Filomat 40:5 (2026), 1727–1740
https://doi.org/10.2298/FIL2605727H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and stability results for a fractional stochastic problem involving Lévy noise and the ψ-Hilfer derivative

Samah Horrigue[a,*], Walid Hammami[a]

[a] *Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis-El Manar, Tunisia*

Abstract. The outlines of this work focused on the existence, uniqueness, and stability of solutions for a fractional stochastic problem that incorporates the L'evy noise and the ψ-Hilfer derivative. More precisely, we use the Picard-Lindelöf successive approximation scheme to prove a solution's existence and uniqueness. After that, some results on the stability theory are used to prove exponential stability in the quadratic mean of the solution. Finally, some examples are provided to illustrate the main results of this paper.

1. Introduction

When modeling physical systems, the stability of differential equation solutions is crucial because even small model deviations, which result from inevitable measurement errors, can cause disproportionately large changes in the solution. The equation's predictive value could be compromised if they are unstable, since they may fail to forecast future behavior accurately. Ulam [35, 36] provided the first results addressing the question of the Cauchy equation's stability in 1940. Following that, Hyers demonstrated significant stability results in 1941 in [20]. For 38 years, the initial theories of Ulam-Hyers' stability were undeveloped. By demonstrating the presence of a unique linear map close to the approximate additive maps, Rassias [30] provided a generalization of Hyers' Theorem. Due to their importance, Ulam-Hyers stability of differential equations has been extensively studied by several researchers. We refer interested readers to the papers [21–23, 29, 30] and references therein.

Fractional calculus has garnered a lot of interest from researchers in recent years. Additionally, it shows up as an application in several fields such as physics, chemistry, and electrodynamics of complex media. For more details and other applications, one can see the papers [3, 24, 26, 28, 31]. As a result, throughout the past few years, several researchers have focused on the development of problems involving various fractional operators, including Riemann-Liouville, Caputo, and Hadamard. Interested readers can consult the works [3, 6, 9, 10, 24, 28, 31, 42]. In his study of an evolution equation in fractional time in physical phenomena, Hilfer [16] produced a more generalized derivative known as the Hilfer fractional derivative, which depends on the order $\alpha \in (0, 1)$ and the type $\beta \in [0, 1]$. We note that if $\beta \to 0$, the Hilfer derivative is

2020 *Mathematics Subject Classification*. Primary 34A08; Secondary 34A12, 60H10, 93D20.
Keywords. ψ-Hilfer fractional stochastic differential equation, Lévy the noise, Existence, Uniqueness, Successive Approximations, Stability, Mittag-Leffler function.
Received: 13 April 2025; Revised: 20 August 2025; Accepted: 07 December 2025
Communicated by Miljana Jovanović
* Corresponding author: Samah Horrigue
Email addresses: samah.horrigue@fst.rnu.tn (Samah Horrigue), Walid.hammami@fst.rnu.tn (Walid Hammami)
ORCID iD: https://orcid.org/0000-0001-9813-4039 (Samah Horrigue)

reduced to the Riemann-Liouville derivative, and if $\beta = 1$, this derivative is reduced to the Caputo fractional derivative.
Very recently, many researchers have focused on the study of a fractional derivatives depending on a given function, among which we mention the fractional derivatives: ψ -Riemann-Liouville, $\psi-$ Caputo and $\psi-$ Hilfer; see [1, 15, 17–19, 24, 38]. The authors in [16] introduced the ψ-Hilfer fractional derivative to unify a class of fractional operators.

Because of the impact of noise, deterministic models frequently fail to effectively represent real-world occurrences. To solve this, these models are expanded into stochastic frameworks, in which Brownian motion or other stochastic processes are used to model parameters. Stochastic differential equations (SDEs) are more effective than deterministic ones at describing several systems in practice, particularly those that are inherently unpredictable. Therefore, examining stochastic effects is essential to improving our comprehension of complicated dynamics, especially in fractional order dynamical systems. Refer to specialized texts on fractional calculus and stochastic calculus for comprehensive methods and applications. We refer to books [2, 40, 41] and the articles [5, 11–13, 27, 39].

Fathalla et al. [8] used successive approximation theory to study the existence and the uniqueness of a mild solution for the following Hilfer fractional stochastic differential equations with Poisson jumps

$$(\mathcal{P}_2)\begin{cases} {}^{H}D^{\alpha,\beta}_{0+}X(t) = AX(t) + g(t, X(t))\frac{dW(t)}{dt} + \int_Z h(t, X(t), \eta)\widetilde{N}(dt, d\eta), \\ I^{(1-\alpha)(1-\beta)}_{0^+}X(0) = X_0, \end{cases}$$

where $t \in (0, T]$, $\alpha, \beta \in (0, 1)$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ in Hilbert space K and $\widetilde{N}(dt, d\eta)$ is the compensating martingale measure. Vanterler and al. [39] studied the following Caputo fractional stochastic differential equations:

$$(\mathcal{P}_1)\begin{cases} {}^{C}D^{\alpha}_{0+}X(t) = f(t, X(t)) + g(t, X(t))\frac{dW(t)}{dt}, \qquad t > 0, \\ X(0) = X_0, \end{cases}$$

where $\alpha \in (0, 1]$, $f, g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $\{W(t),\ t \in [0, +\infty)\}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,+\infty)}, P\right)$.

In [14], the authors study the existence and uniqueness of the solution of the following fractional stochastic differential equation involving the Riemann–Liouville derivative and a noisy environment:

$$\begin{cases} {}^{RL}D^{\alpha}_{\psi}X(t) = b\Big[t, X(t)\Big]dt + \sum_{i=1}^{m} \sigma^i\Big[t, X(t)\Big]dW_i(t), \\ I^{(1-\alpha),\psi}_{0^+}X(0) = x_0, \end{cases}$$

where $t \in [0, T]$, $0 < \alpha < 1$, $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}^m$ are two well-defined maps. Moreover, some properties of the energy growth bound and the asymptotic behavior of the random solution are given for the last problem.

