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# Atomic decomposition on Grand Herz-Hardy spaces on locally compact Vilenkin groups

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**Abstract.** In this paper, we introduce the idea of grand Herz-Hardy spaces on locally compact Vilenkin groups. Then we obtain the central block and atomic decomposition on these spaces.

## 1. Introduction

The idea of Herz-Hardy spaces are introduced in [1, 2]. In this paper, we extend these results to grand Herz spaces and define the concept of grand Herz-Hardy spaces on locally compact Vilenkin group. We obtain the atomic decomposition of these spaces.

In this paper, $Q$ denotes the locally compact abelian group containing a strictly decreasing sequence of compact open subgroups $\{Q_m\}_{m=-\infty}^{\infty}$ such that

1. $\bigcap_{m=-\infty}^{\infty} Q_m = \{0\}$ and $\cup_{m=-\infty}^{\infty} Q_m = Q$,
2. $\sup\left\{\text{order}\left(\frac{Q_m}{Q_{m+1}}\right) : m \in \mathbb{Z}\right\} < \infty$.

Let $m \in \mathbb{Z}$, then $Q'$ denotes the dual group of $Q$ and define as $Q'_m = \{\gamma \in Q' : \gamma(y) = 1 \text{ for all } y \in Q_m\}$. Then $\{Q'_m\}_{m=-\infty}^{\infty}$ is a strictly increasing sequence of open compact subgroups of $Q'$ and

1. $\bigcap_{m=-\infty}^{\infty} Q'_m = \{1\}$ and $\cup_{m=-\infty}^{\infty} Q'_m = Q'$,
2. $\text{order}\left(\frac{Q'_{m+1}}{Q'_m}\right) = \text{order}\left(\frac{Q_m}{Q_{m+1}}\right)$.

Let $\mathbf{1}_A$ denote the characteristic function of a set $A$ and $d\gamma$ and $dx$ denotes the Haar measures of $Q'$ and $Q$ respectively. Let $|A|$ denotes the Haar measure of a measurable subset $A$ of $Q$, or $Q'$ such that $|Q_0| = \left|Q'_0\right| = 1$. Let $m \in \mathbb{Z}$, then $|Q_m|^{-1} = \left|Q'_m\right| := n_m$. Since $2n_m \le n_{m+1}$ where $m \in \mathbb{Z}$, implies that $\sum_{m=t_0}^{\infty} (n_m)^{-\beta} \le C\,(n_{t_0})^{-\beta}$ and $\sum_{m=-\infty}^{t_0} (n_m)^{a} \le C\,(m_{t_0})^{a}$ for any $\beta > 0, t_0 \in \mathbb{Z}$. Define the function $d : Q \times Q \to \mathbb{R}$ by $d(s,t) = 0$ when $s - t = 0$ and $d(s,t) = (n_m)^{-1}$ when $s - t \in Q_m \backslash Q_{m+1}$, then $d$ defines a metric on $Q \times Q$ and the topology on $Q$ induced by this metric is the same as the original topology on $Q$. Let $y \in Q$, we set $|y| = d(y, 0)$. Then $|y| = (n_m)^{-1}$ if

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and only if $y \in Q_m \setminus Q_{m+1}$. Let $\langle y\rangle = \max\left\{1, \left|y\right|\right\}$. Lets define a metric $\bar{d}$ on $Q' \times Q'$ such that $|\gamma| = n_{m+1}$ iff $\gamma \in Q'_{m+1} \backslash Q'_m$. Let $\hat{f}$ and $f^{\vee}$ denote the Fourier transform and inverse Fourier transform, respectively. We have $(\mathbf{1}_{Q_m})^{\wedge}(\gamma) = \left|Q'_m\right|^{-1} \mathbf{1}_{Q'_m}(\gamma)$ and $\left(\mathbf{1}_{Q'_{\mathbf{m}}}\right)^{\vee}(y) = |Q_m|^{-1} \mathbf{1}_{Q_{\mathbf{m}}}(\mathbf{y}) =: \mathbf{\Delta_n}(\mathbf{y})$ for each $m \in \mathbb{Z}$. The spaces $\mathfrak{S}(Q)$ ( or $\mathfrak{S}(Q')$ ) and $\mathfrak{S}'(Q)$ (or $\mathfrak{S}'(Q')$ ) denote the spaces of test functions and distributions on $Q$ (or $Q'$ ), respectively.

