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Generalized Hermite-Hadamard inequalities via (m,n)-fractional integrals involving with applications

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Abstract. The main objective of this article is to develop some new Hermite-Hadamard type inequalities involving the generalized (m,n)-Riemann-Liouville fractional integrals that include the map $\varphi : (0,\infty) \times (0,\infty) \to [0,\infty]$ satisfying the condition $\varphi(m,m) = m$. Some related inequalities that are closely connected to some known results have been established using the convexity of differentiable functions. We also deduce the some known results from our general results. We will derive the inequalities for the arithmetic, geometric and harmonic (m,n)-Riemann-Liouville fractional integral operators.

1. Introduction

Convex function inequalities are important in literature, especially the inequalities. Hermite and Hadamard originally presented (see [15, p. 137]). Scholars have been researching Hermite–Hadamard type inequalities since 1893 [7] and state that:
If $g : Y \to \mathbb{R}$ is convex on the interval Y of real numbers and $l, m \in Y$ with $m < n$, then

$$g\left(\frac{m+n}{2}\right) \le \frac{1}{n-m}\int_m^n g(\mu)d\mu \le \frac{g(m)+g(n)}{2}, \quad \mu \in [m,n].$$

Recent years have seen significant developments in the theory of inequalities. We point out that the concept of convexity could be explained by the Hermite–Hadamard inequality. The Hermite–Hadamard inequality for convex functions has garnered new interest in recent years, and some peculiar variations of the basic and conclusion have been established (see, for example, [8], [9], [10], [17], [20], [22]). The theory related to past inequalities was developed as a result of a greater focus on this topic. Different fractional integrals and convexities have been used to establish various versions of Hermite–Hadamard inequalities. This inequality has been used to solve a number of fractional calculus problems (see [1], [3], [12], [13], [14], [18],

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[19]). By defining convex functions using fractional calculus and drawing certain findings for the extended (m,n)-Riemann-Liouville fractional integral, we continue the study of the Hermite–Hadamard inequality in this paper. Because of this inequality, we apply an application of means to study the generalized integral operator. As a result, many fractional integral inequalities that are helpful in the study of fractional inequality are obtained. Using these fractional inequalities, we also discover new forms of the generalized (m,n)-Riemann-Liouville fractional integral. In this paper, we analyze how these inequities apply to different kinds of means, or averages. Knowing these applications can help us solve real-world problems where means and inequality are important. Let's first discuss some fundamental ideas and past findings below.

Definition 1.1. *A function $f:[l,m]\to\mathbb{R}$ is called convex on an interval $[l,m]\subseteq\mathbb{R}$, if*

$$f(\mu\alpha+(1-\mu)\beta)\le\mu f(\alpha)+(1-\mu)f(\beta), \tag{1}$$

hold for $\alpha,\beta\in[l,m]$ and $\mu\in[0,1]$.

Definition 1.2. *[16] Suppose $f\in L_1[l,m]$. The left and right Riemann-Liouville integrals $J^{\eta}_{l^+}f$ and $J^{\eta}_{m^-}f$ of order $\eta>0$ with $l\ge 0$ are defined by*

$$J^{\eta}_{l^+}f(\mu)=\frac{1}{\Gamma(\eta)}\int_l^{\mu}(\mu-x)^{\eta-1}f(x)dx,\mu>l,$$

and

$$J^{\mu}_{m^-}f(\mu)=\frac{1}{\Gamma(\eta)}\int_{\mu}^{m}(x-\mu)^{\eta-1}f(x)dx,\mu<m,$$

respectively, where Γ is the classical Gamma function defined by $\Gamma(\eta)=\int_0^{\infty}v^{\eta-1}e^{-v}dv$ and $J^0_{l^+}f=J^0_{m^-}f=f$.

In [4] the k-gamma function was introduced by Diaz et al. as fellows.

Definition 1.3. *Suppose $k,\mathbb{R}(r)>0$. Then k-gamma function is defined by following integral:*

$$\Gamma_k(\mu)=\int_0^{\infty}r^{\mu-1}e^{-\frac{r^k}{k}}dr. \tag{2}$$

Definition 1.4. *[11] If $k>0$, suppose $g\in L_1[l,m], l\ge 0$, then k-Riemann-Liouville fractional integral $I^{\eta}_{l^+,k}g$ and $I^{\eta}_{m^-,k}g$ of order $\eta>0$ for a real-valued continuous function g are defined by*

$$I^{\eta}_{l^+,k}g(\mu)=\frac{1}{k\Gamma_k(\eta)}\int_l^{\mu}(\mu-\phi)^{\frac{\eta}{k}-1}g(\phi)d\phi,\quad \mu>l,$$

and

$$I^{\eta}_{m^-,k}g(\mu)=\frac{1}{k\Gamma_k(\eta)}\int_{\mu}^{m}(\phi-\mu)^{\frac{\eta}{k}-1}g(\phi)d\phi,\quad \mu<m,$$

respectively and Γ_k is the k-Gamma function defined in (2).

Definition 1.5. *[2] Let $\beta\in C/lZ^-; p,q\in\mathbb{R}^+-\{0\}$ and $\mathbb{R}(\beta)>0$. Then the integral representation of (m,n)-Gamma function is given by*

$$\Gamma_{(m,n)}(\beta)=\int_0^{\infty}y^{\beta-1}e^{\frac{-y^m}{n}}dy. \tag{3}$$

2. The general (m, n)-Riemann-Liouville fractional integrals

Here we shall describe the general (m, n)-Riemann-Liouville fractional integrals of order a with two exponential parameters, m and n. These integrals generalize the Riemann-Liouville fractional integrals and are motivated by the works mentioned above. We shall emulate the actions of the authors in [5, 6] as part of the investigation. Let $\varphi : (0, +\infty) \times (0, +\infty) \to [0, +\infty]$ be map satisfying the condition $\varphi(m, m) = m$ for example

1. the arithmetic mean $\varphi(m, n) = \frac{m+n}{2}$
2. the geometric mean $\varphi(m, n) = \sqrt{mn}$
3. $\varphi(m, n) = \frac{m^2}{n}$ called the inverse of harmonic case.

Definition 2.1. *[2] Let $[c_1, d_1] \subseteq [0, +\infty], c_1 < d_1, g \in L_1[c_1, d_1]$ and $m, n > 0$. The left and right-sided general $(m, n)-$Riemann-Liouville fractional integral of order $\beta > 0$ are define as*

$$ {}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{m\beta}{\varphi(m,n)})} \int_{c_1}^{y} (y - x)^{\frac{\beta}{\varphi(m,n)}-1} g(x)dx, \;\; c_1 < y \le d_1, \tag{4} $$

$$ {}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{m.\beta}{\varphi(m,n)})} \int_{y}^{d_1} (x - y)^{\frac{\beta}{\varphi(m,n)}-1} g(x)dx, \;\; c_1 \le y < d_1, \tag{5} $$

where $\Gamma_{(m,n)}$ is defined in (3).

Here we shall extract the definitions of the arithmetic, geometric and harmonic $(m, n)-$Riemman-Liouville fractional integrals operators from the Definition 2.1 as special cases.

