Filomat 40:5 (2026), 1789–1796
https://doi.org/10.2298/FIL2605789C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some properties on topological gyrogroups

Tianle Chen[a,b], Jidong Guo[b], Piyu Li[a,*]

[a] *School of Mathematics and Statistics, Xuzhou Institute of Technology, Xuzhou 221018, China*
[b] *School of Mathematics and Statistics, Yili Normal University, Yining 835000, China*

Abstract. In this paper, some properties of topological gyrogroups are discussed. Among all, we show the following results: (1) It is shown that every first-countable left ω-narrow semitopological gyrogroup is separable; (2) let K a compact subset and F a closed subset of a topological gyrogroup G with $K \cap F = \emptyset$. Then there is an open neighborhood V of e in G such that $(K \oplus V) \cap F = \emptyset$, and $(V \oplus K) \cap F = \emptyset$. The two results answer the questions [9, Question 3.12] and [7, Question 4.6], respectively.

1. Introduction

A. A. Ungar [18] introduced the structure of the gyrogroup, which is an algebraic system with a binary operation. The operation process of this system does not satisfy the associative law, a fundamental operation law in the group structure. From the perspective of algebraic operations, the gyrogroup belongs to an algebraic structure type that is more general than a group, and it has many properties similar to the group structure. As an important branch of topological algebra, the topological group has extensive and profound research significance. In 2017, W. Atiponra [2], by analogy with the definition method of the topological group, first proposed the concept of the topological gyrogroup. In recent years, many scholars have carried out in-depth research around this work. In [10], Cai, Lin, and He proved that every topological gyrogroup is a rectifiable space, which implies that every first-countable Hausdorff topological gyrogroup is metrizable. Upon further research, Bao and Lin in [6] introduced the concept of strongly topological gyrogroups, and this concept has been investigated from multiple theoretical dimensions. Atiponrat and Maungchang [3] introduced the concept of paratopological gyrogroups and studied some separation axioms of paratopological gyrogroups.

In the study of topological groups, cardinal has long been used to characterize the basic properties and structural features of topological groups. In Section 3, we investigate the cardinalities of semitoplolgical (paratoplolgical) gyrogroups and show that every first-countable left ω-narrow semitopological gyrogroup is separable.

2020 *Mathematics Subject Classification*. Primary 54A20; Secondary 54E35, 54E20, 54H11, 22A05.
Keywords. topological gyrogroup, semitopological gyrogroup, neutral subgyrogroup, π-character.
Received: 30 July 2025; Revised: 15 December 2025; Accepted: 24 December 2025
Communicated by Ljubiša D. R. Kočinac
Research supported by the National Natural Science Foundation of China (No. 12361012) and University-level Project of Yili Normal University (No. YSPY2022012).
* Corresponding author: Piyu Li
Email addresses: 2657991376@qq.com (Tianle Chen), guojd662@163.com (Jidong Guo), lpy91132006@aliyun.com.cn (Piyu Li)
ORCID iDs: https://orcid.org/0009-0000-5210-7027 (Tianle Chen), https://orcid.org/0009-0007-3745-0733 (Jidong Guo), https://orcid.org/0000-0001-7537-1170 (Piyu Li)

In [5], Bao proved that in a strongly topological gyrogroup G, if K a compact subset and F a closed subset of G with $K \cap F = \emptyset$, then there exists an open neighborhood V of the neutral element in G such that $(K \oplus V) \cap F = \emptyset$, and $(V \oplus K) \cap F = \emptyset$. In Section 4, we show the same result in the case of topological gyrogroups, which gives a positive answer to [7, Question 4.6].

2. Preliminaries

In this section, we recall some necessary definitions and facts about topological gyrogroups. The readers could consult [1, 11] for notations and terminologies not explicitly given here.

Throughout this paper, all topological spaces are assumed to be T_1, unless otherwise is explicitly stated.