Let $\omega(y)$ be a nonnegative locally integrable function on $Q$. By $L^q_{\omega}(Q)$ we denote the weighted Lebesgue space with respect to the weight measure $\omega(y)\mathrm{d}y$ and $0 < q < \infty$. Let $\omega(A) = \int_A \omega(y)\mathrm{d}y$. Let $A_1(Q)$ is the usual Muckenhoupt class of weights. It is easy to verify that $\omega(y) = |y|^{\beta}$ with $-1 < \beta \leq 0$ is an $A_1(Q)$-weight. Let $\omega_1, \omega_2$ denotes the non-negative weights, $0 \leq \beta < \infty$, $\theta > 0$ and $0 < p, q < \infty$. Then the homogeneous grand Herz spaces $\dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q)$ are defined by

$\dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q) = \{g : g$ is a measurable function on $Q$ and $\left\|g\right\|_{\dot{K}^{\beta,p),\theta}_q(\omega_1,\omega_2;Q)} < \infty\Big\}$, where

$$\left\|g\right\|_{\dot{K}^{\beta,p)}_q(\omega_1,\omega_2;Q)} = \sup_{\epsilon>0}\left\{\epsilon^{\theta} \sum_{k=-\infty}^{\infty} \left[\omega_1\left(Q_k\right)\right]^{\beta(1+\epsilon)p} \left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|^{(1+\epsilon)p}_{L^q_{\omega_2}(Q)}\right\}^{1/(1+\epsilon)p}.$$

Let $g \in \mathfrak{S}'(Q)$, define maximal functions $g^*(y)$ and $Q^*_r(y)$ as

$$g^*(y) = \sup_{m\in\mathbb{Z}}\left|g * \Delta_m(y)\right| = \sup_{m\in\mathbb{Z}}\left|n_m \int_{y+Q_m} g(x)\mathrm{d}x\right|,$$

and

$$Q^*_r(y) = \sup_{m\in\mathbb{N}}\left|g * \Delta_m(y)\right| = \sup_{m\in\mathbb{N}}\left|n_m \int_{y+Q_m} g(x)\mathrm{d}x\right|.$$

Let $\omega_1, \omega_2$ denote the non-negative weights, $0 \leq \beta < \infty$, $\theta > 0$ and $0 < p, q < \infty$. Then the homogeneous grand Herz-Hardy spaces $H\dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q)$ are defined by

$$H\dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q) = \left\{g \in \mathfrak{S}'(Q) : g^* \in \dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q)\right\}$$

and

$$h\dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q) = \left\{g \in \mathfrak{S}'(Q) : Q^*_r \in \dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q)\right\}.$$

Numerous studies have explored various extensions and versions of Herz-type spaces. For a comprehensive account of these developments and related results, see [5–18].

## 2. Central atomic decomposition on grand Herz-Hardy spaces

**Definition 2.1.** *Let $\omega_1, \omega_2$ be non-negative weights, $0 \leq a < \infty$ and $0 < q < \infty$.*
*(a) A function $a \in L^q_{\omega_2}(Q)$ is said to be a central $(\beta, q; \omega_1, \omega_2)_Q$-block with the support $Q_m$ for some $m \in \mathbb{Z}$, if i) supp $a \subseteq Q_m$; ii) $\|a\|_{L^q_{\omega_2}(Q)} \leq [\omega_1(Q_m)]^{-\beta}$.*
*(b) A function $a \in L^q_{\omega_2}(Q)$ is said to be a central $(\beta, q; \omega_1, \omega_2)_Q$-atom with the support $Q_m$ for some $m \in \mathbb{Z}$, if it satisfies i) and ii) in (a), and iii) $\int_Q a(y)\mathrm{d}y = 0$.*

**Theorem 2.2.** *Let $\omega_1 \in A_1(Q)$ and $\omega_2$ be any non-negative weight function on $Q$. Assume $0 < \beta < \infty, \theta > 0, 1 \leq q < \infty$ and $0 < p < \infty$. Then $g \in \dot{K}^{\beta,p),\theta}_q(\omega_1, \omega_2; Q)$ if and only if $g(y) = \sum_{k=-\infty}^{\infty} \lambda_k b_k(y)$, where $b_k$ is a central $(\beta, q; \omega_1, \omega_2)_Q$-block supported on $Q_k$ and $\sup_{\epsilon>0} \epsilon^{\theta} \sum_{k=-\infty}^{\infty} |\lambda_k|^{(1+\epsilon)p} < \infty$. Moreover,*

$$\left\|g\right\|_{\dot{K}^{\beta,p),\theta}_q(\omega_1,\omega_2;Q)} \sim \inf\left\{\sup_{\epsilon>0}\left(\epsilon^{\theta} \sum_{k=-\infty}^{\infty} |\lambda_k|^{(1+\epsilon)p}\right)^{1/(1+\epsilon)p}\right\},$$