Case 2.2. *Choose $\varphi_1(m, n) = \frac{m+n}{2}$ in Definition 2.1 we obtain left and right sided the arithmetic (m, n)-Riemann-Liouville fractional respectively i.e.*

$$ {}_{c^+}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{2m\beta}{m+n})} \int_{c_1}^{y} (y - x)^{\frac{2\beta}{m+n}-1} g(x)dx, \;\; c_1 < y \le d_1, \tag{6} $$

$$ {}_{d_1^-}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{2m\beta}{m+n})} \int_{y}^{d_1} (x - y)^{\frac{2\beta}{m+n}-1} g(x)dx, \;\; c_1 \le y < d_1, \tag{7} $$

where $\Gamma_{(m,n)}$ is defined in (3).

Case 2.3. *By taking $\varphi_2(m, n) = \sqrt{mn}$, the Definition 2.1 becomes for the geometric (m, n)-Riemann-Liouville fractional i.e.*

$$ {}_{c_1^+}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}\left(\sqrt{\frac{m}{n}}\beta\right)} \int_{c_1}^{y} (y - x)^{\frac{\beta}{\sqrt{mn}}-1} g(x)dx, \;\; c_1 < y \le d_1, \tag{8} $$

$$ {}_{d_1^-}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}\left(\sqrt{\frac{m}{n}}\beta\right)} \int_{y}^{d_1} (x - y)^{\frac{\beta}{\sqrt{mn}}-1} g(x)dx, \;\; c_1 \le x < d. \tag{9} $$

where $\Gamma_{(m,n)}$ is defined in (3).

Case 2.4. *As special case of Definition 2.1 we can write it for harmonic* $(m,n)-$*Riemann-Liouville fractional by taking* $\varphi_3(m,n) = \frac{m^2}{n}$

$$ {}_{c_1^+}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{n}{m}\beta)} \int_{c_1}^{y} (y-x)^{\frac{m\beta}{n^2}-1} g(x)dx, \; c_1 < y \le d_1, \tag{10} $$

$$ {}_{d_1^-}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(y) = \frac{1}{m\Gamma_{(m,n)}(\frac{n}{m}\beta)} \int_{y}^{d_1} (y-x)^{\frac{m\beta}{n^2}-1} g(x)dx, \; c_1 \le y < d_1, \tag{11} $$

where $\Gamma_{(m,n)}$ *is defined in (3).*

The main motivation of this paper is to generalize the results of [21]. After furnishing the basic concepts in Section 1 and 2, in Section 3 we shall prove the new Hermite–Hadamard type inequalities for general $(m,n)-$ Riemann-Liouvill's fractional integral. We shall derive the related inequalities for arithmetic, geometric and Harmonic fractional integrals. In Section 4, we use mid point convexity to prove the new Hermite–Hadamard type inequalities. Section 5 is dedicated to to the applications of quadrature formulas. At the end we conclude the paper.

3. The main results

Our first result is given in upcoming first theorem.

Theorem 3.1. *Let* $m,n > 0, {}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(x)$ *and* ${}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(x)$ *be the left- and right-sided generated* (m,n)*-Riemann-Liouville fractional integral of order* $\beta > 0$*. Let* $g : [c_1, d_1] \to \mathbb{R}$ *be positive mapping with* $0 \le c_1 < d_1$ *and* $g \in L_1[c_1, d_1]$*. If* g *is convex on* $[c_1, d_1]$*, then*

$$ g\left(\frac{c_1+d_1}{2}\right) \le \frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}} \left[{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d) + {}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c) \right] \le \frac{g(c_1)+g(d_1)}{2}, \tag{12} $$

where Γ *is the classical gamma function.*

Proof. Since g is convex function on $[\xi, \mu]$, we have for $\xi, \mu \in [c_1, d_1]$ with $\eta = \frac{1}{2}$, we write

$$ 2g\left(\frac{\xi+\mu}{2}\right) \le g(\xi) + g(\mu). \tag{13} $$

Put $\xi = \eta c_1 + (1-\eta)d_1$ and $\mu = \eta d_1 + (1-\eta)c_1$ in (13), we get

$$ 2g\left(\frac{c_1+d_1}{2}\right) \le g(\eta c_1 + (1-\eta)d_1) + g(\eta d_1 + (1-\eta)c_1). \tag{14} $$

Multiplying inequality (14) with $\eta^{\frac{\beta}{\varphi(m,n)}-1}$ on both sides and integrating with respect to η over $[0,1]$, we have

$$ \begin{aligned} & 2g\left(\frac{c_1+d_1}{2}\right)\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1}d\eta \\ \le\; & \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1}g(\eta c_1 + (1-\eta)d_1)d\eta + \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1}g(\eta d_1 + (1-\eta)c_1)d\eta \end{aligned} $$

$$= \quad K_1 + K_2 \tag{15}$$

Firstly take

$$K_1 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta c_1 + (1-\eta)d_1)d\eta. \tag{16}$$

Substitute $\eta c_1 + (1-\eta)d_1 = x$ in (16), we get after some calculations

$$K_1 = \left(\frac{1}{(d_1 - c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)\int_c^{d_1} (d_1 - x)^{\frac{\beta}{\varphi(m,n)}-1} g(x)dx. \tag{17}$$

Multiplying and dividing by (17) with $m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)$ and using Definition 2.1, we obtain

$$K_1 = \left(m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\right)\left(\frac{1}{(d_1 - c_1)^{\frac{\beta}{\varphi(m,n)}}}\right) {}_{c_1^+}J^{\beta}_{\varphi(l,m)} g(d_1).$$

Now for

$$K_2 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta d_1 + (1-\eta)c_1)d\eta. \tag{18}$$

Put $\varphi = \eta d_1 + (1-\eta)c_1$ in (18), then after some settings we can write

$$K_2 = \left(\frac{1}{(d_1 - c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)\int_c^{d_1} (\varphi - c_1)^{\frac{\beta}{\varphi(m,n)}-1} g(\varphi)d\varphi. \tag{19}$$

Multiplying and dividing (19) by $m\Gamma_{(m,n)}(\frac{m\beta}{\varphi(m,n)})$ and after some setting we get

$$K_2 = \left(\frac{1}{(d_1 - c_1)^{\frac{\beta}{\varphi(m,n)}}}\right) m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right) {}_{c_1^+}J^{\beta}_{\varphi(l,m)} g(d_1).$$

Putting value of K_1 and K_2 in inequality (15), we get

$$g\left(\frac{c_1 + d_1}{2}\right) \quad \leq \quad \frac{m\beta}{2(\varphi(m,n))}\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\left(\frac{1}{(d_1 - c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)\left[{}_{c_1^+}J^{\beta}_{\varphi(l,m)} g(d_1) + {}_{d_1^-}J^{\beta}_{\varphi(l,m)} g(c_1)\right]. \tag{20}$$

We have the following property (see [2, Proposition 1.5])

$$\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right) = \frac{n^{\frac{m\beta}{m\varphi(m,n)}}}{m}\Gamma\left(\frac{m\beta}{m\varphi(m,n)}\right) = \frac{n^{\frac{\beta}{\varphi(m,n)}}}{m}\Gamma\left(\frac{\beta}{\varphi(m,n)}\right). \tag{21}$$

By using classical Γ property $n\Gamma(n) = \Gamma(n+1)$ we write

$$\frac{m\beta}{2(\varphi(m,n))}\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right) = \frac{n^{\frac{\beta}{\varphi(m,n)}}}{2}\Gamma\left(\frac{\beta}{\varphi(m,n)} + 1\right). \tag{22}$$

