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On graded n-1-absorbing primary ideals

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Abstract. Let G be a group with identity e, and let R be a G-graded commutative ring with $1 \neq 0$. In this paper, we introduce and investigate graded n-1-absorbing primary ideals. We establish some properties of these ideals in graded rings and provide a characterization of graded rings in which every graded semi-primary ideal is a graded 1-absorbing primary ideal.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with $1 \neq 0$. We introduce and study graded n-1-absorbing primary ideals of graded rings with n is a nonzero positive integer. First, we recall some basic properties of graded rings. For more information on graded rings and modules, we refer to [17] and [18].

Let G be a multiplication group with identity e. A ring R is called to be G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and R_e is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $x = \sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$ the sell of all homogeneous elements. Also, if $r \in R_g$ and r is unit, then $r^{-1} \in R_{g^{-1}}$. A G-graded ring $R = \bigoplus_{g\in G} R_g$ is called a crossed product if R_g contain a unit for every $g \in G$. Note that a G-crossed product $R = \bigoplus_{g\in G} R_g$ is a strongly graded ring. Let R be a G-graded ring and I an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g\in G} \left(I \cap R_g\right) = \bigoplus_{g\in G} I_g$ i.e. if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. If $R = \bigoplus_{g\in G} R_g$ and $R' = \bigoplus_{g\in G} R'_g$ are two G-graded rings, then a mapping $f : R \to R'$ with $f(1_R) = 1_{R'}$ is called a gr-homomorphism if $f\left(R_g\right) \subseteq R'_g$ for all $g \in G$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and let I be a graded ideal of R. Then the Quotient ring R/I is also G-graded ring. Indeed $R/I = \bigoplus_{g\in G} (R/I)_g$ where $(R/I)_g = \left\{x + I : x \in R_g\right\}$. With this grading, $(A/I)_e \cong A_e/I_e$. From the $I = \bigoplus_{g\in G} \left(I \cap R_g\right) = \bigoplus_{g\in G} I_g$, it follows that an ideal I of R is a graded ideal if and only if I is generated by homogeneous elements of R.

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Recall that a G-graded ring is called graded integral domain if for element $a \in h(R), a$ is not a divisor of zero. In particular, R is said graded field if every nonzero homogeneous element has a multiplicative inverse.

Let R be G-graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. The ring of fractions $S^{-1}R$ is a G-graded ring which is called the gr-ring of fractions. Indeed $S^{-1}R = \bigoplus_{g\in G} \left(S^{-1}R\right)_g$, where

$$\left(S^{-1}R\right)_g = \left\{\frac{r}{s}, r \in h(R), s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\right\}.$$

A proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$.

Let R be a G-graded ring. The graded radical of a graded ideal I, denoted by $Gr(I)$ the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some positive integer n.

The concept of 1-absorbing primary ideal was introduced by Badawi et al. [5] as a proper ideal I of a commutative ring R is called a 1-absorbing primary ideal, if whenever nonunits $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$.
Additionally, in [20] A. Yassine et al. introduced the notion of 1-absorbing prime ideal of a commutaive ring. A proper ideal is called a 1-absorbing prime, if whenever nonunits $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in I$. For more related notions, see [1]. These concepts have been extended to various versions in graded rings.
Furthermore, in [15] R. Mashhoor and A. Khaldoun defined graded primary ideal of a G-graded ring R. A proper ideal is called a garded primary, if whenever nonunits $a, b \in h(R)$ with $ab \in I$, it holds that $a \in I$ or $b \in Gr(I)$.
Moreover, in [12] M. Issoual introduced the concept of graded 1-absorbing prime ideal. A proper ideal I is called a graded 1-absorbing prime ideal, if whenever nonunits $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $c \in I$. Recently, Badawi et al. are introduced a new generalization of 1-absorbing prime ideals [6]. Let m, n be nonzero positive integers such that $m > n$, a proper ideal I of a ring R is called (m, n)-absorbing prime if whenever nonunit elements $a_1, ..., a_m \in R$ with $a_1...a_m \in I$, then $a_1...a_n \in I$ or $a_{n+1}...a_m \in I$.
In this paper, we define and discuss some properties of graded n-1-absorbing primary ideal of a G-graded ring R. Let n be a nonzero positive integer and I be a graded ideal of a G-graded ring R. We say that I is a graded n-1-absorbing primary ideal of R if whenever $a_1...a_{n+1} \in I$ for some nonunits homogeneous elements $a_1, a_2, ..., a_{n+1} \in h(R)$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in Gr(I)$. Moreover, we investigate graded rings over which every graded semi-primary ideal is graded 1-absorbing primary ideal.

2. Main Results

We start this section by the following definition.

Definition 2.1. *Let n be a nonzero positive integer, $R = \oplus_{g\in G} R_g$ a G-graded ring and $I = \oplus_{g\in G} I_g$ be a graded ideal of R such that $I \neq R$.*

1. *We say that I is a graded n-1-absorbing primary ideal of R if whenever $a_1...a_{n+1} \in I$ for some nonunits homogeneous elements*
   $a_1, a_2, ..., a_{n+1} \in h(R)$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in Gr(I)$.
2. *We say that I is a graded semi-primary ideal of R, if $Gr(I)$ is a graded prime ideal of R.*

Note that a graded 1-1-absorbing primary ideal of R is simply a graded primary ideal. We begin with some elementary results that follow directly from the definition.

