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Nonlinear Lie type centralizers and derivations by local actions on triangular algebras

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Abstract. Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$. Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(\delta(U_1), U_2, U_3, ..., U_n)$$

and

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(U_1, \delta(U_2), U_3, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Then $\delta(U) = ZU + \tau(U)$ for some $Z \in \mathcal{Z}(\mathcal{U})$ and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$. Besides, we characterize nonlinear Lie type derivations by local actions on triangular algebras under certain conditions. Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = \sum_{i=1}^{n} p_n(U_1, ..., U_{i-1}, \delta(U_i), U_{i+1}, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Then $\delta(U) = d(U) + \tau(U)$ for all $U \in \mathcal{U}$, where $d : \mathcal{U} \to \mathcal{U}$ ia an additive derivation and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{A}$ be an algebra over $\mathcal{R}$ with the center $\mathcal{Z}(\mathcal{A})$. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is said to be a derivation if $L(XY) = L(X)Y + XL(Y)$ for all $X, Y \in \mathcal{A}$. An $\mathcal{R}$ linear mapping is said to be a Lie derivation if $L([X, Y]) = [l(X), Y] + [X, L(Y)]$ and a Lie triple derivation if $L([[X, Y], Z]) = [[L(X), Y], Z] + [[X, L(Y)], Z] + [[X, Y], L(Z)]$ for all $X, Y, Z \in \mathcal{A}$, where $[X, Y] = XY - YX$ is the usual Lie product. On can easily check that each derivation is a Lie derivation and each Lie derivation is

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a Lie triple derivation. However, the converse is not true in general (see [17]). Therefore, authors further developed them in a more general way. Let us define the following sequence of left-normed commutators:

$$p_1(X_1) = X_1,$$

$$p_2(X_1, X_2) = [p_1(X_1, X_2)] = [X_1, X_2],$$

$$p_3(X_1, X_2, X_3) = [p_2(X_1, X_2), X_3] = [[X_1, X_2], X_3],$$

......

$$p_n(X_1, X_2, ..., X_n) = [p_{n-1}(X_1, X_2, ..., X_{n-1}), X_n]$$

for all $X_1, X_2, ..., X_n \in \mathcal{A}$. The polynomial $p_n(X_1, X_2, ..., X_n)$ is said to be a commutator of length n ($n \geq 2$). Then Lie n-derivations were introduced by Abdullaev [1]. A Lie n-derivation is an $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ satisfying

$$L(p_n(X_1, X_2, X_3, ..., X_n)) = \sum_{i=1}^{n} p_n(X_1, ..., X_{i-1}, L(X_i), X_{i+1}, ..., X_n). \tag{1}$$

Recently, there have been many results on Lie n-derivations. We refer to [4, 6, 8, 11, 15, 16, 18] and the references therein.

Recall that an $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is said to be a left (right) centralizer if $L(XY) = L(X)Y$ ($L(XY) = XL(Y)$) for all $X, Y \in \mathcal{A}$ and a centralizer if it is a both left and right centralizer. An $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is said to be a Lie centralizer of $\mathcal{A}$ if $L([X, Y]) = [L(X), Y]$ for all $X, Y \in \mathcal{A}$. It is easy to verify that L is a Lie centralizer if and only if $L([X, Y]) = [X, L(Y)]$. More generally, L is called a Lie triple centralizer of $\mathcal{A}$ if $L([[X, Y], Z]) = [[L(X), Y], Z]$ for all $X, Y, Z \in \mathcal{A}$. It is easy to verify that L is a Lie triple centralizer if and only if $L([[X, Y], Z]) = [[X, L(Y)], Z]$. On can easily check that each centralizer is a Lie centralizer and each Lie centralizer is a Lie triple centralizer. However, the converse is not true in general. These years, many authors have paid much attention to centralizer, Lie centralizer and Lie triple centralizer. As Lie n-derivations, authors further developed the definition of Lie n-centralizers. A Lie n-centralizer is an $\mathcal{R}$-linear map $\mathcal{A} \to \mathcal{A}$ satisfying

$$L(p_n(X_1, X_2, ..., X_n)) = p_n(L(X_1), X_2, ..., X_n) \tag{2}$$

for all $X_1, X_2, ..., X_n \in \mathcal{A}$. Similar to Lie centralizers and Lie triple centralizers, one can also easily check that L is a Lie n-centralizer if and only if $L(p_n(X_1, X_2, ..., X_n)) = p_n(X_1, L(X_2), ..., X_n)$. Recently, some authors have paid attentions to Lie n-centralizers. We refer to [4, 14, 19] and the references therein.

Recently, there have been a number of authors studying conditions under which derivations and centralizers of rings or operator algebras can be completely determined by the action on some elements concerning products. Assume that $L : \mathcal{A} \to \mathcal{A}$ is a $\mathcal{R}$-linear (nonlinear) map and $F : \mathcal{A} \times \mathcal{A} \times ... \times \mathcal{A} \to \mathcal{A}$ is a map. If Q is a proper subset of $\mathcal{A}$ and Eq. (1.1) holds for any $X_1, X_2, ..., X_n$ with $F(X_1, X_2, ..., X_n) \in Q$, then L is called a linear (nonlinear) Lie n-derivation by local action of $\mathcal{A}$. Similarly, if Q is a proper subset of $\mathcal{A}$ and Eq. (1.2) holds for any $X_1, X_2, ..., X_n$ with $F(X_1, X_2, ..., X_n) \in Q$, then L is called a linear (nonlinear) Lie n-centralizer by local action of $\mathcal{A}$. These years, some authors are paying attentions to Lie n-derivations and centralizers by local actions on several operator algebras. We refer to [2, 3, 5, 13] and the references therein.

Let $\mathcal{U}$ be a triangular algebra over a commutative ring $\mathcal{R}$. In the note, under some mild assumptions on $\mathcal{U}$, we study nonlinear Lie n-centralizers and Lie n-derivations by local actions on $\mathcal{U}$. It is proved that nonlinear Lie n-centralizers and Lie n-derivations by local actions are both almost additive. Moreover, we characterize the structures of them, respectively.

2. Preliminaries

Triangular algebras were first introduced in [9]. Let $\mathcal{R}$ be a commutative ring with identity. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$, with unit $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$, respectively; and $\mathcal{M}$ is a unital $(\mathcal{A},\mathcal{B})$-bimodule. In this note, assume that $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Under the usual matrix operations,

$$\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\right\}$$

is called a triangular algebra. The main examples of triangular algebras are nest algebras and (block) upper triangular matrix algebras. For more details, see [7, 12] and so on.

Let $\mathcal{Z}(\mathcal{U})$ be the center of $\mathcal{U}$. It follows from [10, Proposition 3] that

$$\mathcal{Z}(\mathcal{U}) = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \quad for \quad all \quad m \in \mathcal{M}\right\}.$$

Let $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ be two maps defined by

$$\pi_{\mathcal{A}} : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \to \mathcal{A},\ \pi_{\mathcal{B}} : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \to \mathcal{B}.$$

Moreover, $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{B})$. And there exists a unique algebra isomorphism η from $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U}))$ to $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U}))$ such that $\eta(b)m = mb$ for all $m \in \mathcal{M}$ (see [10]).

Consider a triangular algebra $\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Let I be the identity of $\mathcal{U}$. Set

$$P_1 = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{B}} \end{pmatrix}$$

and $\mathcal{U}_{ij} = P_i\mathcal{U}P_j$ for all $1 \le i \le j \le 2$. It is clear that $\mathcal{U}$ can be represented as

$$\mathcal{U} = \mathcal{U}_{11} \bigoplus \mathcal{U}_{12} \bigoplus \mathcal{U}_{22}.$$

Finally, we end this section with a proposition, which plays an important role in the proof of the main results.

