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Family of inequalities for Newton-Cotes formulae with their error bounds

Shumaila Asgher[a], Saad Ihsan Butt[b], Waqas Ali Faridi[c,d,e], Muhammad Toseef[f,*], Loredana Ciurdariu[g]

[a]*Department of Mathematics, Riphah International University Sahiwal, Pakistan*
[b]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan*
[c]*Department of Mathematics, University of Management and Technology, Lahore, Pakistan*
[d]*Jadara University Research Center, Jadara University, Jordan*
[e]*Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan*
[f]*School of Science, Nanjing University of Posts and Telecommunications, Nanjing, Jiangsu, 210023 China*
[g]*Department of Mathematics, Politehnica University of Timisoara, 300006 Timisoara, Romania*

Abstract. In numerical analysis, quadrature formulae are essential for approximating definite integrals. This paper introduces a new family of Newton-Cotes-type inequalities, derived from a parameterized identity, and establishes their associated error bounds for several function classes (Convex, Bounded, and Lipschitzian). A key advantage of these inequalities is their generality: they provide a unified framework to derive the error bounds of classical quadrature formulae such as the Midpoint, Simpson's 1/3, Simpson's 3/8, Maclaurin's, and Weddle's approximations. We extend traditional formulae and employ the power mean inequality and Hölder's integral inequality to obtain more general and sharper error estimates. The practical utility of these inequalities is demonstrated through solutions to mathematical problems and a precise numerical assessment comparing their efficiency in realistic examples. Applications to quadrature formulae and numerical examples further illustrate the effectiveness of the proposed inequalities, contributing to the advancement of numerical methods and broadening the applicability of Newton-Cotes-type inequalities within mathematical sciences.

1. Introduction

Newton-Cotes formulae are a set of numerical integration methods through which the integral of a function is approximated by the finite sum of functional values at chosen points in the interval of integration. These formulae are derived by interpolating the integrand $F(\xi)$ using a polynomial that passes through a

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* Corresponding author: Muhammad Toseef
Email addresses: shumaila.asghar@riphahsahiwal.edu.pk (Shumaila Asgher), saadihsanbutt@gmail.com (Saad Ihsan Butt), wa966142@gmail.com (Waqas Ali Faridi), toseefrana95@gmail.com (Muhammad Toseef), loredana.ciurdariu@upt.ro (Loredana Ciurdariu)
ORCID iDs: https://orcid.org/0009-0007-9759-9409 (Shumaila Asgher), https://orcid.org/0000-0002-4761-235X (Saad Ihsan Butt), https://orcid.org/0000-0003-0713-5365 (Waqas Ali Faridi), https://orcid.org/0009-0004-1878-7136 (Muhammad Toseef), https://orcid.org/0000-0002-1234-7939 (Loredana Ciurdariu)

set of equally spaced points and then integrates the interpolating polynomial. Given an interval $[\alpha, \beta]$ the integral of a function $F(\xi)$ over this interval is approximated by

$$\int_{\alpha}^{\beta} F(\xi)\, d\xi \approx \sum_{i=0}^{n} \varpi_i F(\xi_i),$$

where ξ_i are the equally spaced nodes within $[\alpha, \beta]$ and ϖ_i are the weights associated with each node. The nodes ξ_i are typically chosen as:
(*) for a closed Newton-Cotes formula

$$\xi_i = \alpha + i\frac{\beta - \alpha}{n}, \text{ for } i = 0, 1, 2, ..., n.$$

(**) for the open Newton-Cotes formula

$$\xi_i = \alpha + (i+1)\frac{\beta - \alpha}{n+2}, \text{ for } i = 0, 1, 2, ..., n.$$

The weights ϖ_i depend on the specific form of the Newton-Cotes. The formula is determined by integrating the Lagrange interpolating polynomial that passes through these nodes. Types of Newton-Cotes formulae for different choices of n.

1. Midpoint formula ($n = 0$)
2. Trapezoidal rule ($n = 1$)
3. Simpson's 1/3 rule ($n = 2$)
4. Simpson's 3/8 rule ($n = 3$)
5. Higher order Newton-Cotes formula ($n > 3$, Boole's rule, Weddle's rule)

Error analysis is critical in numerical methods, and the error terms in the Newton-Cotes formulae can be analyzed by considering the remainder of the interpolating polynomial. For n-points Newton-Cotes formulae, the error term E is defined as:

$$E(F) = -\frac{(\beta - \alpha)^{n+2}}{(n+2)!} F^{(n+1)}(\xi). \tag{1}$$

From (1), the error term depends upon the choice of n. A higher-order derivative is required to find the error term of the above-described methods. If the function is convex and first-time differentiable, we can find the error bound of the above-described methods. Finding an error term using the first derivative is an excellent achievement in inequality theory because the class of first-time differentiable functions is more extensive than that of bounded functions with higher-order derivatives. I successfully tackled this problem after comprehensively studying the literature described below.

Earliest contacts with convexity can be traced back to what may have been the Egyptian and Babylonian, and the concept may have reached Greek philosophers. While it is rather difficult to name an idea older than numbers, sketching basic geometric shapes, such as triangles and circles, has been performed since the dawn of mankind. Coming to the point of deciding on who is the person who thought for the first time about convexity is not easy. For further history of convexity, the readers are referred to [15]. It was in the late 19th century that a German mathematician, Karl Hermann Amandus Schwarz, discovered what is now known as a convex function. Schwarz's contribution to convex functions has brought drastic changes to the world of mathematical theory and its practical solutions. Some of the uses of the convex function include the solution of optimization problems in areas such as math, computer science, engineering, and economics. Applications of convex functions are in many other mathematical optimization problems, such as convex optimization, quadratic programming, and linear programming. They assist in determining the optimum value of a certain unknown regarding specific values. Optimization theory in finance involves the use of convex functions in the management of portfolios with the view of capturing higher expected returns while at the same time keeping risks to the barest minimum. It was in this way that certain securities, such as bonds, are sure to reprice higher in response to the falling interest rates since they exhibit convexity. The theory of convex functions provides a powerful framework for deriving inequalities and modeling phenomena across diverse fields. Foundational applications in mathematical analysis were firmly established in early works such as that of Green [17]. The scope of these applications continues to expand, with recent research focusing on generalized convexity; for instance, Sarpong and Prah [33] discussed broad applications, while Du et al. have advanced the field by exploring Bullen-type inequalities via fractional integrals [13] and establishing inclusion relations for interval-valued coordinated convex mappings [12].

Definition 1.1 (Convex Set). *[15] A set $I \subset \mathbb{R}^n$ is convex, for any two points $\alpha, \beta \in I$, the entire segment joining α and β lies in I. The points in the segments are of the form*

$$\varpi\alpha + (1-\varpi)\beta \in I,\ \forall\ \varpi \in [0,1]. \tag{2}$$

Definition 1.2 (Convex Function). *[15] Let I be a convex subset of a real vector space and $F : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\varpi F(\alpha) + (1-\varpi)F(\beta) \geq F(\varpi\alpha + (1-\varpi)\beta), \tag{3}$$

for all $\varpi \in [0,1]$, and $\alpha, \beta \in I$.

The Hermite-Hadamard inequality is a fundamental result in convex analysis that establishes precise bounds for the integral mean of convex functions. While bearing the names of Charles Hermite (1822–1901) and Jacques Hadamard, its historical development also involved Henri Poincaré (1854–1912) [26, 31].

- **Charles Hermite**: Pioneered breakthroughs in number theory, elliptic functions, and mathematical physics.
- **Henri Poincaré**: Made foundational contributions to topology, dynamical systems, and celestial mechanics.

For any convex function $F\colon I \subseteq \mathbb{R} \to \mathbb{R}$ defined on an interval I and points $\alpha, \beta \in I$ with $\alpha < \beta$, the inequality states:

$$F\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi \leq \frac{F(\alpha)+F(\beta)}{2}. \tag{4}$$

This double inequality (4) provides both lower and upper bounds for the integral mean of F over $[\alpha, \beta]$. For further historical details and modern extensions, see [26, 30, 31]. Numerical analysis is among the most important areas of today's scientific and engineering computations that offers many procedures for the estimation of otherwise computationally involved mathematical problems. In all the methods of numerical integration, Newton-Cotes formulae are the systematic procedures to approximate definite integrals using polynomial approximations over sub-interval divisions and function interpolation.

The motivation for this work stems from the need to estimate error terms in numerical integration techniques such as the Trapezoidal rule, Midpoint formula, and Simpson's rule. Building upon Dragomir and Agarwal's 1998 introduction of an error term for the Trapezoidal Rule estimable using only the first derivative of a convex function [10], this paper establishes the following principal result:

Lemma 1.3. *[10] Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\alpha, \beta \in I^\circ$ with $\alpha < \beta$. If $F' \in L^1[\alpha, \beta]$, then the following equality holds:*

$$\frac{F(\beta)+F(\alpha)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi = \frac{\beta-\alpha}{2}\int_0^1 (1-2\varpi)F'(\varpi\beta + (1-\varpi)\alpha)\,d\varpi.$$

Theorem 1.4. *[10] Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\alpha, \beta \in I^{\circ}$ with $\alpha < \beta$, and $|F'|$ is convex on $[\alpha, \beta]$, then the following inequality holds:*

$$\left|\frac{F(\beta)+F(\alpha)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(\xi)\,d\xi\right| \leq \frac{(\beta-\alpha)}{8}\left[\left|F'(\beta)\right|+\left|F'(\alpha)\right|\right].$$

This initiative has reinvigorated the field of inequalities. Building upon this foundation, [21] extended the approach to the Midpoint formula in 2004, while [32] established Simpson-type inequalities with error bounds for differentiable convex functions in 2010. Further developments occurred when [29] proved Newton-type inequalities (Simpson's second type) for p-harmonic convex functions in 2017. Most recently, [23] derived $\frac{3}{8}$-Simpson-type inequalities for functions whose first derivative moduli are s-convex in the second sense. These advances are particularly significant as they enable error estimation using only first derivatives, a substantially broader function class than those requiring bounded higher-order derivatives. Collectively, these results represent major progress in inequality theory. The Hermite-Hadamard inequality was extended to the class of preinvex functions by Barani, Ghazanfari, and Dragomir [4]. Early foundational work by Dragomir [9] on mappings associated with Hadamard's inequality has been instrumental in subsequent developments. Recently, Mateen et al. [25] established Milne's rule-type inequalities for convex functions within the quantum calculus framework, supported by computational analysis. New variants of quantum midpoint-type inequalities were investigated by Butt, Budak, and Nonlaopon [6]. In the realm of fractional calculus, Nasir et al. [28] derived Simpson-type integral inequalities via the Riemann-Liouville operator and discussed their applications. A broad exposition on Hermite-Hadamard inequalities and their applications is provided in the selected topics by Dragomir and Pearce [8]. The study of quantum Ostrowski-type inequalities for differentiable convex functions was pursued by Ali et al. [2].