Inspired by the previously cited works, in this paper, we will demonstrate the existence and exponential stability of solutions for the following system that involves the $\psi-$Hilfer fractional derivative and Poisson jumps:

$$(\mathcal{P})\begin{cases} {}^{H}D^{\alpha,\beta}_{\psi}X(t) = AX(t) + b(t, X(t)) + \sigma(t, X(t))\frac{dW(t)}{dt} + \int_Z g(t, X(t), z)\frac{d\tilde{N}(t,z)}{dt}, \\ I^{(1-\alpha)(1-\beta),\psi}_{0^+}X(0) = x_0, \end{cases}$$

where $t \in [0, T] = J$, $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and ψ is increasing such that $\psi'(t) \neq 0$ and $z \in \mathbb{R}^n_0 = \mathbb{R}^n \backslash \{0\}$. As the $\psi-$Hilfer fractional derivative unifies a class of fractional operators, the differential stochastic equation $(\mathcal{P})$ can be viewed as a generalization of several problems in the literature. In particular, for $\psi(x) = x$, $g \equiv 0$, $\alpha \to 0$ and $\beta = 1$ in Equation $(\mathcal{P})$, we obtain Equation $(\mathcal{P}_1)$. On the other hand, for $\psi(x) = x$, $b \equiv 0$ and $h \equiv g$ in Equation $(\mathcal{P})$, we get Equation $(\mathcal{P}_2)$.
To be more precise, in this paper, we establish existence and uniqueness results for solutions of nonlinear

fractional stochastic differential equations involving the ψ-Hilfer derivative and Lévy noise. Proofs are carried out using the Picard–Lindelöf successive approximation scheme. It develops exponential stability analysis in the quadratic mean for the obtained solutions. Finally, our theoretical results are illustrated with examples, demonstrating applicability to fractional stochastic systems with noise.
The remainder of this work is structured as follows: we will introduce some preliminary results on the ψ-Hilfer derivative in Section 2. We will then provide and demonstrate the existence of solutions to problem $(\mathcal{P})$ in Section 3. A few stability results are covered in Section 4. Finally, two examples are provided in Section 5 to demonstrate the efficacy of our main findings.

2. Preliminaries

A few well-known ideas of $\psi-$fractional and stochastic differential equations are presented in this section.

Throughout this paper, let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\rho = \alpha + \beta(1-\alpha)$ and ψ be an increasing function in $C^1(J, [0,\infty))$ such that $\psi^{'}(0) > 0$ and $a \in J$. We define the function ψ_a by

$$\psi_a(t) = \psi(t) - \psi(a),\ \forall t \in J.$$

Definition 2.1. *[33]*

1. *The fractional integral of a function u with respect to a function ψ, is defined by*

$$I^{\alpha;\psi}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \psi^{'}(t)\psi_t^{\alpha-1}(x)u(t)dt, \tag{1}$$

provided that the integral exists.

2. *Denote by ${}_\psi\partial_t = \frac{1}{\psi'(t)}\frac{d}{dt}$. The ψ-Riemann-Liouville fractional derivative ${}^{RL}D^{\alpha,\psi}(.)$ of order α is given by*

$${}^{RL}D^{\alpha,\psi}(u)(t) := {}_\psi\partial_t I^{1-\alpha,\psi}u(t). \tag{2}$$

3. *The ψ-Hilfer fractional derivative ${}^{H}D^{\alpha,\beta,\psi}(.)$ of order α and type β, is defined by*

$${}^{H}D^{\alpha,\beta,\psi}u(t) \quad = \quad I^{\beta(1-\alpha);\psi}\ {}_\psi\partial_t I^{(1-\beta)(1-\alpha);\psi}u(t). \tag{3}$$

Lemma 2.2. *[7, 32] For a positive real υ, we have the following important properties about the $\psi-$Riemann-Liouville fractional integral and derivative of a power function:*

$$I^{\alpha,\psi}\left(\psi_0^{\upsilon-1}\right) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}\psi_0^{\upsilon+\alpha-1} \text{ and } {}^{RL}D^{\alpha,\psi}\left(\psi_0^{\upsilon-1}\right) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)}\psi_0^{\upsilon-\alpha-1}.$$

Definition 2.3. *The well-known Banach space $C[J,\mathbb{R}]$ of all continuous functions from an interval J into $\mathbb{R}$ which is equipped with the uniform norm. The weighted space $C_{\alpha,\psi}[J,\mathbb{R}]$ is given by*

$$C_{\alpha,\psi}[J,\mathbb{R}] = \{u : J \to \mathbb{R} \text{ such that } (\psi_a(.))^\alpha u(.) \in C[J,\mathbb{R}]\},$$

which is equipped with the following norm:

$$\|u\|_{\alpha,\psi} = \sup_{t\in[0,T]} |\psi_0^{1-\alpha}(t)u(t)|.$$

Lemma 2.4. *[32] Let $f \in C^n[J,\mathbb{R}]$, then we have:*

$$I^{\nu,\psi}\left({}^{H}D^{\nu,\mu,\psi}u\right)(t) = u(t) - \frac{\psi_0^{\rho-1}(t)}{\Gamma(\rho)}u(0),\ \forall\ t \in J. \tag{4}$$

We need to recall the definition of the Mittag-Leffler function.

Definition 2.5. *Let $\nu, \mu > 0$ and $z \in \mathbf{C}$. The two-parameter Mittag-Leffler Function is defined as:*

$$E_{\nu,\mu}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\nu n + \mu)}. \tag{5}$$

So, for any matrix $A \in \mathbb{R}^{n\times m}$, the Mittag-Leffler function of A is given by:

$$E_{\nu,\mu}(At) = \sum_{n=0}^{+\infty} \frac{(At)^n}{\Gamma(\nu n + \mu)}. \tag{6}$$

As Grönwall's inequality is an important tool for giving various estimates in the theory of ordinary and stochastic differential equations, we will recall it.

Lemma 2.6. *Let K be a positive real and $u, v : J \to [0, \infty)$ be continuous functions satisfying:*

$$u(t) \le K + \int_0^t v(s)u(s)\,\mathrm{d}s, \ \forall t \in J.$$

Then the usual Gronwall inequality is:

$$u(t) \le Kexp\Big(\int_0^t v(s)\mathrm{d}s\Big), \ \forall t \in J.$$

Definition 2.7. *A Lévy process $X = (X_t)_{t\ge 0}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that starts at zero a.s. satisfying*

1. *X has independent increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for any $0 \le s \le t$.*
2. *X has stationary increments, i.e., for any $0 \le s \le t$ the distribution of $X_{t+s} - X_t$ does not depend on t.*
3. *X is stochastically continuous, i.e., for every $0 \le s \le t$ and $\varepsilon > 0$:*

$$\lim_{s\to t} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0.$$

In particular, the Brownian motion is a Lévy process with continuous paths. It is characterized by

1. *$W_0 = 0$,*
2. *W_t is a.s. continuous,*
3. *W_t has independent increments,*
4. *$W_t - W_s \sim \mathcal{N}(0, t - s)$ for $0 \le s \le t$.*

We will recall the following useful results.