Proposition 2.1. [20] *Let $U_{ii} \in \mathcal{U}_{ii}, i = 1, 2$. If $U_{11}V_{12} = V_{12}U_{22}$ for any $V_{12} \in \mathcal{U}_{12}$, then $U_{11} + U_{22} \in \mathcal{Z}(\mathcal{U})$.*

3. Lie type centralizers by local action

3.1. Almost additivity

Theorem 3.1. *Let $\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies*

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(\delta(U_1), U_2, U_3, ..., U_n)$$

and

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(U_1, \delta(U_2), U_3, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Then

$$\delta(U_1 + U_2) - \delta(U_1) - \delta(U_2) \in \mathcal{Z}(\mathcal{U}).$$

In the following, Theorem 3.1 will be proved by checking several lemmas.

Lemma 3.2. $\delta(0) = 0$.

Proof. Take $U_1 = U_2 = U_3 = ... = U_n = 0$. One can easily check that $\delta(0) = 0$. □

Lemma 3.3. *Let $U_{11} \in \mathcal{U}_{12}$ and $U_{12} \in \mathcal{U}_{12}$. Then*

$$\delta(U_{11} + U_{12}) - \delta(U_{11}) - \delta(U_{12}) \in \mathcal{Z}(\mathcal{U})$$

and

$$\delta(U_{12} + U_{22}) - \delta(U_{12}) - \delta(U_{22}) \in \mathcal{Z}(\mathcal{U})$$

Proof. For any $U_{11} \in \mathcal{U}_{11}$ and $U_{12}, V_{12} \in \mathcal{U}_{12}$, denote $U = \delta(U_{11} + U_{12}) - \delta(U_{11}) - \delta(U_{12})$. Since $V_{12}(U_{11} + U_{12})P_1 = V_{12}U_{11}P_1 = V_{12}U_{12}P_1 = 0$, it follows that

$$\begin{aligned}\delta(U_{11}V_{12}) =&\delta(p_n(V_{12}, U_{11} + U_{12}, P_1, P_2, ..., P_2))\\ =&p_n(V_{12}, \delta(U_{11} + U_{12}), P_1, P_2, ..., P_2)\end{aligned}$$

and

$$\begin{aligned}\delta(U_{11}V_{12}) =&\delta(p_n(V_{12}, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(V_{12}, U_{12}, P_1, P_2, ..., P_2))\\ =&p_n(V_{12}, \delta(U_{11}) + \delta(U_{12}), P_1, P_2, ..., P_2).\end{aligned}$$

Comparing the two equations above, we have

$$p_n(V_{12}, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_1V_{12} = V_{12}P_2UP_2$ for all $V_{12} \in \mathcal{U}_{12}$. Then it follows from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Moreover, since $P_2(U_{11} + U_{12})P_1 = P_2U_{11}P_1 = P_2U_{12}P_1 = 0$, we have

$$\begin{aligned}\delta(U_{12}) =&\delta(p_n(P_2), U_{11} + U_{12}, P_1, P_2, ..., P_2)\\ =&p_n(P_2, \delta(U_{11} + U_{12}), P_1, P_2, ..., P_2)\end{aligned}$$

and

$$\begin{aligned}\delta(U_{12}) =&\delta(p_n(P_2, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(P_2, U_{12}, P_1, P_2, ..., P_2))\\ =&p_n(P_2, \delta(U_{11}) + \delta(U_{12}), P_1, P_2, ..., P_2).\end{aligned}$$

Comparing the equations above, we have

$$p_n(P_2, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore, we have

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

In the similar manner, we can also show that

$$\delta(U_{12} + U_{22}) - \delta(U_{12}) - \delta(U_{22}) \in \mathcal{Z}(\mathcal{U}).$$

□

Lemma 3.4. *For all $U_{12}, V_{12} \in \mathcal{U}_{12}$, we have*

$$\delta(U_{12} + V_{12}) = \delta(U_{12}) + \delta(V_{12}).$$

Proof. Since $(-U_{12} - P_1)(P_2 + V_{12})P_1 = 0$, it follows from Lemmas 3.2 and 3.3 that

$$\begin{aligned}\delta(U_{12} + V_{12}) =&\delta(p_n(-U_{12} - P_1, P_2 + V_{12}, P_1, P_2, ..., P_2))\\ =&\delta(p_n(\delta(-U_{12}) + \delta(-P_1), P_2 + V_{12}, P_1, P_2, ..., P_2))\\ =&\delta(p_n(-U_{12}, P_2, P_1, P_2, ..., P_2)) + \delta(p_n(-P_1, P_2, P_1, P_2, ..., P_2))\\ &+ \delta(p_n(-U_{12}, V_{12}, P_1, P_2, ..., P_2)) + \delta(p_n(-P_1, V_{12}, P_1, P_2, ..., P_2))\\ =&\delta(U_{12}) + \delta(V_{12}).\end{aligned}$$

□

Lemma 3.5. *Let $U_{ii}, V_{ii} \in \mathcal{U}_{ii}$ for $i = 1, 2$. Then $\delta(U_{ii} + V_{ii}) - \delta(U_{ii}) - \delta(V_{ii}) \in \mathcal{Z}(\mathcal{U})$.*

Proof. Here we only give the proof of $i = 1$. The proof of the case $i = 2$ is similar. Denote $U = \delta(U_{11} + V_{11}) - \delta(U_{11}) - \delta(V_{11})$. Since $W_{12}U_{11}P_1 = W_{12}v_{11}P_1 = W_{12}(U_{11} + V_{11})P_1 = 0$, it follows from Lemma 3.4 that

$$\begin{aligned}\delta((U_{11} + V_{11})W_{12}) =&\delta(p_n(W_{12}, U_{11} + V_{11}, P_1, P_2, ..., P_2))\\ =&p_n(W_{12}, \delta(U_{11} + V_{11}), P_1, P_2, ..., P_2)\end{aligned}$$

and

$$\begin{aligned}\delta((U_{11} + V_{11})W_{12}) =&\delta(U_{11}W_{12} + V_{11}W_{12})\\ =&\delta(U_{11}W_{12}) + \delta(V_{11}W_{12})\\ =&\delta(p_n(W_{12}, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(W_{12}, V_{11}, P_1, P_2, ..., P_2))\\ =&p_n(W_{12}, \delta(U_{11}), P_1, P_2, ..., P_2) + p_n(W_{12}, \delta(V_{11}), P_1, P_2, ..., P_2)\\ =&p_n(W_{12}, \delta(U_{11}) + \delta(V_{11}), P_1, P_2, ..., P_2).\end{aligned}$$

Comparing the two equations above, we have

$$p_n(W_{12}, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UW_{12} = W_{12}UP_2$ for all $W_{12} \in \mathcal{U}_{12}$. Therefore, we can obtain from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Since $(U_{11} + V_{11})P_2P_2 = U_{11}P_2P_2 = V_{11}P_2P_2 = 0$, it follows from Lemma 3.2 that

$$\begin{aligned}0 =&\delta(p_n(U_{11} + V_{11}, P_2, P_2, P_1..., P_1))\\ =&p_n(\delta(U_{11} + V_{11}), P_2, P_2, P_1..., P_1)\end{aligned}$$

and

$$\begin{aligned}0 =&\delta(p_n(V_{11}, P_2, P_2, P_1..., P_1)) + \delta(p_n(V_{11}, P_2, P_2, P_1..., P_1))\\ =&p_n(\delta(U_{11}) + \delta(V_{11}), P_2, P_2, P_1..., P_1).\end{aligned}$$

Comparing the equations above, we have

$$p_n(U, P_2, P_2, P_1..., P_1) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore,

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

□

Lemma 3.6. *For any $U_{ij} \in \mathcal{U}_{ij}$ with $1 \le i \le j \le 2$, we have*

$$\delta(\sum_{1\le i\le j\le 2} U_{ij}) - \sum_{1\le i\le j\le 2} \delta(U_{ij}) \in \mathcal{Z}(\mathcal{U}).$$