*where the infimum is taken over all the decompositions of $g$ as above.*

*Proof.* Let $g \in \dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)$, write $g$ as

$$\begin{aligned}
g(y) &= \sum_{k\in\mathbb{Z}} g(y)\mathbf{1}_{Q_k\setminus Q_{k+1}}(y)\\
&= \sum_{k\in\mathbb{Z}} (\omega_1(Q_k))^{\beta}\left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|_{L^q_{\omega_2}} \frac{g(y)\mathbf{1}_{Q_k\setminus Q_{k+1}}(y)}{(\omega_1(Q_k))^{\beta}\left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|_{L^q_{\omega_2}(Q)}}\\
&:= \sum_{k\in\mathbb{Z}} \lambda_k b_k(y),
\end{aligned}$$

where $\lambda_k = (\omega_1(Q_k))^{\beta}\left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|_{L^q_{\omega_2}(Q)}$ and $b_k(y) = \frac{g(y)\mathbf{1}_{Q_k\setminus Q_{k+1}}(y)}{(\omega_1(Q_k))^{\beta}\left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|_{L^q_{\omega_2}(Q)}}$. It is easy to see that $\operatorname{supp} b_k \subset Q_k$ and $|b_k\|_{L^q_{\omega_2}(Q)} = (\omega_1(Q_k))^{-\beta}$. So $b_k(y)$ is a $(\beta,q;\omega_1,\omega_2)_Q$-atom supported in $Q_k$. Thus we have

$$\begin{aligned}
\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}|\lambda_k|^{(1+\epsilon)p}\right)^{1/(1+\epsilon)p} &= \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{\infty}[\omega_1(Q_k)]^{\beta(1+\epsilon)p}\left\|g\mathbf{1}_{Q_k\setminus Q_{k+1}}\right\|^{(1+\epsilon)p}_{L^q_{\omega_2}(Q)}\right\}^{1/(1+\epsilon)p}\\
&= \|g\|_{\dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)} < \infty.
\end{aligned}$$

Conversely, let $g(y) = \lambda_k b_k(y)$ and $\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}|\lambda_k|^{(1+\epsilon)p}\right)^{1/(1+\epsilon)p} < \infty$. For $j \in \mathbb{Z}$, we have

$$\left\|g\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|_{L^q_{\omega_2}(Q)} = \left\|\left(\sum_{k\in\mathbb{Z}}\lambda_k b_k\right)\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|_{L^q_{\omega_2}(Q)} \le \sum_{k\le j}|\lambda_k|\left\|b_k\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|_{L^q_{\omega_2}(Q)}.$$

*Case* 1: *When* $0 < (1+\epsilon)p \le 1$. *We get*

$$\begin{aligned}
\|g\|_{\dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)} &= \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{j=-\infty}^{\infty}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p}\left\|g\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|^{(1+\epsilon)p}_{L^q_{\omega_2}(Q)}\right\}^{1/(1+\epsilon)p}\\
&\le \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{j=-\infty}^{\infty}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p}\left(\sum_{k\le j}|\lambda_k|^{(1+\epsilon)p}\left\|b_k\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|^{(1+\epsilon)p}_{L^q_{\omega_2}(Q)}\right)\right\}^{1/(1+\epsilon)p}\\
&\le \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{j=-\infty}^{\infty}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p}\left(\sum_{k\le j}|\lambda_k|^{(1+\epsilon)p}[\omega_1(Q_k)]^{-\beta(1+\epsilon)p}\right)\right\}^{1/(1+\epsilon)p}\\
&\le \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{\infty}|\lambda_k|^{(1+\epsilon)p}\left(\sum_{j=k}^{\infty}[\omega_1(Q_k)]^{-\beta(1+\epsilon)p}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p}\right)\right\}^{1/(1+\epsilon)p}\\
&\le \sup_{\epsilon>0}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{\infty}|\lambda_k|^{(1+\epsilon)p}\sum_{j=k}^{\infty}\left(\frac{\omega_1\left(Q_j\right)}{\omega_1(Q_k)}\right)^{\beta(1+\epsilon)p}\right\}^{1/(1+\epsilon)p}\\
&\le \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}|\lambda_k|^{(1+\epsilon)p}\right)^{1/(1+\epsilon)p} < \infty.
\end{aligned}$$

*Case* 2: *When* $1 \le (1+\epsilon)p < \infty$:

$$\left\|g\mathbf{1}_{Q_j\setminus Q_{j+1}}\right\|_{L^q_{\omega_2}(Q)} \le C[\omega_1(Q_j)]^{\beta/2}\left(\sum_{k\le j}|\lambda_k|^{(1+\epsilon)p}[\omega_1(Q_k)]^{\beta(1+\epsilon)p/2}\right)^{1/(1+\epsilon)p}.$$

*And therefore*

$$\begin{aligned}\|g\|_{\dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)} &= \sup_{\epsilon>0}\left\{\epsilon^\theta \sum_{j=-\infty}^{\infty}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p}\|g\mathbf{1}_{Q_j\setminus Q_{j+1}}\|^{(1+\epsilon)p}_{L^q_{\omega_2}(Q)}\right\}^{1/(1+\epsilon)p}\\ &\le C\sup_{\epsilon>0}\left\{\epsilon^\theta \sum_{j=-\infty}^{\infty}\left[\omega_1\left(Q_j\right)\right]^{\beta(1+\epsilon)p/2}\left(\sum_{k\le j}|\lambda_k|^{(1+\epsilon)p}\left[\omega_1(Q_k)\right]^{\beta(1+\epsilon)p/2}\right)\right\}^{1/(1+\epsilon)p}\\ &\le \sup_{\epsilon>0}\left\{\epsilon^\theta \sum_{k=-\infty}^{\infty}|\lambda_k|^{(1+\epsilon)p}\sum_{j=k}^{\infty}\left(\frac{\omega_1\left(Q_j\right)}{\omega_1\left(Q_k\right)}\right)^{\beta(1+\epsilon)p}\right\}^{1/(1+\epsilon)p}\\ &\le \sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k\in\mathbb{Z}}|\lambda_k|^{(1+\epsilon)p/2}\right)^{1/(1+\epsilon)p} < \infty.\end{aligned}$$