Therefore (20) can be written as

$$g\left(\frac{c_1+d_1}{2}\right) \le \frac{n^{\frac{\beta}{\varphi(m,n)}}}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)\left[{}_{c_1^+}J^{\beta}_{\varphi(l,m)}g(d_1) + {}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(c_1)\right]. \tag{23}$$

This is prove of the first part of the required inequality. To prove the second inequality, we noted that g is convex function for $\eta \in [0,1]$. By adding both these inequalities $g(\eta c_1 + (1-\eta)d_1) \le \eta.g(c_1) + (1-\eta)g(d_1)$ and $g(\eta d_1 + (1-\eta)c_1) \le \eta.g(d_1) + (1-\eta)g(c_1)$, we get

$$g(\eta c_1 + (1-\eta)d_1) + g(\eta d_1 + (1-\eta)c_1) \le g(c_1) + g(d_1) \tag{24}$$

Multiplying inequality (24) with $\eta^{\frac{\beta}{\varphi(m,n)}-1}$ on both sides and integrating with respect to η over $[0,1]$ we have

$$\begin{aligned} &\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta c_1 + (1-\eta)d_1)d\eta + \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta d_1 + (1-\eta)c_1)d\eta \\ \le\ & \int_0^1 [g(c_1)+g(d_1)]\eta^{\frac{\beta}{\varphi(m,n)}-1}d\eta. \end{aligned}$$

Denote

$$L_1 + L_2 \le [g(c_1)+g(d_1)]\frac{\varphi(m,n)}{\beta}, \tag{25}$$

where

$$L_1 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta c_1 + (1-\eta)d_1)d\eta, \tag{26}$$

and

$$L_2 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\eta d_1 + (1-\eta)c_1)d\eta. \tag{27}$$

Put $x = \eta c_1 + (1-\eta)d_1$ and $\varphi = \eta d_1 + (1-\eta)c_1$ in (26) and (27) respectively, we get after some computations

$$L_1 = \left(m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\right)\left(\frac{1}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right) {}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(c_1),$$

and

$$L_2 = \left(\frac{1}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right) m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right) {}_{c_1^+}J^{\beta}_{\varphi(l,m)}g(d_1).$$

Therefore (25) becomes

$$\begin{aligned} &m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\left(\frac{1}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)\left[{}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(c_1) + {}_{c_1^+}J^{\beta}_{\varphi(l,m)}g(d_1)\right] \\ &\le [g(c_1)+g(d_1)]\frac{\varphi(m,n)}{\beta}, \end{aligned}$$

we can write as

$$\frac{n^{\frac{\beta}{\varphi(m,n)}}}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)\left[{}_{c_1^+}J^{\beta}_{\varphi(l,m)}g(d_1)+{}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(c_1)\right]\leq\frac{[g(c_1)+g(d_1)]}{2}. \tag{28}$$

Combining (23) and (28) we obtain (12). This complete the proof □

We will now determine the result for the geometric (m, n)-Riemann-Liouville fraction integral.

Corollary 3.2. *Choose $\varphi(m, n) = \sqrt{mn}$ in the inequality (12) becomes*

$$g\left(\frac{c_1+d_1}{2}\right)\leq\frac{n^{\frac{\beta}{\sqrt{mn}}}\Gamma(\frac{\beta}{\sqrt{mn}}+1)}{2(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}[{}_{c_1^+}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(c_1)]\leq\frac{g(c_1)+g(d_1)}{2}.$$

Corollary 3.3. *If we choose $\varphi(m, n) = \frac{m+n}{2}$ in the inequality (12), then we get the results for arithmetic (m, n)-Riemann-Liouville fraction integral i.e.*

$$g\left(\frac{c_1+d_1}{2}\right)\leq\frac{n^{\frac{2\beta}{m+n}}\Gamma(\frac{2\beta}{(m+n)}+1)}{2(d_1-c_1)^{\frac{2\beta}{m+n}}}[{}_{c_1^+}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(c_1)]\leq\frac{g(c_1)+g(d_1)}{2}.$$

Corollary 3.4. *If we take $\varphi(m, n) = \frac{m^2}{n}$ in the inequality (12) we get the result for the harmonic Riemann-Liouville fraction integral*

$$g\left(\frac{c_1+d_1}{2}\right)\leq\frac{n^{\frac{m\beta}{n^2}}\Gamma(\frac{m\beta}{n^2}+1)}{2(d_1-c_1)^{\frac{m\beta}{n^2}}}[{}_{c_1^+}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(c_1)]\leq\frac{g(c_1)+g(d_1)}{2}.$$

Remark 3.5. *If we choose $m = n$ then $\varphi(m, m) = m$ and Definition 2.1 reduces to the Definition 1.2 and Theorem 3.1 becomes [21, Theorem 5].*

Theorem 3.6. *Let $m, n > 0$ and ${}_{c_1^+}J^{\beta}_{\varphi(m,n)}g$ and ${}_{d_1^-}J^{\beta}_{\varphi(m,n)}g$ as in Definition 2.1. Let $g : [c_1, d_1] \to \mathbb{R}$ be a differentiable mapping on (c, d). If $g' \in L[c_1, d_1]$, then*

$$\begin{aligned}&\frac{d_1-c_1}{2}\int_0^1\left[(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right][g'(\eta c_1+(1-\eta)d_1)]\,d\eta\\ =\;&\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1)\right].\end{aligned} \tag{29}$$

Proof. Consider

$$J=\int_0^1\left[(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right][g'(\eta c_1+(1-\eta)d_1)]\,d\eta$$

which can be written as

$$\begin{aligned}J\;&=\;\int_0^1\left[(1-\eta)^{\frac{\beta}{\varphi(m,n)}}\right][g'(\eta c_1+(1-\eta)d_1)]\,d\eta+\left(-\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}[g'(\eta c_1+(1-\eta)d_1)]\right)d\eta\\ &=\;J_1+J_2\end{aligned} \tag{30}$$

Evaluating J_1, we get

$$J_1=\frac{g(d_1)}{d_1-c_1}-\frac{\beta}{\varphi(m,n)(d_1-c_1)}\int_0^1(1-\eta)^{\frac{\beta}{\varphi(m,n)}-1}g(\eta c_1+(1-\eta)d_1)d\eta. \tag{31}$$

Put $x=\eta c_1+(1-\eta)d_1$ in (31) and after some calculations we can write

$$J_1=\frac{g(d_1)}{d_1-c_1}-\frac{m\beta}{\varphi(m,n)(d_1-c_1)(x-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\right]{}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1),$$

which becomes

$$J_1=\frac{g(d_1)}{d_1-c_1}-\left(\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}+1}}\right){}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1).$$

Now

$$J_2=-\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}[g'(\eta c_1+(1-\eta)d_1)]d\eta.$$

Integrating J_2 by parts, we get

$$\begin{aligned}J_2&=\frac{g(c_1)}{d_1-c_1}-\frac{\beta}{\varphi(m,n)(d_1-c_1)}\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}-1}g(\eta c_1+(1-\eta)d_1)d\eta\\&=\frac{g(c_1)}{d_1-c_1}-\left(\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}+1}}\right){}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(x).\end{aligned}$$

Thus (30) can be written as

$$\begin{aligned}&\int_0^1[(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}][g'(\eta c_1+(1-\eta)d_1)]\\&=\frac{g(d_1)}{d_1-c_1}+\frac{g(c_1)}{d_1-c_1}-\left(\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}+1}}\right)\left[{}_{d_1^-}J^{\beta}_{\varphi(l,m)}g(c_1)+{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d_1)\right]\end{aligned} \tag{32}$$

Multiply both sides of (32) by $\frac{d_1-c_1}{2}$, we get the required inequality and this complete the proof. □

Remark 3.7. *If we choose $m=n$, then $\varphi(m,m)=m$ and the Theorem 3.6 becomes [21, Theorem 2.1].*

Here we give the related corollaries for geometric, arithmetic and harmonic Riemann-Liouville fraction integrals.

