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Half inverse problem for the discontinuous Dirac operator with the spectral parameter boundary conditions

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Abstract. In this work, we consider the Dirac operator with the spectral parameter boundary conditions and the jump condition at the point $\frac{\pi}{2}$. We investigate the spectral properties and establish a new uniqueness theorem of the half inverse problem for discontinuous impulsive Dirac operator operators by constructing the Weyl function and exploiting its relevant properties. We conclude that if the potential is known on $(0, \frac{\phi(\pi)}{2})$, where $\frac{\phi(\pi)}{2} < \frac{\pi}{2}$, then the potential on $(0, \pi)$, the partial parameters in the boundary conditions and jump conditions can be uniquely determined by only one spectrum.

1. Introduction

Define $\sigma(x) = \begin{cases} 1, & x < \dfrac{\pi}{2} \\ \alpha, & x > \dfrac{\pi}{2} \end{cases}$ $(0 < \alpha < 1)$. Consider the following impulsive Dirac operator:

$$Ly := By'(x) + Q(x)y(x) = \lambda\sigma(x)y(x),\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},\ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

the function $p(x)$ and $q(x)$ are real-valued functions in $L^2(0,\pi)$, λ is the spectral parameter.

We denote by $L = L(p(x), q(x), \sigma(x), R_{ij}(\lambda), a)$ $(i, j = 0, 1)$ the boundary value problem generated by (1) with the boundary conditions

$$\begin{aligned} U(y) :=& R_{01}(\lambda)y_2(0) + R_{00}(\lambda)y_1(0) \\ =& R_{01}(\lambda)\begin{pmatrix} y_{21}(0) \\ y_{22}(0) \end{pmatrix} + R_{00}(\lambda)\begin{pmatrix} y_{11}(0) \\ y_{12}(0) \end{pmatrix} = 0, \end{aligned} \tag{2}$$

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$$\begin{aligned} V(y) :=& R_{11}(\lambda)y_2(\pi) + R_{10}(\lambda)y_1(\pi) \\ =& R_{11}(\lambda)\begin{pmatrix} y_{21}(\pi) \\ y_{22}(\pi) \end{pmatrix} + R_{10}(\lambda)\begin{pmatrix} y_{11}(\pi) \\ y_{12}(\pi) \end{pmatrix} = 0, \end{aligned} \tag{3}$$

and the discontinuous conditions

$$y\Big(\frac{\pi}{2}+0\Big) = Ay\Big(\frac{\pi}{2}-0\Big),\ \ A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \tag{4}$$

where $a \in \mathbb{R}$ and

$$R_{ij}(\lambda) = \sum_{l=0}^{r_{ij}} c_{ijl}\lambda^{r_{ij}-l}\ (r_{i1} = r_{i0} \geq 0,\ c_{i10} = 1,\ i, j = 0, 1)$$

are arbitrary polynomials of degree r_i with real coefficients such that $R_{i1}(\lambda)$ and $R_{i0}(\lambda)$ have no common zeros.

This paper focuses on half inverse problem for the discontinuous Dirac problem. Here the half inverse problem refers to reconstructing the potential function over the entire interval and boundary condition parameters using spectral data and partial potential information from half of the interval. However, the primary problem addressed in this work and its key innovation is investigating whether reconstruction remains possible when given even less information: specifically, whether the discontinuous Dirac operator can still be uniquely determined using just one spectrum when provided with potential function information from less than half of the interval.

The boundary value problem with a discontinuous always arises in mathematics, physics, geophysics, and other aspects of natural sciences. This kind of problem has been studied by many authors (see, e.g., [1–3, 5, 8, 10, 16, 17, 20–22] and the references therein).

In 1978, Hochstadt and Lieberman (see [13]) firstly studied the half inverse problem for the Sturm-Liouville operator, and showed that if the potential is known a priori on half interval, then one spectrum is sufficient to determine the potential on the whole interval. Subsequently, scholars generalized the conclusions of the Sturm-Liouville operator to the Dirac operator. For example, Mochizuki and Trooshin [15] considered the Dirac problem $L = L(p(x), q(x), 1, 0, 1)$, with the separable boundary conditions, where $R_{01}(\lambda) = R_{10}(\lambda) = 0$, they proved the uniqueness theorem by a set of values of eigenfunctions in some internal point and a single spectrum. In 2005, Amirov [1] investigated the problem $L = L(p(x), q(x), 1, 0, a)$, where $R_{01}(\lambda) = R_{10}(\lambda) = 0$, described the express of the Dirac equation, spectral property, and gave the uniqueness theorem by the Weyl function.