Proposition 2.2. *Let n be a nonzero positive integer and $R = \oplus_{g\in G} R_g$ be a G-graded ring and I be a graded ideal of R.*

1. *Let I be a graded n-1-absorbing primary ideal, then I is a graded k-1-absorbing primary ideal of R for all $k \geq n$.*
2. *Let I be a graded prime ideal of R, then I is a graded n-1-absorbing primary ideal of R.*
3. *If J is a proper graded ideal of R with $J \subseteq I$ and I is a graded n-1-absorbing primary ideal of R, then I/J is a graded n-1-absorbing primary ideal of a G-graded ring R/J.*

In the following examples, we show that the converse of (1) and (3) in Proposition 2.2 are not true.

Example 2.3. *Let $A = \mathbb{Z}[X]$ and $L = (2, X)A$ a maximal ideal of A. Let $G = \mathbb{Z}$. Then $A = \oplus_{n\geq 0} A_n$ is a G-graded ring with $A_n = \mathbb{Z}X^n$ for $n \geq 0$. Let $S = h(A) \setminus L$. We defined $R = S^{-1}A = \oplus_{n\geq 0}(S^{-1}A)_n$, R is a G-graded local ring with maximal ideal $M = (2, X)R$ and $(S^{-1}A)_n = \{\frac{r}{s} \ / \ s \in S \ , \ r \in A_{n+deg(s)}\}$. Set $I = (8, 4X, 2X^2)R$ is a graded ideal of R. Then :*

1. *I is a graded 3-1-absorbing primary ideal of R.*
2. *I is not a graded 1-absorbing primary ideal of R.*

Proof. 1. Observe that $Gr(I) = 2R$. Let $a_1, a_2, a_3, a_4 \in h(R)$ the nonunit elements such that $a_1a_2a_3a_4 \in I$. Then, at least one of the a_i is in $2R$. If $a_i \in 2R$ for some i, $1 \leq i \leq 3$, then $a_1a_2a_3 \in I$. If $a_4 \in 2R$ then $a_4 \in Gr(I)$. We conclude that I is a graded 3-1-absorbing primary ideal of R.
2. We have $2.2.X \in I$ and $2.2 \notin I$ and $X \notin Gr(I)$. Then I is not a graded 1-absorbing primary ideal of R. □

Example 2.4. *Let $R = \mathbb{Z}[i], G = \mathbb{Z}_2$. Then R is G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$.*

Let $I = 27R$. I is not a graded prime ideal of R. Because $3, 9 \in R_0$ with $9 \cdot 3 = 27 \in I$ but neither $9 \in I$ nor $3 \in I$. Now, we will show that I is a graded 3-1-absorbing primary ideal. Let $a, b, c, d \in R_0 \cup R_1$ nonunits elements of $h(R)$ with $abcd \in I$. Then there exist $s, t \in \mathbb{Z}$ with $abcd = 27(s + it)$. If $a, b, c, d \in R_0$ we have $t = 0$ and $27 \mid abcd$. Suppose that $d \notin Gr(I) = 3R$. We conclude that $27 \mid abc$; and hence $abc \in I$. Now if $a, b, c \in R_0, d \in R_1$, then $s = 0$ and $27 \mid abck$ where $d = ik$ and $k \in \mathbb{Z}$. So we come back to the first case, as well as the other remaining cases. We conclude that $abc \in I$ or $d \in Gr(I)$.

Proposition 2.5. *Let n be a nonzero positive integer and R be a G-graded ring and I be a graded n-1-absorbing primary ideal of R, then $Gr(I)$ is a graded prime ideal of R, that is, I is graded semi-primary.*

Proof. Let $a, b \in h(R)$ with $ab \in Gr(I)$, since $a, b \in h(R)$, so there exists $g_1, g_2 \in G$ such that $a \in R_{g_1}$ and $b \in R_{g_2}$, then $ab \in R_{g_1g_2} \subseteq h(R)$. Thus $a^mb^m \in I$ for some positive integer m. Since I is a graded n-1-absorbing primary ideal of R and $a^{n-1}a^mb^m \in I$, hence $a^{n+m-1} \in I$ or $b^m \in Gr(I)$. If $a^{n+m-1} \in I$, then $a \in Gr(I)$. Otherwise we have that $b^{mk} \in I$ for some positive integer k, consequently $b \in Gr(I)$. Therefore, $Gr(I)$ is a graded prime ideal of R. □

Now, we show that the converse of this proposition is not true.