Corollary 3.8. *Choose $\varphi(m,n)=\sqrt{mn}$ in the Theorem 3.6 and get*

$$\begin{aligned}&\frac{d_1-c_1}{2}\int_0^1[(1-\eta)^{\frac{\beta}{\sqrt{mn}}}-\eta^{\frac{\beta}{\sqrt{mn}}}][g'(\eta c_1+(1-\eta)d_1)]d\eta\\&=\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\sqrt{mn}}}\Gamma(\frac{\beta}{\sqrt{mn}}+1)}{2(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}\left[{}_{c_1^+}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(c_1)\right].\end{aligned}$$

Corollary 3.9. *If we choose $\varphi(m,n)=\frac{m+n}{2}$ in the Theorem 3.6, then we get the result for arithmetic (m,n)-Riemann-Liouville fraction integral,*

$$\begin{aligned}
&\frac{d_1-c_1}{2}\int_0^1[(1-\eta)^{\frac{2\beta}{(m+n)}}-\eta^{\frac{2\beta}{(m+n)}}][g'(\eta c_1+(1-\eta)d_1)]d\eta\\
=\;&\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{2\beta}{(m+n)}}\Gamma(\frac{2\beta}{(m+n)}+1)}{2(d_1-c_1)^{\frac{2\beta}{(m+n)}}}\left[{}_{c_1^+}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(c_1)\right].
\end{aligned}$$

Corollary 3.10. *If we take $\varphi(m,n)=\frac{m^2}{n}$ in the Theorem 3.6 we get the result for the harmonic (m,n)- Riemann-Liouville fraction integral,*

$$\begin{aligned}
&\frac{d_1-c_1}{2}\int_0^1[(1-\eta)^{\frac{m\beta}{n^2}}-\eta^{\frac{m\beta}{n^2}}][g'(\eta c_1+(1-\eta)d_1)]d\eta\\
=\;&\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{m\beta}{n^2}}\Gamma(\frac{m\beta}{n^2}+1)}{2(d_1-c_1)^{\frac{m\beta}{n^2}}}\left[{}_{c_1^+}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(c_1)\right].
\end{aligned}$$

Theorem 3.11. *Let $m,n>0$, ${}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1)$ and ${}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(c_1)$ be defined in Definition 2.1. Let $g:[c_1,d_1]\to\mathbb{R}$ be a differentiable mapping on (c,d).If $g'\in L'[c_1,d_1]$, then*

$$\begin{aligned}
&\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right|\left[{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1)\right]\\
\leq\;&\frac{d_1-c_1}{2^{\frac{\beta}{\varphi(m,n)}+1}}\left(1-\frac{1}{2^{\frac{\beta}{\varphi(m,n)}}}\right)(|g'(c)|+|g'(d)|)\,. \qquad (33)
\end{aligned}$$

Proof. By Theorem 3.6 and definition of convex function of $|g'|$, we have

$$\begin{aligned}
&\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}[{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1)]\right|\\
\leq\;&\frac{d_1-c_1}{2}\int_0^1\left|(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right|\left|g'(\eta c_1+(1-\eta)d_1)\right|d\eta\\
\leq\;&\frac{d_1-c_1}{2}\int_0^1\left|(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right|(\eta|g'(c)|+(1-\eta)|g'(d)|)\,d\eta\\
=\;&\frac{d_1-c_1}{2}\left[\int_0^1\left|(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right|\eta|g'(c)|d\eta\right.\\
+\;&\left.\frac{d_1-c_1}{2}\int_0^1\left|(1-\eta)^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}}\right|(1-\eta)|g'(d)|d\eta\right]\\
=\;&\frac{d_1-c_1}{2}[I_1+I_2] \qquad (34)
\end{aligned}$$

First we shall calculate I_1,

$$\begin{aligned} I_1 &= |g'(c)|\left[-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}}{\frac{\beta}{\varphi(m,n)}+1}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}+1}{(\frac{\beta}{\varphi(m,n)}+1)(\frac{\beta}{\varphi(m,n)}+2)}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}}{\frac{\beta}{\varphi(m,n)}+2}\right] \\ &+ |g'(d)|\left[-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}+1}{\frac{\beta}{\varphi(m,n)}+2}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}+\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}}{\frac{\beta}{\varphi(m,n)}+2}\right] \\ &= |g'(c)|\left[(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}\left[\frac{-\frac{\beta}{\varphi(m,n)}-2-1-\frac{\beta}{\varphi(m,n)}-1}{(\frac{\beta}{\varphi(m,n)}+2)(\frac{\beta}{\varphi(m,n)}+1)}\right]+\frac{1}{(\frac{\beta}{\varphi(m,n)}+2)(\frac{\beta}{\varphi(m,n)}+1)}\right] \\ &+ |g'(d)|\left[-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}}{\frac{\beta}{\varphi(m,n)}+2}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}+\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+2}}{\frac{\beta}{\varphi(m,n)}+2}-\frac{1}{\frac{\beta}{\varphi(m,n)}+2}\right] \\ &= |g'(c)|\left[\frac{1}{(\frac{\beta}{\varphi(m,n)}+2)(\frac{\beta}{\varphi(m,n)}+1)}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}\right]+|g'(d)|\left[\frac{1}{\frac{\beta}{\varphi(m,n)}+2}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}\right]. \end{aligned}$$

Similarly,

$$I_2 = |g'(d)|\left[\frac{1}{(\frac{\beta}{\varphi(m,n)}+2)(\frac{\beta}{\varphi(m,n)}+1)}+\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}\right]+|g'(c)|\left[\frac{1}{\frac{\beta}{\varphi(m,n)}+2}-\frac{(\frac{1}{2})^{\frac{\beta}{\varphi(m,n)}+1}}{\frac{\beta}{\varphi(m,n)}+1}\right].$$

Putting the values of I_1 and I_2 in (34) we obtain

$$\begin{aligned} &\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{2(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right|\left[{}_{c_1^+}J^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}J^{\beta}_{\varphi(m,n)}g(c_1)\right] \\ \leq\ &\frac{d_1-c_1}{2^{\frac{\beta}{\varphi(m,n)}+1}}\left(1-\frac{1}{2^{\frac{\beta}{\varphi(m,n)}}}\right)[g'(c)+g'(d)]. \end{aligned}$$

This complete the proof. □

Remark 3.12. *Choose $m = n$, then $\varphi(m,m) = m$ and the Theorem 3.11 becomes [21, Theorem 6].*