The research on the Dirac operator with boundary conditions depending on the spectral parameter has also attracted the attention of many scholars (see, e.g., [9, 11, 12]). In 2011, Keskin and Ozkan [14] provided spectral characteristics, the representation of resolvent and proved the inverse problem by three kinds of data: (i) the Weyl function; (ii) the sets of eigenvalues and norming constants; (iii) two different eigenvalues sets. In [7], Güldü investigated the discontinuous Dirac operator with eigenparameter dependent boundary and two transmission conditions, they got some properties of eigenvalues and eigenfunctions, and presented the uniqueness theorem by using the Weyl function and some spectral data. Yang [19] studied the Titchmarsh-Weyl theorem about the discontinuous Dirac equations with boundary conditions depending polynomially on the spectral parameter, and obtained two analogues of Hochstadt-Lieberman theorem and Mochizuki-Trooshin theorem. Guo, Wei and Yao considered the Dirac operator with eigenparameter boundary conditions and a finite number of transmission conditions. They used the Weyl function or two spectra to reconstruct the operator and extended Hochstadt-Lieberman theorem to the above cases (see [6]).

In this paper, we study the half inverse problem for the Dirac operator $L = L(p(x), q(x), \sigma(x), R_{ij}(\lambda), a)$ $(i, j = 0, 1)$ and establish a new half inverse problem. Using the potential function information on the less than half of the interval, the uniqueness theorem can be proved by one spectrum, i.e. if the potential $Q(x)$ is given on $(0, \frac{\phi(\pi)}{2})$ (see Figure 1), where $\frac{\phi(\pi)}{2} < \frac{\pi}{2}$, then only a single spectrum can uniquely determine the potential $Q(x)$ on the whole interval and some parameters in the boundary conditions and jump conditions.

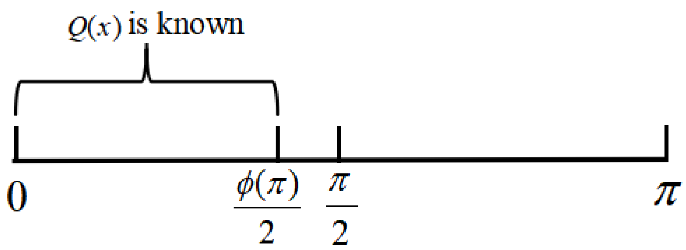

Figure 1: The case of $Q(x)$.

2. Preliminaries

Let $v_1(x,\lambda), v_2(x,\lambda), \varphi(x,\lambda), \psi(x,\lambda)$ be the solutions of the equation (1), satisfying the jump condition (4), and the following initial conditions, respectively:

$$v_1(0,\lambda) = \begin{pmatrix} v_{11}(0) \\ v_{12}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_2(0,\lambda) = \begin{pmatrix} v_{21}(0) \\ v_{22}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\varphi(0,\lambda) = \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} = \begin{pmatrix} R_{01}(\lambda) \\ -R_{00}(\lambda) \end{pmatrix}, \; \psi(\pi,\lambda) = \begin{pmatrix} \psi_1(\pi) \\ \psi_2(\pi) \end{pmatrix} = \begin{pmatrix} R_{11}(\lambda) \\ -R_{10}(\lambda) \end{pmatrix}.$$

It is clearly that these solutions are entire functions with respect to λ, and $U(\varphi) = V(\psi) = 0$. It implies that

$$\varphi(x,\lambda) = R_{01}(\lambda) v_2(x,\lambda) - R_{00}(\lambda) v_1(x,\lambda), \tag{5}$$
$$\psi(x,\lambda) = \Delta_1(\lambda) v_2(x,\lambda) - \Delta_2(\lambda) v_1(x,\lambda). \tag{6}$$

Here $\Delta_1(\lambda) = V(v_1)$, $\Delta_2(\lambda) = V(v_2)$.