□

**Theorem 2.3.** *Let $\omega_1,\omega_2\in A_1(Q), 1<q<\infty, -1<\beta\le 0$ and $0<p\le 1$. Then $g\in H\dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)$ iff $g=\sum_{k=-\infty}^{\infty}\lambda_k a_k$ in distributional sense, where each $a_k$ is a central $(\beta,q;\omega_1,\omega_2)_Q$-atom supported on $Q_k$, and $\sup_{\epsilon>0}\epsilon^\theta\sum_{k=-\infty}^{\infty}|\lambda_k|^{(1+\epsilon)p}<\infty$. Moreover,*

$$\|g\|_{H\dot{K}_q^{\beta,p),\theta}(\omega_1,\omega_2;Q)} \sim \inf\left\{\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{\infty}|\lambda_k|^{(1+\epsilon)p}\right)^{1/(1+\epsilon)p}\right\},$$

*where the infimum is taken over all the decompositions of $g$ as above.*

*Proof.*

$$\begin{aligned}g(z) &= \sum_{t_0=-\infty}^{\infty} g(z)\mathbf{1}_{Q_{t_0}\setminus Q_{t_0+1}}(z)\\ &= \sum_{t_0=-\infty}^{\infty}\left\{g(z)\mathbf{1}_{Q_{t_0}\setminus Q_{t_0+1}}(z) - \left(\int_{Q_{t_0}\setminus Q_{t_0+1}} g(y)dy\right)\frac{\mathbf{1}_{Q_{t_0+1}\setminus Q_{t_0+2}}(z)}{\left|(Q_{t_0+1}\setminus Q_{t_0+2})\right|}\right\}\\ &\quad+\sum_{t_0=-\infty}^{\infty}\left(\int_{Q_{t_0}\setminus Q_{t_0+1}} g(y)dy\right)\frac{\mathbf{1}_{Q_{t_0+1}\setminus Q_{t_0+2}}(z)}{\left|(Q_{t_0+1}\setminus Q_{t_0+2})\right|}\\ &=: I_1+I_2.\end{aligned}$$

*If*

$$b_{t_0}(z) := g(z)\mathbf{1}_{Q_{t_0}\setminus Q_{t_0+1}}(z) - \left(\int_{Q_{t_0}\setminus Q_{t_0+1}} g(y)dy\right)\frac{\mathbf{1}_{Q_{t_0+1}\setminus Q_{t_0+2}}(z)}{\left|(Q_{t_0+1}\setminus Q_{t_0+2})\right|}.$$

*It is easy to note that $\int b_{t_0}(z)dz=0$, where $\operatorname{supp} b_{t_0}\subset Q_{t_0}\setminus Q_{t_0+2}\subset Q_{t_0}$. Let $\beta>-1$, then from Lemma 1 of [3] and (1) in [4] yields*

$$\omega_1\left(Q_{t_0+1}\setminus Q_{t_0+2}\right)\approx\omega_1\left(Q_{t_0+1}\right)\approx\left(m_{t_0+1}\right)^{-(\beta+1)}\approx\left(m_{t_0}\right)^{-(\beta+1)}.$$

*Next, we get*

$$\begin{aligned}\left(\int_{Q_{t_0}} \left|b_{t_0}(z)\right|^{(1+\epsilon)p} d\omega_1(z)\right)^{1/(1+\epsilon)p} \leq& \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|^{(1+\epsilon)p} d\omega_1(z)\right)^{1/(1+\epsilon)p} \\ &+ \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|dz\right) \frac{\omega_1\left(Q_{t_0+1}\backslash Q_{t_0+2}\right)^{1/(1+\epsilon)p}}{\left|(Q_{t_0+1}\backslash Q_{t_0+2})\right|} \\ \leq& \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|^{(1+\epsilon)p} d\omega_1(z)\right)^{1/(1+\epsilon)p} \\ &+ \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|^{(1+\epsilon)p} |z|^{\beta} dz\right)^{1/(1+\epsilon)p} \\ &\times \frac{(m_{t_0})^{\beta/(1+\epsilon)p} \left|(Q_{t_0}\backslash Q_{t_0+1})\right|^{1-1/(1+\epsilon)p} \omega_1\left(Q_{t_0+1}\backslash Q_{t_0+2}\right)^{1/(1+\epsilon)p}}{\left|(Q_{t_0+1}\backslash Q_{t_0+2})\right|} \\ \leq& C_0 \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|^{(1+\epsilon)p} d\omega_1(z)\right)^{1/(1+\epsilon)p}.\end{aligned}$$