There are some parameterized versions of the quadrature formula that were proved by mathematicians working in the theory of inequalities. In 2008, Wenjun Liu first established several error inequalities for a quadrature formula with a parameter and applications [24]. Alomari et al. in [3] proved various error estimations for several Newton-Cotes quadrature formulae in terms of, at most, the first derivative and applications in numerical integration. Du et al. in [11] proved certain quantum estimates of parameterized integral inequalities and their applications. The parameterized fractal integral inequalities and related applications were established by Du et al. in [14]. Ali et al. in [1] proved new parameterized Newton-type inequalities for differentiable functions via fractional integrals. Recently, Toseef et al. in [34] proved a family of quadrature formulae with their error bounds for convex functions and applications. Building upon the framework of fractional calculus, Hezenci and Budak [18] initiated a detailed study on parameterized inequalities for conformable fractional integrals. This line of inquiry was further developed in their subsequent works, which included broader remarks on fractional operators [35] and extended the analysis to functions with higher-order differentiability [5]. Parallel to these developments on parameterized inequalities, significant results have been obtained in refining classical inequality types, such as the Bullen-type inequalities by Fahad et al. [16] and the Newton–Simpson-type inequalities via majorization by Butt et al. [7]. Most recently, Hezenci and Budak [19] have further expanded this landscape by establishing fractional Newton-type inequalities for various function classes. Meftah et al. proved some new local fractional Newton-type inequalities [27]. Hyder et al. give development of fractional Newton-Type inequalities through extended integral operators [20]. Lakhdari et al. established error estimates of Newton-Cotes quadrature rules across diverse function classes [22].

In this context, this paper introduces novel parameterized Newton-Cotes-type inequalities that extend the classical framework using a parameterized identity, resulting in new inequalities with enhanced properties. These inequalities offer a fresh perspective on approximating definite integrals and serve as a versatile tool for addressing various mathematical challenges. The primary focus is to explore the applications of these inequalities in different fields of mathematics, demonstrating their utility and effectiveness in solving practical problems. By presenting a series of examples, this study illustrates the versatility and applicability of these inequalities. It conducts a detailed numerical analysis to evaluate their performance and accuracy in practical scenarios. By comparing the results obtained through these new inequalities with existing methods, this analysis provides insights into the strengths and limitations of the proposed approach, contributing to the advancement of numerical methods and enhancing our understanding of Newton-Cotes-type inequalities' practical implications.

The primary motivation behind the development of the newly established parameterized identity is to provide a unified framework for various quadrature rules. By selecting different parameter values, we can encapsulate a range of specific cases within this generalized identity. In our analysis, we summarize these particular instances, illustrating how distinct parameter choices yield various quadrature rules. Additionally, we investigate the error bounds associated with several numerical integration methods applied to different types of functions, including differentiable convex functions, bounded functions, and Lipschitzian functions. Through this comprehensive examination, we aim to demonstrate the versatility and effectiveness of the parameterized identity in improving the accuracy and reliability of numerical integration techniques across diverse functional classes.

The description of the article is as follows: Section 2 presents the main results, which include parameterized identity and inequalities related to that identity for various classes of functions. In Section 3, we present applications for numerical integration. Section 4 presents a comprehensive computational analysis of newly derived findings. In the last section, concluding remarks and future directions are given.

2. Main Results

In this section, parameterized identity is established for differentiable functions; using the newly established identity, we proved a family of inequalities of the Newton-Cotes formulae for various kinds of functions.

2.1. Differentiable Convex Functions

In this subsection, a parameterized identity for differentiable functions is proved. By using a newly established identity, error bounds for differentiable convex functions are established.

Lemma 2.1. *Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\alpha, \beta \in I^\circ$ with $\alpha < \beta$, and $F' \in L^1[\alpha, \beta]$ then the following equality holds:*

$$\begin{aligned} & \rho_1 F(\alpha) + (\rho_2 - \rho_1) F\left(\frac{\beta + 5\alpha}{6}\right) + (\rho_3 - \rho_2) F\left(\frac{\beta + 2\alpha}{3}\right) + (\rho_4 - \rho_3) F\left(\frac{\alpha + \beta}{2}\right) \\ & + (\rho_5 - \rho_4) F\left(\frac{2\beta + \alpha}{3}\right) + (\rho_6 - \rho_5) F\left(\frac{5\beta + \alpha}{6}\right) + (1 - \rho_6) F(\beta) - \frac{1}{\beta - \alpha} \int_\alpha^\beta F(r)\, dr \\ = \ & (\beta - \alpha) \sum_{i=1}^{6} I_i, \end{aligned} \tag{5}$$

where,

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{6}} (\varpi - \rho_1) F'(\varpi\beta + (1 - \varpi)\alpha)\, d\varpi, \\ I_2 &= \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi - \rho_2) F'(\varpi\beta + (1 - \varpi)\alpha)\, d\varpi, \\ I_3 &= \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi - \rho_3) F'(\varpi\beta + (1 - \varpi)\alpha)\, d\varpi, \\ I_4 &= \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi - \rho_4) F'(\varpi\beta + (1 - \varpi)\alpha)\, d\varpi, \\ I_5 &= \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi - \rho_5) F'(\varpi\beta + (1 - \varpi)\alpha)\, d\varpi, \end{aligned}$$

$$I_6 = \int_{\frac{5}{6}}^{1} (\varpi - \rho_6) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi.$$

Proof. Taking into account the right-hand side of 5, we have:

$$\begin{aligned}
& (\beta-\alpha)\sum_{i=1}^{6} I_i \\
= {} & (\beta-\alpha)\left[I_1 + I_2 + I_3 + I_4 + I_5 + I_6\right] \\
= {} & (\beta-\alpha)\left[\int_0^{\frac{1}{6}} (\varpi - \rho_1) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \right. \\
& + \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi - \rho_2) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi + \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi - \rho_3) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
& + \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi - \rho_4) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi + \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi - \rho_5) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
& \left. + \int_{\frac{5}{6}}^{1} (\varpi - \rho_6) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \right]. \qquad (6)
\end{aligned}$$

Solve these integrals, integration by parts

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{6}} (\varpi - \rho_1) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \frac{1}{\beta-\alpha} (\varpi - \rho_1) F(\varpi\beta + (1-\varpi)\alpha)\Big|_{\varpi=0}^{\frac{1}{6}} - \frac{1}{\beta-\alpha}\int_0^{\frac{1}{6}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \left(\frac{1}{6} - \rho_1\right) F\left(\frac{\beta+5\alpha}{6}\right) + \rho_1 F(\alpha) - \frac{1}{\beta-\alpha}\int_0^{\frac{1}{6}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi - \rho_2) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \frac{1}{\beta-\alpha} (\varpi - \rho_2) F(\varpi\beta + (1-\varpi)\alpha)\Big|_{\varpi=\frac{1}{6}}^{\frac{1}{3}} - \frac{1}{\beta-\alpha}\int_{\frac{1}{6}}^{\frac{1}{3}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \left(\frac{1}{3} - \rho_2\right) F\left(\frac{\beta+2\alpha}{3}\right) + \left(\frac{1}{6} - \rho_2\right) F\left(\frac{\beta+5\alpha}{6}\right) - \frac{1}{\beta-\alpha}\int_{\frac{1}{6}}^{\frac{1}{3}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi - \rho_3) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \frac{1}{\beta-\alpha} (\varpi - \rho_3) F(\varpi\beta + (1-\varpi)\alpha)\Big|_{\varpi=\frac{1}{3}}^{\frac{1}{2}} - \frac{1}{\beta-\alpha}\int_{\frac{1}{3}}^{\frac{1}{2}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi \\
&= \left(\frac{1}{2} - \rho_3\right) F\left(\frac{\beta+\alpha}{2}\right) + \left(\frac{1}{3} - \rho_3\right) F\left(\frac{\beta+2\alpha}{3}\right) - \frac{1}{\beta-\alpha}\int_{\frac{1}{3}}^{\frac{1}{2}} F(\varpi\beta + (1-\varpi)\alpha)\, d\varpi, \qquad (9)
\end{aligned}$$

$$I_4 = \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi - \rho_4) F'(\varpi\beta + (1-\varpi)\alpha)\, d\varpi$$

$$
\begin{aligned}
&= \frac{1}{\beta-\alpha}\left(\varpi-\rho_4\right)F\left(\varpi\beta+(1-\varpi)\alpha\right)\Big|_{\varpi=\frac{1}{2}}^{\frac{2}{3}} - \frac{1}{\beta-\alpha}\int_{\frac{1}{2}}^{\frac{2}{3}} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi \\
&= \left(\frac{2}{3}-\rho_4\right)F\left(\frac{2\beta+\alpha}{3}\right)+\left(\frac{1}{2}-\rho_4\right)F\left(\frac{\beta+\alpha}{2}\right)-\frac{1}{\beta-\alpha}\int_{\frac{1}{2}}^{\frac{2}{3}} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \int_{\frac{2}{3}}^{\frac{5}{6}} \left(\varpi-\rho_5\right)F'\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi \\
&= \frac{1}{\beta-\alpha}\left(\varpi-\rho_5\right)F\left(\varpi\beta+(1-\varpi)\alpha\right)\Big|_{\varpi=\frac{2}{3}}^{\frac{5}{6}} - \frac{1}{\beta-\alpha}\int_{\frac{2}{3}}^{\frac{5}{6}} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi \\
&= \left(\frac{5}{6}-\rho_5\right)F\left(\frac{5\beta+\alpha}{6}\right)+\left(\frac{2}{3}-\rho_5\right)F\left(\frac{2\beta+\alpha}{3}\right)-\frac{1}{\beta-\alpha}\int_{\frac{2}{3}}^{\frac{5}{6}} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi, \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 &= \int_{\frac{5}{6}}^{1} \left(\varpi-\rho_6\right)F'\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi \\
&= \frac{1}{\beta-\alpha}\left(\varpi-\rho_6\right)F\left(\varpi\beta+(1-\varpi)\alpha\right)\Big|_{\varpi=\frac{5}{6}}^{1} - \frac{1}{\beta-\alpha}\int_{\frac{5}{6}}^{1} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi \\
&= \left(1-\rho_6\right)F(\beta)+\left(\frac{5}{6}-\rho_6\right)F\left(\frac{5\beta+\alpha}{6}\right)-\frac{1}{\beta-\alpha}\int_{\frac{5}{6}}^{1} F\left(\varpi\beta+(1-\varpi)\alpha\right)d\varpi. \qquad (12)
\end{aligned}
$$

Replace the values from (7) to (12) in (6), and by change of variables $\xi = \varpi\beta + (1-\varpi)\alpha$, we get the desired result. The proof of Lemma 2.1 is completed. □

It should be noted that the newly established Lemma 2.1, unifies many quadrature rules. We summarize some particular cases in the Table 1.