Example 2.6. *Consider the $\mathbb{Z}$-graded ring $R = k[X, Y, Z]$ with standard grading, where k is any field and $I = \left\langle XYZ, X^2 \right\rangle$ is a graded ideal of R generated by homogeneous elements XYZ, X^2. Also, note that $Gr(I) = \langle X \rangle$, which is graded prime ideal of R, hence I is garded semi-primary ideal of R. Since $XYZ \in I$ but $XY \notin I$ and $Z \notin Gr(I)$, then I is not a graded 1-absorbing primary ideal of R.*

Proposition 2.7. *Let n be a nonzero positive integer and R be a G-graded ring and I be a graded n-1-absorbing primary ideal of R. Let $c \in h(R) \backslash I$. Then $(I : c)$ is a graded $(n-1)$-1-absorbing primary ideal of R.*

Proof. Let $a_1, a_2, ..., a_n \in h(R)$ with $a_1a_2...a_n \in (I : c)$. Suppose that $a_1a_2...a_{n-1} \notin (I : c)$, hence $ca_1a_2...a_{n-1} \notin I$. Since $ca_1a_2...a_n \in I$ and I is a graded n-1-absorbing primary ideal of R, then $a_n \in Gr(I) \subseteq Gr((I : c))$. We conclude that $(I : c)$ is a graded $(n-1)$-1-absorbing primary ideal of R. □

In the following, we present a graded version of the result established in [5, Theorem 3].

Theorem 2.8. *Let n be a positive integer. Let R be a G-graded ring which admits a graded $(n + 1)$-1-absorbing primary ideal that is not a graded n-1-absorbing primary ideal. Then R is a G-graded local ring.*

Proof. Suppose that I is a graded $(n+1)$-1-absorbing primary ideal that is not a graded n-1-absorbing primary ideal of R. Then there exist $a_1, ..., a_{n+1} \in h(R)$ with $a_1a_2...a_{n+1} \in I$ and neither $a_1a_2...a_n \in I$ nor $a_{n+1} \in Gr(I)$. Let c be a nonunit element of R_e hence $ca_1a_2...a_{n+1} \in I$, since $c \in h(R)$ and $a_{n+1} \notin Gr(I)$ then $ca_1a_2...a_n \in I$. Now let $d \in R_e$ a unit element of $h(R)$. Suppose that $d + c \in R_e \subseteq h(R)$ is a nonunit element. Since $(d + c)a_1a_2...a_{n+1} \in I$ and $a_{n+1} \notin Gr(I)$ and I is a graded $(n + 1)$-1-absorbing primary ideal, then $(d + c)a_1a_2...a_n \in I$. We know that $a_1a_2...a_nc \in I$, hence $da_1a_2...a_n \in I$. Since d is a unit, therefore $a_1a_2...a_n \in I$, a contradiction. Thus $d + c$ is a unit. By [5, Lemma 1] and [12, Lemma 2.7], we deduce that R is a G-graded local ring. □

As a consequence, we obtain a graded version of the corresponding corollary given in [5, Theorem 4].

Corollary 2.9. *Let n be a positive integer such that $n \geq 2$. Suppose that R is G-graded ring which is not a G-graded local ring. Let I be a graded ideal of R. Then the followings are equivalent:*

1. *I is a graded n-1-absorbing primary ideal of R.*
2. *I is a graded primary ideal of R.*

Next, we give a method to construct a graded n-1-absorbing primary ideal that are not a graded $(n-1)$-1-absorbing primary ideal.

Theorem 2.10. *Let n be a nonzero positive integer such that $n \geq 2$ and R be a G-graded local ring with graded maximal ideal M. Let x be a nonzero homogeneous prime element of R such that $Gr(xM^{n-1}) \subsetneq M$. If $x \in Gr(xM^{n-1})$, then xM^{n-1} is a graded n-1-absorbing primary ideal of R. If $xM^{n-1} \neq xM^{n-2}$, then xM^{n-1} is not a graded $(n-1)$-1-absorbing primary ideal of R.*

Proof. First, we will show that xM^{n-1} is a graded n-1-absorbing primary ideal of R. Assume that $a_1a_2...a_{n+1} \in xM^{n-1}$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in h(R)$. If $a_1a_2...a_n \notin xM^{n-1}$, then for all $1 \leq i \leq n$, $a_i \notin xR$. So $a_1a_2...a_n \notin xR$ because x is a homogeneous prime element of R. Moreover, the fact that $a_1a_2...a_{n+1} \in xR$ and $a_1a_2...a_n \notin xR$ implies that $a_{n+1} \in xR \subseteq Gr(xM^{n-1})$. Now, we prove that xM^{n-1} is not a graded $(n$-1)-1-absorbing primary ideal of R. Since, $xM^{n-1} \neq xM^{n-2}$, there exist some nonunit elements $a_2, ..., a_{n-1} \in h(R)$ such that $xa_2...a_{n-1} \notin xM^{n-1}$. Let $m \in M \backslash Gr(xM^{n-1})$, then $xa_2...a_{n-1}m \in xM^{n-1}$, therefore xM^{n-1} is not a graded $(n-1)$-1-absorbing primary ideal of R. □

Now, we give a characterization of graded n-1-absorbing primary ideals.