Corollary 3.13. *Choose $\varphi(m,n) = \sqrt{mn}$, then the inequality (33) becomes*

$$\begin{aligned} &\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{\beta}{\sqrt{mn}}}\Gamma(\frac{\beta}{\sqrt{mn}}+1)}{2(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}\right|\left[{}_{c_1^+}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{G}^{\beta}_{\varphi(m,n)}g(c_1)\right] \\ \leq\ &\frac{d_1-c_1}{2^{\frac{\beta}{\sqrt{mn}}+1}}\left(1-\frac{1}{2^{\frac{\beta}{\sqrt{mn}}}}\right)[g'(c)+g'(d)]. \end{aligned}$$

Corollary 3.14. *If we choose $\varphi(m,n) = \frac{m+n}{2}$ in the inequality (33), then we get*

$$\begin{aligned} &\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{2\beta}{m+n}}\Gamma(\frac{2\beta}{m+n}+1)}{2(d_1-c_1)^{\frac{2\beta}{m+n}}}\right|\left[{}_{c_1^+}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{A}^{\beta}_{\varphi(m,n)}g(c_1)\right] \\ \leq\ &\frac{d_1-c_1}{2^{\frac{2\beta}{m+n}+1}}\left(1-\frac{1}{2^{\frac{2\beta}{m+n}}}\right)[g'(c)+g'(d)]. \end{aligned}$$

Corollary 3.15. *If we take $\varphi(m,n)=\frac{m^2}{n}$ in the inequality (33) we get*

$$\left|\frac{g(c_1)+g(d_1)}{2}-\frac{n^{\frac{m\beta}{n^2}}\Gamma(\frac{m\beta}{n^2}+1)}{2(d_1-c_1)^{\frac{m\beta}{n^2}}}\right|\left[{}_{c_1^+}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(d_1)+{}_{d_1^-}\mathfrak{H}^{\beta}_{\varphi(m,n)}g(c_1)\right]$$
$$\leq \frac{d_1-c_1}{2^{\frac{m\beta}{n^2}+1}}\left(1-\frac{1}{2^{\frac{m\beta}{n^2}}}\right)[g'(c)+g'(d)].$$

4. Some more fractional integral inequalities

Definition 4.1. *Let $[c_1,d_1]\subseteq[0,+\infty]$, $c_1<d_1$, $g\in L_1[c_1,d_1]$ and $m,n>0$. The $J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g$ and $J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)}g$ be fractional integrals of order $\beta>0$ defined as*

$$J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}(\frac{m\beta}{\varphi(m,n)})}\int_{\frac{(+d}{2}}^{x}(x-y)^{\frac{\beta}{\varphi(m,n)}-1}g(y)dy,\ x>(\frac{c_1+d_1}{2}), \tag{35}$$

$$J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}(\frac{m\beta}{\varphi(m,n)})}\int_{x}^{\frac{c_1+d_1}{2}}(y-x)^{\frac{\beta}{\varphi(m,n)}-1}g(y)dy,\ x<\frac{c_1+d_1}{2}, \tag{36}$$

where $\Gamma_{(m,n)}$ is defined in (3).

Here we shall extract the definitions of the arithmetic, geometric and harmonic (m,n)−Riemman-Liouville fractional integrals operators from the Definition 4.1 as special cases.

- Choose $\varphi(m,n)=\frac{m+n}{2}$ in Definition 4.1 we obtain left and right sided the arithmetic mean (m,n)-Riemann-Liouville fractional respectively.

$$\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}(\frac{2m\beta}{m+n})}\int_{\frac{c_1+d_1}{2}}^{x}(x-y)^{\frac{2\beta}{m+n}-1}g(y)dy,\ x>\frac{c_1+d_1}{2}, \tag{37}$$

$$\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}(\frac{2m\beta}{m+n})}\int_{x}^{\frac{c_1+d_1}{2}}(y-x)^{\frac{2\beta}{m+n}-1}g(y)dy,\ x<\frac{c_1+d_1}{2}, \tag{38}$$

- By taking $\varphi(l,m)=\sqrt{l.m}$, the Definition 4.1 becomes for the geometric (m,n)-Riemann-Liouville fractional i.e.

$$\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}\left(\sqrt{\frac{m\beta}{n}}\right)}\int_{\frac{c_1+d_1}{2}}^{x}(x-y)^{\frac{\beta}{\sqrt{mn}}-1}g(y)dy,\ x>\frac{c_1+d_1}{2}, \tag{39}$$

$$\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)}g(x)=\frac{1}{m\Gamma_{(m,n)}\left(\sqrt{\frac{m\beta}{n}}\right)}\int_{x}^{\frac{c_1+d_1}{2}}(y-x)^{\frac{\beta}{\sqrt{mn}}-1}g(y)dy,\ x<\frac{c_1+d_1}{2}, \tag{40}$$

- As special case of Definition 4.1 we can write it for harmonic mean (m,n)-Riemann-Liouville fractional by taking $\varphi(m,n) = \frac{m^2}{n}$

$$\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(x) = \frac{1}{m\Gamma_{(m,n)}(\frac{n}{m}\beta)} \int_{\frac{c_1+d_1}{2}}^{x} (x-y)^{\frac{m\beta}{n^2}-1} g(y)dy, \; x > \frac{c_1+d_1}{2}, \tag{41}$$

$$\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(x) = \frac{1}{m\Gamma_{(m,n)}(\frac{n}{m}\beta)} \int_{x}^{\frac{c_1+d_1}{2}} (y-x)^{\frac{m\beta}{n^2}-1} g(y)dy, \; x < \frac{c_1+d_1}{2}, \tag{42}$$

where $\Gamma_{(m,n)}$ is defined in (3).

Theorem 4.2. *Let $m, n > 0$, $J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g$ and $J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g$ be defined in Definition 4.1. Let $g : [c_1, d_1] \to \mathbb{R}$ be a positive mapping with $0 \le c_1 < d_1$ and $g \in L_1[c_1, d_1]$. If g is convex function on $[c_1, d_1]$, then*

$$\begin{aligned} g\left(\frac{c_1+d_1}{2}\right) &\le \frac{2^{\frac{\beta}{\varphi(m,n)}-1} n^{\frac{\beta}{\varphi(m,n)}} \Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}} [J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)} g(c_1)] \\ &\le \frac{g(c_1)+g(d_1)}{2}, \end{aligned} \tag{43}$$

where Γ is the classical gamma function.