Lemma 2.1. *[6] The following asymptotic relations hold as $|\lambda| \to \infty$,*

$$\varphi_1(x,\lambda) = \begin{cases} \lambda^{r_{01}}\Big[\cos\lambda\phi(x) + O\Big(e^{|\Im\lambda|\phi(x)}\Big)\Big], & 0 < x < \dfrac{\pi}{2}, \\ \lambda^{r_{01}}\Big[(\alpha_1^+ \cos\lambda\phi(x) + \alpha_1^- \cos\lambda(\pi - \phi(x))) + O\Big(e^{|\Im\lambda|\phi(x)}\Big)\Big], & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

$$\varphi_2(x,\lambda) = \begin{cases} \lambda^{r_{01}}\Big[\sin\lambda\phi(x) + O\Big(e^{|\Im\lambda|\phi(x)}\Big)\Big], & 0 < x < \dfrac{\pi}{2}, \\ \lambda^{r_{01}}\Big[(\alpha_1^+ \sin\lambda\phi(x) + \alpha_1^- \sin\lambda(\pi - \phi(x))) + O\Big(e^{|\Im\lambda|\phi(x)}\Big)\Big], & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

$$\psi_1(x,\lambda) = \begin{cases} \lambda^{r_{11}}\Big[\cos\lambda(\phi(\pi) - \phi(x)) + O\Big(e^{|\Im\lambda|(\phi(\pi)-\phi(x))}\Big)\Big], & 0 < x < \dfrac{\pi}{2}, \\ \lambda^{r_{11}}\Big[(\alpha_2^+ \cos\lambda(\phi(\pi) - \phi(x)) + \alpha_2^- \cos\lambda(\phi(\pi) + \phi(x) - \pi)) + O\Big(e^{|\Im\lambda|(\phi(\pi)-\phi(x))}\Big)\Big], & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

$$\psi_2(x,\lambda) = \begin{cases} \lambda^{r_{11}}\Big[\sin\lambda(\phi(\pi) - \phi(x)) + O\Big(e^{|\Im\lambda|(\phi(\pi)-\phi(x))}\Big)\Big], & 0 < x < \dfrac{\pi}{2}, \\ \lambda^{r_{11}}\Big[(\alpha_2^+ \sin\lambda(\phi(\pi) - \phi(x)) + \alpha_2^- \sin\lambda(\phi(\pi) + \phi(x) - \pi)) + O\Big(e^{|\Im\lambda|(\phi(\pi)-\phi(x))}\Big)\Big], & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

here $\phi(x) = \int_0^x \sigma(t)dt$, $\alpha_1^\pm = \frac{1}{2}(a \pm \frac{1}{a\alpha})$ and $\alpha_2^\pm = \frac{1}{2}(\frac{1}{a} \pm a\alpha)$.

Denote

$$\Delta(\lambda) := [\varphi(x), \psi(x)], \tag{7}$$

where

$$[\varphi(x), \psi(x)] = \varphi_1(x)\psi_2(x) - \varphi_2(x)\psi_1(x)$$

is the Wronskian of $\varphi(x)$ and $\psi(x)$, which is independent of x and satisfies the following formula:

$$[\varphi(x), \psi(x)]\Big|_{x=\frac{\pi}{2}-0} = [\varphi(x), \psi(x)]\Big|_{x=\frac{\pi}{2}+0}.$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$ (counting with multiplicities). It follows from (5), (6) and (7) that

$$\Delta(\lambda) = V(\varphi) = -U(\psi) = R_{01}(\lambda)\Delta_2(\lambda) - R_{00}(\lambda)\Delta_1(\lambda). \tag{8}$$

Together Lemma 2.1 with (5)-(8), we have that

$$\Delta(\lambda) = \lambda^{r_{11}+r_{01}}\Big[(\alpha_1^+ \sin\lambda\phi(\pi) + \alpha_1^- \sin\lambda(\pi-\phi(\pi))) + O\Big(e^{|\Im\lambda|\phi(\pi)}\Big)\Big],$$
$$|\lambda| \to \infty. \tag{9}$$

Define the sector $S_{\varepsilon,\lambda^*} := \{\lambda \in \mathbb{C}: |\lambda| \geq \lambda^*, \varepsilon < \arg\lambda < \pi - \varepsilon\}$ for $\varepsilon > 0$, $\lambda^* > 0$. The asymptotic formula (9) implies

$$|\Delta(\lambda)| \geq C_{\varepsilon,\lambda^*}|\lambda|^{r_{11}+r_{01}} e^{|\Im\lambda|\phi(\pi)}, \quad \lambda \in S_{\varepsilon,\lambda^*}, \tag{10}$$

where $C_{\varepsilon,\lambda^*}$ is a constant.

