\le|x|+|y|}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac12}\mathrm{d}y\\
&\le \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac12}\mathrm{d}y\\
&\le \sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac12}\mathrm{d}y.
\end{aligned}$$

When $x\in B$, $y\in 2^{k+1}B\setminus 2^kB$, $k\ge1$, $k\in N^*$, we have

$$\begin{aligned}
\left(\frac{1}{|x-y|^{2\rho}}\right)^2+\left(\frac{1}{(|x|+|y|)^{2\rho}}\right)^2 &\ge 2\left(\frac{1}{|x-y|^{2\rho}}\frac{1}{(|x|+|y|)^{2\rho}}\right),\\
\left(\frac{1}{|x-y|^{2\rho}}\right)^2+\left(\frac{1}{(|x|+|y|)^{2\rho}}\right)^2 &\ge \frac{1}{|x-y|^{2\rho}}\frac{1}{(|x|+|y|)^{2\rho}},\\
\left(\frac{1}{|x-y|^{2\rho}}\right)^2(|x|+|y|)^{2\rho}+\frac{1}{(|x|+|y|)^{2\rho}} &\ge \frac{1}{|x-y|^{2\rho}},\\
\left(\frac{1}{|x-y|^{2\rho}}\right)^2(|x|+|y|)^{2\rho} &\ge \frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}},
\end{aligned}$$

so

$$\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\le\left(\frac{1}{|x-y|^{2\rho}}\right)^2(|x|+|y|)^{2\rho}. \tag{3.2}$$

According to the above formula, we can get that

$$\begin{aligned}
J_1 &\le \sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{(|x|+|y|)^\rho}{|x-y|^{2\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}|\Omega(x-y)||f(y)|\frac{(|x|+|y|)^\rho}{|x-y|^{n+\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^\infty\left(R+2^{k+1}R\right)^\rho\left(2^{k-1}R\right)^{-n-\rho}\int_{2^{k+1}B\setminus 2^kB}|\Omega(x-y)||f(y)|\mathrm{d}y.
\end{aligned}$$

Noticing that $1/p_1'(\cdot)=1/s+1/p_2(\cdot)$, we have

$$\begin{aligned}&\int_{2^{k+1}B\setminus 2^kB}|\Omega(x-y)||f(y)|\mathrm{d}y\\ &\quad\le C\left\|\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\quad\le C\left\|\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^s(\mathbb{R}^n)}\|\chi_{2^{k+1}B}(y)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{3.3}$$

For $x\in B$ and $y\in 2^{k+1}B$, $x-y\in 2^{k+2}B$. Noticing that Ω is homogeneous of degree zero and $\Omega\in L^s(S^{n-1})$, we obtain

$$\begin{aligned}\left\|\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^s(\mathbb{R}^n)}&=\left(\int_{2^{k+1}B}|\Omega(x-y)|^s\mathrm{d}y\right)^{1/s}\\ &\le\left(\int_{2^{k+2}B}|\Omega(z)|^s\mathrm{d}y\right)^{1/s}\\ &=\left(\int_0^{2^{k+2}R}\int_{S^{n-1}}|\Omega(z')|^s\mathrm{d}\sigma(z')r^{n-1}\mathrm{d}r\right)^{1/s}\\ &=C\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}.\end{aligned}\tag{3.4}$$

Thus, we have

$$\begin{aligned}J_1&\le C\sum_{k=1}^{\infty}\left(R+2^{k+1}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\quad\cdot\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}\\ &\le C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}|2^{k+1}B||2^{k+1}B|^{-\frac{1}{s}}\\ &\le C\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_1}\sum_{k=1}^{\infty}2^{-(n+\rho)(k-1)+n(\lambda_1+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}+(k+2)\rho}\\ &\le C\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_1}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+1)(k+1)-\frac{n}{s}+3\rho}\\ &\le C|B|^{\lambda_1}\left\|f\right\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.\end{aligned}$$

When $|2^{k+1}B|\le 2^n$ and $x\in 2^{k+1}B$, by Lemma 2.7 and $1/p_2(\cdot)+1/s=1/p_1'(\cdot)$, we have

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\approx|2^{k+1}B|^{\frac{1}{p_2(x)}}\approx\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

When $|2^{k+1}B|\ge 1$, we have

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\approx|2^{k+1}B|^{\frac{1}{p_2(\infty)}}\approx\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