Proof. Denote $U = \delta(\sum_{1\le i\le j\le 2} U_{ij}) - \sum_{1\le i\le j\le 2} \delta(U_{ij})$. Since $\sum_{1\le i\le j\le 2} U_{ij}V_{12}P_1 = 0$, it follows from Lemma 3.4 that

$$\begin{aligned}\delta(V_{12}U_{22} - U_{11}V_{12}) =&\delta(p_n(\sum_{1\le i\le j\le 2} U_{ij}, V_{12}, P_1, P_2, ..., P_2))\\ =&p_n(\delta(\sum_{1\le i\le j\le 2} U_{ij}), V_{12}, P_1, P_2, ..., P_2)\end{aligned}$$

and

$$\begin{aligned}\delta(V_{12}U_{22} - U_{11}V_{12}) =&\delta(V_{12}U_{22}) + \delta(-U_{11}V_{12})\\ =&\sum_{1\le i\le j\le 2} \delta(p_n(U_{ij}, V_{12}, P_1, P_2, ..., P_2))\\ =&p_n(\sum_{1\le i\le j\le 2} \delta(U_{ij}), V_{12}, P_1, P_2, ..., P_2).\end{aligned}$$

Comparing the equations above, we have

$$p_n(U, V_{12}, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_1V_{12} = V_{12}P_2UP_2$ for all $V_{12} \in \mathcal{U}_{12}$. Then it follows from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Since $\sum_{1\le i\le j\le 2} U_{ij}(-P_1)P_2 = 0$, we have

$$\begin{aligned}\delta((-1)^{n+1}U_{12}) =&\delta(p_n(\sum_{1\le i\le j\le 2} U_{ij}, -P_1, P_2, P_1, ..., P_1))\\ =&p_n(\delta(\sum_{1\le i\le j\le 2} U_{ij}), -P_1, P_2, P_1, ..., P_1).\end{aligned}$$

On the other hand, it follows from Lemma 3.4 that

$$\begin{aligned}\delta((-1)^{n+1}U_{12}) =&\sum_{1\le i\le j\le 2} \delta(p_n(U_{ij}, -P_1, P_2, P_1, ..., P_1))\\ =&p_n(\sum_{1\le i\le j\le 2} \delta(U_{ij}), -P_1, P_2, P_1, ..., P_1).\end{aligned}$$

Comparing the two equations above, we have

$$P_n(U, -P_1, P_2, P_1, ..., P_1) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore,

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

□

Now we give the proof of Theorem 3.1.

Proof. Let $U = \sum_{1\leq i\leq j\leq 2} U_{ij}$ and $V = \sum_{1\leq i\leq j\leq 2} V_{ij}$. Then we have from Lemmas 3.4-3.6 that there exist $Z_1, Z_2, Z_3, Z_4, Z_5 \in \mathcal{Z}(\mathcal{U})$ such that

$$\begin{aligned}\delta(U+V) =&\delta(\sum_{1\leq i\leq j\leq 2} U_{ij} + \sum_{1\leq i\leq j\leq 2} V_{ij})\\ =&\delta(\sum_{1\leq i\leq j\leq 2} (U_{ij} + V_{ij}))\\ =&\sum_{1\leq i\leq j\leq 2} \delta(U_{ij} + V_{ij}) + Z_1\\ =&\sum_{1\leq i\leq j\leq 2} \delta(U_{ij}) + \sum_{1\leq i\leq j\leq 2} \delta(V_{ij}) + Z_1 + Z_2 + Z_3\\ =&\delta(\sum_{1\leq i\leq j\leq 2} U_{ij}) + \delta(\sum_{1\leq i\leq j\leq 2} V_{ij}) + Z_1 + Z_2 + Z_3 - Z_4 - Z_5\\ =&\delta(U) + \delta(V) + Z_1 + Z_2 + Z_3 - Z_4 - Z_5,\end{aligned}$$

which means that $\delta(U + V) - \delta(U) - \delta(V) \in \mathcal{Z}(\mathcal{U})$. □

3.2. Structure

Theorem 3.7. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$. Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies*

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(\delta(U_1), U_2, U_3, ..., U_n)$$

and

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = p_n(U_1, \delta(U_2), U_3, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Then $\delta(U) = ZU + \tau(U)$ for some $Z \in \mathcal{Z}(\mathcal{U})$ and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$.

To complete the proof of Theorem 3.8, we need to prove several lemmas.

Lemma 3.8. $\delta(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

Proof. Since $U_{12}P_1P_1 = 0$, it follows that

$$\begin{aligned}\delta(U_{12}) =&\delta(p_n(U_{12}, P_1, P_1, P_2..., P_2))\\ =&P_n(\delta(U_{12}), P_1, P_1, P_2..., P_2)\\ =&P_1\delta(U_{12})P_2,\end{aligned}$$

which means that $\delta(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$. □

Lemma 3.9. *There exist $f_i : \mathcal{U}_{ii} \to \mathcal{Z}(\mathcal{U})$ such that*

$$\delta(U_{ii}) - f_i(U_{ii}) \in \mathcal{U}_{ii}$$

for $i = 1, 2$.

Proof. Here we only give the proof for $i = 1$. Since $U_{11}P_2P_2 = 0$, It follows from Lemma 3.2 that

$$\begin{aligned}0 =&\delta(p_n(U_{11}, P_2, P_2, P_2, ..., p_2))\\ =&p_n(\delta(U_{11}), P_2, P_2, P_2, ..., p_2)\\ =&P_1\delta(U_{11})P_2.\end{aligned}$$

Consequently, $\delta(U_{11}) = P_1\delta(U_{11})P_1 + P_2\delta(U_{11})P_2$.

Since $U_{11}U_{22}U_{12} = 0$, it follows that

$$\begin{aligned}0 =&\delta(p_n(U_{11}, U_{22}, U_{12}, P_2, ..., P_2))\\ =&p_n(\delta(U_{11}), U_{22}, U_{12}, P_2, ..., P_2)\\ =&[[\delta(U_{11}), U_{22}], U_{12}]\\ =&[[P_2\delta(U_{11})P_2, U_{22}], U_{12}]\\ =&-U_{12}[P_2\delta(U_{11})P_2, U_{22}].\end{aligned}$$

Hence $U_{12}[P_2\delta(U_{11})P_2, U_{22}] = 0$ for all $U_{12} \in \mathcal{U}_{12}$. Since $\mathcal{U}_{12}$ is left faithful $\mathcal{U}_{22}$ module, it follows that $[P_2\delta(U_{11})P_2, U_{22}] = 0$ for all $U_{22} \in \mathcal{U}_{22}$. Thus $P_2\delta(U_{11})P_2 \in \mathcal{Z}(\mathcal{U}_{22})$. Employing the condition $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, we can write

$$\delta(U_{11}) = P_1\delta(U_{11})P_1 - \eta^{-1}(P_2\delta(U_{11})P_2) + \eta^{-1}(P_2\delta(U_{11})P_2) + P_2\delta(U_{11})P_2.$$

Thus, we can define a map $f_1 : \mathcal{U}_{11} \to \mathcal{Z}(\mathcal{U})$ as

$$f_1(U_{11}) = \eta^{-1}(P_2\delta(U_{11})P_2) + P_2\delta(U_{11})P_2.$$

Obviously,

$$\delta(U_{11}) - f_1(U_{11}) \in \mathcal{U}_{11}.$$

Similarly, we can show that

$$\delta(U_{22}) = P_1\delta(U_{22})P_1 + \eta(P_1\delta(U_{22})P_1) + P_2\delta(U_{22})P_2 - \eta(P_1\delta(U_{22})P_1).$$

Then we can define a map $f_2 : \mathcal{U}_{22} \to \mathcal{Z}(\mathcal{U})$ as

$$f_1(U_{22}) = P_1\delta(U_{22})P_1 + \eta(P_1\delta(U_{22})P_1).$$

Obviously,

$$\delta(U_{22}) - f_2(U_{22}) \in \mathcal{U}_{22}.$$

□

Remark 3.10. *Inspired by Lemmas 3.8 and 3.9, we define a new map* $\Delta : \mathcal{U} \to \mathcal{U}$ *as*

$$\Delta(U) = \sum_{1\le i\le j\le 2} \delta(U_{ij}) - f_1(U_{11}) - f_2(U_{22})$$

for any $U \in \mathcal{U}$. *Then we can immediately obtain the following result:*

Lemma 3.11. *For any* $U_{ij} \in \mathcal{U}_{ij}$ *with* $1 \le i \le j \le 2$, *the following statements hold:*

(1) $\Delta(U_{ij}) \in \mathcal{U}_{ij}$;

(2) $\Delta(U_{12}) = \delta(U_{12})$ *and* Δ *is additive on* $\mathcal{U}_{12}$.