Proposition 2.11. *Let n be a nonzero positive integer and R be a G-graded ring, I be a proper graded ideal of R. If I is a graded n-1-absorbing primary ideal of R, then either I is a graded semi-primary ideal of R or R is graded local with graded maximal ideal M, such that $M^n \subseteq I$.*

Proof. If R is not graded local, then Corollary 2.9 implies that I is graded primary, in particular I is a graded semi-primary ideal of R. Now, assume that R is graded local with graded maximal ideal M such that I is not a graded semi-primary ideal of R, then there exist some nonunits elements of $h(R)$, $a_1, a_2 \in M$ such that $a_1a_2 \in I$ and $a_1 \notin Gr(I)$ and $a_2 \notin Gr(I)$. To prove that $M^n \subseteq I$, it suffices to show that $x_1x_2...x_n \in I$ for every $x_1, x_2, ..., x_n \in M$. Let $x_1, x_2, ..., x_n \in M$. Then $x_1x_2...x_na_1a_2 \in I$. Since $a_2 \notin Gr(I)$ and I is a graded n-1-absorbing primary ideal, we conclude that $x_1x_2...x_na_1 \in I$. Again, since $a_1 \notin Gr(I)$ and I is a graded n-1-absorbing primary ideal, we have that $x_1x_2...x_n \in I$. □

Recall that a ring R is a G-graded chained ring if the set of all graded ideals of R is linearly ordered by inclusion. Now, we determine the graded n-1-absorbing primary ideals of a graded chained ring.

Theorem 2.12. *Let R be a G-graded chained ring with graded maximal ideal M, I be a graded n-1-absorbing primary ideal for some positive integer $n \geq 2$. If I is not graded semi-primary, then $I = M^k$ for some positive integer k, $2 \leq k \leq n$.*

Proof. By Proposition 2.11 R is a G-graded local with graded maximal ideal M such that $M^n \subseteq I$. Let $k = min\{i/M^i \subseteq I\}$, we show that $I = M^k$. Suppose that $M^k \subsetneq I$. Thus there is $a \in M^{k-1} \backslash I$ and $b \in I \backslash M^k$ with $a, b \in h(R)$. Since R is a G-graded chained ring, we conclude that $b \in aR$. Hence $b = ar$ for some $r \in M$. Thus $b \in M^k$, a contradiction. Hence $I = M^k$. □

Recall that a G-graded ring R is called divided if for every graded prime ideal P of R and for every $x \in h(R) \setminus P$, we have $x|p$ for every $p \in P$.
In the next theorem, we extend the corresponding result in [5, Theorem 10] to the context of graded rings.

Theorem 2.13. *Let R be a G-graded divided ring and n be nonzero positive integer. Then a proper graded ideal I of R is a graded n-1-absorbing primary ideal of R if and only if I is a graded primary ideal of R*

Proof. It is clear that every graded primary ideal of R is a graded n-1-absorbing primary ideal of R. Hence, assume that I is a graded n-1-absorbing primary ideal of R. Suppose that $xy \in I$ for some nonunit elements x, y of $h(R)$ such that $y \notin Gr(I)$. Since $Gr(I)$ is a graded prime ideal of R by Proposition 2.5, we conclude that $x \in Gr(I)$. As R is divided, then $y^{n-1}|x$. Thus $x = y^{n-1}w$ for some $w \in R$. Since $y^{n-1} \notin Gr(I)$ and $x \in Gr(I)$, we conclude that w is a nonunit element of R. Since $xy = y^{n-1}wy \in I$ and I is a graded n-1-absorbing primary ideal of R. Then $x = y^{n-1}w \in I$. □

Proposition 2.14. *Let n be a nonzero positive integer and R be a G-graded ring, I a proper graded ideal of R. Then I is a graded n-1-absorbing primary ideal if and only if whenever $I_1I_2...I_{n+1} \subseteq I$ for some proper graded ideals $I_1, I_2, ..., I_{n+1}$ of R, then $I_1...I_n \subseteq I$ or $I_{n+1} \subseteq Gr(I)$.*

Proof. It suffices to prove the "if" assertion. Suppose that I is a graded n-1-absorbing primary ideal and let I_1, I_2, ..., I_{n+1} be proper graded ideals of R such that $I_1I_2...I_{n+1} \subseteq I$ and $I_{n+1} \nsubseteq Gr(I)$. Thus $a_1a_2...a_{n+1} \in I$ for every $a_i \in I_i$, with $1 \leq i \leq n$ and $a_{n+1} \in I_{n+1} \setminus Gr(I)$. Since I is a graded n-1-absorbing primary ideal, we then have $I_1I_2...I_n \subseteq I$, as desired. □

Proposition 2.15. *Let n be a nonzero positive integer and R be a G-graded ring. Then the following statements are equivalent.*

(1) *Every proper graded principal ideal is a graded n-1-absorbing primary ideal of R.*
(2) *Every proper graded ideal is a graded n-1-absorbing primary ideal of R.*

Proof. Assume that (1) holds and let I be a proper graded ideal of R. Let $a_1, a_2, ..., a_{n+1}$ be nonunit elements of $h(R)$ such that $a_1a_2...a_{n+1} \in I$. Hence $a_1a_2...a_{n+1} \in a_1a_2...a_{n+1}R$ which implies that $a_1a_2...a_n \in a_1a_2...a_{n+1}R \subseteq I$ or $a_{n+1} \in Gr(a_1a_2...a_{n+1}R) \subseteq Gr(I)$. Therefore I is a graded n-1-absorbing primary ideal of R. The converse is clear. □

Let R be a G-graded ring, recall that the jacobson radical, denote by $J(R)$, is the intersection of all graded maximal ideals of R.