$(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)$	**n**	$\mathbf{Q}(\alpha,\beta)$	**Formula**
$(0,0,0,1,1,1)$	0	$F\left(\frac{\alpha+\beta}{2}\right)$	Midpoint
$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	1	$\frac{F(\alpha)+F(\beta)}{2}$	Trapezoidal
$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6}\right)$	2	$\frac{1}{6}\left[F(\alpha)+4F\left(\frac{\alpha+\beta}{2}\right)+F(\beta)\right]$	Simpson's 1/3
$\left(\frac{1}{8},\frac{1}{8},\frac{4}{8},\frac{4}{8},\frac{7}{8},\frac{7}{8}\right)$	3	$\frac{1}{8}\left[F(\alpha)+3F\left(\frac{2\alpha+\beta}{3}\right)+3F\left(\frac{\alpha+2\beta}{3}\right)+F(\beta)\right]$	Simpson's 3/8
$\left(0,\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},0\right)$	3	$\frac{1}{8}\left[3F\left(\frac{5\alpha+\beta}{6}\right)+2F\left(\frac{\alpha+\beta}{2}\right)+3F\left(\frac{\alpha+5\beta}{6}\right)\right]$	Maclaurin's
$\left(\frac{1}{20},\frac{6}{20},\frac{7}{20},\frac{13}{20},\frac{14}{20},\frac{19}{20}\right)$	6	$\frac{1}{20}\left[F(\alpha)+5F\left(\frac{5\alpha+\beta}{6}\right)+F\left(\frac{2\alpha+\beta}{3}\right)+6F\left(\frac{\alpha+\beta}{2}\right)+F\left(\frac{\alpha+2\beta}{3}\right)+5F\left(\frac{\alpha+5\beta}{6}\right)+F(\beta)\right]$	Weddle's

Table 1: Derived formulae with different choices of parameters.

From Table 1, it is evident that different parameter choices in the newly established parameterized identity can yield a variety of quadrature rules. Using Lemma 2.1, we will demonstrate the error bounds associated with these quadrature rules for various types of functions.

Theorem 2.2. *Assume that $F : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $F' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in I^\circ$. If $|F'|$ is convex on $[\alpha,\beta]$, then we have the following inequality:*

$$
\left|\rho_1 F(\alpha) + (\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right) + (\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right) + (\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\right.
$$

$$+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|$$
$$\leq (\beta-\alpha)\left[M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\beta)\right|+N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\alpha)\right|\right],$$

where

$$\begin{aligned} M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= \frac{\rho_4^3}{3}-\frac{25\rho_4}{72}+\frac{\rho_5^3}{3}-\frac{41\rho_5}{72}+\frac{\rho_6^3}{3}-\frac{61\rho_6}{72}+\frac{\rho_1^3}{3}\\ &\quad -\frac{\rho_1}{72}+\frac{\rho_2^3}{6}+\frac{\rho_2\rho_2\rho_3^2}{2}-\frac{5\rho_2}{72}-\frac{\rho_2\rho_3^3}{3}+\frac{\rho_3^3}{3}-\frac{13\rho_3}{72}+\frac{37}{36},\\ N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= -\frac{\rho_4^3}{3}+\rho_4^2-\frac{59\rho_4}{72}-\frac{\rho_5^3}{3}+\rho_5^2-\frac{67\rho_5}{72}-\frac{\rho_6^3}{3}+\rho_6^2-\frac{71\rho_6}{72}\\ &\quad -\frac{\rho_1^3}{3}+\rho_1^2-\frac{11\rho_1}{72}-\frac{\rho_2^3}{3}+\rho_2^2-\frac{31\rho_2}{72}-\frac{\rho_3^3}{3}+\rho_3^2-\frac{47\rho_3}{72}+1. \end{aligned}$$

Proof. By Lemma 2.1 and absolute property, we have:

$$\Bigg|\rho_1F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)$$
$$+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|$$
$$\begin{aligned} \leq (\beta-\alpha)\Bigg[&\int_0^{\rho_1}(\rho_1-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\\ &+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\\ &+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\\ &+\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\\ &+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\\ &+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi+\int_{\rho_6}^{1}(\varpi-\rho_6)\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|d\varpi\Bigg]. \end{aligned} \tag{13}$$

Since $|F'|$ is convex on $[\alpha,\beta]$, we get

$$\Bigg|\rho_1F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)$$
$$+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|$$
$$\begin{aligned} \leq (\beta-\alpha)\Bigg[&\int_0^{\rho_1}(\rho_1-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_0^{\rho_1}(\rho_1-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\ &+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)(1-\varpi)\left|F'(\alpha)\right|d\varpi \end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)\varpi\left|F'(\beta)\right|d\varpi+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)(1-\varpi)\left|F'(\alpha)\right|d\varpi\\
&+\left.\int_{\rho_6}^{1}(\varpi-\rho_6)\varpi\left|F'(\beta)\right|d\varpi+\int_{\rho_6}^{1}(\varpi-\rho_6)(1-\varpi)\left|F'(\alpha)\right|d\varpi\right]\\
=\;&(\beta-\alpha)\left[M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\beta)\right|+N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\alpha)\right|\right].
\end{aligned}
$$

Hence, the proof of Theorem 2.2 is completed. □

Remark 2.3. *If we replace $\rho_1=\rho_2=\rho_3=\rho_4=\rho_5=\rho_6=\frac{1}{2}$ in Theorem 2.2, we get error bounds of the Trapezoidal rule proved by Dragomir and Agarwal in [10].*

Remark 2.4. *If we replace $\rho_1=\rho_2=\rho_3=0,\rho_4=\rho_5=\rho_6=1$ in Theorem 2.2, we get error bounds of the Midpoint formula proved by Kirmaci in [21].*

Remark 2.5. *If we replace $\rho_1=\frac{1}{20},\rho_2=\frac{6}{20},\rho_3=\frac{7}{20},\rho_4=\frac{13}{20},\rho_5=\frac{14}{20}$ and $\rho_6=\frac{19}{20}$ in Theorem 2.2, we get error bounds Weddle's rule.*

Remark 2.6. *For different suitable values of parameters in Theorem 2.2, interested readers can get the error bounds of Simpson's 1/3, Simpson's 3/8, and Maclaurin's rules.*

Theorem 2.7. *Assume that $F: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable on I° such that $F'\in L_1[\alpha,\beta]$, where $\alpha,\beta\in I^\circ$. If $|F'|^q$ is convex on $[\alpha,\beta]$, and $p,q>1$, then we have the following inequality:*

$$
\begin{aligned}
&\left|\rho_1F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\right.\\
&\left.+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\right|\\
\leq\;&(\beta-\alpha)\left(\frac{\left(\frac{2}{3}-\rho_4\right)^{p+1}}{p+1}+\frac{\left(\rho_4-\frac{1}{2}\right)^{p+1}}{p+1}+\frac{\left(\frac{5}{6}-\rho_5\right)^{p+1}}{p+1}+\frac{\left(\rho_5-\frac{2}{3}\right)^{p+1}}{p+1}\right.
\end{aligned}
$$

$$+\frac{(1-\rho_6)^{p+1}}{p+1}+\frac{\left(\rho_6-\frac{5}{6}\right)^{p+1}}{p+1}+\frac{\rho_1^{p+1}}{p+1}+\frac{\left(\frac{1}{6}-\rho_1\right)^{p+1}}{p+1}+\frac{\left(\frac{1}{3}-\rho_2\right)^{p+1}}{p+1}$$

$$+\frac{\left(\rho_2-\frac{1}{6}\right)^{p+1}}{p+1}+\frac{\left(\frac{1}{2}-\rho_3\right)^{p+1}}{p+1}+\frac{\left(\rho_3-\frac{1}{3}\right)^{p+1}}{p+1}\Bigg)^{\frac{1}{p}}\left[\frac{\left|F'(\beta)\right|^q+\left|F'(\alpha)\right|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}$.