Proof. As g is convex function on $[c_1, d_1]$, we have for $\xi, \mu \in [c_1, d_1]$ with $\eta = \frac{1}{2}$

$$g(\frac{\xi+\mu}{2}) \le \frac{g(\xi)+g(\mu)}{2}. \tag{44}$$

Put $\xi = \frac{\eta c}{2} + \frac{(2-\eta).d}{2}$ and $\mu = \frac{\eta d}{2} + \frac{(2-\eta).c}{2}$ in inequality (44) and get

$$2g\left(\frac{c_1+d_1}{2}\right) \le g\left(\frac{\eta c}{2} + \frac{(2-\eta).d}{2}\right) + g\left(\frac{\eta d}{2} + \frac{(2-\eta).c}{2}\right). \tag{45}$$

Multiply (45) by $\eta^{\frac{\beta}{\varphi(m,n)}-1}$on both sides and integrating with respect to η over $[0,1]$ we have

$$\begin{aligned} & 2g\left(\frac{c_1+d_1}{2}\right) \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} d\eta \\ \le & \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g\left(\frac{\eta c}{2} + \frac{(2-\eta).d}{2}\right) d\eta + \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g\left(\frac{\eta d}{2} + \frac{(2-\eta).c}{2}\right) d\eta \\ = & F_1 + F_2. \end{aligned} \tag{46}$$

First we take

$$F_1 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g\left(\frac{\eta c}{2} + \frac{(2-\eta).d}{2}\right) d\eta. \tag{47}$$

Substitute $\frac{\eta c}{2}+\frac{(2-\eta).d}{2}=x$, in (47) and after some settings we get

$$F_1 = m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)\left(\frac{2^{\frac{\beta}{\varphi(m,n)}}}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)}g(c_1)$$

Now for

$$F_2 = \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}-1} g(\frac{\eta d}{2}+\frac{(2-\eta).c}{2})d\eta \tag{48}$$

Put $\varphi = \frac{\eta d}{2}+\frac{(2-\eta).c}{2}$in equation (48) and after some computations we get

$$F_2 = \left(\frac{2^{\frac{\beta}{\varphi(m,n)}}}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)m\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right)J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(d_1).$$

Putting value of F_1 and F_2 in (46) we get

$$\begin{aligned} g\left(\frac{c_1+d_1}{2}\right) &\leq \left(\frac{2^{\frac{\beta}{\varphi(m,n)}}}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\right)\left(\frac{m\beta}{\varphi(m,n)}\right)\Gamma_{(m,n)}\left(\frac{m\beta}{\varphi(m,n)}\right) \\ &\times \left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)}g(x)\right]. \end{aligned} \tag{49}$$

Using the property stated in (21)and (22) the inequality (49) can be written as

$$g\left(\frac{c_1+d_1}{2}\right) \leq \frac{2^{\frac{\beta}{\varphi(m,n)}-1}.n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)}g(c_1)\right]. \tag{50}$$

This is prove of the ist part of this given inequality. To, prove the second inequality we noted that g is convex function for $\eta\in[0,1]$ and by he addition of $g(\frac{\eta c}{2}+\frac{(2-\eta).d}{2})\leq\frac{\eta}{2}.g(c_1)+\frac{(1-\eta)}{2}g(d_1)$ and $g(\frac{\eta d}{2}+\frac{(1-\eta)c}{2})\leq \frac{\eta.}{2}g(d_1)+\frac{(1-\eta)}{2}g(c_1)$, we get

$$g(\frac{\eta}{2}.c+\frac{(2-\eta)}{2}.d)+g(\frac{\eta}{2}.d+\frac{(1-\eta)}{2}.c)\leq g(c_1)+g(d_1) \tag{51}$$

Multiplying inequality (51) with $\eta^{\frac{\beta}{\varphi(m,n)}-1}$on both sides and integrating with respect to η over$[0,1]$ we have after some necessary cacluation and settings

$$\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)}g(c_1)\right]\leq\frac{g(c_1)+g(d_1)}{2}. \tag{52}$$

Combiing(50) and (52) obtain (43). □

Remark 4.3. *For $m=n$ we have $\varphi(m,m)=m$ and Definition 4.1 reduces to [21, Definition 5] and Theorem 4.2 becomes [21, Theorem 7].*

Corollary 4.4. *Choose $\varphi(m,n)=\sqrt{mn}$ in the inequality (43) we get*

$$\begin{aligned} g\left(\frac{c_1+d_1}{2}\right) &\leq \frac{2^{\frac{\beta}{\sqrt{mn}}-1}n^{\frac{\beta}{\sqrt{mn}}}\Gamma(\frac{\beta}{\sqrt{mn}}+1)}{(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}\left[\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)}g(d_1)+\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)}g(c_1)\right] \\ &\leq \frac{g(c_1)+g(d_1)}{2}. \end{aligned}$$

Corollary 4.5. *Choose $\varphi(m,n)=\frac{m+n}{2}$ in the the inequality (43), then we get*

$$\begin{aligned} g\left(\frac{c_1+d_1}{2}\right) &\leq \frac{2^{\frac{2\beta}{(m+n)}-1}n^{\frac{2\beta}{(m+n)}}\Gamma\left(\frac{2\beta}{(m+n)}+1\right)}{(d_1-c_1)^{\frac{2\beta}{(m+n)}}}\left[\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{-},\varphi(m,n)}g(c_1)\right] \\ &\leq \frac{g(c_1)+g(d_1)}{2}, \end{aligned}$$

Corollary 4.6. *Take $\varphi(m,n)=\frac{m^2}{n}$ in the inequality (43) we get*

$$\begin{aligned} g\left(\frac{c_1+d_1}{2}\right) &\leq \frac{2^{\frac{m\beta}{n^2}-1}n^{\frac{m\beta}{n^2}}\Gamma(\frac{m\beta}{n^2}+1)}{(d_1-c_1)^{\frac{m\beta}{n^2}}}\left[\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{-},\varphi(m,n)}g(c_1)\right] \\ &\leq \frac{g(c_1)+g(d_1)}{2}. \end{aligned}$$

Lemma 4.7. *Let $m,n>0$, and $J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g$, $J^{\beta}_{(\frac{c_1+d_1}{2})^{-},\varphi(m,n)}g$ be given in Definition 4.1. Let $g:[c_1,d_1]\to\mathbb{R}$ be a differentiable mapping on (c_1,d_1). If $g'\in L'[c_1,d_1]$, then*

$$\begin{aligned} &\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma\left(\frac{\beta}{\varphi(m,n)}+1\right)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^{-},\varphi(m,n)}g(c_1)\right] \\ &= g\left(\frac{c_1+d_1}{2}\right)+\frac{d_1-c_1}{4}\left[\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)d\eta-\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}d+\frac{(2-\eta)}{2}c_1\right)d\eta\right]. \end{aligned} \tag{53}$$

Proof. Let us take

$$L=\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)d\eta-\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}d+\frac{(2-\eta)}{2}c_1\right)d\eta=C_1-C_2, \tag{54}$$

where

$$C_1=\int_0^1\eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)d\eta.$$

After integrating and some simplification we obtain

$$C_1=-g\left(\frac{c_1+d_1}{2}\right)\left(\frac{2}{d_1-c_1}\right)+\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}J^{\beta}_{(\frac{c_1+d_1}{2})^{-},\varphi(m,n)}g(c_1). \tag{55}$$

Likewise

$$C_2=g\left(\frac{c_1+d_1}{2}\right)\left(\frac{2}{d_1-c_1}\right)-\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1). \tag{56}$$

By inequalities (55)and (56) we can write (54)

$$\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^{-},\varphi(m,n)}g(c_1)\right]-g\left(\frac{c_1+d_1}{2}\right)$$

$$= \frac{d_1 - c_1}{4}\left[\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}} g'\left(\frac{\eta}{2}c + \frac{(2-\eta)}{2}d\right)d\eta - \int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}} g'\left(\frac{\eta}{2}d + \frac{(2-\eta)}{2}c_1\right)d\eta\right].$$