Definition 2.1. ([2]) *Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called magma or groupoid. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a groupoid homomorphism if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a groupoid automorphism. We write $Aut(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.*

Definition 2.2. ([21]) *Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a gyrogroup, if its binary operation satisfies the following conditions:*

(1) *There exists a unique identity element $e \in G$ such that $e \oplus a = a = a \oplus e$ for all $a \in G$.*

(2) *For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = e = x \oplus (\ominus x)$.*

(3) *For all $x, y \in G$, there exists $gyr[x, y] \in Aut(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus gyr[x, y](z)$ for all $z \in G$.*

(4) *For any $x, y \in G$, $gyr[x \oplus y, y] = gyr[x, y]$.*

Notice that a group is a gyrogroup $(G, \oplus)$ such that $gyr[x, y]$ is the identity function for all $x, y \in G$. The definition of a subgyrogroup is given as follows.

Definition 2.3. ([17]) *Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a subgyrogroup, denoted by $H \leq G$, if the following statements hold:*

(1) *The restriction $\oplus|_{H\times H}$ is a binary operation on H, i.e. $(H, \oplus|_{H\times H})$ is a groupoid.*

(2) *For any $x, y \in H$, the restriction of $gyr[x, y]$ to H, $gyr[x, y]|_H : H \to gyr[x, y](H)$ is a bijective homomorphism.*

(3) *$(H, \oplus|_{H\times H})$ is a gyrogroup.*

Furthermore, a subgyrogroup H of G is said to be an L-subgyrogroup, denoted by $H \leq_L G$, if $gyr[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Definition 2.4. ([8]) *A subgyrogroup H of a topological gyrogroup is called strong subgyrogroup if for any $x, y \in G$, we have $gyr[x, y](H) = (H)$.*

Lemma 2.5. ([20]) *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*

(1) $(\ominus x) \oplus (x \oplus y) = y$.

(2) $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$.

(3) $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$.

(4) $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$.

(5) $(\ominus x \oplus y) \oplus gyr[\ominus x, y](\ominus y \oplus z) = \ominus x \oplus z$.

(6) $\ominus(x \oplus y) = gyr[x, y](\ominus y \ominus x)$.

Definition 2.6. ([2]) *A triple $(G, \tau, \oplus)$ is called a topological gyrogroup if the following statements hold:*

(1) *(G, τ) is a topological space.*

(2) *$(G, \oplus)$ is a gyrogroup.*

(3) *The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e., $x \to \ominus x$ is also continuous.*

If a triple $(G, \tau, \oplus)$ satisfies the first two conditions and its binary operation is continuous, we call such triple a paratopological gyrogroup ([3]).

It is easy to see that every topological gyrogroup is a paratopological gyrogroup.

Definition 2.7. ([6]) *Let G be a topological gyrogroup. We say that G is a strongly topological gyrogroup if there exists a neighborhood base $\mathcal{U}$ of the identity element e such that, for every $U \in \mathcal{U}$, $gyr[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathcal{U}$ at e.*

3. Cardinal invariants in semitopological gyrogroups

Recall that a family γ of nonempty open subsets of a space X is called a *π-base* of a point $x \in X$ if, for any nonempty open subset V of X there is $U \in \gamma$ such that $U \subset V$. The π-character of x in X is defined by $\pi_\chi(x, X)$ =min{$|\gamma|$: γ is a π-base of the point x}. If $\sup\{\pi_\chi(x, X) : x \in X\}$ is countable, then X is said to have *countable π-character*. We denote by $\pi\chi(X)$ and $\pi w(X)$ the π-character and π-weight of a space X, respectively (see [1, p.296]).

Definition 3.1. ([14]) *A triple $(G, \tau, \oplus)$ is called a left (respectively, right) topological gyrogroup if and only if*

(1) *(G, τ) is a topological space.*

(2) *$(G, \oplus)$ is a gyrogroup.*

(2) *For all $a \in G$, the left action $L_a : G \to G$, where $L_a(x) = a \oplus x$ (respectively, right action $R_a : G \to G$, where $R_a(x) = x \oplus a$) for each $a \in G$, is a continuous mapping.*

A semitopological gyrogroup is a left topological gyrogroup which is also a right topological gyrogroup. It is clear that every paratopological gyrogroup is a semitopological gyrogroup.