Lemma 2.8. *Let Y be a random variable and $X, W : J \times \Omega \to \mathbb{R}$ be, respectively, an adapted stochastic process (meaning that it is F_t^W-measurable for each $t \in J$) and a Brownian motion. Then, we have*

Chebyshev's Inequality: *For any $a > 0$ and $1 \le n < \infty$:*

$$\mathbb{P}(|Y| \ge a) \le \frac{\mathbb{E}(Y^n)}{a^n}.$$

Itô isometry: $\mathbb{E}\left[\left(\int_0^T X_t\,\mathrm{d}W_t\right)^2\right] = \mathrm{E}\left[\int_0^T X_t^2\,\mathrm{d}t\right].$

Lemma 2.9 (Borel Cantelli Lemma). *Let A_n be a sequence of events, if $\sum_{n=1}^{+\infty} A_n < \infty$, then we have*

$$\mathbb{P}(\lim_{n\to+\infty} supA_n) = 0.$$

Throughout this paper, we consider $W(t)$ the m-dimensional Brownian motion and $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ be the l-dimensional compensated jump measure of $\eta(.)$, which is also an independent compensated Poisson random measure on a complete probability space $(\Omega, F, \mathbb{P})$, where $N(dt,dz)$ is the l-dimensional jump measure (or Poisson measure) and $\nu(dz)dt$ is the Lévy-measure of l-dimensional Lévy process $\eta(.)$. unambiguously, we will write $X(t)$ instead of $X(t,\omega)$ for all $t \geq 0$ and $\omega \in \Omega$. Assume that $A \in \mathbb{R}^{n\times n}$ is a given matrix and $b : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{nm}$, $g : J \times \mathbb{R}^n \times \mathbb{R}_0^n \to \mathbb{R}^{nl}$ are given functions satisfying for all $t \in J$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}_0^n$ $b(t,X(t))$, $\sigma(t,X(t))$ and $g(t,X(t),z)$ are $\mathcal{F}_t$ measurable. Consider the stochastic fractional differential equation with Lévy noise of the form:

$$(\mathcal{P})\begin{cases} {}^H D_\psi^{\alpha,\beta} X(t) = AX(t) + b(t,X(t)) + \sigma(t,X(t))\frac{dW(t)}{dt} + \int_z g(t,X(t),z)\frac{d\tilde{N}(t,z)}{dt}, \\ I^{(1-\alpha)(1-\beta)}X(0) = x_0. \end{cases}$$

Using Lemma 2.4, the equation $(\mathcal{P})$ is equivalent to the integral form:

$$\begin{aligned} X(t) &= \frac{\psi_0^{(\alpha-1)(1-\beta)}(t)}{\Gamma(\alpha+\beta(1-\alpha))}c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{\alpha-1}(s)\psi'(s)AX(s)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{\alpha-1}(s)\psi'(s)b(s,X(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{\alpha-1}(s)\psi'(s)\int_Z g(s,X(s),z)d\tilde{N}(ds,dz) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{\alpha-1}(s)\psi'(s)\sigma(s,X(s))dW. \end{aligned}$$

Using Lemma 2.2, and the fact that $I^{(1-\alpha)(1-\beta)}X(0) = x_0$, we get

$$X(t) = x_0 + I^\alpha\left(AX(s) + b(s,X(s)) + \sigma(s,X(s))\frac{dW(s)}{ds} + \int_z g(s,X(s),z)\frac{d\tilde{N}(s,z)}{ds}\right)(t). \tag{7}$$

3. Proof of the main results

In this section, by employing the classical Picard-Lindelöf method of successive approximation scheme (see [34, 37]), we will prove the existence and uniqueness of the solution of nonlinear stochastic ψ-Hilfer fractional differential equations with Lévy noise. More precisely, we shall prove the following theorem. For this purpose, we define the following hypothesis for all $(t,x,z) \in J \times \mathbb{R}^n \times \mathbb{R}_0^n$:
(i) Linear growth condition: there exists a positive constant K such that

$$|Ax|^2 + |b(t,x)|^2 + |\sigma(t,x)|^2 + \int_Z |g(t,x,z)|^2\nu(dz) \leq \frac{K^2(1+|x|^2)}{\psi'(t)}. \tag{8}$$

(ii) The Lipschitz condition: there exists a positive constant L such that

$$|A\cdot|^2 + |\tau_{x,y}b(t,\cdot)|^2 + |\tau_{x,y}\sigma(t,\cdot)|^2 + \int_Z |\tau_{x,y}g(t,\cdot,z)|^2\nu(dz) \leq \frac{L^2(|x|^2+|y|^2)}{\sqrt{\psi'(t)}}, \tag{9}$$

where $\tau_{x,y}$ is defined by

$$\tau_{x,y}f(t,.) = f(x,t) - f(t,y).$$

Theorem 3.1. *Let us assume that* (8) *and* (9) *are true. A random variable defined in* $(\Omega, \mathcal{F}, \mathbb{P})$ *for given* x_0 *is independent of the* σ*-algebra* $\mathcal{F}_s^t \subset \mathcal{F}$ *produced by* $\{W(t); t \geq s \geq 0\}$ *and such that* $\mathbb{E}(|x_0|^2) < \infty$. *Then, there is a unique solution to the initial value problem* $(\mathcal{P})$ *that satisfies*

$$\mathbb{E}(|X(t)|^2) < \infty. \tag{10}$$

Furthermore, this solution is t-continuous with the property that $X(t, w)$ *is adapted to the filtration* $\mathcal{F}_t^{x_0}$ *generated by* x_0 *and* $\{W(s)(.); t \geq s \geq 0\}$.

Proof. We will start by proving the existence of a solution. For this purpose, we will define the sequence as follows: For $k = 0, 1, 2, \ldots$, we define $X^{(0)}(t) = x_0$ and $X^{(k)}(t) = X^{(k)}(t; \omega)$ inductively as follows:

$$\begin{aligned} X^{(k+1)}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi_s^{(1-\alpha)}(t)\psi'(s)\Big[AX^{(k)}(s) + b(s, X^{(k)}(s))ds. \\ + \int_z g(s, X^{(k)}(s), z)d\tilde{N}(ds, dz) + \sigma(s, X^{(k)}(s))dW(s)\Big]. \end{aligned} \tag{11}$$

It is clear that $X^{(0)}(t) = x_0$ is $\mathcal{F}_t$-measurable and continuous on J. So, for each k, $X^{(k)}(t)$ is also $\mathcal{F}_t$-measurable and continuous on J. Moreover, it is easy to see that

$$\sup_{0 \leq t \leq T} \mathbb{E}(|X^{(0)}(t)|^2) = \sup_{0 \leq t \leq T} \mathbb{E}(|x_0|^2) < \infty.$$

Applying the algebraic inequality $\left(\sum_{i=1}^{p} a_i\right)^2 \leq p \sum_{i=0}^{p} a_i^2$, the Cauchy-Schwartz inequality, the Itô isometry and the linear growth condition (8), we obtain from (11) that for each k, we have

$$\begin{aligned} &\mathbb{E}(|X^{(k+1)}(t)|)^2 \\ &\leq 4\mathbb{E}(x_0)^2 + \frac{4\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \Bigg[\mathbb{E}\Bigg(\int_0^t [\psi'(s)\Big\{|AX^{(k)}(s))|^2 + b(s, X^{(k)}(s))|^2 \\ &+ |\sigma(s, X^{(k)}(s))|^2 ds + \int_z |g(s; X^{(k)}(s); z)|^2 \nu(dz)\Big\} ds\Bigg)\Bigg] \\ &\leq 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \mathbb{E}\left(\int_0^T (1 + |X^{(k)}(s)|^2) ds\right). \end{aligned}$$