Theorem 2.16. *Let n be a nonzero positive integer and R be a G-graded ring. Suppose that $Gr(xI) = xGr(I)$ for every proper graded ideal I of R and every $x \in h(R)$. The following statements are equivalent.*
(i) Every proper graded principal ideal is a graded n-1-absorbing primary ideal of R.
(ii) Every proper graded ideal is a graded n-1-absorbing primary ideal of R.
(iii) R is local with $Jac(R)^n = (0)$.

Proof. (*i*) ⇔ (*ii*) Follows from Proposition 2.15.
(*ii*) ⇒ (*iii*) We will show that R is a graded local ring. Choose graded maximal ideals M_1, M_2 of R. Now, put $I = M_1 \cap M_2$. Since $M_1^n M_2 \subseteq I$ and I is a graded n-1-absorbing primary ideal, we have either $M_1^n \subseteq I \subseteq M_2$ or $M_2 \subseteq Gr(I) \subseteq Gr(M_1)$.
Case 1: Suppose that $M_1^n \subseteq M_2$. Since M_2 is a graded prime, clearly we have $M_1 \subseteq M_2$ which implies that $M_1 = M_2$.
Case 2: Suppose that $M_2 \subseteq Gr(M_1)$. Since $Gr(M_1)$ is proper. As $M_1 \subseteq Gr(M_1)$ and M_1 is a graded maximal ideal, we have $M_1 = Gr(M_1)$. Then we get $M_2 \subseteq M_1$, which implies that $M_1 = M_2$. Therefore, R is a graded local ring.
Now, we will prove that $Jac(R)^n = (0)$. We may assume that $Jac(R) \neq 0$. So, let $x_1, x_2, ..., x_n \in Jac(R)$ and we choose $z \in Jac(R) \setminus (0)$. Since $x_1x_2...x_nz \in (x_1x_2...x_nz)$ and $(x_1x_2...x_nz)$ is a graded n-1-absorbing primary ideal of

R, we get $x_1x_2...x_n \in (x_1x_2...x_nz)$ or $z \in Gr(x_1x_2...x_nz) = zGr(x_1x_2...x_n)$. First, assume that $x_1x_2...x_n \in (x_1x_2...x_nz)$. Then there exists $r \in R$ such that $x_1...x_n = rx_1...x_nz$, which implies that $x_1...x_n(1-rz) = 0$. Since $1-rz$ is unit, we have $x_1x_2...x_n = 0$ which completes the proof. Now, assume that $x_1...x_n \notin (x_1...x_nz)$, that is, $z \in zGr(x_1x_2...x_n)$. Then there exists $a \in Gr(x_1x_2...x_n) \subseteq Jac(R)$ such that $z = za$. This implies that $z(1-a) = 0$ so that $z = 0$, a contradiction. Therefore, $Jac(R) = (0)$.

$(iii) \Rightarrow (i)$ Suppose that R is a graded local ring with $Jac(R)^n = (0)$. Let I be a proper graded ideal of R and $a_1a_2...a_{n+1} \in I$ for some nonunits $a_1, a_2, ..., a_{n+1} \in h(R)$. Then $a_1, a_2, ..., a_{n+1} \in Jac(R)$ since R is local. As $Jac(R)^n = (0)$, we have $a_1a_2...a_n = 0 \in I$. Therefore, I is a graded n-1-absorbing primary ideal of R. □

For G-graded rings R and R', a G-graded ring homomorphism $\varphi : R \to R'$ is a ring homomorphism such that $\varphi\left(R_g\right) \subseteq R'_g$ for every $g \in G$.

Theorem 2.17. *Let n be a nonzero positive integer and R, R' be two G-graded rings and $\varphi : R \to R'$ be a G-graded ring homorphism with $\varphi(1) = 1$. Then the following statements hold:*

1. *Assume that for every nonunits homogeneous elements r of $h(R)$, we have $\varphi(r)$ is nonunit element of $h(R')$. Then if I' is graded n-1-absorbing primary ideal of R', then $\varphi^{-1}(I')$ is a graded n-1-absorbing primary ideal of R.*
2. *If φ is a graded epimorphism and I is a graded n-1-absorbing primary ideal of R containing $\ker(\varphi)$, then $\varphi(I)$ is a graded n-1-absorbing primary ideal of R'.*