Proof. From Lemma 2.1, and Hölder integral inequality, we have:

$$\begin{aligned}
&\Bigg|\rho_1 F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\
&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|\\
\le\ &(\beta-\alpha)\Bigg(\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|^p d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|^p d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|^p d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|^p d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|^p d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|^p d\varpi\Bigg)^{\frac{1}{p}}\left(\int_0^1\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|F'|^q$ is convex on $[\alpha,\beta]$, we get

$$\begin{aligned}
&\Bigg|\rho_1 F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\
&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|\\
\le\ &(\beta-\alpha)\Bigg(\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|^p d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|^p d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|^p d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|^p d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|^p d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|^p d\varpi\Bigg)^{\frac{1}{p}}\left(\int_0^1\varpi\left|F'(\beta)\right|^q+(1-\varpi)\left|F'(\alpha)\right|^q d\varpi\right)^{\frac{1}{q}}\\
=\ &(\beta-\alpha)\Bigg(\int_0^{\rho_1}(\rho_1-\varpi)^p d\varpi+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)^p d\varpi+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)^p d\varpi+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)^p d\varpi\\
&+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)^p d\varpi+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)^p d\varpi++\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)^p d\varpi+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)^p d\varpi\\
&+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)^p d\varpi+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)^p d\varpi+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)^p d\varpi+\int_{\rho_6}^{1}(\varpi-\rho_6)^p d\varpi\Bigg)^{\frac{1}{p}}\\
&\times\left(\int_0^1\varpi\left|F'(\beta)\right|^q+(1-\varpi)\left|F'(\alpha)\right|^q d\varpi\right)^{\frac{1}{q}}\\
=\ &(\beta-\alpha)\Bigg(\frac{\left(\frac{2}{3}-\rho_4\right)^{p+1}}{p+1}+\frac{\left(\rho_4-\frac{1}{2}\right)^{p+1}}{p+1}+\frac{\left(\frac{5}{6}-\rho_5\right)^{p+1}}{p+1}+\frac{\left(\rho_5-\frac{2}{3}\right)^{p+1}}{p+1}\\
&+\frac{(1-\rho_6)^{p+1}}{p+1}+\frac{\left(\rho_6-\frac{5}{6}\right)^{p+1}}{p+1}+\frac{\rho_1^{p+1}}{p+1}+\frac{\left(\frac{1}{6}-\rho_1\right)^{p+1}}{p+1}+\frac{\left(\frac{1}{3}-\rho_2\right)^{p+1}}{p+1}
\end{aligned}$$

$$+\frac{\left(\rho_2-\frac{1}{6}\right)^{p+1}}{p+1}+\frac{\left(\frac{1}{2}-\rho_3\right)^{p+1}}{p+1}+\frac{\left(\rho_3-\frac{1}{3}\right)^{p+1}}{p+1}\Bigg)^{\frac{1}{p}}\left[\frac{\left|F'(\beta)\right|^q+|F'(\alpha)|^q}{2}\right]^{\frac{1}{q}}.$$

Hence, proof of the Theorem 2.7 is completed. □

Theorem 2.8. *Assume that $F: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $F' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in I^\circ$. If $|F'|^q$ is convex on $[\alpha,\beta]$, and $q \geq 1$, then we have the following inequality:*

$$\begin{aligned}&\Bigg|\rho_1 F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|\\ \leq\;&(\beta-\alpha)\left(R(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right)^{1-\frac{1}{q}}\left[\frac{M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\beta)\right|^q+N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)|F'(\alpha)|^q}{2}\right]^{\frac{1}{q}},\end{aligned}$$

where $M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)$, $N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)$ is defined in Theorem 2.2, and $R(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)$ is defined as:

$$\begin{aligned}R(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) \;=\;& -\rho_4^2-\left(\frac{2}{3}-\rho_4\right)\rho_4+\left(\rho_4-\frac{1}{2}\right)\rho_4-\rho_5^2-\left(\frac{5}{6}-\rho_5\right)\rho_5+\rho_5\left(\rho_5-\frac{2}{3}\right)\\&-\rho_6^2-(1-\rho_6)\rho_6+\rho_6\left(\rho_6-\frac{5}{6}\right)-\left(\frac{1}{6}-\rho_1\right)\rho_1-\rho_2^2-\left(\frac{1}{3}-\rho_2\right)\rho_2\\&+\rho_2\left(\rho_2-\frac{1}{6}\right)-\rho_3^2-\left(\frac{1}{2}-\rho_3\right)\rho_3+\rho_3\left(\rho_3-\frac{1}{3}\right)+\frac{73}{36}.\end{aligned}$$

Proof. From Lemma 2.1, and power mean integral inequality, we have:

$$\begin{aligned}&\Bigg|\rho_1 F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|\\ \leq\;&(\beta-\alpha)\Bigg(\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|d\varpi\\&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|d\varpi\Bigg)^{1-\frac{1}{q}}\\&\Bigg(\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi\\&+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi\\&+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi\Bigg)^{\frac{1}{q}}.\end{aligned}\tag{14}$$

$$\Bigg[\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|d\varpi$$

$$
\begin{aligned}
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|d\varpi\Bigg] \\
= \;& -\rho_4^2-\left(\frac{2}{3}-\rho_4\right)\rho_4+\left(\rho_4-\frac{1}{2}\right)\rho_4-\rho_5^2-\left(\frac{5}{6}-\rho_5\right)\rho_5+\rho_5\left(\rho_5-\frac{2}{3}\right) \\
&-\rho_6^2-(1-\rho_6)\rho_6+\rho_6\left(\rho_6-\frac{5}{6}\right)-\left(\frac{1}{6}-\rho_1\right)\rho_1-\rho_2^2-\left(\frac{1}{3}-\rho_2\right)\rho_2 \\
&+\rho_2\left(\rho_2-\frac{1}{6}\right)-\rho_3^2-\left(\frac{1}{2}-\rho_3\right)\rho_3+\rho_3\left(\rho_3-\frac{1}{3}\right)+\frac{73}{36}. \qquad (15)
\end{aligned}
$$

Since $|F'|^q$ is convex on $[\alpha,\beta]$, we get

$$
\begin{aligned}
&\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi \\
&+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi \\
&+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)\right|^q d\varpi \\
= \;& \Bigg[\int_0^{\rho_1}(\rho_1-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_0^{\rho_1}(\rho_1-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_1}^{\frac{1}{6}}(\varpi-\rho_1)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\frac{1}{6}}^{\rho_2}(\rho_2-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_2}^{\frac{1}{3}}(\varpi-\rho_2)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\frac{1}{3}}^{\rho_3}(\rho_3-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_3}^{\frac{1}{2}}(\varpi-\rho_3)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\frac{1}{2}}^{\rho_4}(\rho_4-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_4}^{\frac{2}{3}}(\varpi-\rho_4)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\frac{2}{3}}^{\rho_5}(\rho_5-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_5}^{\frac{5}{6}}(\varpi-\rho_5)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\frac{5}{6}}^{\rho_6}(\rho_6-\varpi)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi \\
&+\int_{\rho_6}^{1}(\varpi-\rho_6)\varpi\left|F'(\beta)\right|^q d\varpi+\int_{\rho_6}^{1}(\varpi-\rho_6)(1-\varpi)\left|F'(\alpha)\right|^q d\varpi\Bigg] \\
= \;& \left[M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\beta)\right|^q+N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\alpha)\right|^q\right]. \qquad (16)
\end{aligned}
$$

Replace the values of (15) and (16) in (14), we get the desired result. Hence, proof of the Theorem 2.8 is completed. □

2.2. Bounded Functions

A function with a bounded first derivative has a derivative that remains within fixed limits for all points in its domain. This property ensures that the function does not change too rapidly, providing a measure of smoothness and predictability. Functions with bounded first derivatives are crucial in numerical analysis and optimization, as they guarantee stability and convergence in various algorithms.

Theorem 2.9. *Assume that $F : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $F' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in I^\circ$. If there exists constants $m, M \in \mathbb{R}$, such that $m \le F'(\xi) \le M$, for all $\xi \in [\alpha,\beta]$, then we have the following inequality:*

$$\begin{aligned}
&\Bigg|\rho_1 F(\alpha) + (\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right) + (\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right) + (\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\
&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right) + (\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right) + (1-\rho_6)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|\\
\le\ & \frac{(\beta-\alpha)}{2}\left[\frac{(M-m)}{72}B_1(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) + (m+M)B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right],
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
B_1(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= 146 + 72\rho_1^2 - 12\rho_1 + 72\rho_4^2 - 84\rho_4 + 72\rho_5^2 - 108\rho_5\\
&\quad +72\rho_6^2 - 132\rho_6 + 72\rho_2^2 - 36\rho_2 + 72\rho_3^2 - 60\rho_3,\\
B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= \frac{1}{2} - \frac{\rho_1}{6} - \frac{\rho_2}{6} - \frac{\rho_3}{6} - \frac{\rho_4}{6} - \frac{\rho_5}{6} - \frac{\rho_6}{6}.
\end{aligned}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\rho_1 F(\alpha) + (\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right) + (\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right) + (\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\
&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right) + (\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right) + (1-\rho_6)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\\
=\ & (\beta-\alpha)\Bigg[\int_0^{\frac{1}{6}} (\varpi-\rho_1)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\\
&+\int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi-\rho_2)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi + \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi-\rho_3)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi-\rho_4)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi + \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi-\rho_5)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\\
&+\int_{\frac{5}{6}}^{1} (\varpi-\rho_6)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\Bigg]\\
=\ & (\beta-\alpha)\Bigg[\int_0^{\frac{1}{6}} \left((\varpi-\rho_1)F'(\varpi\beta+(1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2}\right)d\varpi\\
&+\int_{\frac{1}{6}}^{\frac{1}{3}} \left((\varpi-\rho_2)F'(\varpi\beta+(1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2}\right)d\varpi\\
&+\int_{\frac{1}{3}}^{\frac{1}{2}} \left((\varpi-\rho_3)F'(\varpi\beta+(1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2}\right)d\varpi
\end{aligned}$$

$$
\begin{aligned}
& + \int_{\frac{1}{2}}^{\frac{2}{3}} \left((\varpi - \rho_4) F'(\varpi\beta + (1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{2}{3}}^{\frac{5}{6}} \left((\varpi - \rho_5) F'(\varpi\beta + (1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{5}{6}}^{1} \left((\varpi - \rho_6) F'(\varpi\beta + (1-\varpi)\alpha) + \frac{m+M}{2} - \frac{m+M}{2} \right) d\varpi \Bigg] \\
= \; & (\beta - \alpha) \Bigg[\int_0^{\frac{1}{6}} \left((\varpi - \rho_1) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{1}{6}}^{\frac{1}{3}} \left((\varpi - \rho_2) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{1}{3}}^{\frac{1}{2}} \left((\varpi - \rho_3) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{1}{2}}^{\frac{2}{3}} \left((\varpi - \rho_4) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{2}{3}}^{\frac{5}{6}} \left((\varpi - \rho_5) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \\
& + \int_{\frac{5}{6}}^{1} \left((\varpi - \rho_6) F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right) d\varpi \Bigg] \\
& + \frac{m+M}{2} \Bigg[\int_0^{\frac{1}{6}} (\varpi - \rho_1) d\varpi + \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi - \rho_2) d\varpi + \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi - \rho_3) d\varpi \\
& + \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi - \rho_4) d\varpi + \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi - \rho_5) d\varpi + \int_{\frac{5}{6}}^{1} (\varpi - \rho_6) d\varpi \Bigg].
\end{aligned}
\tag{18}
$$