It follows for $k = 0$, that

$$\begin{aligned} \mathbb{E}(|X^{(1)}(t)|)^2 &\leq 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \mathbb{E}\left(\int_0^T (1 + |X^{(0)}(s)|^2) ds\right) \\ &\leq 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \left(1 + \mathbb{E}(x_0^2)\right) T \end{aligned}$$

So, we get if $k = 1$:

$$\begin{aligned} \mathbb{E}(|X^{(2)}(t)|)^2 &\leq 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \mathbb{E}\left(\int_0^T (1 + |X^{(1)}(s)|^2) ds\right) \\ &\leq 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \left(1 + 4\mathbb{E}(x_0^2) + \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} \left(1 + \mathbb{E}(x_0^2)\right) T\right) T. \end{aligned}$$

By induction, we deduce that

$$\sup_{0 \leq t \leq T} \mathbb{E}(|X^{(k)}(t)|^2) \leq M < \infty, \text{ for } k = 0, 1, 2, \ldots,$$

where

$$M = 4\mathbb{E}(x_0^2) + (1 + 4\mathbb{E}(x_0^2))\frac{\frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)} - \left(\frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)}\right)^{(k-1)}}{1 - \frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)}} + \left(\frac{4K^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)}T\right)^k.$$

Put $\mathbb{E}(|X^{(k+1)}(t) - X^{(k)}(t)|^2) = d^{(k+1)}(t)$. Using the Lipschitz condition (9), Itô isometry, and the Schwartz inequality, we derive from equation (11)

$$\begin{aligned}
d^{(1)}(t) &= \mathbb{E}(|X^{(1)}(t) - X^{(0)}(t)|^2)\\
&\leq \frac{1}{\Gamma(\alpha)}\mathbb{E}\left(\int_0^t \psi_s^{\alpha-1}(t)\psi'(s)\Big[AX^{(0)}(s) + b(s, X^{(0)}(s))ds + \sigma(s, X^{(0)}(s))dW\right.\\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^t \psi_s^{\alpha-1}(t)\psi'(s)\int_z g(s, X^{(0)}(s), z)d\tilde{N}(ds, dz)\Big]\right)\\
&\leq \frac{5\psi_0^{2\alpha-1}(t)}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left(\int_0^t\left[\psi'(s)\left\{|AX^{(0)}(s)|^2 + |b(s, X^{(0)}(s))|^2 + |\sigma(s, X^{(0)}(s))|^2 + \int_z |g(s; X^{(0)}(s); z)|^2\nu(dz)\right\}\right]ds\right)\\
&\leq \frac{5K^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left(\int_0^t K^2(1 + |X^{(0)}(s)|^2)ds\right)\\
&\leq \frac{5K^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left[(1 + |x_0|^2)(\int_0^t ds)\right]\\
&\leq \frac{5K^2t\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\left(1 + \mathbb{E}(|x_0|^2)\right)t,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
d^{(2)}(t) &= \mathbb{E}\Big(|X^{(2)}(t) - X^{(1)}(t)|^2\Big)\\
&\leq \frac{5L^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^T \mathbb{E}\Big(|X^{(1)}(s) - X^{(0)}(s)|^2\Big)ds\\
&\leq K^2\left[\frac{5L^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\right]^2 \mathbb{E}\left[\Big(1 + |x_0|^2\Big)\left(\int_0^t sds\right)\right]\\
&\leq K^2(1 + \mathbb{E}(|x_0|^2))\left[\frac{5L^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}\right]^2\frac{t^2}{2!}.
\end{aligned}$$

Moreover, for all integer k, by the same argument, we get

$$\begin{aligned}
d^{(k+1)}(t) &= \mathbb{E}(|X^{(k+1)}(t) - X^{(k)}(t)|^2)\\
&\leq \frac{5\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)}\left[\int_0^t \psi'(s)\mathbb{E}\Big(|A(X^{(k)}(s) - X^{(k-1)}(s))|^2 + |\tau_{X^{(k)},X^{(k-1)}}b(s,\cdot)|^2\right.\\
&\quad \left. + |\tau_{X^{(k)},X^{(k-1)}}\sigma(s,\cdot)|^2 + \int_z |\tau_{X^{(k)},X^{(k-1)}}g(s,\cdot,z)|^2\nu(dz)\Big)ds\right]\\
&\leq \frac{5L^2\psi_0^{2\alpha-1}(T)}{\Gamma^2(\alpha)(2\alpha-1)}\int_0^t \mathbb{E}\Big(|X^{(k)}(s) - X^{(k-1)}(s)|^2\Big)ds.
\end{aligned} \tag{13}$$

Thus, by induction, for $k = 0, 1, 2, \ldots,$ and $0 \leq t \leq T$, we have

$$d^{(k+1)}(t) = \mathbb{E}\Big(|X^{(k+1)}(t) - X^{(k)}(t)|^2\Big) \leq CD^{k+1}\frac{t^{k+1}}{(k+1)!},$$

where

$$C = K^2(1 + \mathbb{E}(|x_0|^2)), \quad \text{and} \quad D = \frac{5L^2\psi_0^{2\alpha-1}(T)}{(2\alpha-1)\Gamma^2(\alpha)}.$$

We note that, from hypothesis (9), Cauchy-Schwartz inequality and Theorem 3.1, we have

$$\begin{aligned}
&\mathbb{E}\left[\max_{0\le t\le T}\left(\left|X^{(k+1)}(t) - X^{(k)}(t)\right|^2\right)\right] \\
\le& \frac{5\psi_0^{2\alpha-1}(t)}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left[\max_{0\le t\le T}\left(\int_0^t \psi'(s)\left[|A(X^{(k)}(s) - X^{(k-1)}(s))|^2 + |\tau_{X^{(k)},X^{(k-1)}}b(s,\cdot(s))|^2\right.\right.\right. \\
&\left.\left.\left. +|\tau_{X^{(k)},X^{(k-1)}}\sigma(s,\cdot)|^2 + \int_z |\tau_{X^{(k)},X^{(k-1)}}g(s,\cdot,z)|^2\nu(dz)\right]ds\right)\right] \\
\le& \frac{5L^2\psi_0^{2\alpha-1}(t)}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left[\max_{0\le t\le T}\left(\int_0^t \left(\left|X^{(k)}(s) - X^{(k-1)}(s)\right|^2\right)ds\right)\right].
\end{aligned}$$

By induction, it is easy to see that

$$\mathbb{E}\left[\max_{0\le t\le T}\left(\left|X^{(k+1)}(t) - X^{(k)}(t)\right|^2\right)\right] \le CD^{k+1}\frac{T^{k+1}}{(k+1)!}.$$