(3) $\Delta(\sum_{1\le i\le j\le 2} U_{ij}) = \sum_{1\le i\le j\le 2} \Delta(U_{ij})$;

Lemma 3.12. Δ *is additive.*

Proof. By Lemma 3.11, it is sufficient to show that Δ is additive on $\mathcal{U}_{ii}$, $i = 1, 2$. In fact, for any $U_{11}, V_{11} \in \mathcal{U}_{11}$, in view of Theorem 3.1 and Lemma 3.11, we have

$$\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}) \in \mathcal{Z}(\mathcal{U}) \cap \mathcal{U}_{11}.$$

Assume that

$$Z = W_{11} = \Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11})$$

for some $Z \in \mathcal{Z}(\mathcal{U})$ and $W_{11} \in \mathcal{U}_{11}$. Thus $P_2Z = 0$ and then by the definition of $\mathcal{Z}(\mathcal{U})$, it follows that $Z = 0$, which means that Δ is additive on $\mathcal{U}_{11}$. Similarly, we can also show that Δ is additive on $\mathcal{U}_{22}$. The proof is completed. □

Lemma 3.13. (1) $\Delta(U_{11}U_{12}) = \Delta(U_{11})U_{12} = U_{11}\Delta(U_{12})$;

(2) $\Delta(U_{12}U_{22}) = \Delta(U_{12})U_{22} = U_{12}\Delta(U_{22})$.

Proof. Here we only give the proof of (1). The proof of (2) is similar. Since $U_{12}U_{11}P_1 = 0$, we have from Lemma 3.11 that

$$\begin{aligned}\Delta(U_{11}U_{12}) =&\delta(U_{11}U_{12})\\ =&\delta(p_n(U_{12}, U_{11}, P_1, P_2, ..., P_2))\\ =&p_n(\delta(U_{12}), U_{11}, P_1, P_2, ..., P_2)\\ =&p_n(\Delta(U_{12}), U_{11}, P_1, P_2, ..., P_2)\\ =&U_{11}\Delta(U_{12}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta(U_{11}U_{12}) =&\delta(U_{11}U_{12})\\ =&\delta(p_n(U_{12}, U_{11}, P_1, P_2, ..., P_2))\\ =&p_n(U_{12}, \delta(U_{11}), P_1, P_2, ..., P_2)\\ =&p_n(U_{12}, \Delta(U_{11}), P_1, P_2, ..., P_2)\\ =&\Delta(U_{11})U_{12}.\end{aligned}$$

□

Lemma 3.14. *Let* $U_{ii}, V_{ii} \in U_{ii}$ *for* $i = 1, 2$. *Then*

$$\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} = U_{ii}\Delta(V_{ii}).$$

Proof. Let $W_{12} \in \mathcal{U}_{12}$. In view of Lemma 3.13, we have

$$\Delta(U_{11}V_{11}W_{12}) = \Delta(U_{11}V_{11})W_{12};$$

$$\Delta(U_{11}V_{11}W_{12}) = \Delta(U_{11})V_{11}W_{12};$$

$$\Delta(U_{11}V_{11}W_{12}) = U_{11}\Delta(V_{11}W_{12}) = U_{11}\Delta(V_{11})W_{12}.$$

Thus $\Delta(U_{11}V_{11})W_{12} = \Delta(U_{11})V_{11}W_{12} = U_{11}\Delta(V_{11})W_{12}$ for all $W_{12} \in \mathcal{U}_{12}$. Since $\mathcal{U}_{12}$ is right faithful $\mathcal{U}_{11}$ module, it follows that

$$\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} = U_{11}\Delta(V_{11}).$$

Similarly, we can also show that

$$\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} = U_{22}\Delta(V_{22}).$$

□

Finally. we give the proof of Theorem 3.7.

Proof. Using Lemmas 3.12-3.14, one can easily show that $\Delta(UV) = U\Delta(V) = \Delta(U)V$ for all $U, V \in \mathcal{U}$. Set $Z = \Delta(1)$. Then, for any $U \in \mathcal{U}$, we can see that

$$\Delta(U) = \Delta(1U) = \Delta(1)U = ZU$$

and

$$\Delta(U) = \Delta(U1) = U\Delta(1) = UZ.$$

Hence $Z \in \mathcal{Z}(\mathcal{U})$. Up to now, by Remark 3.10, we can define a map $\tau : U \to \mathcal{Z}(\mathcal{U})$ as

$$\tau(U) = \delta(U) - \Delta(U)$$

for any $U \in \mathcal{U}$. Now, to prove Theorem 3.7, it is sufficient to show that τ vanishes on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$. Indeed, suppose that $p_n(U_1, U_2, U_3, ..., U_n)$ satisfies $U_1U_2U_3 = 0$. Then

$$\begin{aligned}
&\tau(p_n(U_1, U_2, U_3, ..., U_n))\\
=&\delta(p_n(U_1, U_2, U_3, ..., U_n)) - \Delta(p_n(U_1, U_2, U_3, ..., U_n))\\
=&p_n(\delta(U_1), U_2, U_3, ..., U_n)) - p_n(\Delta(U_1), U_2, U_3, ..., U_n)\\
=&p_n(\Delta(U_1), U_2, U_3, ..., U_n) - p_n(\Delta(U_1), U_2, U_3, ..., U_n) = 0.
\end{aligned}$$

The proof is completed. □

Corollary 3.15. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$. Suppose that $\delta : \mathcal{U} \to \mathcal{U}$ is a nonlinear Lie n-centralizer. Then $\delta(U) = ZU + \tau(U)$ for some $Z \in \mathcal{Z}(\mathcal{U})$ and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator.*

4. Lie type derivations by local action

4.1. Almost additivity

Theorem 4.1. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies*

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = \sum_{i=1}^{n} p_n(U_1, ..., U_{i-1}, \delta(U_i), U_{i+1}, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Then

$$\delta(U_1 + U_2) - \delta(U_1) - \delta(U_2) \in \mathcal{Z}(\mathcal{U}).$$

In the following, Theorem 4.1 will be proved by checking several lemmas.

Lemma 4.2. *$\delta(0) = 0$.*

Proof. Take $U_1 = U_2 = U_3 = ... = U_n = 0$. One can easily check that $\delta(0) = 0$. □

Lemma 4.3. *Let $U_{ij} \in \mathcal{U}_{ij}$, $1 \le i \le j \le 2$. Then*

$$\delta(U_{11} + U_{12}) - \delta(U_{11}) - \delta(U_{12}) \in \mathcal{Z}(\mathcal{U})$$

and

$$\delta(U_{12} + U_{22}) - \delta(U_{12}) - \delta(U_{22}) \in \mathcal{Z}(\mathcal{U}).$$

Proof. For any $U_{11} \in \mathcal{U}_{11}$ and $U_{12}, V_{12} \in \mathcal{U}_{12}$, denote $U = \delta(U_{11} + U_{12}) - \delta(U_{11}) - \delta(U_{12})$. Since $V_{12}(U_{11} + U_{12})P_1 = V_{12}U_{11}P_1 = V_{12}U_{12}P_1 = 0$, it follows that

$$\begin{aligned}\delta(U_{11}V_{12}) =& \delta(p_n(V_{12}, U_{11} + U_{12}, P_1, P_2, ..., P_2))\\ =& p_n(\delta(V_{12}), U_{11} + U_{12}, P_1, P_2, ..., P_2)\\ &+ p_n(V_{12}, \delta(U_{11} + U_{12}), P_1, P_2, ..., P_2)\\ &+ p_n(V_{12}, U_{11} + U_{12}, \delta(P_1), P_2, ..., P_2)\\ &+ \sum_{k=4}^{n} p_n(V_{12}, U_{11} + U_{12}, P_1, P_2, ..., \delta(P_2), ...P_2)\end{aligned}$$

and

$$\begin{aligned}\delta(U_{11}V_{12}) =& \delta(p_n(V_{12}, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(V_{12}, U_{12}, P_1, P_2, ..., P_2))\\ =& p_n(\delta(V_{12}), U_{11} + U_{12}, P_1, P_2, ..., P_2)\\ &+ p_n(V_{12}, \delta(U_{11}) + \delta(U_{12}), P_1, P_2, ..., P_2)\\ &+ p_n(V_{12}, U_{11} + U_{12}, \delta(P_1), P_2, ..., P_2)\\ &+ \sum_{k=4}^{n} p_n(V_{12}, U_{11} + U_{12}, P_1, P_2, ..., \delta(P_2), ...P_2).\end{aligned}$$