*Hence we get*

$$a_{t_0}(z) := \left\{C_0 \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(y)|^{(1+\epsilon)p} d\omega_1(y)\right)^{1/(1+\epsilon)p}\right\}^{-1} \omega_1\left(Q_{t_0}\right)^{1/(1+\epsilon)p-1} b_{t_0}(z),$$

*is a central* $(\beta, q; \omega_1, \omega_2)_Q$*-atom. Let*

$$\lambda_{t_0} := C_0 \omega_1\left(Q_{t_0}\right)^{1-1/(1+\epsilon)p} \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} |g(z)|^{(1+\epsilon)p} d\omega_1(z)\right)^{1/(1+\epsilon)p}.$$

*Then* $I_1 = \sum\limits_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}(z)$*, and*

$$\begin{aligned}\sup_{\epsilon>0} \epsilon^{\theta} \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} &\leq C_0 \sup_{\epsilon>0} \epsilon^{\theta} \sum_{t_0=-\infty}^{\infty} \omega_1\left(Q_{t_0}\right)^{1/((1+\epsilon)p)'} \left\|g \mathbf{1}_{Q_{t_0}\backslash Q_{t_0+1}}\right\|_{L^q_{\omega_2}(Q)} \\ &\leq C_0 \|g\|_{\dot{K}_q^{\beta,1),\theta}(\omega_1,\omega_2;Q)} \leq C_0 \left\|g^*\right\|_{\dot{K}_q^{\beta,1),\theta}(\omega_1,\omega_2;Q)} = C_0 \left\|g\right\|_{H\dot{K}_q^{\beta,1),\theta}(\omega_1,\omega_2;Q)}.\end{aligned}$$

*To estimate* $I_2$*, we write* $\widetilde{\mathbf{1}}_{Q_{t_0}\backslash Q_{t_0+1}}(z) = \left|(Q_{t_0}\backslash Q_{t_0+1})\right|^{-1} \mathbf{1}_{Q_{t_0}\backslash Q_{t_0+1}}(z)$*; then*

$$\begin{aligned} I_2 &= \sum_{t_0=-\infty}^{\infty} \left(\int_{Q_{t_0}\backslash Q_{t_0+1}} g(y)dy\right) \widetilde{\mathbf{1}}_{Q_{t_0+1}\backslash Q_{t_0+2}}(z) \\ &= \sum_{t_0=-\infty}^{\infty} \left\{\sum_{j=t_0}^{\infty} \int_{Q_j\backslash Q_{j+1}} g(y)dy\right\} \left(\widetilde{\mathbf{1}}_{Q_{t_0+1}\backslash Q_{t_0+2}}(z) - \widetilde{\mathbf{1}}_{Q_{t_0}\backslash Q_{t_0+1}}(z)\right) \\ &=: \sum_{t_0=-\infty}^{\infty} h_{t_0}(z),\end{aligned}$$

*where*

$$h_{t_0}(z) : = \left(\sum_{j=t_0}^{\infty} \int_{Q_j\backslash Q_{j+1}} g(y)dy\right) \left(\widetilde{\mathbf{1}}_{Q_{t_0+1}\backslash Q_{t_0+2}}(z) - \widetilde{\mathbf{1}}_{Q_{t_0}\backslash Q_{t_0+1}}(z)\right)$$

$$= \left( \int_{Q_{t_0}} g(y)dy \right) \left( \widetilde{\mathbf{1}}_{Q_{t_0+1} \backslash Q_{t_0+2}}(z) - \widetilde{\mathbf{1}}_{Q_{t_0} \backslash Q_{t_0+1}}(z) \right).$$

*Hence supp* $h_{t_0} \subset Q_{t_0} \backslash Q_{t_0+2} \subset Q_{t_0}$ *and* $\int h_{t_0}(z)dz = 0$. *Consequently we get*

$$\begin{aligned} \left| h_{t_0}(z) \right| &\leq C_1 \left| (Q_{t_0}) \right| g^*(z) \mathbf{1}_{Q_{t_0}}(z) \left( \left| (Q_{t_0}) \right|^{-1} \mathbf{1}_{Q_{t_0} \backslash Q_{t_0+1}}(z) + \left| (Q_{t_0}+1) \right|^{-1} \mathbf{1}_{Q_{t_0+1} \backslash Q_{t_0+2}}(z) \right) \\ &\leq C_1 g^*(z) \mathbf{1}_{Q_{t_0} \backslash Q_{t_0+1}}(z) + C_1 g^*(z) \mathbf{1}_{Q_{t_0+1} \backslash Q_{t_0+2}}(z) \Big). \end{aligned}$$