In this context, we consider the scenarios in which we take into account the fact that

$$
\begin{aligned}
& \frac{m+M}{2} \Bigg[\int_0^{\frac{1}{6}} (\varpi - \rho_1) d\varpi + \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi - \rho_2) d\varpi + \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi - \rho_3) d\varpi \\
& + \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi - \rho_4) d\varpi + \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi - \rho_5) d\varpi + \int_{\frac{5}{6}}^{1} (\varpi - \rho_6) d\varpi \Bigg] \\
= \; & \frac{(m+M)}{2} \left[\frac{1}{2} - \frac{\rho_1}{6} - \frac{\rho_2}{6} - \frac{\rho_3}{6} - \frac{\rho_4}{6} - \frac{\rho_5}{6} - \frac{\rho_6}{6} \right].
\end{aligned}
$$

Using the absolute property on each side of 18, we have

$$
\begin{aligned}
& \Bigg| \frac{1}{20} \left[F(\alpha) + 5F\left(\frac{\beta + 5\alpha}{6}\right) + F\left(\frac{\beta + 2\alpha}{3}\right) + 6F\left(\frac{\beta + \alpha}{2}\right) + F\left(\frac{2\beta + \alpha}{3}\right) \right. \\
& \left. + 5F\left(\frac{5\beta + \alpha}{6}\right) + F(\beta) \right] - \frac{1}{\beta - \alpha} \int_\alpha^\beta F(\xi) d\xi \Bigg| \\
\leq \; & (\beta - \alpha) \Bigg[\int_0^{\frac{1}{6}} |\varpi - \rho_1| \left| F'(\varpi\beta + (1-\varpi)\alpha) - \frac{m+M}{2} \right| d\varpi
\end{aligned}
$$

$$+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|F'\left|\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|d\varpi$$
$$+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\left|F'\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|d\varpi$$
$$+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\left|F'\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|d\varpi$$
$$+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\left|F'\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|d\varpi$$
$$\left.+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\left|F'\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|d\varpi\right]$$
$$+\frac{(m+M)(\beta-\alpha)}{2}\left|\frac{1}{2}-\frac{\rho_1}{6}-\frac{\rho_2}{6}-\frac{\rho_3}{6}-\frac{\rho_4}{6}-\frac{\rho_5}{6}-\frac{\rho_6}{6}\right|.$$

Since $m\leq F'(\xi)\leq M$, for all $\xi\in[\alpha,\beta]$, we can conclude

$$\left|F'\left(\varpi\beta+(1-\varpi)\alpha\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}.$$

So,

$$\begin{aligned}
&\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.\\
&\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(\xi)\,d\xi\right|\\
\leq\;&\frac{(M-m)(\beta-\alpha)}{2}\left[\int_{0}^{\frac{1}{6}}\left|\varpi-\rho_1\right|d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|d\varpi\right.\\
&\left.+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|d\varpi\right]\\
&+\frac{(m+M)(\beta-\alpha)}{2}\left|\frac{1}{2}-\frac{\rho_1}{6}-\frac{\rho_2}{6}-\frac{\rho_3}{6}-\frac{\rho_4}{6}-\frac{\rho_5}{6}-\frac{\rho_6}{6}\right|\\
=\;&\frac{(M-m)(\beta-\alpha)}{2\times 72}\left[146+72\rho_1^2-12\rho_1+72\rho_4^2-84\rho_4+72\rho_5^2\right.\\
&\left.-108\rho_5+72\rho_6^2-132\rho_6+72\rho_2^2-36\rho_2+72\rho_3^2-60\rho_3\right]\\
&+\frac{(m+M)(\beta-\alpha)}{2}\left|\frac{1}{2}-\frac{\rho_1}{6}-\frac{\rho_2}{6}-\frac{\rho_3}{6}-\frac{\rho_4}{6}-\frac{\rho_5}{6}-\frac{\rho_6}{6}\right|.\\
=\;&\frac{(\beta-\alpha)}{2}\left[\frac{(M-m)}{72}B_1(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)+(m+M)B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right].
\end{aligned}$$

Hence, the proof of Theorem 2.9 is completed. □

Remark 2.10. *If we replace $\rho_1=\rho_2=\rho_3=\rho_4=\rho_5=\rho_6=\frac{1}{2}$ in Theorem 2.9, we get error bounds of the Trapezoidal rule.*

Remark 2.11. *If we replace $\rho_1=\rho_2=\rho_3=0$, $\rho_4=\rho_5=\rho_6=1$ in Theorem 2.9, we get error bounds of the Midpoint formula.*

Remark 2.12. *If we replace* $\rho_1 = \frac{1}{20}, \rho_2 = \frac{6}{20}, \rho_3 = \frac{7}{20}, \rho_4 = \frac{13}{20}, \rho_5 = \frac{14}{20}$ *and* $\rho_6 = \frac{19}{20}$ *in Theorem 2.9, we get error bounds Weddle's rule.*

Remark 2.13. *For different suitable values of parameters in Theorem 2.9, interested readers can get the error bounds of Simpson's* 1/3*, Simpson's* 3/8*, and Maclaurin's rules.*

2.3. Lipschitzian Functions

Lipschitzian functions are characterized by a property that bounds the rate at which they can change. Lipschitzian functions are widely used in optimization, differential equations, and numerical analysis due to their favourable convergence properties and robustness.

Theorem 2.14. *Assume that* $F : I \subset [0,\infty) \to \mathbb{R}$ *be differentiable on* I° *such that* $F' \in L_1[\alpha,\beta]$*, where* $\alpha,\beta \in I^\circ$*. If* F' *is L-Lipschitz function on* $[\alpha,\beta]$ *then we have the following inequality:*

$$\begin{aligned} & \left|\rho_1 F(\alpha) + (\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right) + (\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right) + (\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\right. \\ & \left. + (\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right) + (\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right) + (1-\rho_6)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\right| \\ \leq\ & (\beta-\alpha)L\left[(\beta-\alpha)B_3(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) + \alpha B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right], \end{aligned} \tag{19}$$

where

$$\begin{aligned} B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= \frac{1}{2} - \frac{\rho_1}{6} - \frac{\rho_2}{6} - \frac{\rho_3}{6} - \frac{\rho_4}{6} - \frac{\rho_5}{6} - \frac{\rho_6}{6}, \\ B_3(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) &= \frac{1}{648}\left[533 + 216\rho_1^3 - 9\rho_1 + 216\rho_4^3 - 225\rho_4 - 132\rho_5\right. \\ & \left. + 108\rho_5^3 + 216\rho_6^3 - 549\rho_6 + 108\rho_2^3 - 9\rho_2 + 216\rho_3^3 - 117\rho_3\right]. \end{aligned}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned} & \rho_1 F(\alpha) + (\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right) + (\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right) + (\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right) \\ & + (\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right) + (\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right) + (1-\rho_6)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr \\ =\ & (\beta-\alpha)\left[\int_0^{\frac{1}{6}} (\varpi-\rho_1)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\right. \\ & + \int_{\frac{1}{6}}^{\frac{1}{3}} (\varpi-\rho_2)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi + \int_{\frac{1}{3}}^{\frac{1}{2}} (\varpi-\rho_3)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi \\ & + \int_{\frac{1}{2}}^{\frac{2}{3}} (\varpi-\rho_4)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi + \int_{\frac{2}{3}}^{\frac{5}{6}} (\varpi-\rho_5)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi \\ & \left. + \int_{\frac{5}{6}}^{1} (\varpi-\rho_6)F'(\varpi\beta+(1-\varpi)\alpha)\,d\varpi\right] \\ =\ & (\beta-\alpha)\left[\int_0^{\frac{1}{6}} ((\varpi-\rho_1)F'(\varpi\beta+(1-\varpi)\alpha) + F'(\alpha) - F'(\alpha))\,d\varpi\right. \\ & + \int_{\frac{1}{6}}^{\frac{1}{3}} ((\varpi-\rho_2)F'(\varpi\beta+(1-\varpi)\alpha) + F'(\alpha) - F'(\alpha))\,d\varpi \end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{1}{3}}^{\frac{1}{2}}\left((\varpi-\rho_3)F'(\varpi\beta+(1-\varpi)\alpha)+F'(\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left((\varpi-\rho_4)F'(\varpi\beta+(1-\varpi)\alpha)+F'(\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{2}{3}}^{\frac{5}{6}}\left((\varpi-\rho_5)F'(\varpi\beta+(1-\varpi)\alpha)+F'(\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{5}{6}}^{1}\left((\varpi-\rho_6)F'(\varpi\beta+(1-\varpi)\alpha)+F'(\alpha)-F'(\alpha)\right)d\varpi\Bigg]\\
=\;&(\beta-\alpha)\Bigg[\int_0^{\frac{1}{6}}\left((\varpi-\rho_1)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{1}{6}}^{\frac{1}{3}}\left((\varpi-\rho_2)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{1}{3}}^{\frac{1}{2}}\left((\varpi-\rho_3)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left((\varpi-\rho_4)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{2}{3}}^{\frac{5}{6}}\left((\varpi-\rho_5)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\\
&+\int_{\frac{5}{6}}^{1}\left((\varpi-\rho_6)F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right)d\varpi\Bigg]\\
&+F'(\alpha)\Bigg[\int_0^{\frac{1}{6}}(\varpi-\rho_1)\,d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}(\varpi-\rho_2)\,d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}(\varpi-\rho_3)\,d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}(\varpi-\rho_4)\,d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}(\varpi-\rho_5)\,d\varpi+\int_{\frac{5}{6}}^{1}(\varpi-\rho_6)\,d\varpi\Bigg]. \qquad (20)
\end{aligned}
$$