Proof.
1. Suppose that I' is a graded n-1-absorbing primary ideal of R' and let $x_1, x_2, ..., x_{n+1}$ a nonunits homogeneous elements of $h(R)$ such that $x_1...x_{n+1} \in \varphi^{-1}(I')$, then $\varphi(x_1...x_{n+1}) = \varphi(x_1)...\varphi(x_{n+1}) \in I'$. Since $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{n+1})$ are a nonunits homogeneous elements of $h(R')$ by assumption and I' is a graded n-1-absorbing primary ideal of R', we conclude that either $\varphi(x_1)...\varphi(x_n) \in I'$ or $\varphi(x_{n+1}) \in Gr(I')$, and hence either $x_1...x_n \in \varphi^{-1}(I')$ or $x_{n+1} \in Gr(\varphi^{-1}(I'))$. Therefore $\varphi^{-1}(I')$ is a graded n-1-absorbing primary ideal of R.
2. Suppose that I is a graded n-1-absorbing primary ideal of R containing $\ker(\varphi)$ and $x'_1x'_2...x'_{n+1} \in \varphi(I)$ for some nonunit $x'_1, ..., x'_{n+1} \in h(R')$. Since φ is a graded epimorphism, there exist $x_1, x_2, ...x_{n+1} \in h(R)$ such that $x'_1 = \varphi(x_1), x'_2 = \varphi(x_2), ..., x'_{n+1} = \varphi(x_{n+1})$. Since $\varphi(1) = 1$, we have $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{n+1})$ are a nonunits homogeneous elements of $h(R')$. Now we have $\varphi(x_1...x_{n+1}) = x'_1...x'_{n+1} \in \varphi(I)$. Since $\ker(\varphi) \subseteq I$, we have $x_1...x_{n+1} \in I$. Since I is a graded n-1-absorbing primary ideal of R, we obtain either $x_1...x_n \in I$ or $x_{n+1} \in Gr(I)$. So either $x'_1...x'_n \in \varphi(I)$ or $x'_{n+1} \in Gr(\varphi(I))$. Thus $\varphi(I)$ is a graded n-1-absorbing primary ideal of R'.

□

Theorem 2.18. *Let n be a nonzero positive integer and R be a G-graded ring, $S \subseteq h(R)$ be a multiplicatively closed subset of R, and I be a proper graded ideal of R. Then the following hold:*

1. *If I is a graded n-1-absorbing primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a graded n-1-absorbing primary ideal of a graded ring $S^{-1}R$.*
2. *If $S^{-1}I$ is a graded n-1-absorbing primary ideal of a graded ring $S^{-1}R$ such that $S \cap G - Z_I(R) = \emptyset$, then I is a graded n-1-absorbing primary ideal of R.*

Proof.
1. Suppose that $\frac{r_1}{s_1}\frac{r_2}{s_2}...\frac{r_{n+1}}{s_{n+1}} \in S^{-1}I$ for some nonunits homogeneous elements of $S^{-1}R$. Then there exist $v \in S$ such that $0 \neq vr_1r_2...r_{n+1} \in I$. Since I is a graded n-1-absorbing primary ideal of R and $vr_1 \in h(R)$, then either $vr_1...r_n \in I$ or $r_{n+1} \in Gr(I)$, then $\frac{vr_1...r_n}{vs_1...s_n} \in S^{-1}I$ or $\frac{r_{n+1}}{s_{n+1}} \in Gr(S^{-1}I)$. Hence $S^{-1}I$ is a graded n-1-absorbing primary ideal of $S^{-1}R$.
2. Suppose that $r_1...r_{n+1} \in I$ for some nonunits homogeneous elements of R. Then $\frac{r_1}{1}\frac{r_2}{1}...\frac{r_{n+1}}{1} \in S^{-1}I$. Since $S^{-1}I$ is a graded n-1-absorbing primary ideal of $S^{-1}R$, we deduce that either $\frac{r_1}{1}...\frac{r_n}{1} \in S^{-1}I$ or $\frac{r_{n+1}}{1} \in Gr(S^{-1}I)$. If $\frac{r_1}{1}...\frac{r_n}{1} \in S^{-1}I$, then $vr_1...r_n \in I$ for some $v \in S$. Since $S \cap G - Z_I(R) = \emptyset$ and $v \in S$, we conclude that $r_1...r_n \in I$. If $\frac{r_{n+1}}{1} \in Gr(S^{-1}I)$, then $tr_{n+1} \in I$ for some $t \in S$. The fact that $t \in S$, gives $t \notin G - Z_I(R)$ and hence $r_{n+1} \in I$. Therefore, I is a graded n-1-absorbing primary ideal of R.

□

3. Graded rings over which every graded semi-primary ideal is graded 1-absorbing primary

In this section, we provide a characterization of graded minimal prime ideals in a graded ring R. This serves as the graded counterpart to [16, Lemma 6 p.106]. We will utilize this result to characterize graded rings over which every graded semi-primary ideal is graded 1-absorbing primary. We say that a G-graded ring R is local graded if R is a local ring such that the maximal ideal is a graded ideal.

Definition 3.1. *We say that M is a graded multiplicative system of a G-graded ring R, if $ab \in M$ for every $a, b \in h(R) \cap M$.*

Lemma 3.2. *Let R be a G-graded ring, I a graded ideal and M be a graded multiplicative system of R.*

1. *Assume that M does not meet I. Then, M is contained in a graded maximal multiplicative system M^* which does not meet I, that is, if N is a graded multiplicative system such that $M^* \subset N$, then N contains an element of I.*
2. *Assume that I does not meet M. Then, I is contained in a graded maximal ideal P^* which does not meet M, that is, if J is an ideal such that $P^* \subset J$, then J contains an element of M. Such an graded ideal P is necessarily graded prime.*

Proof. 1) Let A be the set of all graded multiplicative systems in R which contain M but do not meet I. If B is a set of elements of A comparable under inclusion, it is clear that B is included in A. Set U is a union of all elements of B, clearly U is a graded multiplicative system contains M and does not meet I, hence is an element of A. The Maximum Principle then asserts that A has a maximal element M^*, as required in the statement of the lemma. If then M^* is properly contained in a multiplicative system N, N can not be in A and hence must contain an element of I.