*We have*

$$\left\| h_{t_0} \right\|_{L^q_{\omega_2}(Q)} \leq C_1 \left\| g^* \mathbf{1}_{Q_{t_0} \backslash Q_{t_0+1}} \right\|_{L^q_{\omega_2}(Q)} + C_1 \left\| g^* \mathbf{1}_{Q_{t_0+1} \backslash Q_{t_0+2}} \right\|.$$

*Thus*

$$a_{t_0}(z) := \left\{ C_1 \left\| g^* \mathbf{1}_{Q_{t_0} \backslash Q_{t_0+1}} \right\|_{L^q_{\omega_2}(Q)} + C_1 \left\| g^* \mathbf{1}_{Q_{t_0+1} \backslash Q_{t_0+2}} \right\|_{L^q_{\omega_2}(Q)} \right\}^{-1} \omega_1 \left( Q_{t_0} \right)^{1/(1+\epsilon)p-1} h_{t_0}(z),$$

*is a central* $(\beta, q; \omega_1, \omega_2)_Q$*-atom. If we write*

$$\lambda_{t_0} := \left\{ C_1 \left\| g^* \mathbf{1}_{Q_{t_0} \backslash Q_{t_0+1}} \right\|_{L^q_{\omega_2}(Q)} + C_1 \left\| g^* \mathbf{1}_{Q_{t_0+1} \backslash Q_{t_0+2}} \right\|_{L^q_{\omega_2}(Q)} \right\} \omega_1 \left( Q_{t_0} \right)^{1-1/(1+\epsilon)p}.$$

*Then* $I_2 = \sum \lambda_{t_0} a_{t_0}$, *and*

$$\sup_{\epsilon > 0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} \left| \lambda_{t_0} \right| \leq C_1 \left\| g^* \right\|_{\dot{K}_q^{\beta,1),\theta}(\omega_1, \omega_2; Q)} = C_1 \left\| g \right\|_{H\dot{K}_q^{\beta,1),\theta}(\omega_1, \omega_2; Q)}.$$

*Next we will prove that* $g = \sum \lambda_{t_0} a_{t_0} \in S'(Q)$. *Firstly we will have*

$$\lambda_{t_0} \int_{Q_{t_0+1} \backslash Q_{t_0+2}} a_{t_0}(z)dz \to 0 \quad \text{as } t_0 \to \infty. \tag{1}$$

*Thus, we get*

$$\int_{Q_{t_0+1} \backslash Q_{t_0+2}} b_{t_0}(z)dz \to 0 \quad \text{and} \quad \int_{Q_{t_0+1} \backslash Q_{t_0+2}} h_{t_0}(z)dz \to 0.$$

*As* $t_0 \to \infty$. *Next by using the definitions of* $b_{t_0}$, $h_{t_0}$, *and* (1), *we get*

$$\int_{Q_{t_0} \backslash Q_{t_0+1}} g(y)dy \to 0 \quad \text{and} \quad \int_{Q_{t_0}} g(y)dy \to 0. \tag{2}$$

*As* $t_0 \to \infty$. *Let* $\beta = 0$, $g \in L^1(Q)$ *then* $0 < \left| Q_{t_0} \backslash Q_{t_0+1} \right| \leq \left| Q_{t_0} \right| \to 0$ *as* $t_0 \to \infty$, (2) *holds. If* $-1 < \beta < 0$, $g \in L^1_\beta(Q)$, *then* $|y| \leq (m_{t_0})^{-1}$ *for* $y \in Q_{t_0}$, *and* $|y| = (m_{t_0})^{-1}$ *for* $y \in Q_{t_0} \backslash Q_{t_0+1}$, *we obtain*

$$\left| \int_{Q_{t_0}} g(y)dy \right| \leq (m_{t_0})^\beta \int_{Q_{t_0}} \left| g(y) \right| \left| y \right|^\beta dy \leq (m_{t_0})^\beta \left\| g \right\|_{L^1_{\omega_2}(Q)} \to 0.$$

*If* $t_0 \to \infty$, *and*

$$\left| \int_{Q_{t_0} \backslash Q_{t_0+1}} g(y)dy \right| \leq \int_{Q_{t_0} \backslash Q_{t_0+1}} |g(y)|dy \leq \int_{Q_{t_0}} |g(y)|dy \to 0.$$

*If $t_0 \to \infty$, then* (2) *also holds.*
*If $\varphi \in S(Q)$, $\varphi$ is constant on the cosets of $Q_{t_0}$ in $Q$ but not on the cosets of $Q_{t_0-1}$ (unless $\varphi(z) \equiv 0$ ) and* $\operatorname{supp} \varphi \subset Q_m$*. Obviously, $t_0 \geq m$. Because $g(z) = \sum \lambda_{t_0} a_{t_0}(z)$ pointwise, from* (1) *and* $\int_{Q_{t_0} \setminus Q_{t_0+2}} a_{t_0}(z) d\mu(z) = 0$*, it follows*

$$\begin{aligned}\langle g, \varphi \rangle &= \int \left( \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}(z) \right) \varphi(z) dz \\ &= \sum_{t=m}^{\infty} \int_{Q_t \setminus Q_{t+1}} \left( \sum_{t_0=t-1}^{t} \lambda_{t_0} a_{t_0}(z) \right) \varphi(z) dz \\ &= \sum_{t=m}^{\infty} \left( \lambda_{t-1} \int_{Q_t \setminus Q_{t+1}} a_{t-1} \varphi + \lambda_t \int_{Q_t \setminus Q_{t+1}} a_t \varphi \right).\end{aligned}$$