Comparing the two equations above, we have

$$p_n(V_{12}, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_1V_{12} = V_{12}P_2UP_2$ for all $V_{12} \in \mathcal{U}_{12}$. Then it follows from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Moreover, since $P_2(U_{11} + U_{12})P_1 = P_2U_{11}P_1 = P_2U_{12}P_1 = 0$, it follows that

$$\begin{aligned}\delta(U_{12}) =& \delta(p_n(P_2, U_{11} + U_{12}, P_1, P_2, ..., P_2))\\ =& p_n(\delta(P_2), U_{11} + U_{12}, P_1, P_2, ..., P_2)\\ &+ p_n(P_2, \delta(U_{11} + U_{12}), P_1, P_2, ..., P_2)\\ &+ p_n(P_2, U_{11} + U_{12}, \delta(P_1), P_2, ..., P_2)\\ &+ \sum_{k=4}^{n} p_n(P_2, U_{11} + U_{12}, P_1, P_2, ..., \delta(P_2), ..., P_2)\end{aligned}$$

and

$$\begin{aligned}\delta(U_{12}) =& \delta(p_n(P_2, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(P_2, U_{12}, P_1, P_2, ..., P_2))\\ =& p_n(\delta(P_2), U_{11} + U_{12}, P_1, P_2, ..., P_2)\\ &+ p_n(P_2, \delta(U_{11}) + \delta(U_{12}), P_1, P_2, ..., P_2)\\ &+ p_n(P_2, U_{11} + U_{12}, \delta(P_1), P_2, ..., P_2)\\ &+ \sum_{k=4}^{n} p_n(P_2, U_{11} + U_{12}, P_1, P_2, ..., \delta(P_2), ..., P_2)\end{aligned}$$

Comparing the equations above, we have

$$p_n(P_2, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore, we finally get

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

In the similar manner, we can also show that

$$\delta(U_{12} + U_{22}) - \delta(U_{12}) - \delta(U_{22}) \in \mathcal{Z}(\mathcal{U}).$$

□

Lemma 4.4. *For all $U_{12}, V_{12} \in \mathcal{U}_{12}$, we have*

$$\delta(U_{12} + V_{12}) = \delta(U_{12}) + \delta(V_{12}).$$

Proof. Since $(-U_{12} - P_1)(P_2 + V_{12})P_1 = 0$, it follows from Lemmas 4.2 and 4.3 that

$$\begin{aligned}
\delta(U_{12} + V_{12}) =& \delta(p_n(-U_{12} - P_1, P_2 + V_{12}, P_1, P_2, ..., P_2))\\
=& p_n(\delta(-U_{12}) + \delta(-P_1), P_2 + V_{12}, P_1, P_2, ..., P_2)\\
&+ p_n(-U_{12} - P_1, \delta(P_2) + \delta(V_{12}), P_1, P_2, ..., P_2)\\
&+ p_n(-U_{12} - P_1, P_2 + V_{12}, \delta(P_1), P_2, ..., P_2)\\
&+ \sum_{k=4}^{n} p_n(-U_{12} - P_1, P_2 + V_{12}, P_1, P_2, ..., \delta(P_2), ..., P_2)\\
=& \delta(p_n(-U_{12}, P_2, P_1, P_2, ..., P_2)) + \delta(p_n(-P_1, P_2, P_1, P_2, ..., P_2))\\
&+ \delta(p_n(-U_{12}, V_{12}, P_1, P_2, ..., P_2)) + \delta(p_n(-P_1, V_{12}, P_1, P_2, ..., P_2))\\
=& \delta(U_{12}) + \delta(V_{12}).
\end{aligned}$$

□

Lemma 4.5. *Let $U_{ii}, V_{ii} \in \mathcal{U}_{ii}$ for $i = 1, 2$. Then $\delta(U_{ii} + V_{ii}) - \delta(U_{ii}) - \delta(V_{ii}) \in \mathcal{Z}(\mathcal{U})$.*

Proof. Here we only give the proof for $i = 1$. The proof for $i = 2$ is similar. Denote $U = \delta(U_{11} + V_{11}) - \delta(U_{11}) - \delta(V_{11})$. Since $W_{12}U_{11}P_1 = W_{12}V_{11}P_1 = W_{12}(U_{11} + V_{11})P_1 = 0$, it follows from Lemma 4.4 that

$$\begin{aligned}
\delta((U_{11} + V_{11})W_{12}) =& \delta(p_n(W_{12}, U_{11} + V_{11}, P_1, P_2, ..., P_2))\\
=& p_n(\delta(W_{12}), U_{11} + V_{11}, P_1, P_2, ..., P_2)\\
&+ p_n(W_{12}, \delta(U_{11} + V_{11}), P_1, P_2, ..., P_2)\\
&+ p_n(W_{12}, U_{11} + V_{11}, \delta(P_1), P_2, ..., P_2)\\
&+ \sum_{k=4}^{n} p_n(W_{12}, U_{11} + V_{11}, P_1, P_2, ..., \delta(P_2), ..., P_2)
\end{aligned}$$

and

$$\begin{aligned}
\delta((U_{11} + V_{11})W_{12}) =& \delta(U_{11}W_{12} + V_{11}W_{12})\\
=& \delta(U_{11}W_{12}) + \delta(V_{11}W_{12})\\
=& \delta(p_n(W_{12}, U_{11}, P_1, P_2, ..., P_2)) + \delta(p_n(W_{12}, V_{11}, P_1, P_2, ..., P_2))\\
=& p_n(\delta(W_{12}), U_{11} + V_{11}, P_1, P_2, ..., P_2)\\
&+ p_n(W_{12}, \delta(U_{11}) + \delta(V_{11}), P_1, P_2, ..., P_2)\\
&+ p_n(W_{12}, U_{11} + V_{11}, \delta(P_1), P_2, ..., P_2)\\
&+ \sum_{k=4}^{n} p_n(W_{12}, U_{11} + V_{11}, P_1, P_2, ..., \delta(P_2), ..., P_2).
\end{aligned}$$

Comparing the two equations above, we have

$$p_n(W_{12}, U, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UW_{12} = W_{12}UP_2$ for all $W_{12} \in \mathcal{U}_{12}$. Therefore, we can obtain from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Since $(U_{11} + V_{11})P_2P_2 = U_{11}P_2P_2 = V_{11}P_2P_2 = 0$, it follows from Lemma 4.2 that

$$\begin{aligned}
0 =& \delta(p_n(U_{11} + V_{11}, P_2, P_2, P_1 ..., P_1)) \\
=& p_n(\delta(U_{11} + V_{11}), P_2, P_2, P_1 ..., P_1) \\
&+ p_n(U_{11} + V_{11}, \delta(P_2), P_2, P_1 ..., P_1) \\
&+ p_n(U_{11} + V_{11}, P_2, \delta(P_2), P_1 ..., P_1) \\
&+ \sum_{k=4}^{n} p_n(U_{11} + V_{11}, P_2, P_2, P_1 ..., \delta(P_1), ..., P_1)
\end{aligned}$$

and

$$\begin{aligned}
0 =& \delta(p_n(V_{11}, P_2, P_2, P_1 ..., P_1)) + \delta(p_n(V_{11}, P_2, P_2, P_1 ..., P_1)) \\
=& p_n(\delta(U_{11}) + \delta(V_{11}), P_2, P_2, P_1 ..., P_1) \\
&+ p_n(U_{11} + V_{11}, \delta(P_2), P_2, P_1 ..., P_1) \\
&+ p_n(U_{11} + V_{11}, P_2, \delta(P_2), P_1 ..., P_1) \\
&+ \sum_{k=4}^{n} p_n(U_{11} + V_{11}, P_2, P_2, P_1 ..., \delta(P_1), ..., P_1).
\end{aligned}$$