2) The existence of an ideal P^* with the required maximal property follows from an application of the Maximum Principle to the set A of all graded ideals which contain I but do not meet M. There remains to show that P^* is necessarily a graded prime ideal. To this end, We show that $ab \notin P^*$ with $a \notin P^*, b \notin P^*$ and $a, b \in h(R)$. Since $a \notin P^*$, it is clear that $P^* \subset (P^*, (a))$, and because of the maximal property of P^* this implies that $(P^*, (a))$ contains an element m_1 of M. Hence $m_1 = p_1 + r_1 a$, $(p_1 \in P^*, r_1 \in h(R))$. Likewise $(P^*, (b))$ contains an element m_2 of M with $m_2 = p_2 + r_2 b$, $(p_2 \in P^*, r_2 \in h(R))$.
Since $m_1 m_2 = p_1 p_2 + p_1 r_2 b + p_2 r_1 a + r_1 r_2 ab$ and $m_1 m_2 \notin P^*$, then $ab \notin P^*$. □

Lemma 3.3. *A graded ideal P of a G-graded ring is graded minimal over a graded ideal I if and only if $C(P) = h(R) \backslash P$ is a maximal graded multiplicative system of R.*

Proof. First, Suppose that $M = C(P)$ is a maximal graded multiplicative system which does not meet I. By Lemma 3.2(2), there is a graded prime ideal P^* which contains I and does not meet M. Hence $C(P^*)$ is a multiplicative system which does not meet I and contains M. From the maximal property of M, it follows that $C(P^*) = M = C(P)$, and hence $P = P^*$. Also this maximal property shows that there is no graded prime ideal P_1 such that $I \subseteq P_1 \subset P$, as otherwise $C(P_1)$ would be a graded multiplicative system which does not meet I and contains M as proper subset. We have therefore shown that P is a graded minimal prime ideal belonging to I.
For the converse, suppose that P is a minimal graded prime ideal belonging to I. Then $M = C(P)$ is a graded multiplicative system which does not meet I, and Lemma 3.2(1) shows the existence of a maximal graded multiplicative system M' which contains M and does not meet I. By the case just proved, $C(M') = P'$ is a graded minimal prime ideal belonging to I. But since $M' \supseteq M$, it follows that $P' \subseteq P$, and the minimal property of P then shows that $P = P'$. Hence $M = M'$, and $C(P) = h(R) \backslash P$ is a maximal graded multiplicative system of R. □

Lemma 3.4. *Let (R, M) be a local G-graded ring over which every graded semi-primary ideal is graded 1-absorbing primary. Let I be a graded semi-primary ideal with $Gr(I) = P$ and P is non-graded maximal. Then, $PM \subseteq I$. In particular $P^2 \subseteq I$.*

Proof. Let M be a the unique graded maximal ideal of R and consider $m \in h(R)$ such that $m \in M \backslash P$.
Let $p \in P \cap h(R)$, we have $Gr(I + (pm^2)) = P$. Then, $I + (pm^2)$ is graded 1-absorbing primary ideal of R. Now, since $pm^2 \in I + (pm^2)$ and $m \notin P$, we obtain that $pm \in I + (pm^2)$. Hence, for some $r \in R$ we have $pm(1 - mr) \in I$. Since $1 - mr$ is unit, we conclude that $pm \in I$ and thus $PM \subseteq I$. □

Lemma 3.5. *Let R be a G-graded ring over which every graded semi-primary ideal is graded* 2-1*-absorbing primary. If r is a nonunit regular element of h(R). Suppose that P is a graded minimal prime ideal over the graded principal ideal (r), then P is graded maximal.*

Proof. Suppose that P is not graded maximal. Set

$$A = \left\{x \in h(R) \mid \text{ there exists } m \in h(R)\backslash P \text{ such that } mx \in \left(r^3\right)\right\}.$$

Since P is graded prime, A is an graded ideal and $A \subseteq P$. Let $g \in G$ and $p \in P_g$, set

$$N_p = \left\{p^i m \mid i \geq 0 \text{ and } m \in h(R)\backslash P\right\}.$$

Note that N_p is a graded multiplicative system containing $C(P) = R\backslash P$ and p. Following Lemma 3.3, $C(P)$ is a maximal graded multiplicative system which does not meet $\left(r^3\right)$(Since P is a graded minimal prime ideal of $\left(r^3\right)$) . Hence, N_p meets $\left(r^3\right)$ in an elements $p^i m = xr^3$ for some integer $i \geq 1, m \in h(R)\backslash P$, and $x \in h(R)$. Therefore, $p^i \in A$. So, $p \in Gr(A)$. Consequently, $P \subseteq Gr(A)$, and so $Gr(A) = P$. Using Lemma 3.4, we have $P_g^2 \subseteq A$. Since $r \in h(R)$, so $r^2 \in A$ ($r \in P_g$ for some $g \in G$). Thus, there exist $m \in h(R)\backslash P$ and $x \in h(R)$ such that $r^2 m = r^3 x$. Hence, since r is regular, we get $m = xr \in P$, a contradiction. Consequently, P is graded maximal. □