In this context, we consider the scenarios in which we take into account the fact that

$$
\begin{aligned}
&F'(\alpha)\Bigg[\int_0^{\frac{1}{6}}\left(\varpi-\frac{1}{20}\right)d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left(\varpi-\frac{6}{20}\right)d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left(\varpi-\frac{7}{20}\right)d\varpi\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}\left(\varpi-\frac{13}{20}\right)d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left(\varpi-\frac{14}{20}\right)d\varpi+\int_{\frac{5}{6}}^{1}\left(\varpi-\frac{19}{20}\right)d\varpi\Bigg]\\
=\;&F'(\alpha)\,B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6),
\end{aligned}
$$

where

$$
B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)=\frac{1}{2}-\frac{\rho_1}{6}-\frac{\rho_2}{6}-\frac{\rho_3}{6}-\frac{\rho_4}{6}-\frac{\rho_5}{6}-\frac{\rho_6}{6}.
$$

Using the absolute property on each side of 20, and using the property F' is Lipschitzian on $[\alpha,\beta]$, we have

$$
\begin{aligned}
&\Bigg|\rho_1F(\alpha)+(\rho_2-\rho_1)F\left(\frac{\beta+5\alpha}{6}\right)+(\rho_3-\rho_2)F\left(\frac{\beta+2\alpha}{3}\right)+(\rho_4-\rho_3)F\left(\frac{\alpha+\beta}{2}\right)\\
&+(\rho_5-\rho_4)F\left(\frac{2\beta+\alpha}{3}\right)+(\rho_6-\rho_5)F\left(\frac{5\beta+\alpha}{6}\right)+(1-\rho_6)F(\beta)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\beta-\alpha)\left[\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|F'\left|(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi\right.\\
&\quad+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi\\
&\quad\left.+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\left|F'(\varpi\beta+(1-\varpi)\alpha)-F'(\alpha)\right|d\varpi\right]\\
&\quad+F'(\alpha)B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\\
&\leq (\beta-\alpha)L\left[\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi\right.\\
&\quad+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi\\
&\quad+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi\\
&\quad\left.+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\left|(\varpi\beta+(1-\varpi)\alpha)-\alpha\right|d\varpi\right]\\
&\quad+\alpha(\beta-\alpha)LB_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\\
&= (\beta-\alpha)^2L\left[\int_0^{\frac{1}{6}}\left|\varpi-\rho_1\right|\varpi d\varpi+\int_{\frac{1}{6}}^{\frac{1}{3}}\left|\varpi-\rho_2\right|\varpi d\varpi+\int_{\frac{1}{3}}^{\frac{1}{2}}\left|\varpi-\rho_3\right|\varpi d\varpi\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{2}{3}}\left|\varpi-\rho_4\right|\varpi d\varpi+\int_{\frac{2}{3}}^{\frac{5}{6}}\left|\varpi-\rho_5\right|\varpi d\varpi+\int_{\frac{5}{6}}^{1}\left|\varpi-\rho_6\right|\varpi d\varpi\right]\\
&\quad+\alpha(\beta-\alpha)LB_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\\
&= (\beta-\alpha)^2LB_3(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)+\alpha(\beta-\alpha)LB_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\\
&= (\beta-\alpha)L\left[(\beta-\alpha)B_3(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)+\alpha B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
B_3(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6) = \frac{1}{648}\Big[&533+216\rho_1^3-9\rho_1+216\rho_4^3-225\rho_4-132\rho_5\\
&+108\rho_5^3+216\rho_6^3-549\rho_6+108\rho_2^3-9\rho_2+216\rho_3^3-117\rho_3\Big].
\end{aligned}
$$

Hence, the proof of Theorem 2.14 is completed. □

Remark 2.15. *If we replace $\rho_1=\rho_2=\rho_3=\rho_4=\rho_5=\rho_6=\frac{1}{2}$ in Theorem 2.14, we get error bounds of the Trapezoidal rule.*

Remark 2.16. *If we replace $\rho_1=\rho_2=\rho_3=0,\rho_4=\rho_5=\rho_6=1$ in Theorem 2.14, we get error bounds of the Midpoint formula.*

Remark 2.17. *If we replace $\rho_1=\frac{1}{20},\rho_2=\frac{6}{20},\rho_3=\frac{7}{20},\rho_4=\frac{13}{20},\rho_5=\frac{14}{20}$ and $\rho_6=\frac{19}{20}$ in Theorem 2.14, we get error bounds Weddle's rule.*

Remark 2.18. *For different suitable values of parameters in Theorem 2.14, interested readers can get the error bounds of Simpson's 1/3, Simpson's 3/8, and Maclaurin's rules.*

3. Applications in Numerical Integration

In this section, we explore the application of the derived parameterized inequalities to the Newton-Cotes formula. Applying newly established results to the Newton-Cotes formula enhances the accuracy and efficiency of numerical integration techniques. By incorporating these inequalities, we can refine the approximation process in the Newton-Cotes formula, particularly for functions with specific properties such as convexity. Newly established inequalities systematically optimize the selection of sampling points and weights in the Newton-Cotes formula, improving precision in estimating definite integrals. This application extends the theoretical understanding of these inequalities and demonstrates their practical relevance in numerical analysis and computational mathematics.

Assume that I_n be the partition given by

$$I_n : \alpha = \xi_0 < \xi_1 < \xi_2 < ... < \xi_{n-1} < \xi_n = \beta,$$

$$h_i = \frac{\xi_{i+1} - \xi_i}{\mathbf{n}},\ i = 0, 1, 2, ..., n-1,$$

where the value of $\mathbf{n}$ depends upon the value of n is Newton-Cotes formula. Then we have

$$\int_\alpha^\beta F(\xi)\, d\xi = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where

$$\begin{aligned} S_n(I_n, F) &= \sum_{i=0}^{n-1} (\xi_{i+1} - \xi_i) \left[\rho_1 F(\xi_i) + (\rho_2 - \rho_1) F(\xi_i + h) + (\rho_3 - \rho_2) F(\xi_i + 2h)\right. \\ &\quad \left. + (\rho_4 - \rho_3) F(\xi_i + 3h) + (\rho_5 - \rho_4) F(\xi_i + 4h) + (\rho_6 - \rho_5) F(\xi_i + 5h) + (1 - \rho_6) F(\xi_{i+1})\right], \end{aligned}$$

and remainder term satisfies the estimation

$$|\mathbb{R}_n(I_n, F)| \le \sum_{i=0}^{n-1} \frac{(\xi_{i+1} - \xi_i)^2}{\mathbf{n}} \left[|F'(\xi_i)| + |F'(\xi_{i+1})|\right].$$

We prove the following general Propositions for the error bounds of family of Newton-Cotes formula.

Proposition 3.1. *Suppose that $F : [\alpha, \beta] \to \mathbb{R}$ be a differentiable mapping on (α, β) then we have:*

$$\int_\alpha^\beta F(\xi)\, d\xi = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where

$$\begin{aligned} S_n(I_n, F) &= \sum_{i=0}^{n-1} (\xi_{i+1} - \xi_i) \left[\rho_1 F(\xi_i) + (\rho_2 - \rho_1) F(\xi_i + h) + (\rho_3 - \rho_2) F(\xi_i + 2h)\right. \\ &\quad \left. + (\rho_4 - \rho_3) F(\xi_i + 3h) + (\rho_5 - \rho_4) F(\xi_i + 4h) + (\rho_6 - \rho_5) F(\xi_i + 5h) + (1 - \rho_6) F(\xi_{i+1})\right], \end{aligned}$$

I_n be the partition given by

$$I_n : \alpha = \xi_0 < \xi_1 < \xi_2 < ... < \xi_{n-1} < \xi_n = \beta,$$

$$h_i = \frac{\xi_{i+1} - \xi_i}{\mathbf{n}},\ i = 0, 1, 2, ..., n-1.$$

The remainder term satisfies the condition

$$|\mathbb{R}_n(I_n, F)| \le \sum_{i=0}^{n-1} \frac{(\xi_{i+1} - \xi_i)^2}{\mathbf{n}} \left[M(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_i)| + N(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_{i+1})|\right],$$

for all $i = 0, 1, 2, ..., n-1$, and value of $M(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)$, $N(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)$ is defined in Theorem 2.2.

Proof. Let us set things according to the situation

$$\alpha = \xi_i, \beta = \xi_{i+1}, h_i = \frac{\xi_{i+1} - \xi_i}{\mathbf{n}},$$

where $i = 0, 1, 2, ..., n-1$. Then we have the following estimation

$$\begin{aligned} & \Big| (\xi_{i+1} - \xi_i) \left[\rho_1 F(\xi_i) + (\rho_2 - \rho_1) F(\xi_i + h) + (\rho_3 - \rho_2) F(\xi_i + 2h) \right. \\ & \left. + (\rho_4 - \rho_3) F(\xi_i + 3h) + (\rho_5 - \rho_4) F(\xi_i + 4h) + (\rho_6 - \rho_5) F(\xi_i + 5h) + (1 - \rho_6) F(\xi_{i+1}) \right] \\ & - \int_{\xi_i}^{\xi_{i+1}} F(\varpi)\, d\varpi \Big| \\ \leq & \left(\frac{(\xi_{i+1} - \xi_i)^2}{\mathbf{n}} \right) \left[M(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_i)| + N(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_{i+1})| \right], \end{aligned}$$

for all $i = 0, 1, 2, ..., n-1$. After summing and by triangular inequality, we have

$$\begin{aligned} & \Big| \sum_{i=0}^{n-1} (\xi_{i+1} - \xi_i) \left[\rho_1 F(\xi_i) + (\rho_2 - \rho_1) F(\xi_i + h) + (\rho_3 - \rho_2) F(\xi_i + 2h) \right. \\ & \left. + (\rho_4 - \rho_3) F(\xi_i + 3h) + (\rho_5 - \rho_4) F(\xi_i + 4h) + (\rho_6 - \rho_5) F(\xi_i + 5h) + (1 - \rho_6) F(\xi_{i+1}) \right] \\ & - \int_{\xi_i}^{\xi_{i+1}} F(\varpi)\, d\varpi \Big| \\ \leq & \sum_{i=0}^{n-1} \left(\frac{(\xi_{i+1} - \xi_i)^2}{\mathbf{n}} \right) \left[M(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_i)| + N(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) |F'(\xi_{i+1})| \right], \end{aligned}$$

for all $i = 0, 1, 2, ..., n-1$. Which is the required proof of the Proposition 3.1. □