Comparing the equations above, we have

$$p_n(U, P_2, P_2, P_1 ..., P_1) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore,

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

□

Lemma 4.6. *For all $U_{ij} \in \mathcal{U}_{ij}$ with $1 \le i \le j \le 2$, we have*

$$\delta(\sum_{1 \le i \le j \le 2} U_{ij}) - \sum_{1 \le i \le j \le 2} \delta(U_{ij}) \in \mathcal{Z}(\mathcal{U}).$$

Proof. Denote $U = \delta(\sum_{1 \le i \le j \le 2} U_{ij}) - \sum_{1 \le i \le j \le 2} \delta(U_{ij})$. Since $\sum_{1 \le i \le j \le 2} U_{ij} V_{12} P_1 = 0$, it follows from Lemma 4.4 that

$$\begin{aligned}
\delta(V_{12}U_{22} - U_{11}V_{12}) =& \delta(p_n(\sum_{1 \le i \le j \le 2} U_{ij}, V_{12}, P_1, P_2, ..., P_2)) \\
=& p_n(\delta(\sum_{1 \le i \le j \le 2} U_{ij}), V_{12}, P_1, P_2, ..., P_2) \\
&+ p_n(\sum_{1 \le i \le j \le 2} U_{ij}, \delta(V_{12}), P_1, P_2, ..., P_2) \\
&+ p_n(\sum_{1 \le i \le j \le 2} U_{ij}, V_{12}, \delta(P_1), P_2, ..., P_2) \\
&+ \sum_{k=4}^{n} p_n(\sum_{1 \le i \le j \le 2} U_{ij}, V_{12}, P_1, P_2, ..., \delta(P_2), ..., P_2)
\end{aligned}$$

and

$$\begin{aligned}\delta(V_{12}U_{22} - U_{11}V_{12}) =&\delta(V_{12}U_{22}) + \delta(-U_{11}V_{12})\\ =& \sum_{1\leq i\leq j\leq 2} \delta(p_n(U_{ij}, V_{12}, P_1, P_2, ..., P_2))\\ =&p_n(\sum_{1\leq i\leq j\leq 2} \delta(U_{ij}), V_{12}, P_1, P_2, ..., P_2)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, \delta(V_{12}), P_1, P_2, ..., P_2)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, V_{12}, \delta(P_1), P_2, ..., P_2)\\ &+ \sum_{k=4}^{n} p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, V_{12}, P_1, P_2, ..., \delta(P_2), ..., P_2).\end{aligned}$$

Comparing the equations above, we have

$$p_n(U, V_{12}, P_1, P_2, ..., P_2) = 0,$$

which implies that $P_1UP_1V_{12} = V_{12}P_2UP_2$ for all $V_{12} \in \mathcal{U}_{12}$. Then it follows from Proposition 2.1 that

$$P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

Since $\sum_{1\leq i\leq j\leq 3} U_{ij}(-P_1)P_2 = 0$, it follows that

$$\begin{aligned}\delta((-1)^{n+1}U_{12}) =&\delta(p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, -P_1, P_2, P_1, ..., P_1))\\ =&p_n(\delta(\sum_{1\leq i\leq j\leq 2} U_{ij}), -P_1, P_2, P_1, ..., P_1)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, \delta(-P_1), P_2, P_1, ..., P_1)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, -P_1, \delta(P_2), P_1, ..., P_1)\\ &+ \sum_{k=4}^{n} p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, -P_1, P_2, P_1, ..., \delta(P_1), ..., P_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta((-1)^{n+1}U_{12}) =& \sum_{1\leq i\leq j\leq 2} \delta(p_n(U_{ij}, -P_1, P_2, P_1, ..., P_1))\\ =&p_n(\sum_{1\leq i\leq j\leq 2} \delta(U_{ij}), -P_1, P_2, P_1, ..., P_1)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, \delta(-P_1), P_2, P_1, ..., P_1)\\ &+ p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, -P_1, \delta(P_2), P_1, ..., P_1)\\ &+ \sum_{k=4}^{n} p_n(\sum_{1\leq i\leq j\leq 2} U_{ij}, -P_1, P_2, P_1, ..., \delta(P_1), ..., P_1).\end{aligned}$$

Comparing the two equations above, we have

$$P_n(U, -P_1, P_2, P_1, ..., P_1) = 0,$$

which implies that $P_1UP_2 = 0$. Therefore,

$$U = P_1UP_1 + P_2UP_2 \in \mathcal{Z}(\mathcal{U}).$$

□

Now we give the proof of Theorem 4.1.

Proof. Let $U = \sum_{1\leq i\leq j\leq 2} U_{ij}$ and $V = \sum_{1\leq i\leq j\leq 2} V_{ij}$. Then we have from Lemmas 4.4-4.6 that there exist $Z_1, Z_2, Z_3, Z_4, Z_5 \in \mathcal{Z}(\mathcal{U})$ such that

$$\begin{aligned}
\delta(U+V) =&\delta(\sum_{1\leq i\leq j\leq 2} U_{ij} + \sum_{1\leq i\leq j\leq 2} V_{ij})\\
=&\delta(\sum_{1\leq i\leq j\leq 2} (U_{ij} + V_{ij}))\\
=&\sum_{1\leq i\leq j\leq 2} \delta(U_{ij} + V_{ij}) + Z_1\\
=&\sum_{1\leq i\leq j\leq 2} \delta(U_{ij}) + \sum_{1\leq i\leq j\leq 2} \delta(V_{ij}) + Z_1 + Z_2 + Z_3\\
=&\delta(\sum_{1\leq i\leq j\leq 2} U_{ij}) + \delta(\sum_{1\leq i\leq j\leq 2} V_{ij}) + Z_1 + Z_2 + Z_3 - Z_4 - Z_5\\
=&\delta(U) + \delta(V) + Z_1 + Z_2 + Z_3 - Z_4 - Z_5,
\end{aligned}$$

which means that $\delta(U+V) - \delta(U) - \delta(V) \in \mathcal{Z}(\mathcal{U})$. □

4.2. Structure

Theorem 4.7. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying the conditions:*

(★) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}), \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$;

(★) $\mathcal{Z}(\mathcal{A}) = \{A \mid [[A, X], Y] = 0, \forall X, Y \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \mid [[B, X], Y] = 0, \forall X, Y \in \mathcal{B}\}$.

Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = \sum_{i=1}^{n} p_n(U_1, ..., U_{i-1}, \delta(U_i), U_{i+1}, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$ and $n \geq 4$. Then $\delta(U) = d(U) + \tau(U)$ for all $U \in \mathcal{U}$, where $d : \mathcal{U} \to \mathcal{U}$ ia an additive derivation and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$.

To complete the proof of Theorem 4.7, we need to prove several lemmas.

Lemma 4.8. *$P_1\delta(P_i)P_1 + P_2\delta(P_i)P_2 \in \mathcal{Z}(\mathcal{U})$, $i = 1, 2$.*

Proof. Since $U_{12}P_1P_1 = 0$, it follows from Lemmas 4.2 and 4.4 that

$$\begin{aligned}
(-1)^{n+1}\delta(U_{12}) =&\delta((-1)^{n+1}U_{12})\\
=&\delta(p_n(U_{12}, P_1, P_1, P_1 ..., P_1))\\
=&P_n(\delta(U_{12}), P_1, P_1, P_1 ..., P_1) + \sum_{k=2}^{n} p_n(U_{12}, P_1, ..., \delta(P_1) ..., P_1)\\
=&(-1)^{n+1}P_1\delta(U_{12})P_2 + (-1)^{n+1}P_1\delta(P_1)U_{12} - (-1)^{n+1}U_{12}\delta(P_1)P_2.
\end{aligned}$$

Multiplying by P_1 on the left side and P_2 on the right side, we obtain that

$$P_1\delta(P_1)P_1U_{12} = U_{12}P_2\delta(P_1)P_2.$$

By Proposition 2.1, we have $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}(\mathcal{U})$.