Definition 3.6. *The gr-dimension of a G-graded ring R is defined as the supremum of the lengths of all chains of distinct graded prime ideals of R and is denoted by gr-dim(R).*

Theorem 3.7. *Let R be a G-graded domain. Then, the following are equivalent:*

1. *Every graded semi-primary ideal is graded* 1*-absorbing primary.*
2. *gr-dim(R)=1.*
3. *Every graded semi-primary ideal of R is graded primary.*

Proof. (1) ⇒ (2) Let P be a nonzero graded prime ideal. Consider $0 \neq x \in P$. There is a minimal graded prime ideal P_1 over (x) contained in P. Using Lemma 3.4, P_1 is maximal, and so $P = P_1$. Then gr-dim(R)=1
(2) ⇒ (3) Let I be a graded semi-primary ideal of R. If $I = (0)$ then I is graded prime, and if $I \neq (0)$ then $Gr(I)$ is maximal, and so I is graded primary. □

As a consequence of Theorem 3.7, we have the next result.

Proposition 3.8. *Let R be a G- graded ring over which every graded semi-primary ideal is graded* 1*-absorbing primary. Then, every graded prime ideal of R is either graded minimal or graded maximal.*

Proof. Let P be a graded prime ideal of R which is not graded minimal. Hence, there is a graded prime ideal P_1 of R such that $P_1 \subsetneq P$. Its is easy to see that every graded semi-primary ideal of R/P_1 is graded 1-absorbing primary. Since P/P_1 is a nonzero graded prime ideal of R/P_1 and by using Theorem 3.7, we conclude that P/P_1 is a graded maximal ideal of R/P_1, and then P is a graded maximal ideal of R. □

Let R be a graded ring. R is said to be a G-UN-ring if $Gr(0)$ is the unique graded maximal ideal of R.

Theorem 3.9. *Let R be a local graded ring with the graded maximal ideal M. Then the following statements are equivalents :*

1. *Every graded semi-primary ideal is a graded* 1*-absorbing primary ideal.*

2. *Either R is a G-UN-ring or G-Spec(R)=* $\{Gr(0), M\}$ *such that* $Gr(0)M = 0$.

Proof. (1) $\Rightarrow$ (2) Assume that R is not a *G-UN*-ring. Thus, R admits a graded nonmaximal prime ideal P. Let $\{I_i\}_i$ be the set of all graded primary ideals of R such that $Gr(I_i) = M$. It is clear that $Gr(I_iP) = P$ for each i. Then, $P^2 \subseteq I_iP \subseteq I_i$ by Lemma 3.4, and hence $P^2 \subseteq \cap_i I_i$. Consider $x \in H := \cap_i I_i$. Suppose that $x \notin P$. Thus, $P \nsubseteq P + \left(x^2\right) \subseteq M$. Let P' be a graded minimal prime ideal over $P + \left(x^2\right)$. Then, $P \subsetneq P' \subseteq M$ and so $P' = M$ by Proposition 3.8. Thus, M is the unique graded minimal prime ideal over $P + \left(x^2\right)$. Hence, $GrP + \left(x^2\right) = M$ and thus $x \in P + \left(x^2\right)$ because $P + \left(x^2\right) \in \{I_i\}_i$. Thus, there exist $a \in R$ such that $x(1 - ax) \in P$. Thus, $x \in P$ since $1 - ax$ is unit, a contradiction. Hence, we conclude that $P^2 \subseteq J \subseteq P$. Likewise, if there is an other graded nonmaximal prime ideal Q, we get $Q^2 \subseteq J \subseteq Q$. Thus, $Q^2 \subseteq J \subseteq P$ and $P^2 \subseteq J \subseteq Q$. Hence, $P = Q$, a contradiction. This implies that R admits a unique nonmaximal prime ideal which is necessarily $Gr(0)$, and then $Gr(0)$ and M are the only graded prime ideals of R. Moereover, we have $Gr(0)M = \{0\}$ by using Lemma 3.4.

(2) $\Rightarrow$ (1) If R is a *G-UN*-ring then $Gr(0$ is the only graded prime ideal of R. Thus, if I is graded semi-primary, then $Gr(I) = Gr(0)$ is maximal, which implies that I is primary. Now, assume that R is not a *G-UN*-ring. Let I be a graded semi-primary ideal of R. If $Gr(I) = M$ then I is primary. Then, assume that $Gr(I) = Gr(0)$. Let $a, b, c \in h(R)$ such that $abc \in I$ and $c \notin Gr(0)$. Hence, $a \in Gr(0)$ or $b \in Gr(0)$. Therefore, $ab = 0 \in I$ and so I is a graded 1-absorbing primary ideal of R. □

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