So we obtain

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

For J_2, using the Minkowski's inequality, by the formula (3.3), (3.4), Definition 2.1 and Lemma 2.5, we can obtain

$$\begin{aligned}
J_2 &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x|+|y|}^{\infty}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}y\\
&\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}|\Omega(x-y)||f(y)|\frac{1}{|x-y|^{n}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\int_{2^{k+1}B\backslash 2^kB}|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\left\|f\right\|_{\mathscr{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}|2^{k+1}B||2^{k+1}B|^{-\frac{1}{s}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_1}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_1}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+1)(k+1)-\frac{n}{s}}\\
&\leq C|B|^{\lambda_1}\left\|f\right\|_{\mathscr{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.
\end{aligned}$$

Summarizing the estimates of J_1 and J_2, we conclude that

$$|\mu_{\Omega}^{\rho}\left(f\chi_{(2B)^c}(x)\right)| \leq C|B|^{\lambda_1}\left\|f\right\|_{\mathscr{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

So, we have

$$\begin{aligned}
U_2 &= \|(b-b_B)^m(\mu_{\Omega}^{\rho}(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C|B|^{\lambda_1}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|(b-b_B)^m\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C|B|^{\lambda_1}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Estimate U_4. We divide $\left|\mu_{\Omega}^{\rho}\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right|$ into the following two parts

$$
\begin{aligned}
&\left|\mu_{\Omega}^{\rho}\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right|\\
&\quad=\left(\int_0^{\infty}\left|\int_{|x-y|\le t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}}\\
&\quad\le\left(\int_0^{|x|+|y|}\left|\int_{|x-y|\le t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}}\\
&\quad+\left(\int_{|x|+|y|}^{\infty}\left|\int_{|x-y|\le t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}}\\
&\quad=H_1+H_2.
\end{aligned}
$$

Firstly, we estimate H_1. According to Minkowski's inequality and the formula (3.2), we have

$$
\begin{aligned}
H_1 &\le \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x-y|\le t\le|x|+|y|}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}y\\
&\le \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac{1}{2}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac{1}{2}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{(|x|+|y|)^{\rho}}{|x-y|^{2\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\frac{(|x|+|y|)^{\rho}}{|x-y|^{n+\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\left(R+2^{k+1}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y.
\end{aligned}
$$

In order to continue the calculation, we need to estimate the value of $\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y$. By Lemma 3.1, Lemma 3.2 and $1/p_1'(\cdot)=1/s+1/p_2(\cdot)$, we have

$$
\begin{aligned}
&\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\quad\le C\left\|(b(y)-b_B)^m\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C\left\|\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^s(\mathbb{R}^n)}\|(b(y)-b_B)^m\chi_{2^{k+1}B}(y)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C\left\|\Omega(x-y)\chi_{2^{k+1}B}(y)\right\|_{L^s(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \qquad (3.5)
\end{aligned}
$$

Summing up the above estimates, by Definition 2.1 and Lemma 2.5, we can obtain

$$\begin{aligned}
H_1 &\leq C\sum_{k=1}^{\infty}\left(R+2^{k+1}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\qquad\cdot|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\\
&\quad\cdot\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\\
&\quad\cdot|2^{k+1}B||2^{k+1}B|^{-\frac{1}{s}}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}|B|^{\lambda_1+m\beta/n}\sum_{k=1}^{\infty}2^{-(n+\rho)(k-1)+n(\lambda_1+m\beta/n+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}+(k+2)\rho}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}|B|^{\lambda_1+m\beta/n}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+m\beta/n+1)(k+1)-\frac{n}{s}+3\rho}\\
&\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.
\end{aligned}$$

For H_2, using the Minkowski's inequality, by the formula (3.4), (3.5) and Definition 2.1, we can obtain

$$\begin{aligned}
H_2 &\leq \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x|+|y|}^{\infty}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}y\\
&\leq \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\frac{1}{|x-y|^{n}}\mathrm{d}y\\
&\leq \sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\\
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda_1}\\
&\quad\cdot|2^{k+1}B||2^{k+1}B|^{-\frac{1}{s}}|2^{k+1}B|^{m\beta/n}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}|B|^{\lambda_1+m\beta/n}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+m\beta/n+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|^m_{\mathrm{Lip}_{\beta(\mathbb{R}^n)}}|B|^{\lambda_1+m\beta/n}\sum_{k=1}^{\infty}2^{-n(k-1)+n(\lambda_1+m\beta/n+1)(k+1)-\frac{n}{s}}\\
&\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.
\end{aligned}
$$

So

$$\left|\mu^\rho_\Omega\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right|\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

We have

$$
\begin{aligned}
U_4 &= \|\mu^\rho_\Omega((b-b_B)^m(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Summarizing all the estimates of U_1, U_2, U_3 and U_4, we conclude that

$$\|[b,\mu^\rho_\Omega]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C|B|^{\lambda_1+m\beta/n}\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

so

$$\|[b,\mu^\rho_\Omega]f\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|b\|^m_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Therefore, we complete the proof of Theorem 3.3.