3. Weyl function and related properties

This section characterizes the representation and related properties of the Weyl function, and subsequently employs these results to establish uniqueness theorems for Dirac operators with eigenvalue dependent boundary conditions. The developed theory serves as the mathematical foundation for proving the half inverse problems in Section 4.

Consider a boundary value problem $\tilde{L} = L(\tilde{p}(x), \tilde{q}(x), \tilde{\sigma}(x), \tilde{R}_{ij}(\lambda), \tilde{a})$ of the same form but with different coefficients $\tilde{p}(x), \tilde{q}(x), \tilde{\sigma}(x), \tilde{R}_{ij}(\lambda), \tilde{a}$. We agree that if a certain symbol v denotes an object related to L, then $\tilde{v}$ denotes the analogous object related to $\tilde{L}$.

Let $\Phi(x,\lambda) = \begin{pmatrix}\Phi_1(x,\lambda)\\ \Phi_2(x,\lambda)\end{pmatrix}$ be the solution of the equation (1), satisfying the jump condition (4), and the following boundary conditions $U(\Phi) = 1$, $V(\Phi) = 0$. It follows from $U(\psi) = -\Delta(\lambda)$, $V(\psi) = 0$ that

$$\Phi(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{11}$$

Together (8) with (11), it can be deduced that $[\varphi(x,\lambda), \Phi(x,\lambda)] \equiv 1$. By virtue of (8), we obtain that

$$\Phi(x,\lambda) = \frac{1}{R_{01}(\lambda)}(v_1(x,\lambda) + M(\lambda)\varphi(x,\lambda)). \tag{12}$$

It follows from (12) that

$$M(\lambda) = -\frac{V(v_1)}{V(\varphi)} = -\frac{\Delta_1(\lambda)}{\Delta(\lambda)}. \tag{13}$$

The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are respectively referred to the Weyl-solution and Weyl-type function for the operator L. It is different from the classic Dirac operator, here $\Delta_1(\lambda)$ and $\Delta(\lambda)$ have common zeros.

Next, the following lemma is given by analyzing and discussing the relationship between the functions $\Delta_1(\lambda)$ and $\Delta(\lambda)$.

Lemma 3.1. *(i)* $\Delta_1(\lambda)$ *and* $\Delta_2(\lambda)$ *have no zeros.*
(ii) Let λ^* *is the zero of* $\Delta(\lambda)$*, i.e.* $\Delta(\lambda^*) = 0$*. Then* $\Delta_1(\lambda^\star)R_{01}(\lambda^\star) \neq 0$ *or* $\Delta_1(\lambda^\star) = R_{01}(\lambda^\star) = 0$*.*

Proof. The proof of (i): In virtue of (3), if $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ have common zero λ^*, i.e. $\Delta_1(\lambda^*) = \Delta_2(\lambda^*) = 0$, then

$$R_{11}(\lambda^*)v_{12}(\pi,\lambda^*) + R_{10}(\lambda^*)v_{11}(\pi,\lambda^*) = 0, \tag{14}$$

$$R_{11}(\lambda^*)v_{22}(\pi,\lambda^*) + R_{10}(\lambda^*)v_{21}(\pi,\lambda^*) = 0. \tag{15}$$

Since $[v_2(\pi,\lambda^\star), v_1(\pi,\lambda^\star)] \equiv 1$, i.e. the determinant of $v_2(\pi,\lambda^\star)$ and $v_1(\pi,\lambda^\star)$ is 1 in (14) and (15). That means $R_{11}(\lambda^\star) = R_{10}(\lambda^\star) = 0$. This conclusion contradicts the assumption that $R_{10}(\lambda)$ and $R_{11}(\lambda)$ have no common zero point.

The proof of (ii): It follows from the result of (i) that if $\Delta(\lambda^\star) = \Delta_1(\lambda^\star) = 0$, then $\Delta_2(\lambda^\star) \neq 0$. By virtue of (8), we get $R_{01}(\lambda^\star) = 0$. Similarly, if $\Delta(\lambda^\star) = R_{01}(\lambda^\star) = 0$, then $R_{00}(\lambda^\star) \neq 0$. That means $\Delta_1(\lambda^\star) = 0$. □

Define

$$\Psi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) & \Phi_1(x,\lambda) \\ \varphi_2(x,\lambda) & \Phi_2(x,\lambda) \end{pmatrix}.$$

we can obtain that $\det \Psi(x,\lambda) \equiv 1$.