Proposition 3.2. *Suppose that $F : [\alpha, \beta] \to \mathbb{R}$ be a differentiable mapping on (α, β) then we have:*

$$\int_\alpha^\beta F(\xi)\, d\xi = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where $S_n(I_n, F)$ is defined in Proposition 3.1, and the remainder term satisfies the condition

$$\begin{aligned} |\mathbb{R}_n(I_n, F)| \leq & \sum_{i=0}^{n-1} \frac{(\xi_{i+1} - \xi_i)^2}{\mathbf{n}} \left(\frac{\left(\frac{2}{3} - \rho_4\right)^{p+1}}{p+1} + \frac{\left(\rho_4 - \frac{1}{2}\right)^{p+1}}{p+1} + \frac{\left(\frac{5}{6} - \rho_5\right)^{p+1}}{p+1} + \frac{\left(\rho_5 - \frac{2}{3}\right)^{p+1}}{p+1} \right. \\ & + \frac{(1 - \rho_6)^{p+1}}{p+1} + \frac{\left(\rho_6 - \frac{5}{6}\right)^{p+1}}{p+1} + \frac{\rho_1^{p+1}}{p+1} + \frac{\left(\frac{1}{6} - \rho_1\right)^{p+1}}{p+1} + \frac{\left(\frac{1}{3} - \rho_2\right)^{p+1}}{p+1} \\ & \left. + \frac{\left(\rho_2 - \frac{1}{6}\right)^{p+1}}{p+1} + \frac{\left(\frac{1}{2} - \rho_3\right)^{p+1}}{p+1} + \frac{\left(\rho_3 - \frac{1}{3}\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|F'(\xi_i)|^q + |F'(\xi_{i+1})|^q}{2} \right)^{\frac{1}{q}}, \end{aligned}$$

for all $i = 0, 1, 2, ..., n-1$.

Proof. The proof is similar to the Proposition 3.1, by using the inequality of Theorem 2.7, we get the desired result. □

Proposition 3.3. *Suppose that $F : [\alpha, \beta] \to \mathbb{R}$ be a differentiable mapping on (α, β) then we have:*

$$\int_\alpha^\beta F(\xi)\, d\xi = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where $S_n(I_n, F)$ is defined in Proposition 3.1, and the remainder term satisfies the condition

$$\begin{aligned}
&|\mathbb{R}_n(I_n, F)| \\
\leq\ & \sum_{i=0}^{n-1} \frac{(\xi_{i+1}-\xi_i)^2}{\mathbf{n}} \left(R(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right)^{1-\frac{1}{q}} \\
&\times \left[\frac{M(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\beta)\right|^q + N(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\left|F'(\alpha)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

for all $i = 0, 1, 2, ..., n-1$.

Proof. The proof is similar to the Proposition 3.1; we get the desired result by using the inequality of Theorem 2.8. □

Remark 3.4. *For different choices of these unknown parameters and* **n** *in Newton-Cotes formula, interested readers can find error bounds of Trapezoidal, Midpoint, Simpson's, Newton's, Maclaurin's, and Weddle's rule.*

4. Computational Analysis

We conducted a comprehensive numerical analysis to validate the effectiveness of newly derived results. In this Section, we present the results of our computational experiments, providing empirical evidence of the practical utility of the introduced inequalities. Through various numerical tests and comparisons, we assess the accuracy and efficiency of the proposed approach in approximating integrals of differentiable convex functions. We are going to describe different cases for different choices of unknown parameters:

(1) If we choose $\rho_1 = \frac{1}{20}, \rho_2 = \frac{6}{20}, \rho_3 = \frac{7}{20}, \rho_4 = \frac{13}{20}, \rho_5 = \frac{14}{20}$ and $\rho_6 = \frac{19}{20}$, we get Weddle's type inequalities.

Example 4.1. *Let $F(\xi) = e^{\xi}$ is differentiable convex function, and if we fixed values $\alpha = 1, \beta = 2$ then we have:*

$$\begin{aligned}
&\left|\frac{1}{20}\left[F(\alpha) + 5F\left(\frac{\beta+5\alpha}{6}\right) + F\left(\frac{\beta+2\alpha}{3}\right) + 6F\left(\frac{\beta+\alpha}{2}\right) + F\left(\frac{2\beta+\alpha}{3}\right)\right.\right. \\
&\left.\left. +5F\left(\frac{5\beta+\alpha}{6}\right) + F(\beta)\right] - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 1.18029 \times 10^{-7}, \quad (21)
\end{aligned}$$

$$\frac{13(\beta-\alpha)}{450}\left[\left|F'(\beta)\right| + \left|F'(\alpha)\right|\right] = 0.29199. \quad (22)$$

From (21) and (22), Theorem 2.2 is verified.

Example 4.2. *Let $F(\xi) = e^{2\xi}$ is differentiable convex function, and if we fixed values $\alpha = 1, \beta = 2$, and $p = q = 2$, then we have:*

$$\begin{aligned}
&\left|\frac{1}{20}\left[F(\alpha) + 5F\left(\frac{\beta+5\alpha}{6}\right) + F\left(\frac{\beta+2\alpha}{3}\right) + 6F\left(\frac{\beta+\alpha}{2}\right) + F\left(\frac{2\beta+\alpha}{3}\right)\right.\right. \\
&\left.\left. +5F\left(\frac{5\beta+\alpha}{6}\right) + F(\beta)\right] - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 0.00003709, \quad (23)
\end{aligned}$$

$$(\beta-\alpha)\left(\frac{2^{-1-2p}\times 15^{-1-p}\left(1+2^{1+p}+3^{1+p}+7^{1+p}+8^{1+p}+9^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{\left|F'(\beta)\right|^q + \left|F'(\alpha)\right|^q}{2}\right)^{\frac{1}{q}} = 5.5096. \quad (24)$$

From (23) and (24), Theorem 2.7 is verified.

Example 4.3. *Let $F(\xi)=\xi^3$ is differentiable convex function, and if we fixed values $\alpha=1,\beta=2$ and $q=2$, then we have:*

$$\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)+F\left(\frac{2\beta+\alpha}{3}\right)\right.\right.$$
$$\left.\left.+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right|=0, \tag{25}$$

$$\frac{13(\beta-\alpha)}{225}\left[\frac{|F'(\beta)|^q+|F'(\alpha)|^q}{2}\right]^{\frac{1}{q}}=0.5053. \tag{26}$$

From (25) and (26) Theorem 2.8, is verified.

Example 4.4. *Let $F(\xi)=\sin\xi$ is a bounded function with $m=-1$ and $M=1$, if we fixed values $\alpha=1,\beta=2$ then we have:*

$$\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.$$
$$\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right|=2.4643\times10^{-8}, \tag{27}$$

$$\begin{aligned}&\frac{(\beta-\alpha)}{2}\left[\frac{(M-m)}{72}B_1(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)+(m+M)B_2(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6)\right]\\ =\;&\frac{13}{450}(2)=0.0578.\end{aligned} \tag{28}$$

From (27) and (28), Theorem 2.9 is verified.

Example 4.5. *Let $F(\xi)=\xi^6$ is a differentiable function with $F'(\xi)=6\xi^5$is Lipschitzian function with $L=6$, if we fixed values $\alpha=1,\beta=2$ then we have:*

$$\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.$$
$$\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right|=0.00001837, \tag{29}$$

$$\frac{13(\beta-\alpha)^2}{450}L=\frac{13}{450}(6)=0.1733. \tag{30}$$

From (29) and (30), Theorem 2.14 is verified.

(2) If we choose $\rho_1=\rho_2=\rho_3=\rho_4=\rho_5=\rho_6=\frac{1}{2}$, we get Trapezoidal type inequalities.

Example 4.6. *Let $F(\xi)=e^\xi$ is differentiable convex function, and if we fixed values $\alpha=1,\beta=2$ then we have:*

$$\left|\frac{F(\beta)+F(\alpha)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)\,d\xi\right|=0.3829,$$

$$\frac{19(\beta-\alpha)}{36}\left[\frac{|F'(\beta)|+|F'(\alpha)|}{2}\right]=2.6672.$$

Example 4.7. *Let $F(\xi) = \xi^3$ is differentiable convex function, and if we fixed values $\alpha = 1, \beta = 2$ and $q = 2$, then we have:*

$$\left|\frac{F(\beta) + F(\alpha)}{2} - \frac{1}{\beta - \alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 0.75,$$

$$\frac{19(\beta - \alpha)}{36}\left(\frac{19}{36}\right)^{1-\frac{1}{q}}\left[\frac{|F'(\beta)|^q + |F'(\alpha)|^q}{2}\right]^{\frac{1}{q}} = 3.35.$$

Example 4.8. *Let $F(\xi) = e^{2\xi}$ is differentiable convex function, and if we fixed values $\alpha = 1, \beta = 2$, and $p = q = 2$, then we have:*

$$\left|\frac{F(\beta) + F(\alpha)}{2} - \frac{1}{\beta - \alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 7.3890,$$

$$(\beta - \alpha)\left(\frac{2\left(-\frac{1}{3}\right)^{p+1}}{p+1} + \frac{(-1)^{p+1}2^{-p}3^{-p-1}}{p+1} + \frac{2^{-p}3^{-p-1}}{p+1} + \frac{2\,3^{-p-1}}{p+1} + \frac{2^{-p}}{p+1}\right)^{\frac{1}{p}}\left[\frac{|F'(\beta)|^q + |F'(\alpha)|^q}{2}\right]^{\frac{1}{q}} = 22.4928.$$

Example 4.9. *Let $F(\xi) = \sin\xi$ is a bounded function with $m = -1$ and $M = 1$, if we fixed values $\alpha = 1, \beta = 2$ then we have:*

$$\left|\frac{F(\beta) + F(\alpha)}{2} - \frac{1}{\beta - \alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 2.4643 \times 10^{-8}, \tag{31}$$

$$\begin{aligned}&\frac{(\beta - \alpha)}{2}\left[\frac{(M - m)}{72}B_1(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) + (m + M)B_2(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)\right]\\ =\ &\frac{13}{450}(2) = 0.0578.\end{aligned} \tag{32}$$

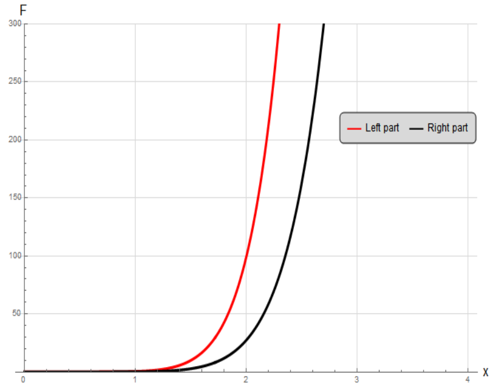
From (31) and (32), Theorem 2.9 is verified.