On the other hand, Since $U_{12}P_1P_1 = 0$, it follows from $P_1\delta(P_1)P_1U_{12} = U_{12}P_2\delta(P_1)P_2$ that

$$\begin{aligned}\delta(U_{12}) =&\delta(p_n(U_{12}, P_1, P_1, P_2..., P_2))\\ =&p_n(\delta(U_{12}), P_1, P_1, P_2..., P_2) + p_n(U_{12}, \delta(P_1), P_1, P_2..., P_2)\\ &+ p_n(U_{12}, P_1, \delta(P_1), P_2..., P_2) + \sum_{k=4}^{n} p_n(U_{12}, P_1, P_1, P_2..., \delta(P_2), ..., P_2)\\ =&P_1\delta(U_{12})P_2 + U_{12}\delta(P_2)P_2 - \delta(P_2)U_{12}.\end{aligned}$$

Multiplying by P_1 on the left side and P_2 on the right side, we obtain that

$$P_1\delta(P_2)P_1U_{12} = U_{12}P_2\delta(P_2)P_2.$$

By Proposition 2.1, we have $P_1\delta(P_2)P_1 + P_2\delta(P_2)P_2 \in \mathcal{Z}(\mathcal{U})$. □

Remark 4.9. *Inspired by Lemma 4.8, we can define a map $\phi : \mathcal{U} \to \mathcal{U}$ by $\phi(U) = \delta(U) - [U, P_1\delta(P_1)P_2]$. One can easily check that $\phi(P_1) \in \mathcal{Z}(\mathcal{U})$ and ϕ satisfies that*

$$\phi(p_n(U_1, U_2, U_3, ..., U_n)) = \sum_{i=1}^{n} p_n(U_1, ..., U_{i-1}, \phi(U_i), U_{i+1}, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$. Therefore, ϕ satisfies all the properties that δ satisfies.

Lemma 4.10. $\phi(P_2) \in \mathcal{Z}(\mathcal{U})$.

Proof. Since $P_2P_1P_1 = 0$ and $\phi(P_1) \in \mathcal{Z}(\mathcal{U})$, it follows that

$$\begin{aligned}0 =&\phi(p_n(P_2, P_1, ..., P_1))\\ =&p_n(\phi(P_2), P_1, ..., P_1)\\ =&(-1)^{n+1}P_1\phi(P_2)P_2,\end{aligned}$$

which means that $P_1\phi(P_2)P_2 = 0$. Then we have from Remark 4.9 that $\phi(P_2) = \delta(P_2) + P_1\delta(P_2)P_2$. Thus $P_1\phi(P_2)P_1 = P_1\delta(P_2)P_1$ and $P_2\phi(P_2)P_2 = P_2\delta(P_2)P_2$. Hence,

$$\phi(P_2) = P_1\phi(P_2)P_1 + P_2\phi(P_2)P_2 = P_1\delta(P_2)P_1 + P_2\delta(P_2)P_2 \in \mathcal{Z}(\mathcal{U}).$$

□

Lemma 4.11. $\phi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

Proof. Let $U_{12} \in \mathcal{U}_{12}$. Since $\phi(P_1), \phi(P_2) \in \mathcal{Z}(\mathcal{U})$ and $U_{12}P_1P_1 = 0$, we have

$$\begin{aligned}\phi(U_{12}) =&\phi(p_n(U_{12}, P_1, P_1, P_2, ..., P_2))\\ =&p_n(\phi(U_{12}), P_1, P_1, P_2, ..., P_2)\\ =&P_1\phi(U_{12})P_2.\end{aligned}$$

□

Lemma 4.12. *There exist* $f_i : \mathcal{U}_{ii} \to \mathcal{Z}(\mathcal{U})$ *such that*

$$\phi(U_{ii}) - f_i(U_{ii}) \in \mathcal{U}_{ii}$$

for $i = 1, 2$.

Proof. Here we only give the proof for $i = 1$. Let $U_{11} \in \mathcal{U}_{11}$. Since $U_{11}P_2P_2 = 0$ and $\phi(P_2) \in \mathcal{Z}(\mathcal{U})$, It follows that

$$\begin{aligned}0 =&\phi(p_n(U_{11}, P_2, P_2, P_2, ..., P_2))\\ =&p_n(\phi(U_{11}), P_2, P_2, P_2, ..., P_2)\\ =&P_1\phi(U_{11})P_2.\end{aligned}$$

Consequently, $\phi(U_{11}) = P_1\phi(U_{11})P_1 + P_2\phi(U_{11})P_2$. Similarly, we can also show that $\phi(U_{22}) = P_1\phi(U_{22})P_1 + P_2\phi(U_{22})P_2$.

Since $U_{11}U_{22}U_{12} = 0$ and $\phi(P_2) \in \mathcal{Z}(\mathcal{U})$, we have

$$\begin{aligned}0 =&\phi(p_n(U_{11}, U_{22}, U_{12}, P_2, ..., P_2))\\ =&p_n(\phi(U_{11}), U_{22}, U_{12}, P_2, ..., P_2) + p_n(U_{11}, \phi(U_{22}, U_{12}, P_2, ..., P_2))\\ =&[[\phi(U_{11}), U_{22}], U_{12}] + [[U_{11}, \phi(U_{22})], U_{12}]\\ =&[[\phi(U_{11}), U_{22}] + [U_{11}, \phi(U_{22})], U_{12}].\end{aligned}$$

Thus we have from Proposition 2.1 that

$$[\phi(U_{11}), U_{22}] + [U_{11}, \phi(U_{22})] = Z \in \mathcal{Z}(\mathcal{U}).$$

Hence, $[P_2\phi(U_{11})P_2, U_{22}] \in ZP_2$ and $[U_{11}, P_1\phi(U_{22})P_1] \in ZP_1$. Employing the conditions

$$\mathcal{Z}(\mathcal{A}) = \{A \mid [[A, X], Y] = 0, \forall X, Y \in \mathcal{A}\}$$

and

$$\mathcal{Z}(\mathcal{B}) = \{B \mid [[B, X], Y] = 0, \forall X, Y \in \mathcal{B}\},$$

we can conclude that $P_2\phi(U_{11})P_2 \in \mathcal{Z}(\mathcal{U}_{22})$ and $P_1\phi(U_{22})P_1 \in \mathcal{Z}(\mathcal{U}_{11})$. Employing the condition $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, we can write

$$\phi(U_{11}) = P_1\phi(U_{11})P_1 - \eta^{-1}(P_2\phi(U_{11})P_2) + \eta^{-1}(P_2\phi(U_{11})P_2) + P_2\phi(U_{11})P_2.$$

Thus, we can define a map $f_1 : \mathcal{U}_{11} \to \mathcal{Z}(\mathcal{U})$ as

$$f_1(U_{11}) = \eta^{-1}(P_2\phi(U_{11})P_2) + P_2\phi(U_{11})P_2.$$

Obviously,

$$\phi(U_{11}) - f_1(U_{11}) \in \mathcal{U}_{11}.$$

Similarly, we can show that

$$\phi(U_{22}) = P_1\phi(U_{22})P_1 + \eta(P_1\phi(U_{22})P_1) + P_2\phi(U_{22})P_2 - \eta(P_1\phi(U_{22})P_1).$$

Then we can define a map $f_2 : \mathcal{U}_{22} \to \mathcal{Z}(\mathcal{U})$ as

$$f_1(U_{22}) = P_1\phi(U_{22})P_1 + \eta(P_1\phi(U_{22})P_1).$$

Obviously,

$$\phi(U_{22}) - f_2(U_{22}) \in \mathcal{U}_{22}.$$

□

Remark 4.13. *Inspired by Lemmas 4.11 and 4.12, we define a new map* $\Delta : \mathcal{U} \to \mathcal{U}$ *as*

$$\Delta(U) = \sum_{1\leq i\leq j\leq 2} \phi(U_{ij}) - f_1(U_{11}) - f_2(U_{22})$$

for any $U \in \mathcal{U}$*. Then we can immediately obtain the following result:*