Theorem 3.2. *Let* $R_{0i}(\lambda) = \tilde{R}_{0i}(\lambda)$ $(i = 0,1)$*. If* $M(\lambda) = \tilde{M}(\lambda)$*, then* $Q(x) = \tilde{Q}(x)$ *a.e.* $x \in (0,\pi)$ *and* $a = \tilde{a}$*.*

Proof. It follows from the expressions of $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ along with (9) that

$$|\Psi_{(i)}(x,\lambda)| \leq C_{\delta,\rho^*}|\lambda|^{-r_{01}}e^{-|\Im\lambda|\phi(x)}, \quad \rho \in S_{\delta,\lambda^*}, \quad i = 1,2.$$

Introduce the matrix

$$P(x,\lambda) = [P_{ij(x,\lambda)}]_{i,j=1,2},$$

which satisfies

$$P(x,\lambda)\tilde{\Psi}(x,\lambda) = \Psi(x,\lambda). \tag{16}$$

By calculating, we have

$$\begin{aligned} P_{11}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\Phi_1(x,\lambda), \\ P_{12}(x,\lambda) &= -\varphi_1(x,\lambda)\tilde{\Phi}_1(x,\lambda) + \tilde{\varphi}_1(x,\lambda)\Phi_1(x,\lambda), \\ \varphi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}_2(x,\lambda), \\ \Phi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}_2(x,\lambda). \end{aligned} \tag{17}$$

Since $M(\lambda) = \tilde{M}(\lambda)$ and together with (12),

$$P_{11}(x,\lambda) - 1 = (\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda))\tilde{\Phi}_2(x,\lambda) - (\Phi_1(x,\lambda) - \tilde{\Phi}_1(x,\lambda))\tilde{\varphi}_2(x,\lambda), \tag{18}$$

$$P_{12}(x,\lambda) = -(\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda))\tilde{\Phi}_1(x,\lambda) + (\Phi_1(x,\lambda) - \tilde{\Phi}_1(x,\lambda))\tilde{\varphi}_1(x,\lambda). \tag{19}$$

Substituting (12) into (17), we have

$$\begin{aligned} P_{11}(x,\lambda) = &\frac{1}{\tilde{R}_{01}(\lambda)}\varphi_1(x,\lambda)(\tilde{v}_{12}(x,\lambda) + \tilde{M}(\lambda)\tilde{\varphi}_2(x,\lambda)) \\ &- \frac{1}{R_{01}(\lambda)}\tilde{\varphi}_2(x,\lambda)(v_1(x,\lambda) + M(\lambda)\varphi_1(x,\lambda)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tilde{R}_{01}(\lambda)} \varphi_1(x,\lambda)\tilde{v}_{12}(x,\lambda) + \frac{1}{\tilde{R}_{01}(\lambda)} \varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda)\tilde{M}(\lambda) \\
&\quad - \frac{1}{R_{01}(\lambda)} \tilde{\varphi}_2(x,\lambda) v_{11}(x,\lambda) - \frac{1}{R_{01}(\lambda)} \tilde{\varphi}_2(x,\lambda)\varphi_1(x,\lambda)M(\lambda).
\end{aligned}
$$

Considering the conditions $M(\lambda) = \tilde{M}(\lambda)$ and $R_{0i} = \tilde{R}_{0i} (i = 0, 1)$ and substituting (2) into the above formula, we get

$$
\begin{aligned}
P_{11}(x,\lambda) &= \frac{1}{\tilde{R}_{01}(\lambda)} (\varphi_1(x,\lambda)\tilde{v}_{12}(x,\lambda) - \tilde{\varphi}_2(x,\lambda) v_{11}(x,\lambda)) \\
&= \frac{1}{\tilde{R}_{01}(\lambda)} \Big[(\tilde{R}_{01}(\lambda) v_{21}(x,\lambda) - \tilde{R}_{00}(\lambda) v_{11}(x,\lambda))\tilde{v}_{12}(x,\lambda) \\
&\quad - (\tilde{R}_{01}(\lambda)\tilde{v}_{22}(x,\lambda) - \tilde{R}_{00}(\lambda)\tilde{v}_{12}(x,\lambda)) v_{11}(x,\lambda) \Big] \\
&= v_{21}(x,\lambda)\tilde{v}_{12}(x,\lambda) - \tilde{v}_{22}(x,\lambda) v_{11}(x,\lambda).
\end{aligned}
$$

Similarly, we can obtain

$$
P_{12}(x,\lambda) = \tilde{v}_{21}(x,\lambda) v_{11}(x,\lambda) - v_{21}(x,\lambda)\tilde{v}_{11}(x,\lambda).
$$

From the expressions of $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$, it is known that when $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire function with respect to λ.