Example 4.10. *Let $F(\xi) = \xi^2$ is a differentiable function with $F'(\xi) = 2\xi$ is Lipschitzian function with $L = 2$, if we fixed values $\alpha = 0, \beta = 2$ then we have*

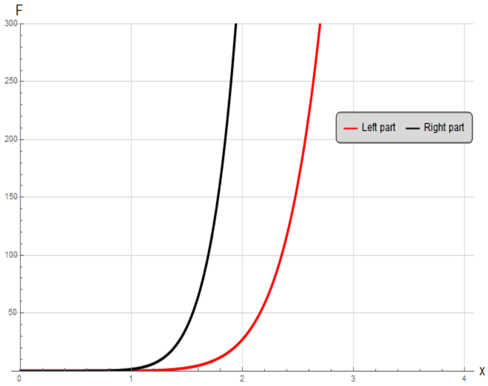
$$\left|\frac{F(\beta) + F(\alpha)}{2} - \frac{1}{\beta - \alpha}\int_\alpha^\beta F(\xi)\,d\xi\right| = 1.2919, \tag{33}$$

$$(\beta - \alpha)L\left[(\beta - \alpha)B_3(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6) + \alpha B_2(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)\right] = 1.7469. \tag{34}$$

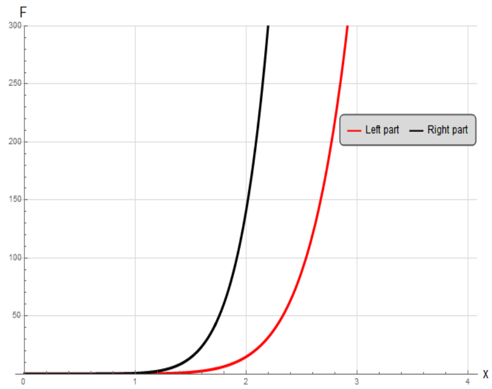
From (33) and (34), Theorem 2.14 is verified.



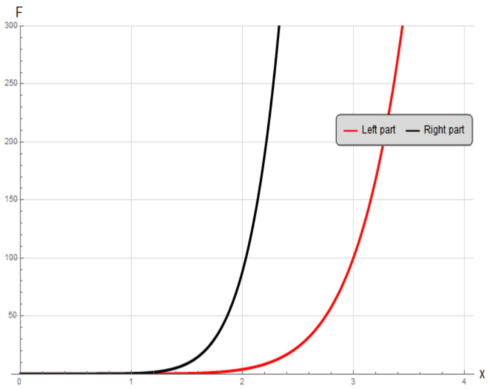
(a) For Trapezoidal rule: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$



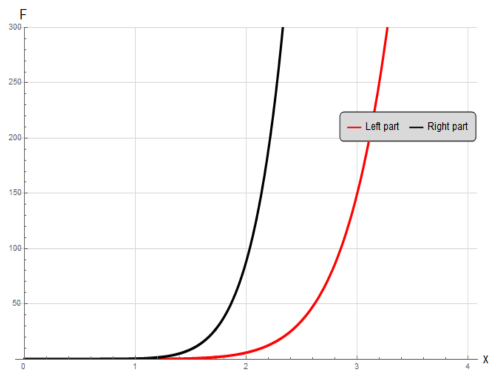
(b) For Midpoint formula: $(0, 0, 0, 1, 1, 1)$



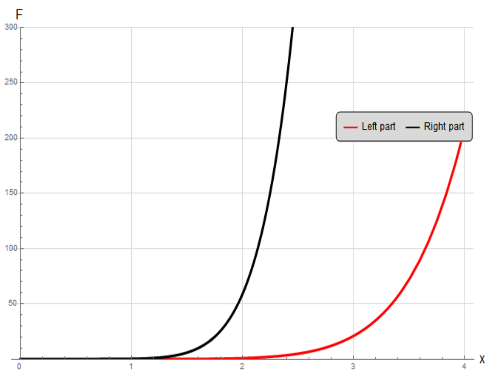
(c) For Simpson's 1/3 formula: $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}\right)$



(d) For Simpson 3/8 formula: $\left(\frac{1}{8}, \frac{1}{8}, \frac{4}{8}, \frac{4}{8}, \frac{7}{8}, \frac{7}{8}\right)$



(e) For Maclaurin's formula: $\left(0, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, 0\right)$



(f) For Weddle's rule: $\left(\frac{1}{20}, \frac{6}{20}, \frac{7}{20}, \frac{13}{20}, \frac{14}{20}, \frac{19}{20}\right)$

Figure 1: Comparative analysis of inequalities by utilizing Example taking $F(x) = x^8$ for Theorem 2.2 with different choices of parameters.

n	Newton-Cotes formulae	Exact value	Approximated value	Absolute Error
6	Weddle's rule	0.111111	0.11432	0.00320933
3	Simpson's rule 3/8	0.111111	0.0881898	0.0229213
3	Maclaurin rule	0.111111	0.126471	0.0153595
2	Simpson's 1/3 rule	0.111111	0.169271	0.0581597
1	Trapezoidal rule	0.111111	0.5	0.388889
0	Midpoint formula	0.111111	0.00390625	0.107205

Table 2: Comparison of Absolute Error, taking Example taking $F(x) = x^8$ for Theorem 2.2, for different choices of n in Newton-Cotes formulae

4.1. Discussion

The comparative analysis presented in Figure 1 and Table 2 evaluates the efficacy of Newton-Cotes formulae in approximating the integral of $x^8 = \frac{1}{9}$, a high-degree polynomial function. This case study highlights critical trade-offs between theoretical precision and practical error performance across six quadrature methods. Figure 1 visualizes the distinct weight distributions (parameters) for each method, which directly influence their approximation behavior. For instance: The uniform weights in the Trapezoidal rule (Fig. 1a) and the asymmetric weights $(0, 0, 0, 1, 1, 1)$ in the Midpoint formula (Fig. 1b) reflect their lower-order designs, leading to significant errors (Table 2). Conversely, Weddle's rule (Fig. 1f) employs a balanced yet non-uniform distribution, which mitigates error accumulation for high-order polynomials. Table 2 quantifies the absolute errors observed, revealing a clear hierarchy: Weddle's rule ($n = 6$) achieves the lowest error (0.00320933), outperforming all others due to its higher algebraic degree of precision (7th order). Maclaurin's rule ($n = 3$) and Simpson's 3/8 rule ($n = 3$) exhibit moderate errors (0.0153595 and 0.0229213, respectively), constrained by their limited degrees of precision (cubic polynomials).

Simpson's 1/3 rule ($n = 2$), Midpoint formula ($n = 0$), and Trapezoidal rule ($n = 1$) incur the largest errors (> 0.05), as their lower-order formulations fail to capture the curvature of the function. Notably, the superior performance of Weddle's rule underscores its suitability for high-degree integrands, where higher-order methods reduce error propagation. However, its computational complexity may be a practical constraint for large-scale applications. Conversely, while lower-order methods (e.g., Midpoint, n=0) exhibit larger errors, their simplicity remains advantageous for rapid approximations of smoother functions.

Notably, the superior performance of Weddle's rule underscores its suitability for high-degree integrands, where higher-order methods reduce error propagation. However, this accuracy is achieved at the expense of computational efficiency, as Weddle's rule requires more function evaluations per subinterval than lower-order rules. This illustrates the classic trade-off in numerical integration between computational cost and precision. Conversely, while lower-order methods (e.g., Midpoint, n=0) exhibit larger errors, their simplicity and lower computational demand remain advantageous for rapid approximations of smoother functions.

5. Concluding Remarks and Future Direction

In conclusion, the presented research on a new family of Newton-Cotes-type inequalities brought a presentation from the paper concerning a new theory in numerical analysis and mathematical computation as an important contribution. With the help of a parameterized identity offered as a reference point, the study has broadened the theoretical discourse about these inequalities and introduced new possibilities for estimating definite integrals and solving numerous mathematical problems. The theoretical ideas of the outlined inequalities are supported by numerous examples of both particular branches of mathematics and real-life situations, which prove the applicability of the inequalities in question. This research provides the application and realism of these inequalities in real-world conditions by performing detailed calculations and proving the effectiveness of these notions for improving numerical schemes and computations. This work stimulates more investigation and creativity in the area by revealing the possible uses of Newton-Cotes-type inequalities and stressing their importance in mathematical research. The findings contained in this research enrich the science of numerical analysis and create the basis for further study in the application of these inequalities as efficient tools in overcoming various mathematical problems.

The presented research on Newton-Cotes-type inequalities opens several promising avenues for future investigation. Below are some potential directions that could further enrich the field of numerical analysis and mathematical computation:

- Explore higher-order Newton-Cotes-type inequalities by considering more complex weight functions or additional parameters. Investigate multivariate or multidimensional analogues of these inequalities for applications in partial differential equations (PDEs) and computational physics. Study fractional or quantum-calculus-based variants to extend applicability to non-classical settings.

- Derive optimal error bounds for the proposed inequalities to determine their sharpness compared to classical Newton-Cotes rules. Investigate adaptive schemes where the parameterization is dynamically adjusted to minimize approximation errors.

- Design and analyze fast computational algorithms that leverage these inequalities for high-performance computing applications. Implement parallelized or GPU-accelerated versions for large-scale numerical simulations.

- Apply these inequalities to real-world problems in physics (e.g., finite element methods, computational fluid dynamics), engineering (signal processing, control theory), and economics (stochastic modeling, risk analysis). Study their utility in financial mathematics for option pricing or risk assessment models involving integral approximations.

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