4. Boundedness of higher-order BMO commutator of Parameterized Marcinkiewicz integrals

In this part, we will give the BMO estimate for the commutators$\left[b^m,\mu^\rho_\Omega\right]$ on central Morry spaces with variable exponent. Therefore, let us first recall the space BMO($\mathbb{R}^n$).

Lemma 4.1[24] Let us first recall that the space BMO($\mathbb{R}^n$) consists of all locally integrable f such that

$$\|f\|_*=\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B|f(x)-f_B|\mathrm{d}x,$$

here, B is a circle with zero as the center and R as the radius, $f_B=\frac{1}{|B|}\int_B f(x)\mathrm{d}x$, is called the average on B. In 1961, John and Nirenberg [6] proved that BMO functions satisfy John-Nirenberg inequality. The following is an important corollary related to John-Nirenberg inequality.

for $1<q<\infty$, we have

$$\|f\|_{\mathrm{BMO_q}}:=\left(\frac{1}{|B|}\int_B|f(x)-f_B|^q\mathrm{d}x\right)^{1/q}\approx\|f\|_*.$$

Lemma 4.2[22] Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$, $B_j \in B_k$, we have

(1) $C^{-1}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^m\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$;

(2) $\|(b-b_{B_j})^m\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(k-j)^m\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

where

$$B_k = B(0,2^k) = \{x \in \mathbb{R} : |x| \le 2^k\}, \;\; B_j = B(0,2^j) = \{x \in \mathbb{R} : |x| \le 2^j\}.$$

Theorem 4.3 Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, m is a positive integer, $\Omega \in L^s(S^{n-1})(1 \le s < \infty)$ is a homogeneous function of degree zero and satisfies (1.1). Suppose that

$$1/p_2(\cdot) + 1/s = 1/p_1'(\cdot), \quad p_1(\cdot),\ p_2(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH,$$

$$\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}, \qquad \lambda < -1,\ 0 < \rho < \frac{n}{3s},$$

Then $[b^m, \mu^\rho_\Omega]$ is bounded from $\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)$ into $\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and the following inequality holds:

$$\left\|[b^m, \mu^\rho_\Omega]f\right\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} \le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.$$

Proof of Theorem 4.3

Let $f \in \dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)$, the decomposition of $f(x)$ is the same as that in the third part. The meanings represented by symbol B and symbol b_B are also the same as those in the third part. Write

$$\begin{aligned}
\|[b^m, \mu^\rho_\Omega]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le\ & \|(b-b_B)^m(\mu^\rho_\Omega(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
& +\|(b-b_B)^m(\mu^\rho_\Omega(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
& +\|\mu^\rho_\Omega((b-b_B)^m(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
& +\|\mu^\rho_\Omega((b-b_B)^m(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
=\ & W_1 + W_2 + W_3 + W_4.
\end{aligned}$$

Firstly, we estimate W_1, by Definition 2.1, Lemma 3.1 and Lemma 4.2. At the same time, μ^ρ_Ω was bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. We can obtain

$$\begin{aligned}
W_1 &\le C\left\|\mu^\rho_\Omega(f\chi_{2B})\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|(b-b_B)^m\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\left\|f\chi_{2B}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2B|^\lambda\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2B|^\lambda\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C|B|^\lambda\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, we estimate W_3, by Definition 2.1, Lemma 3.1, Lemma 4.2 and μ^ρ_Ω was bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. We have

$$\begin{aligned}
W_3 &\le C\|(b-b_B)^m f\chi_{2B}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\|(b-b_B)^m\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\left\|f\chi_{2B}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2B|^\lambda\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2B|^\lambda\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\le C|B|^\lambda\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, we estimate W_2 and W_4. When estimating U_2 in the second part, we get

$$|\mu_\Omega^\rho\left(f\chi_{(2B)^c}(x)\right)| \leq C|B|^\lambda \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.$$

Using this formula and Lemma 4.2, we have

$$\begin{aligned}
W_2 &= \|(b-b_B)^m(\mu_\Omega^\rho(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C|B|^\lambda \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)} \|(b-b_B)^m\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C|B|^\lambda \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\leq C|B|^\lambda\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, let us estimate W_4, the process is similar to the calculation process of U_4. Similarly, we first need to estimate $\left|\mu_\Omega^\rho\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right|$. We divide it into two parts, G_1 and G_2.