Lemma 4.14. *For any* $U_{ij} \in \mathcal{U}_{ij}$ *with* $1 \leq i \leq j \leq 2$*, the following statements hold:*

(1) $\Delta(U_{ij}) \in \mathcal{U}_{ij}$;

(2) Δ *is additive on* $\mathcal{U}_{12}$ *and* $\Delta(U_{12}) = \phi(U_{12})$;

(3) $\Delta(\sum_{1\leq i\leq j\leq 2} U_{ij}) = \sum_{1\leq i\leq j\leq 2} \Delta(U_{ij})$;

Lemma 4.15. Δ *is additive.*

Proof. By Lemma 4.14, it is sufficient to show that Δ is additive on $\mathcal{U}_{ii}$, $i = 1, 2$. In fact, for any $U_{11}, V_{11} \in \mathcal{U}_{11}$, in view of Theorem 4.1, Remark 4.9 and Lemma 4.14, we have

$$\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}) \in \mathcal{Z}(\mathcal{U}) \cap \mathcal{U}_{11}.$$

Assume that

$$Z = W_{11} = \Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11})$$

for some $Z \in \mathcal{Z}(\mathcal{U})$ and $W_{11} \in \mathcal{U}_{11}$. Thus $P_2Z = 0$ and then by the definition of $\mathcal{Z}(\mathcal{U})$, we have $Z = 0$, which means that Δ is additive on $\mathcal{U}_{11}$. Similarly, we can also show that Δ is additive on $\mathcal{U}_{22}$. The proof is completed. □

Lemma 4.16. (1) $\Delta(U_{11}U_{12}) = \Delta(U_{11})U_{12} + U_{11}\Delta(U_{12})$;

(2) $\Delta(U_{12}U_{22}) = \Delta(U_{12})U_{22} + U_{12}\Delta(U_{22})$;

Proof. Here we only give the proof of (1). The proof of (2) is similar. Since $U_{12}U_{11}P_1 = 0$ and $\phi(P_1), \phi(P_2) \in \mathcal{Z}(\mathcal{U})$, it follows from Remark 4.9 and Lemma 4.14 that

$$\begin{aligned}\Delta(U_{11}U_{12}) =&\phi(U_{11}U_{12})\\ =&\phi(p_n(U_{12}, U_{11}, P_1, P_2, ..., P_2))\\ =&p_n(\phi(U_{12}), U_{11}, P_1, P_2, ..., P_2) + p_n(U_{12}, \phi(U_{11}), P_1, P_2, ..., P_2)\\ =&p_n(\Delta(U_{12}), U_{11}, P_1, P_2, ..., P_2) + p_n(U_{12}, \Delta(U_{11}), P_1, P_2, ..., P_2)\\ =&U_{11}\Delta(U_{12}) + \Delta(U_{11})U_{12}.\end{aligned}$$

□

Lemma 4.17. *Let* $U_{ii}, V_{ii} \in U_{ii}$ *for* $i = 1, 2$*. Then*

$$\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} + U_{ii}\Delta(V_{ii}).$$

Proof. Let $W_{12} \in \mathcal{U}_{12}$. In view of Lemma 4.16, on the one hand, we have

$$\Delta(U_{11}V_{11}W_{12}) = \Delta(U_{11}V_{11})W_{12} + U_{11}V_{11}\Delta(W_{12}).$$

On the other hand,

$$\begin{aligned}\Delta(U_{11}V_{11}W_{12}) =&\Delta(U_{11})V_{11}W_{12} + +U_{11}\Delta(V_{11}W_{12})\\ =&\Delta(U_{11})V_{11}W_{12} + U_{11}V_{11}\Delta(W_{12}) + U_{11}\Delta(V_{11})W_{12}.\end{aligned}$$

Comparing the two equations above, we have

$$\Delta(U_{11}V_{11})W_{12} = \Delta(U_{11})V_{11}W_{12} + U_{11}\Delta(V_{11})W_{12}.$$

Then

$$(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11}))W_{12} = 0$$

for all $W_{12} \in \mathcal{U}_{12}$. Since $\mathcal{U}_{12}$ is right faithful $\mathcal{U}_{11}$ module, it follows that

$$\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11}).$$

Similarly, we can also show that

$$\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22}).$$

□

Finally. we give the proof of Theorem 4.7.

Proof. Using Lemmas 4.15-4.17, one can easily show that $\Delta(UV) = U\Delta(V)+\Delta(U)V$ for all $U, V \in \mathcal{U}$. That is, Δ is an additive derivation of $\mathcal{U}$. Besides, by Theorem 4.1 and Remark 4.9, we can define a map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ by $\tau(U) = \phi(U) - \Delta(U)$. Thus

$$\begin{aligned}\delta(U) =&\phi(U) + [U, P_1\delta(P_1)P_2]\\ =&\Delta(U) + [U, P_1\delta(P_1)P_2] + \tau(U).\end{aligned}$$

Denote $d(U) = \Delta(U) + [U, P_1\delta(P_1)P_2]$. Then $d : \mathcal{U} \to \mathcal{U}$ is an additive derivation of $\mathcal{U}$ and $\delta(U) = d(U)+\tau(U)$. Now, to prove Theorem 4.7, it is sufficient to show that τ vanishes on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$. Indeed, suppose that $p_n(U_1, U_2, U_3, ..., U_n)$ satisfies $U_1U_2U_3 = 0$. Then

$$\begin{aligned}&\tau(p_n(U_1, U_2, U_3, ..., U_n))\\ =&\delta(p_n(U_1, U_2, U_3, ..., U_n)) - d(p_n(U_1, U_2, U_3, ..., U_n))\\ =&\sum_{k=1}^{n} p_n(U_1, U_2, U_3, ..., \delta(U_k)..., U_n)) - \sum_{k=1}^{n} p_n(U_1, U_2, U_3, ..., d(U_k)..., U_n))\\ =&\sum_{k=1}^{n} p_n(U_1, U_2, U_3, ..., \delta(U_k)..., U_n)) - \sum_{k=1}^{n} p_n(U_1, U_2, U_3, ..., \delta(U_k)..., U_n)) = 0.\end{aligned}$$

The proof is completed. □

Combining Theorem 4.7 with Theorem 2.2 in [20], we can obtain the following result.

Corollary 4.18. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying the conditions:*
($\star$) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}), \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$;
($\star$) $\mathcal{Z}(\mathcal{A}) = \{A \mid [[A, X], Y] = 0, \forall X, Y \in \mathcal{A}\}, \mathcal{Z}(\mathcal{B}) = \{B \mid [[B, X], Y] = 0, \forall X, Y \in \mathcal{B}\}$.
Suppose that a nonlinear map $\delta : \mathcal{U} \to \mathcal{U}$ satisfies

$$\delta(p_n(U_1, U_2, U_3, ..., U_n)) = \sum_{i=1}^{n} p_n(U_1, ..., U_{i-1}, \delta(U_i), U_{i+1}, ..., U_n)$$

for all $U_1, U_2, ..., U_n \in \mathcal{U}$ with $U_1U_2U_3 = 0$ and $n \geq 3$. Then $\delta(U) = d(U) + \tau(U)$ for all $U \in \mathcal{U}$, where $d : \mathcal{U} \to \mathcal{U}$ is an additive derivation and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator $p_n(U_1, U_2, U_3, ..., U_n)$ with $U_1U_2U_3 = 0$.

Finally, we have the result as follows.

Corollary 4.19. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule satisfying the conditions:*

($\star$) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}), \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$;

($\star$) $\mathcal{Z}(\mathcal{A}) = \{A \mid [[A, X], Y] = 0, \forall X, Y \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \mid [[B, X], Y] = 0, \forall X, Y \in \mathcal{B}\}$.

Suppose that $\delta : \mathcal{U} \to \mathcal{U}$ is a Lie n-derivations and $n \geq 3$. Then $\delta(U) = d(U) + \tau(U)$ for all $U \in \mathcal{U}$, where $d : \mathcal{U} \to \mathcal{U}$ is an additive derivation and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map vanishing on each $(n-1)$-th commutator.

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Declarations

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