This complete the proof. □

Corollary 4.8. *Choose $\varphi(m,n) = \sqrt{mn}$, then (53) in Lemma 4.7 becomes*

$$\frac{2^{\frac{\beta}{\sqrt{mn}}-1} n^{\frac{\beta}{\sqrt{mn}}} \Gamma\left(\frac{\beta}{\sqrt{mn}}+1\right)}{(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}\left[\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + \mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(c_1)\right] - g\left(\frac{c_1+d_1}{2}\right)$$
$$= \frac{d_1 - c_1}{4}\left[\int_0^1 \eta^{\frac{\beta}{\sqrt{mn}}} g'\left(\frac{\eta}{2}c + \frac{(2-\eta)}{2}d\right)d\eta - \int_0^1 \eta^{\frac{\beta}{\sqrt{mn}}} g'\left(\frac{\eta}{2}d + \frac{(2-\eta)}{2}c_1\right)d\eta\right].$$

Corollary 4.9. *If we choose $\varphi(m,n) = \frac{m+n}{2}$ in (53), then we get*

$$\frac{2^{\frac{2\beta}{(m+n)}-1}.n^{\frac{2\beta}{m+n}} \Gamma\left(\frac{2\beta}{(m+n)}+1\right)}{(d_1-c_1)^{\frac{2\beta}{m+n}}}[\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + \mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(c_1)] - g\left(\frac{c_1+d_1}{2}\right)$$
$$= \frac{d_1 - c_1}{4}\left[\int_0^1 \eta^{\frac{2\beta}{m+n}} g'\left(\frac{\eta}{2}c + \frac{(2-\eta)}{2}d\right)d\eta - \int_0^1 \eta^{\frac{2\beta}{m+n}} g'\left(\frac{\eta}{2}d + \frac{(2-\eta)}{2}c_1\right)d\eta\right].$$

Corollary 4.10. *If we take $\varphi(m,n) = \frac{m^2}{n}$ in (53), we get*

$$\frac{2^{\frac{m\beta}{n^2}-1} n^{\frac{m\beta}{n^2}} \Gamma\left(\frac{m\beta}{n^2}+1\right)}{(d_1-c_1)^{\frac{m\beta}{n^2}}}\left[\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + \mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(c_1)\right] - g\left(\frac{c_1+d_1}{2}\right)$$
$$= \frac{d_1 - c_1}{4}\left[\int_0^1 \eta^{\frac{m\beta}{n^2}} g'\left(\frac{\eta}{2}c + \frac{(2-\eta)}{2}d\right)d\eta - \int_0^1 \eta^{\frac{m\beta}{n^2}} g'\left(\frac{\eta}{2}d + \frac{(2-\eta)}{2}c_1\right)d\eta\right].$$

Theorem 4.11. *Let $m,n > 0$, $J^{\beta}_{\frac{(c_1+d_1)^+}{2},\varphi(m,n)} g$ and $J^{\beta}_{\frac{(c_1+d_1)^+}{2},\varphi(m,n)} g$ be given in Definition 4.1. Let $g : [c_1, d_1] \to \mathbb{R}$ be a differentiable mapping on (c_1, d_1). If $g' \in L'[c_1, d_1]$, then*

$$\left|\frac{2^{\frac{\beta}{\varphi(m,n)}-1} n^{\frac{\beta}{\varphi(m,n)}} \Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)} g(c_1)]\right|$$
$$\leq \frac{d_1-c_1}{4\left(\frac{\beta}{\varphi(m,n)}+1\right)}\left(\frac{1}{2(\frac{\beta}{\varphi(m,n)}+2)}\right)^{\frac{1}{\ell}}\left[\left(\left(\frac{\beta}{\varphi(m,n)}+1\right)|g'(c)|^{\ell} + \left(\frac{\beta}{\varphi(m,n)}+3\right)|g'(d)|^{\ell}\right)^{\frac{1}{\ell}}\right.$$
$$\left. + \left(\frac{\beta}{\varphi(m,n)}+3\right)|g'(c)|^{\ell} + \left(\frac{\beta}{\varphi(m,n)}+1\right)|g'(d)|^{\ell}\right]^{\frac{1}{\ell}}. \tag{57}$$

Proof. : First we suppose case $\ell = 1$ by using Theorem 4.2 and definition of convex function of $|g'|$, we get

$$\frac{2^{\frac{\beta}{\varphi(m,n)}-1} n^{\frac{\beta}{\varphi(m,n)}} \Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1) + J^{\beta}_{(\frac{c_1+d_1}{2})^-,\varphi(m,n)} g(c_1)] - g\left(\frac{c_1+d_1}{2}\right)$$

$$
\begin{aligned}
&\leq \frac{d_1-c_1}{4}\left[\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}\left(\left|g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)\right|-\left|g'\left(\frac{\eta}{2}d+\frac{(2-\eta)}{2}c_1\right)\right|d\eta\right)\right]\\
&\leq \frac{d_1-c_1}{4}\left(\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}\left[\frac{\eta}{2}\left|g'(c)\right|+\frac{2-\eta}{2}|g'(d)|+\frac{\eta}{2}\left|g'(d)\right|+\frac{2-\eta}{2}|g'(c)|\right]d\eta\right)\\
&= \frac{d_1-c_1}{4}\left(\int_0^1 \frac{\eta^{\frac{\beta}{\varphi(m,n)}+1}}{2}|g'(c)|+\frac{2\eta^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}+1}}{2}|g'(d)|\frac{\eta^{\frac{\beta}{\varphi(m,n)}+1}}{2}|g'(d)|+\frac{2\eta^{\frac{\beta}{\varphi(m,n)}}-\eta^{\frac{\beta}{\varphi(m,n)}+1}}{2}|g'(c)|d\eta\right)\\
&= \frac{d_1-c_1}{4\left(\frac{\beta}{\varphi(m,n)}+1\right)}\left[|g'(c)|+|g'(d)|\right]. \qquad \text{(after some computations)}
\end{aligned}
$$

Now for ℓ is greater than 1. By using Lemma 4.7, the Holder inequality and the definition of convex function $|g'|^{\ell}$, we get

$$
\begin{aligned}
&\left|\frac{2^{\frac{\beta}{\varphi(m,n)}-1}n^{\frac{\beta}{\varphi(m,n)}}\Gamma(\frac{\beta}{\varphi(m,n)}+1)}{(d_1-c_1)^{\frac{\beta}{\varphi(m,n)}}}\left[J^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+J^{\beta}_{(\frac{c_1+d_1}{2})^{-},\varphi(m,n)}g(c_1)\right]-g\left(\frac{c_1+d_1}{2}\right)\right|\\
&\leq \frac{d_1-c_1}{4}\left[\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)d\eta-\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}g'\left(\frac{\eta}{2}d+\frac{(2-\eta)}{2}c_1\right)d\eta\right]\\
&\leq \frac{d_1-c_1}{4}\left[\int_0^1\left(\eta^{\frac{\beta}{\varphi(m,n)}}\left|g'\left(\frac{\eta}{2}c+\frac{(2-\eta)}{2}d\right)\right|^{\ell}d\eta\right)^{\frac{1}{\ell}}\right.\\
&-\left.\left(\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}\left|g'\left(\frac{\eta}{2}d+\frac{(2-\eta)}{2}c_1\right)\right|^{\ell}d\eta\right)^{\frac{1}{\ell}}\right]\left(\int_0^1 \eta^{\frac{\beta}{\varphi(m,n)}}d\eta\right)^{1-\frac{1}{\ell}}\\
&\leq \frac{d_1-c_1}{4}\left(\frac{1}{\frac{\beta}{\varphi(m,n)}+1}\right)\left(\frac{1}{2\left(\frac{\beta}{\varphi(m,n)}+2\right)}\right)^{\frac{1}{\ell}}\left[\left(\left(\frac{\beta}{\varphi(m,n)}+1\right)|g'(c)|^{\ell}+\left(\frac{\beta}{\varphi(m,n)}+3\right)|g'(d)|^{\ell}\right)^{\frac{1}{\ell}}\right.\\
&+\left.\left(\frac{\beta}{\varphi(m,n)}+3\right)|g'(c)|^{\ell}+\left(\frac{\beta}{\varphi(m,n)}+1\right)|g'(d)|^{\ell}\right]^{\frac{1}{\ell}}. \text{(after necessary calculations)}
\end{aligned}
$$