*Hence we have*

$$\begin{aligned}&= \begin{cases} \lambda_{m-1} \int_{Q_m \setminus Q_{m+1}} a_{m-1} \varphi, & t_0 = m, \\ \lambda_{m-1} \int_{Q_m \setminus Q_{m+1}} a_{m-1} \varphi + \sum\limits_{t=m}^{t_0-1} \lambda_t \int a_t \varphi, & t_0 > m, \end{cases} \\ &= \lim_{\substack{m_1 \to \infty \\ m_2 \to \infty}} \int \left( \sum_{t_0=-m_1}^{m_2} \lambda_{t_0} a_{t_0} \right) \varphi,\end{aligned}$$

*that is, $g = \sum \lambda_{t_0} a_{t_0}$ in $S'(Q)$.*

*Let $0 < (1+\epsilon)p < 1$ yields $\lim_{n \to \infty} Q_n = \lim_{n \to \infty} g * \Delta_n = g \in S'(Q)$ and $Q_n(z) = g * \Delta_n(z)$ is a function on $Q$ which is constant on the cosets of $Q_n$ in $Q$. It is easy to note that*

$$|Q_n(z)| \leq g^*(z) \tag{3}$$

$$(Q_n)^*(z) \leq g^*(z), \tag{4}$$

$$\int_{Q_{t_0} \setminus Q_{t_0+1}} Q_n(z) dz \to 0 \quad \text{and} \quad \int_{Q_{t_0}} Q_n(z) dz \to 0 \quad \text{as } t_0 \to \infty. \tag{5}$$

*Using* (3),(4) *and* (5) *for $Q_n$, similarly we get*

$$Q_n(z) = \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}^n(z) \tag{6}$$

*in $S'(Q)$ pointwise, where $\sup\limits_{\epsilon>0} \epsilon^{\theta} \sum\limits_{t_0=-\infty}^{\infty} \left| \lambda_{t_0} \right|^{(1+\epsilon)p} \leq C \left\| g \right\|_{H\dot{K}_q^{\beta, p), \theta}(Q)}^{(1+\epsilon)p}$, and each $a_{t_0}^n(z)$ is a central $(\beta, q; \omega_1, \omega_2)_Q$-atom supported in $Q_{t_0}$, note that* $\operatorname{supp} a_{t_0}^n \subset Q_{t_0} \setminus Q_{t_0+2}$*. Hence we get*

$$\sup_{n \in \mathbb{N}} \left\| a_0^n(z) \right\|_{L_{\omega_2}^q(Q)} \leq \omega_1 (Q_0)^{1/q - 1/(1+\epsilon)p}.$$

*Then using the Banach-Alaoglu theorem yields a subsequence $\left\{ a_0^{n_{v_0}} \right\}$ of $\left\{ a_0^n \right\}$ converging in the weak* topology of $L_{\omega_2}^q(Q)$ to some $a_0 \in L_{\omega_2}^q(Q)$. It is easy to verify that $a_0$ is a central $(\beta, q; \omega_1, \omega_2)_Q$-atom supported in $Q_0$. Next, since*

$$\sup_{n_{v_0}} \left\| a_1^{n_{v_0}}(z) \right\|_{L_{\omega_2}^q(Q)} \leq \omega_1 (Q_1)^{1/q - 1/(1+\epsilon)p}.$$

*Another application of the Banach-Alaoglu theorem yields a subsequence $\left\{ a_1^{n_{v_1}} \right\}$ of $\left\{ a_1^{n_{v_0}} \right\}$ and a central $(\beta, q; \omega_1, \omega_2)_Q$-atom $a_1$ with* $\operatorname{supp} a_1 \subset Q_1$ *which converges weak* in $L_{\omega_2}^q(Q)$ to $a_1$. Hence we get*

$$\sup_{n_{v_1}} \left\| a_{-1}^{n_{v_1}}(z) \right\|_{L_{\omega_2}^q(Q)} \leq \omega_1 (Q_{-1})^{1/q - 1/(1+\epsilon)p}.$$