Let G be a semitopological gyrogroup with an open neighborhood base $\mathcal{U}$ at the identity element e. The *left index* (respectively, *right index*) of narrowness of G, denoted by $In_l(G)$ (respectively, $In_r(G)$), is the minimal cardinal $\tau \geq \omega$ such that for every $U \in \mathcal{U}$, there exists $F \subseteq G$ satisfying $F \oplus U = G$ (respectively, $U \oplus F = G$) and $|F| \leq \tau$. If $In_l(G)In_r(G) \leq \omega$, we say that the semitopological gyrogroup G is ω-narrow.

Question 3.2. ([9, Question 3.12]) *Is each first-countable left ω-narrow topological gyrogroup G separable? What if the topological gyrogroup is a strongly topological gyrogroup?*

In [9], the authors gave a positive answer for the above question in the case of strongly topological gyrogroups. In the following, we give a positive answer to the above question in the case of topological gyrogroups and even more. First we establish the following theorem.

Theorem 3.3. *The equation $\pi\omega(G) = In_l(G)\pi\chi(G)$ holds for every semitopological gyrogroup G.*

Proof. Clearly, $\pi\omega(G) \geq \pi\chi(G)$ and $\pi\omega(G) \geq d(G) \geq In_l(G)$, whence $\pi\omega(G) \geq In_l(G)\pi\chi(G)$.

First, we prove that $d(G) \leq In_l(G)\pi\chi(G)$. Let γ be a local π-base at the identity element e in G, with $|\gamma| = \pi\chi(G)$. For every $V \in \gamma$, there exists $A_V \subseteq G$ such that $A_V \oplus V = G$ and $|A_V| \leq In_l(G)$. Put $A = \bigcup_{V\in\gamma} A_V$. Obviously, $|A| \leq In_l(G)\pi\chi(G)$. Then we show that $\ominus A$ is dense in G. Let U be an arbitrary nonempty open subset in G. Take $x \in U$; there exists $V \in \gamma$ such that $V \subseteq U \oplus (\ominus x)$. It follows from the definition of A that $\ominus x \in a \oplus V$, for some $a \in A$. Then there exists $v \in V$ such that $\ominus x = a \oplus v$, equivalently, $\ominus a \oplus (\ominus x) = v$. We conclude that

$$\begin{aligned}\ominus a &= \ominus a \oplus ((\ominus x) \oplus x)\\ &= (\ominus a \oplus (\ominus x)) \oplus gyr[\ominus a, \ominus x](x)\\ &= (\ominus a \oplus (\ominus x)) \oplus gyr[\ominus a \oplus (\ominus x), \ominus x](x)\\ &= v \oplus gyr[v, \ominus x](x).\end{aligned}$$

For the above $v \in V$, there exists $u \in U$ such that $v = u \oplus (\ominus x)$. Then

$$\begin{aligned}v \oplus gyr[v, \ominus x](x) &= (u \oplus (\ominus x)) \oplus gyr[u \oplus (\ominus x), \ominus x](x)\\ &= (u \oplus (\ominus x)) \oplus gyr[u, \ominus x](x)\\ &= u \oplus ((\ominus x) \oplus x)\\ &= u \in U.\end{aligned}$$

It follows that $\ominus a \in U$ which implies that $\ominus A$ is dense in G. It is clear that $|\ominus A| = |A| \leq In_l(G)\pi\chi(G)$, we have that $d(G) \leq In_l(G)\pi\chi(G)$.

Choose $D \subseteq G$ such that D is dense in G and $|D| = d(G)$. For every $d \in D$, let $\mathcal{B}_d$ be a local π-base at d in G, where $|\mathcal{B}_d| = \pi\chi(G)$, for each $d \in D$. Put $\mathcal{B} = \bigcup_{d\in D} \mathcal{B}_d$. Clearly $|\mathcal{B}| \leq \pi\chi(G)d(G)$. So $|\mathcal{B}| \leq In_l(G)\pi\chi(G)$. It is easy to see that $\mathcal{B}$ is a π-base for G. It follows that $\pi\omega(G) = In_l(G)\pi\chi(G)$. □

Since the inequation $d(G) \leq \pi\omega(G)$ holds for every topological space, we give the following corollary.