Using Chebyshev's inequality, we obtain

$$\begin{aligned}
\mathbb{P}\left(\max_{0\le t\le T}\left(|X^{(k+1)}(t) - X^{(k)}(t)|^2\right) > \frac{1}{k}\right) &\le \frac{1}{(\frac{1}{k})^2}\mathbb{E}\left(\max_{0\le t\le T}\left[|X^{(k+1)}(t) - X^{(k)}(t)|^2\right]\right) \\
&\le Ck^2D^{k+1}\frac{T^{k+1}}{(k+1)!}.
\end{aligned}$$

The sum of the resulting inequalities yields

$$\sum_{k=0}^{+\infty}\mathbb{P}\Big(\max_{0\le t\le T}(|X^{(k+1)}(t) - X^{(k)}(t)|^2) > \frac{1}{k}\Big) \le \sum_{k=0}^{+\infty} CD^{k+1}\frac{k^2T^{k+1}}{(k+1)!}.$$

The series on the left side of the last inequality converges since the series on the right side does, according to the ratio test. Accordingly, $\max_{0\le t\le T}(\mathbb{E}(|X^{(k+1)}(t) - X^{(k)}(t)|^2))$ almost surely converges to 0 by the Borel-Cantelli Lemma. Thus, it is almost surely that the consecutive approximations $X^{(k)}(t)$ converge uniformly on J to a limit $X(t)$ defined by

$$X(t) = \lim_{n\longrightarrow+\infty}\Big(X^{(0)}(t) + \sum_{k=1}^{n}(X^{(k)}(t) - X^{(k-1)}(t)\Big) = \lim_{n\longrightarrow+\infty}X^{(n)}(t).$$

From (11), we have

$$X(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{(\alpha-1)}(t)\psi'(s)\Big[\Big(AX(s)ds + b(s,X(s))\Big)ds + \sigma(s,X(s))dW(s) + \int_z g(s,X(s),z)d\tilde{N}(ds,dz)\Big],$$

for all t ∈ J . This concludes the demonstration that there is a solution to ($\mathcal{P}$).

We will now demonstrate the solution's uniqueness. To do this, we will apply the Itô isometry and the Lipschitz conditions (8). In fact, given $X(t;\omega)$ and $Y(t;\omega)$ two solution processes through the initial data $(0;x_0)$ and $(0;y_0)$ respectively, that is, $X(0;\omega) = x_0(\omega)$ and $Y(0;\omega) = y_0(\omega)$, $\omega \in \Omega$. Then, under the Schwarz inequality and the Itô isometry, we have

$$\mathbb{E}(|X(t) - Y(t)|^2)$$

$$\leq 5\mathbb{E}(|x_0 - y_0|^2) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi_0^{(\alpha-1)}(t)\psi'(s)\Big[A|X(s) - Y(s)|ds + \tau_{X,Y}b(s,\cdot)ds$$
$$+ \tau_{X,Y}\sigma(s,\cdot)dW(s) + \int_z \tau_{X,Y}g(s,\cdot,z)d\tilde{N}(ds,dz)\Big]$$
$$\leq 5\mathbb{E}(|x_0 - y_0|^2) + \frac{5L^2\psi_0^{2\alpha-1}(t)}{(2\alpha-1)\Gamma^2(\alpha)} \int_0^t \mathbb{E}\Big(|X(s) - Y(s)|^2\Big)ds.$$

Put $\delta(t) = \mathbb{E}(|X(t) - Y(t)|^2)$, then the function δ satisfies

$$0 \leq \delta(t) \leq I_0 + D \int_0^t \delta(s)ds,$$

where

$$I_0 = 5\mathbb{E}(|x_0 - y_0|^2).$$

From the Gronwall inequality, we deduce that

$$0 \leq \delta(t) \leq I_0 e^{Dt}.$$

As $x_0 = y_0$, then $I_0 = 0$ and $\delta(t) = 0$ for all $t \geq 0$. So, we have $\mathbb{E}(|X(t) - Y(t)|^2) = 0$. This implies that

$$\int_0^t |X(s) - Y(s)|^2 d\mathbb{P} = 0.$$

This yields to $X(t) = Y(t)$ a.s for all $t \in J$.
Therefore, we obtain

$$\mathbb{P}\left\{|X(t,\omega) - Y(t,\omega)| = 0, \text{ for all } t \in J\right\} = 1.$$

In conclusion, $(\mathcal{P})$ has a unique solution. The solution to the given stochastic fractional differential equation $(\mathcal{P})$ is proven to exist and be unique. □

Now, we will give an integral representation of the solution of problem $(\mathcal{P})$ via the Mittag-Leffler function.

Theorem 3.2. *The stochastic fractional differential equation $(\mathcal{P})$ has a unique solution given by:*

$$X(t) = E_\alpha\left(\psi_0^\alpha(t)A\right)x_0 + \int_0^t \frac{\psi_s^{\alpha-1}(t)\psi'(s)}{\Gamma(\alpha)} E_{\alpha,\alpha}\left(A\psi_s^\alpha(t)\right)\Big[b(s,X(s))$$
$$+ \sigma(s,X(s))dW(s) + \int_z g(s,X(s),z)d\tilde{N}(ds,dz)\Big]ds. \tag{14}$$

provided that A commutes with the fractional integral operator $I^{\alpha,\psi}$ and the hypothesis:

$$(\mathcal{H}): \quad \|A\| \leq \frac{\Gamma(\alpha+1)}{\psi_0^\alpha(T)},$$

yields.

Proof. Let $X \in C[J,\mathbb{R}]$, using the hypothesis $\mathcal{H}$, we obtain

$$\|I^{\alpha,\psi}(AX(\cdot))(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi_s^{\alpha-1}(t)\psi'(s)\|AX(s)\|ds$$
$$\leq \frac{\|AX\|}{\Gamma(\alpha)} \int_0^t \psi_s^{\alpha-1}(t)\psi'(s)ds$$
$$\leq \frac{\|AX\|}{\Gamma(\alpha+1)}\psi_0^\alpha(T) \leq \|X\|.$$