Using the expressions of $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, (9) and (18), it can be calculated that

$$
|P_{11}(x,\lambda) - 1| \leq \frac{C}{|\lambda|}, \qquad |P_{12}(x,\lambda)| \leq \frac{C}{|\lambda|}, \quad \rho \in S_{\varepsilon,\lambda^*}.
$$

We can also get that

$$
|P_{22}(x,\lambda) - 1| \leq \frac{C}{|\lambda|}, \qquad |P_{21}(x,\lambda)| \leq \frac{C}{|\lambda|}, \quad \rho \in S_{\varepsilon,\lambda^*}.
$$

By Liouville's theorems and Phragmen-Lindelöf's in [4], we deduce that

$$
P_{11}(x,\lambda) \equiv P_{22}(x,\lambda) \equiv 1, \quad P_{12}(x,\lambda) \equiv P_{21}(x,\lambda) \equiv 0.
$$

By virtue of (16), we obtain $\varphi_k(x,\lambda) = \tilde{\varphi}_k(x,\lambda)$, $\Phi_k(x,\lambda) = \tilde{\Phi}_k(x,\lambda)$, $k = 1, 2$. That is $Q(x) = \tilde{Q}(x)$ a.e. $x \in (0,\pi)$ and $a = \tilde{a}$. □

4. Half inverse problem

In this section, considering the Dirac operator $\tilde{L} = L(p(x), q(x), \sigma(x),$ $R_{ij}(\lambda), a)$ given in section 3, the half inverse problem for the Dirac operator $L = L(p(x), q(x), \sigma(x), R_{ij}(\lambda), a)$ can be investigated. Let $R_{00}(\lambda)$ and $R_{01}(\lambda)$ be known a priori.

Theorem 4.1. *Assume $r_{11} < r_{01}$. If $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$ (counting with multiplicities), $Q(x) = \tilde{Q}(x)$ on $(0, \frac{\phi(\pi)}{2})$, $R_{00}(\lambda) = \tilde{R}_{00}(\lambda)$, $R_{01}(\lambda) = \tilde{R}_{01}(\lambda)$ and $\alpha = \tilde{\alpha}$, then $Q(x) = \tilde{Q}(x)$ a.e. on $(0,\pi)$ and $a = \tilde{a}$.*

In order to prove Theorem 4.1, we need the help of following Lemma.

Lemma 4.2. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$ (counting with multiplicities) and $\alpha = \tilde{\alpha}$, then $a = \tilde{a}$.*

Proof. It is known that $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ are entire functions of λ of order 1. So the characteristic function can be uniquely determined by the eigenvalues up to an exponential factor. Since $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$, we obtain that $\Delta(\lambda) = C_1 e^{C_2\lambda}\tilde{\Delta}(\lambda)$, where $C_1, C_2 \neq 0$ are some constant. Together (9) with $\alpha = \tilde{\alpha}$, we have $\alpha^+ = C_1 e^{C_2\lambda}\tilde{\alpha}^+$ and $\alpha^- = C_1 e^{C_2\lambda}\tilde{\alpha}^-$, that is

$$\frac{1}{2}(a \pm \frac{1}{a\alpha}) = \frac{C_1 e^{C_2\lambda}}{2}(\tilde{a} \pm \frac{1}{\tilde{a}\alpha}).$$

It is clearly that

$$a = C_1 e^{C_2\lambda}\tilde{a}, \quad a^{-1} = C_1 e^{C_2\lambda}\tilde{a}^{-1}. \tag{20}$$

In view of $a, \tilde{a} > 0$ and dividing two formulas in (20), so $a = \tilde{a}$. □

Proof. [**Proof of Theorem 4.1**] For convenience, denote $d = \frac{\phi(\pi)}{2}$. Since $\psi_1(\pi,\lambda) = R_{11}(\lambda)$, $\psi_2(\pi,\lambda) = -R_{10}(\lambda)$, and using Green formula, we can obtain

$$\begin{aligned}\int_0^{\pi}(Q(x) - \tilde{Q}(x))\psi(x,\lambda)\tilde{\psi}^T(x,\lambda)dx &= [\psi,\tilde{\psi}](0,\lambda) - [\psi,\tilde{\psi}](\pi,\lambda)\\ &= \zeta(0,\lambda) - \zeta(\pi,\lambda),\end{aligned} \tag{21}$$

where $\zeta(x,\lambda) = [\psi,\tilde{\psi}](x,\lambda)$.