$$\begin{aligned}
&\left|\mu_\Omega^\rho\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right| \\
&= \left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{|x|+|y|}\left|\int_{|x-y|\leq t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}} \\
&+ \left(\int_{|x|+|y|}^\infty\left|\int_{|x-y|\leq t}\frac{(b(y)-b_B)^m\Omega(x-y)}{|x-y|^{n-\rho}}f\chi_{(2B)^c}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{\frac{1}{2}} \\
&= G_1 + G_2.
\end{aligned}$$

Firstly, we estimate G_1. According to Minkowski's inequality, the formula (3.2), Definition 2.1 , Lemma 2.5 and Lemma 4.2, we have

$$\begin{aligned}
G_1 &\leq \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x-y|\leq t\leq|x|+|y|}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}y \\
&\leq \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac{1}{2}}\mathrm{d}y \\
&\leq \sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x|+|y|)^{2\rho}}\right|^{\frac{1}{2}}\mathrm{d}y \\
&\leq \sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{(|x|+|y|)^\rho}{|x-y|^{2\rho}}\mathrm{d}y \\
&\leq \sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\frac{(|x|+|y|)^\rho}{|x-y|^{n+\rho}}\mathrm{d}y \\
&\leq \sum_{k=1}^\infty\left(R+2^{k+1}\right)^\rho\left(2^{k-1}R\right)^{-n-\rho}\int_{2^{k+1}B\backslash 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y,
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{\infty}\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot (k+1)^m\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{k=1}^{\infty}(k+1)^m\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\quad\cdot\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{k=1}^{\infty}(k+1)^m\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\le C\sum_{k=1}^{\infty}(k+1)^m\left(2^{k+2}R\right)^{\rho}\left(2^{k-1}R\right)^{-n-\rho}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\\
&\quad\cdot|2^{k+1}B|^{\lambda}|2^{k+1}B||2^{k+1}B|^{-\frac{1}{s}}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\le C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}|B|^{\lambda}\sum_{k=1}^{\infty}(k+1)^m2^{-(n+\rho)(k-1)+n(\lambda+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}+(k+2)\rho}\\
&\le C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}|B|^{\lambda}\sum_{k=1}^{\infty}(k+1)^m2^{-n(k-1)+n(\lambda+1)(k+1)-\frac{n}{s}+3\rho}\\
&\le C|B|^{\lambda}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

Next, we estimate G_2, according to Minkowski's inequality, Definition 2.1, Lemma 2.5 and Lemma 4.2, we have

$$
\begin{aligned}
G_2 &\le \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\left(\int_{|x|+|y|}^{\infty}\frac{1}{t^{2\rho+1}}\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}y\\
&\le \int_{\mathbb{R}^n}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f\chi_{(2B)^c}|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&= \int_{(2B)^c}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|b(y)-b_B|^m|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\frac{1}{|x-y|^{\rho}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\frac{1}{|x-y|^{n}}\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\int_{2^{k+1}B\setminus 2^kB}|b(y)-b_B|^m|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\le C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}\|\Omega\|_{L^s(S^{n-1})}|2^kB|^{1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot(k+1)^m\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\sum_{k=1}^{\infty}\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2^{k+1}B|^{\lambda}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\cdot(k+1)^m\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^{\infty}(k+1)^m\left(2^{k-1}R\right)^{-n}|2^kB|^{1/s}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}|2^{k+1}B||2^{k+1}B|^{\lambda}|2^{k+1}B|^{-\frac{1}{s}}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}|B|^{\lambda}\sum_{k=1}^{\infty}(k+1)^m2^{-n(k-1)+n(\lambda+1)(k+1)-\frac{n}{s}(k+1)+\frac{nk}{s}}\\
&\leq C\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}|B|^{\lambda}\sum_{k=1}^{\infty}(k+1)^m2^{-n(k-1)+n(\lambda+1)(k+1)-\frac{n}{s}}\\
&\leq C|B|^{\lambda}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

So

$$\left|\mu_{\Omega}^{\rho}\left((b-b_B)^m f\chi_{(2B)^c}\right)(x)\right|\leq C|B|^{\lambda}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.$$

We have

$$\begin{aligned}
U_4 &= \|\mu_{\Omega}^{\rho}((b-b_B)^m(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C|B|^{\lambda}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So

$$\|[b,\mu_{\Omega}^{\rho}]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C|B|^{\lambda}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

So

$$\|[b,\mu_{\Omega}^{\rho}]f\|_{\dot{\mathscr{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\left\|f\right\|_{\dot{\mathscr{B}}^{p_1(\cdot),\lambda}(\mathbb{R}^n)}.$$

Therefore, we complete the proof of Therorem 4.3.

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