So, $\|I^{\alpha,\psi}A\| \leq 1$. Using Theorem 7.3.1 in [25], we conclude that $(I - I^{\alpha,\psi}A)^{-1}$ is a bounded linear operator satisfying $(I - I^{\alpha,\psi}A)^{-1} = \sum_{k=0}^{\infty}(I^{\alpha}A)^k$, where I denoted the identity operator. On the other hand, from Equation (1), we have

$$\begin{aligned} X(t) =& x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi_0^{(\alpha-1)}(t)\psi'(s)\Big[AX(s)ds + b(s,X(s))ds + \sigma(s,X(s))dW(s) + \int_z g(s,X(s),z)d\tilde{N}(ds,dz)\Big] \\ =& x_0 + I^{\alpha,\psi}(AX(\cdot))(t) + I^{\alpha,\psi}(f(\cdot))(t), \end{aligned}$$

where $f(s) = b(s,X(s))\sigma(s,X(s))dW(s) + \int_z g(s,X(s),z)d\tilde{N}(ds,dz)$. It follows that

$$\begin{aligned} X(t) &= (I - I^{\alpha,\psi}A)^{-1}\left(x_0 + I^{\alpha,\psi}(f(\cdot))(t)\right) \\ &= \sum_{k=0}^{\infty}(I^{\alpha}A)^k\left(x_0 + I^{\alpha,\psi}(f(\cdot))(t)\right) \\ &= \sum_{k=0}^{\infty}\int_0^t \frac{\psi_s^{k\alpha-1}(t)\psi'(s)}{\Gamma(k\alpha)}A^k x_0 ds + \sum_{k=0}^{\infty}\int_0^t \frac{\psi_s^{k\alpha-1}(t)\psi'(s)}{\Gamma(k\alpha)}A^k\int_0^s \frac{\psi_\tau^{\alpha-1}(s)\psi'(\tau)}{\Gamma(\alpha)}f(\tau)d\tau ds \\ &= \sum_{k=0}^{\infty}\frac{\left(\psi_0^{\alpha}(t)A\right)^k}{\Gamma(k\alpha+1)}x_0 + \int_0^t \frac{\psi_s^{\alpha-1}(t)\psi'(s)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\frac{\left[\psi_s^{\alpha-1}(t)A\right]^k}{\Gamma(k\alpha+\alpha)}\right)f(s)ds \\ &= E_\alpha\left(\psi_0^{\alpha}(t)A\right)x_0 + \int_0^t \frac{\psi_s^{\alpha-1}(t)\psi'(s)}{\Gamma(\alpha)}E_{\alpha,\alpha}\left(A\psi_s^{\alpha}(t)\right)f(s)ds. \end{aligned}$$

This completes the proof. □

4. Exponentially Asymptotic Stability Analysis

The exponential asymptotic stability of the quadratic mean of a trivial solution of a stochastic fractional nonlinear system with Lévy noise $(\mathcal{P})$ will be examined in this section.

Definition 4.1. *The trivial solution of a stochastic fractional nonlinear system with Lévy noise $(\mathcal{P})$ is $\psi-$exponentially stable in the quadratic mean if there are two positive constants C and λ such that*

$$\mathbb{E}|X(t)|^2) \leq C\mathbb{E}|x_0|^2 exp(-\lambda\psi_0(t)), \quad t \geq 0.$$

To prove the exponentially asymptotic stability, we need the following lemmas.

Lemma 4.2. *[4] Let $0 < \alpha < 1$ and $\beta = 1, 2, \alpha$. If A is a diagonalizable stability matrix then there exists a positive constant θ such that*

$$\|E_{\alpha,\beta}(At^{\alpha})\| \leq \theta e^{-at}, \tag{15}$$

where a is the largest eigenvalue of the matrix A and $\|.\|$ denotes any matrix norm.

Remark 4.3. *It follows from the above lemma that, under the same hypothesis, we have*

$$\|E_{\alpha,\beta}(A\psi_s^{\alpha}(t))\| \leq \theta e^{-a\psi_s^{\alpha}(t)}. \tag{16}$$

It follows the following Lemmas.

Lemma 4.4. *[4] For each strongly measurable stochastic process $F : [0; \infty) \to \mathbb{R}^n$ such that*

$$\int_0^T \mathbb{E}|F(t)|^2 dt < \infty,$$

the following inequality holds

$$\mathbb{E}\left|\int_0^t \psi'(s) E_{\alpha,\beta}(A\psi_s(t))F(s)ds\right|^2 \leq \frac{\theta^2}{|a|}\int_0^t \psi'(s)exp(-2a\psi_0(t))\mathbb{E}|F(s)|^2 ds,$$

where $t \in (0, T]$.

Lemma 4.5. *[4] Let $\phi : [0; \infty) \to \mathbb{R}^n$ be an B_t - adapted predictable process with $\int_0^T \mathbb{E}|\phi(t)|^2 dt < \infty$ for all positive T. Then, for all $0 < t \leq T$, we have*

$$\mathbb{E}\left|\int_0^t \psi'(s) E_{\alpha,\beta}(A\psi_s(t))\phi(s)dW(s)\right|^2 \leq \theta^2 \int_0^t \psi'(s)exp(-2a\psi_0(t))\mathbb{E}|\phi(s)|^2 ds.$$

Using the above Lemmas and some technical properties, the following theorem shows.

Theorem 4.6. *Assume that the assumptions of Theorem (3.1) are satisfied. In the quadratic mean, the solution of Equation ($\mathcal{P}$) is exponentially stable.*

Proof. From Theorem 3.2, the integral form of the equation ($\mathcal{P}$) is given by

$$\begin{aligned} X(t) &= E_\alpha(A(\psi_0^\alpha(T)))x_0 + \int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t))b(s, X(s))ds \\ &+ \int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t))\sigma(s, X(s))dW(s) \\ &+ \int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t)) \int_z g(s, X(s), z)d\tilde{N}(ds, dz). \end{aligned} \tag{17}$$

Applying the algebraic inequality $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, Inequality (16) and Cauchy-Schwartz inequality, we get

$$\begin{aligned} |X(t)|^2 &\leq 4|E_{\alpha,\beta}(A\psi_0^\alpha(t))x_0|^2 + 4\left|\int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t))b(s, X(s))ds\right|^2 \\ &+ 4\left|\int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t))\sigma(s, X(s))dW(s)\right|^2 \\ &+ 4\left|\int_0^t \psi_s^{\alpha-1}(t)\psi'(s)E_{\alpha,\alpha}(A\psi_s^\alpha(t)) \int_z g(s, X(s), z)d\tilde{N}(ds, dz)\right|^2 \\ &\leq 4\theta^2 e^{-2a\psi_0^\alpha(t)}|x_0|^2 \\ &+ 4\theta^2\left|\int_0^t \psi'(s)\psi_s^{2\alpha-2}(t)ds \int_0^t \psi'(s)E_{\alpha,\alpha}^2(A\psi_s^\alpha(t))b^2(s, X(s))ds\right| \\ &+ 4\theta^2\left|\int_0^t \psi'(s)\psi_s^{2\alpha-2}(t)ds \int_0^t \psi'(s)E_{\alpha,\alpha}^2(A\psi_s^\alpha(t))\sigma^2(s, X(s))dW(s)\right| \\ &+ 4\theta^2\left|\int_0^t \psi'(s)\psi_s^{2\alpha-2}(t)ds \int_0^t \psi'(s)E_{\alpha,\alpha}^2(A\psi_s^\alpha(t))\left(\int_z g(s, X(s), z)\right)^2 d\tilde{N}(ds, dz)\right| \end{aligned}$$

$$\leq 4\theta^2\Bigg(e^{-2a\psi_0^\alpha(t)}|x_0|^2+\frac{\psi_0^{2\alpha-1}(t)}{2\alpha-1}\Bigg[\int_0^t\psi'(s)E^2_{\alpha,\alpha}(A\psi_s^\alpha(t))b^2(s,X(s))ds$$
$$+\int_0^t\psi'(s)E^2_{\alpha,\alpha}(A\psi_s^\alpha(t))\sigma^2(s,X(s))dW(s)ds$$
$$+\int_0^t\psi'(s)E^2_{\alpha,\alpha}(A\psi_s^\alpha(t))\left(\int_z g(s,X(s),z)\right)^2 d\tilde{N}(ds,dz)\Bigg]\Bigg).$$