In virtue of $Q(x) = \tilde{Q}(x)$ $(x \in (0,d))$, we have

$$\zeta(0,\lambda) = \zeta(d,\lambda) = \zeta(\pi,\lambda) + \int_d^{\pi}(Q(x) - \tilde{Q}(x))\psi(x,\lambda)\tilde{\psi}^T(x,\lambda)dx. \tag{22}$$

According to the expression $\zeta(x,\lambda)$ and (2), we can calculate

$$\begin{aligned}\zeta(0,\lambda) &= \psi_1(0,\lambda)\tilde{\psi}_2(0,\lambda) - \psi_2(0,\lambda)\tilde{\psi}_1(0,\lambda)\\ &= -\frac{1}{R_{00}(\lambda)}(\psi_2(0,\lambda)U(\tilde{\psi}) - \tilde{\psi}_2(0,\lambda)U(\psi)) \qquad (23)\\ &= -\frac{1}{R_{01}(\lambda)}(\tilde{\psi}_1(0,\lambda)U(\varphi) - \psi_1(0,\lambda)U(\tilde{\psi})). \qquad (24)\end{aligned}$$

Substituting (6) into (23) and (24), it follows that

$$\begin{aligned}\zeta(0,\lambda) &= -\frac{1}{R_{00}(\lambda)}[\tilde{\Delta}(\lambda)V(v_2) - \Delta(\lambda)V(\tilde{v}_2)], \qquad (25)\\ &= \frac{1}{R_{01}(\lambda)}[\Delta(\lambda)V(\tilde{v}_1) - \tilde{\Delta}(\lambda)V(v_1)], \qquad (26)\end{aligned}$$

Since $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$ (counting with multiplicities), so $\Delta(\lambda) = \tilde{\Delta}(\lambda)$, i.e.

$$\begin{aligned}\zeta(0,\lambda) &= -\frac{1}{R_{00}(\lambda)}\Delta(\lambda)[V(v_2) - V(\tilde{v}_2)] \qquad (27)\\ &= \frac{1}{R_{01}(\lambda)}\Delta(\lambda)[V(\tilde{v}_1) - V(v_1)]. \qquad (28)\end{aligned}$$

It follows from Lemma 3.1 that if λ_n is a zero of multiplicity m_n for $\Delta(\lambda)$, then λ_n is a zero of multiplicity m_n for $\zeta(0,\lambda)$, i.e. $\Delta(\lambda)$ and $\zeta(0,\lambda)$ have the same common zeros. We have

$$\zeta(0,\lambda_n) = \zeta(d,\lambda_n) = \psi_1(d,\lambda_n)\tilde{\psi}_2(d,\lambda_n) - \psi_2(d,\lambda_n)\tilde{\psi}_1(d,\lambda_n) \tag{29}$$

Next, we shall prove that (29) is valid for all $\lambda \in \mathbb{C}$.

In view of [18] and using Schwarz inequality, it follows from Lemma 2.1 and (22) that

$$\begin{aligned}|\zeta(0,\lambda)| = |\zeta(d,\lambda)| &\le |\zeta(\pi,\lambda)| + \Big|\int_d^\pi (Q(x)-\tilde{Q}(x))\psi(x,\lambda)\tilde{\psi}^T(x,\lambda)dx\Big| \\ &\le |\zeta(\pi,\lambda)| + \| Q(x)-\tilde{Q}(x)\|_2 \Big(\int_d^\pi |\psi(x,\lambda)\tilde{\psi}^T(x,\lambda)|^2 dx\Big)^{\frac{1}{2}} \\ &\le C_1|\lambda|^{r_{11}+r_{11}} + C_2 \| Q(x)-\tilde{Q}(x)\|_2 \, |\lambda|^{r_{11}+r_{11}} \mathrm{e}^{2|\Im\lambda|\left(\phi(\pi)-\mathrm{d}\right)} \\ &= C_1|\lambda|^{r_{11}+r_{11}} + C_2 \| Q(x)-\tilde{Q}(x)\|_2 \, |\lambda|^{r_{11}+r_{11}} \mathrm{e}^{|\Im\lambda|\phi(\pi)} \\ &= O(|\lambda|^{r_{11}+r_{11}} \mathrm{e}^{|\Im|\sigma(\pi)}). \end{aligned} \tag{30}$$