This complete the proof. □

Remark 4.12. *Choose $m=n$, then $\varphi(m,m)=m$ and the Theorem 4.11 becomes [21, Theorem 8].*

Corollary 4.13. *We shall give the result for the geometric (m,n)-Riemann-Liouville fraction integral. For this we choose $\varphi(m,n)=\sqrt{mn}$, then the inequality (57) becomes*

$$
\begin{aligned}
&\left|\frac{2^{\frac{\beta}{\sqrt{mn}}-1}n^{\frac{\beta}{\sqrt{mn}}}\Gamma(\frac{\beta}{\sqrt{mn}}+1)}{(d_1-c_1)^{\frac{\beta}{\sqrt{mn}}}}\left[\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{+},\varphi(m,n)}g(d_1)+\mathfrak{G}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^{-},\varphi(m,n)}g(c_1)\right]\right|\\
&\leq \frac{d_1-c_1}{4}\left(\frac{1}{\frac{\beta}{\sqrt{mn}}+1}\right)\left(\frac{1}{2(\frac{\beta}{\sqrt{mn}}+2)}\right)^{\frac{1}{\ell}}\left[\left(\left(\frac{\beta}{\sqrt{mn}}+1\right)|g'(c)|^{\ell}+\left(\frac{\beta}{\sqrt{mn}}+3\right)|g'(d)|^{\ell}\right)^{\frac{1}{\ell}}\right.\\
&+\left.\left(\frac{\beta}{\sqrt{mn}}+3\right)|g'(c)|^{\ell}+\left(\frac{\beta}{\sqrt{mn}}+1\right)|g'(d)|^{\ell}\right]^{\frac{1}{\ell}}
\end{aligned}
$$

Corollary 4.14. *If we choose $\varphi(m,n) = \frac{m+n}{2}$ in the inequality (57), then we get the results for arithmetic (m,n)-Riemann-Liouville fraction integral i.e.*

$$\begin{aligned}
&\left|\frac{2^{\frac{2\beta}{(m+n)}-1} n^{\frac{2\beta}{(m+n)}} \Gamma(\frac{2\beta}{(m+n)}+1)}{(d_1-c_1)^{\frac{2\beta}{(m+n)}}}\left[\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1)+\mathfrak{A}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(c_1)\right]\right| \\
\le\ & \frac{d_1-c_1}{4}\left(\frac{1}{\frac{2\beta}{(m+n)}+1}\right)\left(\frac{1}{2(\frac{2\beta}{(m+n)}+2)}\right)^{\frac{1}{\ell}}\left[\left(\left(\frac{2\beta}{(p+q)}+1\right)|g'(c)|^{\ell}+\left(\frac{2\beta}{(p+q)}+3\right)|g'(d)|^{\ell}\right)^{\frac{1}{\ell}}\right. \\
+\ & \left.\left(\frac{2\beta}{(m+n)}+3\right)|g'(c)|^{\ell}+\left(\frac{2\beta}{(m+n)}+1\right)|g'(d)|^{\ell}\right]^{\frac{1}{\ell}}.
\end{aligned}$$

Corollary 4.15. *If we take $\varphi(m,n) = \frac{m^2}{n}$ in the inequality (57), we get the result for the harmonic (m,n)- Riemann-Liouville fraction integral*

$$\begin{aligned}
&\left|\frac{2^{\frac{m\beta}{n^2}-1} n^{\frac{m\beta}{n^2}} \Gamma(\frac{m\beta}{n^2}+1)}{(d_1-c_1)^{\frac{m\beta}{n^2}}}\left[\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^+,\varphi(m,n)} g(d_1)+\mathfrak{H}^{\beta}_{\left(\frac{c_1+d_1}{2}\right)^-,\varphi(m,n)} g(c_1)\right]\right| \\
\le\ & \frac{d_1-c_1}{4}\left(\frac{1}{\frac{m\beta}{n^2}+1}\right)\left(\frac{1}{2\left(\frac{m\beta}{n^2}+2\right)}\right)^{\frac{1}{\ell}}\left[\left(\left(\frac{m\beta}{n^2}+1\right)|g'(c)|^{h}+\left(\frac{m\beta}{n^2}+3\right)|g'(d)|^{\ell}\right)^{\frac{1}{\ell}}\right. \\
+\ & \left.\left(\frac{m\beta}{n^2}+3\right)|g'(c)|^{\ell}+\left(\frac{m\beta}{n^2}+1\right)|g'(d)|^{\ell}\right]^{\frac{1}{\ell}}.
\end{aligned}$$

5. Applications to quadrature formulas

In this section, we use the results to estimate the quadrature formulas' errors. It is demonstrated that particular situations like mid-point inequality and trapezoid inequality are included in our primary results. Furthermore, our primary findings immediately lead to the Hermite-Hadamard inequality.

Proposition 5.1. *(***Hermite-Hadamard inequality***) By using the assumptions of Theorem 3.1 with $m = 1$ and $n = 1$, we get the following*

$$g\left(\frac{c_1+d_1}{2}\right) \le \frac{1}{d_1-c_1}\int_c^{d_1} g(\eta)d\eta \le \frac{g(c_1)+g(d_1)}{2}.$$

Proposition 5.2. *(***Mid-point inequality***) By using the assumptions of Theorem 4.11 with $m = 1$, $\ell = 1$ and $n = 1$, we get the following*

$$\left|\frac{1}{d_1-c_1}\int_c^{d_1} g(\eta)d\eta - g\left(\frac{c_1+d_1}{2}\right)\right| \le \frac{d_1-c_1}{8}[|g'(c)|+|g'(d)|].$$

Proposition 5.3. *(***Trapezoid inequality***) By using the assumptions of Theorem 3.11 with $m = 1$, and $n = 1$, we get the following*

$$\left|\frac{g(c_1)+g(d_1)}{2} - \frac{1}{d_1-c_1}\int_c^{d_1} g(\eta)d\eta\right| \le \frac{d_1-c_1}{8}\left[|g'(c)|+|g'(d)|\right].$$

6. Conclusion

We have used the convexity of differentiable functions to analyze various Hermite-Hadamard type inequalities for (m, n)-Riemann Liouville fractional integrals. By using Theorems 3.1, 3.11, 4.2, and 8, we presented our primary findings and demonstrated how some of the results from other studies are included in ours as special circumstances. Two inequalities involving the error estimates of quadrature formulas have been established as applications.

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