By using Hölder inequality and Lemmas 4.4 and 4.5, we get

$$\mathbb{E}|X(t)|^2\leq 4\theta^2\Big(\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2$$
$$+\frac{\psi_0^{2\alpha-1}(t)}{2\alpha-1}\Bigg[\frac{1}{|a|}\int_0^t\psi'(s)\exp(-2a\psi_s(t))\mathbb{E}|b(s,X(s))|^2ds$$
$$+\int_0^t\psi'(s)\exp(-2a\psi_s(t))\mathbb{E}|\sigma(s,X(s))|^2ds$$
$$+\frac{1}{|a|}\int_0^t\psi'(s)\exp(-2a\psi_s(t))\mathbb{E}|\int_z g(s,X(s),z)\nu(dz)|^2ds\Bigg]\Big)$$
$$\leq 4\theta^2\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2+\frac{4\max(1,\frac{1}{|a|})\theta^2\psi_0^{2\alpha-1}(t)}{2\alpha-1}$$
$$\times\left[\int_0^t\psi'(s)\exp(-2a\psi_s(t))\left\{\mathbb{E}\left(|b(s,X(s))|^2+|\sigma(s,X(s))|^2+|\int_z g(s,X(s),z)\nu(dz)|^2\right\}\right)ds\right].$$

From the linear growth assumption (8), we get

$$\mathbb{E}|X(t)|^2\leq 4\theta^2\Big(\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2$$
$$+\frac{4\max(1,\frac{1}{|a|})\theta^2\psi_0^{2\alpha-1}(t)}{2\alpha-1}\int_0^t\psi'(s)\exp(-2a\psi_s(t))E(1+|X(s)|^2)ds\Big)$$
$$\leq 4\theta^2\Big(\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2-\frac{\max(1,\frac{1}{|a|})[1-e^{-a\psi_0^\alpha(t)}]\psi_0^{2\alpha-1}(t)}{2a(2\alpha-1)}$$
$$+\frac{\max(1,\frac{1}{|a|})\psi_0^{2\alpha-1}(t)}{2\alpha-1}\int_0^t\psi'(s)\exp(-2a\psi_s(t))E|X(s)|^2ds\Big)$$
$$\leq 4\theta^2\Big(\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2+\frac{\max(1,\frac{1}{|a|})\psi_0^{2\alpha-1}(t)}{2\alpha-1}\int_0^t\psi'(s)\exp(-2a\psi_s(t))E|X(s)|^2ds\Big). \tag{18}$$

We obtain

$$\exp(2a\psi(t))\mathbb{E}|X(t)|^2\leq 4\theta^2\left(\exp(-2a\psi_0(t))\mathbb{E}|x_0|^2+\frac{\max(1,\frac{1}{|a|})\psi_0^{2\alpha-1}(t)}{2\alpha-1}\int_0^t\psi'(s)\exp(2a\psi(s))E|X(s)|^2ds\right).$$

Using Gronwall's inequality, we get

$$\exp(2a\psi(t))\mathbb{E}|X(t)|^2\leq C\mathbb{E}|x_0|^2\exp\left(-\Lambda\psi_0^\alpha(t)\right),$$

where

$$C=4\theta^2\exp(-2a\psi_0(t)),$$

and

$$\Lambda=\frac{4\theta^2\max(1,\frac{1}{|a|})\psi_0^{2\alpha-1}(t)}{2\alpha-1}.$$

Which completes the proof. □

5. Examples

5.1. Example 1:

Let r, $\sigma > 0$. We consider the one-dimensional fractional Langevin equation on $t \in [0, T]$:

$$\begin{cases} {}^{H}D_{\psi}^{\alpha,\beta} X(t) &= -rX(t) + \sigma \frac{dW(t)}{dt}, \\ I^{(1-\alpha)(1-\beta)} X(0) &= x_0. \end{cases} \tag{19}$$

As ψ is a positive continuous function on $[0, T]$, having a continuous derivative ψ' on $[0, T]$, it follows that ψ' admit a maximum M. So for ψ satisfying $M \leq 1$, we deduce that the one-dimensional fractional Langevin equation (19) has a unique ψ−exponentially stable solution given by

$$X(t) = E_{\alpha}\left(-r\psi_0^{\alpha}(t)\right) x_0 + \sigma \int_0^t \frac{\psi_s^{\alpha-1}(t)\psi'(s)}{\Gamma(\alpha)} E_{\alpha,\alpha}\left(-r\psi_s^{\alpha}(t)\right) dW(s). \tag{20}$$

5.2. Example 2:

Study the following L évy noise in a stochastic fractional differential equation of the form

$$\begin{cases} {}^{H}D_{\psi}^{\alpha,\beta} X(t) = 0.2X(t) + \sin(X(t)) + X(t)\frac{dW(t)}{dt} + \int_z (1 + e^{-t}) \cos(X(t)) \frac{d\tilde{N}(t,z)}{dt}, \\ I^{(1-\alpha)(1-\beta)} X(0) = x_0. \end{cases} \tag{21}$$

Here $A = 0.2$, $b(t, X(t)) = \sin(X(t))$, $\sigma(t, X(t)) = X(t)$ and $g(t, X(t), z) = (1 + e^{-t}) \cos(X(t))$. It is easy to see that A, $b(t, X(t))$, $\sigma(t, X(t))$ and $g(t, X(t), z)$ satisfy the assumptions of (8) and (9) of Theorem (3.1). Therefore, the stochastic fractional differential equation (21) has a unique exponentially stable solution according to Theorems 3.1 and 4.6.

Conclusion

In this work, the existence, the uniqueness, and the stability of a solution for a fractional stochastic problem involving the Lévy noise and the ψ-Hilfer derivative are studied. The main tool use to prove the main results is the well known Picard-Lindelöf successive approximation scheme to prove a solution's existence and uniqueness. To validate the main results, some examples are presented. We aim to extend this study to a more general problem by considering a double phase equation with the p-Laplacian operator by using an appropriate fixed-point Theorem.

References

[1] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci Numer Simul, **44** (2017), 460-481.

[2] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus: models and numerical methods*, Vol. 3, (2012), World Scientific.