*Again repeating the process we get the subsequence* $\left\{a_{-1}^{n_{v-1}}\right\}$ *of* $\left\{a_{-1}^{n_{v_1}}\right\}$ *which converges weak* in* $L^q_{\omega_2}(Q)$ *to some* $a_{-1} \in L^q_{\omega_2}(Q)$*, and* $a_{-1}$ *is a central* $(\beta, q; \omega_1, \omega_2)_Q$*-atom supported in* $Q_{-1}$*. Repeating the above process, for each* $t_0 \in \mathbb{Z}$*, we can find a subsequence* $\left\{a_{t_0}^{n_{v_{t_0}}}\right\}$ *of* $\left\{a_{t_0}^{n}\right\}$ *converging weak* in* $L^q_{\omega_2}(Q)$ *to some* $a_{t_0} \in L^q_{\omega_2}(Q)$*, and* $a_{t_0}$ *is a central* $(\beta, q; \omega_1, \omega_2)_Q$*-atom supported in* $Q_{t_0}$*. Then the diagonal method yields a sequence* $\{n_v\}$ *of natural numbers such that for every* $t_0 \in \mathbb{Z}$*,* $\lim_{v\to\infty} a_{t_0}^{n_v} = a_{t_0}$ *in the weak* topology of* $L^q_{\omega_2}(Q)$*, and therefore, in* $S'(Q)$*. Next we will prove that*

$$g = \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0} \tag{7}$$

*in* $S'(Q)$*. Next, we get*

$$\begin{aligned}
\langle g, \varphi\rangle &= \lim_{n_v\to\infty} \langle Q_{n_v}, \varphi\rangle = \lim_{n_v\to\infty} \left\langle \sum_{t_0=-\infty}^{\infty} \lambda_{t_0} a_{t_0}^{n_v}, \varphi \right\rangle \\
&= \begin{cases} \lim_{n_v\to\infty} \lambda_{m-1} \int a_{m-1}^{n_v}\varphi, & t_0 = m, \\ \lim_{n_v\to\infty} \left\{ \lambda_{m-1} \int a_{m-1}^{n_v}\varphi + \sum\limits_{t_0=m}^{t_0-1} \lambda_{t_0} \int a_{t_0}^{n_v}\varphi \right\}, & t_0 > m, \end{cases} \\
&= \begin{cases} \lambda_{m-1} \int a_{m-1}\varphi, & t_0 = m, \\ \lambda_{m-1} \int a_{m-1}\varphi + \sum\limits_{t_0=m}^{t_0-1} \lambda_{t_0} \int a_{t_0}\varphi, & t_0 > m, \end{cases} \\
&= \lim_{\substack{m_1\to\infty \\ m_2\to\infty}} \int \left( \sum_{t_0=-m_1}^{m_2} \lambda_{t_0} a_{t_0} \right)\varphi,
\end{aligned}$$

*that is,* (7) *holds in* $S'(Q)$*.*

*Conversely, suppose* $g = \sum\limits_{j=-\infty}^{\infty} \lambda_j a_j$ *in* $S'(Q)$*, where* $a_j$ *is a central* $(\beta, q; \omega_1, \omega_2)_Q$*-atom. Then* $g^*(z) \le \sum\limits_{j=-\infty}^{\infty} \left|\lambda_j\right| a_j^*(z)$ *and*

$$\left\|g^*\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)} \le \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^{(1+\epsilon)p} \left\|a_j^*\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)}.$$

*Next it is remaining to prove that*

$$\left\|a_j^*\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)} \le C,$$

*where C is independent of* $a_j$*. Let supp* $a_j \subset Q_{n_j}$*, we need to prove that* $\operatorname{supp} a_j^* \subset Q_{n_j}$*. Hence we get*

$$a_j * \Delta_n(z) = m_n \int_{Q_{n_j}\cap(z+Q_n)} a_j(y)dy.$$

*Thus, if* $z \notin Q_{n_j}$ *then for* $n_j \ge n, Q_{n_j} \subseteq Q_n$ *and* $Q_{n_j} \cap (z + Q_n) \ne \emptyset$ *implies* $Q_{n_j} \cap (z + Q_n) \subseteq Q_n$*, so* $a_j * \Delta_n(z) = 0$*. If* $n > n_j$*, then* $Q_n \subseteq Q_{n_j}$ *which leads to* $Q_{n_j} \cap (z + Q_n) = \emptyset$*. Note* $a_j^*(z) = \sup_{n\in\mathbb{Z}} \left(a_j * \Delta_n\right)(z)$ *and* $\operatorname{supp} a_j^* \subset Q_{n_j}$*. Hence we get*

$$\begin{aligned}
\left\|a_j^*\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)} &= \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} \omega_1(Q_{t_0})^{1-(1+\epsilon)p/q} \left\|a_j^* \mathbf{1}_{Q_{t_0}\setminus Q_{t_0+1}}\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)} \\
&\le C \sup_{\epsilon>0} \epsilon^\theta \sum_{t_0=-\infty}^{\infty} \omega_1(Q_{t_0})^{1-(1+\epsilon)p/q} \left\|a_j\right\|^{(1+\epsilon)p}_{\dot{K}_q^{\beta,p),\theta}(Q)} \\
&\le C.
\end{aligned}$$

*Which completes the proof.* □

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