Denote

$$\chi(\lambda) = \frac{\zeta(0,\lambda)}{\Delta(\lambda)}.$$

From the above conclusion, it can be known that $\zeta(0,\lambda)$ and $\Delta(\lambda)$ have the common zeros. By (10) and (30), we have

$$|\chi(\lambda)| = \Big|\frac{\zeta(0,\lambda)}{\Delta(\lambda)}\Big| \le C_{\delta,\rho^*}|\lambda|^{r_{11}-r_{01}}, \quad \rho \in S_{\delta,\rho^*},$$

where $r_{11} < r_{01}$. Using Phragmen-Lindelöf's theorem and Liouville's theorem, it can be calculated that $\zeta(0,\lambda) = \zeta(d,\lambda) \equiv 0$. That is,

$$\psi_1(d,\lambda)\tilde{\psi}_2(d,\lambda) - \psi_2(d,\lambda)\tilde{\psi}_1(d,\lambda) = 0. \tag{31}$$

So (29) is valid for all $\lambda \in \mathbb{C}$. Through transformation of (31), we obtain

$$\frac{\psi_1(d,\lambda)}{\psi_2(d,\lambda)} = \frac{\tilde{\psi}_1(d,\lambda)}{\tilde{\psi}_2(d,\lambda)}. \tag{32}$$

We rigorously demonstrate that the fraction $\frac{\psi_1(d,\lambda)}{\psi_2(d,\lambda)}$ is the Weyl function of the boundary value problem $L_1(p(x), q(x), \sigma(x), R_{ij}(\lambda), a, b)$ for the equation (1) on (d,π) with the boundary conditions $\psi_2(d,\lambda) = 0$, $V(\psi) = 0$ without the discontinuity.

Let $y(x,\lambda)$ be the solutions of the equation (1) ($x \in (d,\pi)$), satisfying the the boundary conditions $y_2(d,\lambda) = 1$, $V(y) = 0$. By applying the same line of reasoning as above, we can derive that the characteristic function of L_1 is $\Upsilon(\lambda) := \psi_2(d,\lambda)$. In virtue of $V(\psi) = 0$, we can get the Weyl solution for L_1:

$$y(x,\lambda) = \frac{\psi(x,\lambda)}{\Upsilon(\lambda)} = \frac{\psi(x,\lambda)}{\psi_2(d,\lambda)}. \tag{33}$$

It follows from (33) that the Weyl function of L_1 is

$$M_0(\lambda) := \frac{\psi_1(d,\lambda)}{\psi_2(d,\lambda)}.$$

In view of (32), $M_0(\lambda) = \tilde{M}_0(\lambda)$. According to Theorem 3.1, Weyl function can uniquely determine the potential function, so $Q(x) = \tilde{Q}(x)$ ($x \in (d,\pi)$).

Since $M(\lambda) = \tilde{M}(\lambda)$, we can get $\Phi(x,\lambda) = \tilde{\Phi}(x,\lambda)$, $x \in (\frac{\phi(\pi)}{2}, \pi)$ though Theorem 3.1. Analogously, because of $\lambda_n = \tilde{\lambda}_n$, $n \ge 0$ (counting with multiplicities), it follows from the proof in Lemma 4.2 that $\Delta(\lambda) = \tilde{\Delta}(\lambda)$. Comparing with (11), it can be deduced that $\psi(x,\lambda) = \tilde{\psi}(x,\lambda)$. Based on the initial value condition of $\psi(x,\lambda)$ at the right endpoint π, it can be proved that $R_{10}(\lambda) = \tilde{R}_{10}(\lambda)$, $R_{11}(\lambda) = \tilde{R}_{11}(\lambda)$. □

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Availability of data and materials. The data described in the manuscript, including all relevant raw data, will be openly available.

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