[3] M. El A. Bengrine and Z. Dahmani, *For fractional differential equations*, Int. J. Open problems compt. Math, 5, **4**, (2012).

[4] W. Ran-Chao, H. Xin-Dong and C. Li-Ping, *Finite-time stability of fractional-order neural networks with delay*, Commun. Theor. Phys., **60** (2013), 189-193.

[5] A. Chadha, D. Bahuguna and DN. Pandey, *Existence of a mild solution for a neutral stochastic fractional integrodifferential inclusion with a nonlocal condition*, Journal of Integral Equations and Applications, **30(2)**(2018), 257-91.

[6] M. Chamekh, A. Ghanmi and S. Horrigue, *Iterative approximation of positive solutions for fractional boundary value problem on the half-line*, Filomat, **32(18)** (2018), 6177-6187.

[7] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, Springer, New York, (2010).

[8] A. R. Fathalla, R. Chinnathambi and M. Palanisamy, *Fractional stochastic differential equations with Hilfer fractional derivative: Poisson jumps and optimal control*, Hindawi Discrete Dynamics in Nature and Society Volume 2017, Article ID 5394528, (2017), 11 pages, https://doi.org/10.1155/2017/5394528.

[9] A. Ghanmi and S. Horrigue, *Existence results for nonlinear boundary value problems*, Filomat, **32(2)** (2018), 609–618.

[10] A. Ghanmi and S. Horrigue, *Existence of positive solutions for a coupled system of nonlinear fractional differential equations*, Ukr. Math. J., **71 (1)**(2019), 39–49.

[11] G. Gokul and R. Udhayakumar, *Optimal Feedback Control of ψ-Hilfer Fractional Differential System with Hemivariational Inequalities*, Complex Anal. Oper. Theory 19, **114** (2025). https://doi.org/10.1007/s11785-025-01741-3

[12] G. Gokul and R. Udhayakumar, *Non-instantaneous impulsive ψ-Hilfer fractional stochastic system with Rosenblatt process and approximate controllability*, Journal of Control and Decision, (2025), 1–13. https://doi.org/10.1080/23307706.2025.2492651

[13] Z. Guo, F. Hongbo and W. Wenya, *An averaging principle for Caputo fractional stochastic differential equations with compensated Poisson random measure*, J. Partial. Differ. Equ., **35** (2022) 1-10., (2022)

[14] W. Hammami, S. Horrigue and S. Gasmi, *Existence and uniqueness of solution for a ψ-Riemann-Liouville fractional stochastic differential equation*, Complex Anal. Oper. Theory 19, **44** (2025), https://doi.org/10.1007/s11785-025-01668-9.

[15] R. Herrmann, *Fractional calculus: an introduction for physicists*, World Scientific, Singapore, (2011).

[16] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, (2000).

[17] S. Horrigue, M. Alsulami and B. A. Alsaeedi, *Existence result to a Kirchhoff ψ-Hilfer fractional equations with p-Laplacian operator via Nehari method*, 2023 International Conference on Fractional Differentiation and Its Applications (ICFDA), IEEE, (2023).

[18] S. Horrigue, M. Alsulami and B. A. Alsaeedi, *The Nehari manifold for a class of singular ψ-Riemann-Liouville fractional with p-Laplacian operator differential equations*, Adv. Appl. Math. Mech., **16** (5) (2024), 1104–1120.

[19] S. Horrigue, H. Almuashi and A. A. Alnashry, *Existence results for some ψ-Hilfer iterative approximation*, Mathematical Foundations of Computing, 7, **4** (2024), 531-543. https://doi.org/10.3934/mfc.2023029.

[20] DH. Hyers, *On the stability of the linear functional equation*, Proc Natl Acad Sci USA.; **27**(4): 222, (1941).

[21] K-W. Jun and H-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. ; 274(2), (2002) 867-878.

[22] S-M. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett., 17(10), (2004) 1135-1140.

[23] S-M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Vol 48. Springer Science & Business Media, (2011).

[24] S. H. M. Kilbas, A. Anatoliĭ and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204. Elsevier, (2006).

[25] E. Kreyszig, *Introductory functional analysis with applications*, New York: Wiley, (1978).

[26] M. Martínez-García, Y. Zhang and T. Gordon, *Memory pattern identification for feedback tracking control in human-machine systems*, Human factors, vol. 63, no. 2, (2021), 210-226.

[27] K. Nandhaprasadh and R. Udhayakumar, *Approximate controllability of a semilinear impulsive fractional stochastic integro-differential inclusions with Poisson jumps*, J. Math. Computer Sci., (2025), 38, 263–280.https://doi.org/10.22436/jmcs.038.02.08

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press: New York, NY, USA, (1999).

[29] TM. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc Amer Math Soc., (1992), 989-993.

[30] TM. Rassias, *On the stability of functional equations in Banach spaces*, J Math Anal Appl.; 251(1), (2000), 264-284.

[31] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional derivatives and integrals: Theory and Applications*, Gordon and Breach, Yverdon, (1993).

[32] Da C. Sousa Vanterler and J., E. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. 60, (2018), 72-91.

[33] J. V. d. C. Sousa, L. S. Tavares, and C. E. T. Ledesma, *A variational approach for a problem involving a psi-Hilfer fractional operator, Journal of Applied Analysis and Computation*, vol. 11, no. 3, (2021), 1610-1630.

[34] A. Siva Ranjani, M. Suvinthra, K. Balachandran and Y. K. Ma, *Analysis of stochastic neutral fractional functional differential equations*, Boundary Value Problems, 2022(1), 49, (2022).

[35] SM. Ulam *A collection of mathematical problems*, Vol 29. New York: Interscience Publishers, (1960).

[36] SM. Ulam, *Problems in modern mathematics*, p. xvii+ 150. New York, NY, USA: John Wiley & Sons, (1964).

[37] P. Umamaheswari, K. Balachandran and N. Annapoorani, *Existence of solutions of stochastic fractional integrodifferential equations*, Discontinuity Nonlinearity and Complexity **7.1**, (2018) 55-65.

[38] C. Vanterler da, J. Sousa, M.N.N. dos Santos, L.A. Magna and E. Capelas de Oliveira, *Validation of a fractional model for erythrocyte sedimentation rate*, Comput. Appl. Math., (2018), https://doi.org/10.1007/s40314-018-0717-0.

[39] G. Xiao and J. R. Wang, *Stability of solutions of Caputo fractional stochastic differential equations*, Nonlinear Analysis: Modelling and Control, Vol. 26, No. 4, (2021), 581-596, https://doi.org/10.15388/namc.2021.26.22421.

[40] Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, (2014).

[41] Y. Zhou, *Fractional evolution equations and inclusions: analysis and control*, New York: Elsevier, (2015).

[42] Y. Zhou, J. Wang and L. Zhang, *Basic theory of fractional differential equations*, world scientific publishing company, Singapore, New Jersey, London and Hong Kong, (2014).