Corollary 3.4. *Every left ω-narrow semitopological gyrogroup with countable π-character is separable.*

It is known that if a semitopological gyrogroup G is first-countable, then G has a countable π-character. Therefore, we have following corollary which gives a positive answer to [9, Question 3.12].

Corollary 3.5. *Every first-countable left ω-narrow semitopological gyrogroup is separable.*

In [15, Proposition 3.4], the authors proved that every first-countable right ω-narrow topological gyrogroup is separable. In the following, we give such theorem which extended this result.

Theorem 3.6. *The equation $\pi\omega(G) = In_r(G)\pi\chi(G)$ holds for every semitopological gyrogroup G.*

Proof. Clearly, $\pi\omega(G) \geq \pi\chi(G)$ and $\pi\omega(G) \geq d(G) \geq In_r(G)$, whence $\pi\omega(G) \geq In_r(G)\pi\chi(G)$.

First, we prove that $d(G) \leq In_r(G)\pi\chi(G)$. Let γ be a local π-base at the identity element e in G, with $|\gamma| = \pi\chi(G)$. For every $V \in \gamma$, there exists $A_V \subseteq G$ such that $V \oplus A_V = G$ and $|A_V| \leq In_r(G)$. Put $A = \bigcup_{V\in\gamma} A_V$. Obviously, $|A| \leq In_r(G)\pi\chi(G)$. Let us show that A is dense in G. Let U be an arbitrary nonempty open subset in G. Fix a point $x \in U$. Then there exists $V \in \gamma$ such that $V \oplus x \subseteq U$. Since $V \oplus A_V = G$, then there exists $v \in V$ and $a \in A$ such that $x = v \oplus a$. We conclude that

$$a = \ominus v \oplus x \in V \oplus x \subseteq U.$$

It follows that A is dense in G and $|A| \leq In_r(G)\pi\chi(G)$. This implies that $d(G) \leq In_r(G)\pi\chi(G)$.

Choose $D \subseteq G$ such that D is dense in G and $|D| = d(G)$. For every $d \in D$, let $\mathcal{B}_d$ be a local π-base at d in G, where $|\mathcal{B}_d| = \pi\chi(G)$, for each $d \in D$. Put $\mathcal{B} = \bigcup_{d\in D} \mathcal{B}_d$. Clearly $|\mathcal{B}| \leq \pi\chi(G)d(G)$. So $|\mathcal{B}| \leq In_r(G)\pi\chi(G)$. It is easy to see that $\mathcal{B}$ is a π-base for G. It follows that $\pi\omega(G) = In_r(G)\pi\chi(G)$. □

It follows from the above theorem, one can easily get the following corollary.

Corollary 3.7. *Every first-countable right ω-narrow semitopological gyrogroup is separable.*

In the following, we consider the left ω-narrowness of a subgyrogroup in a strongly topological gyrogroup.

Theorem 3.8. *Let G be a left ω-narrow strongly topological gyrogroup. Then every subgyrogroup H of G is left ω-narrow.*

Proof. Let W be an open neighborhood of the identity e in H. Choose an open symmetric neighborhood V of e in G such that $(V \oplus V) \cap H \subset W$. Since G is left ω-narrow, there exists a countable subset B of G such that $B \oplus V = G$. Let C be the set of all $c \in B$ such that $(c \oplus V) \cap H \neq \emptyset$. Then $|C| \leq |B| \leq \omega$ and obviously $H \subset C \oplus V$. For each $c \in C$ fix $a_c \in (c \oplus V) \cap H$, and put $A = \{a_c : c \in C\}$. Since C is countable, A is a countable subset of H. We claim that $A \oplus W = H$.

Indeed, $A \oplus W \subset H$. Since H is a subgyrogroup of G and $(V \oplus V) \cap H \subset W \subset H$, we have $(A \oplus (V \oplus V)) \cap H \subset A \oplus W$. It remains to show that $H \subset A \oplus (V \oplus V)$. Clearly, $A \subset H \subset C \oplus V$, equivalently, $\ominus C \oplus A \subset V$. Then we have that

$$\begin{aligned}\ominus(\ominus A \oplus C) &= \bigcup_{a \in A, c \in C} \{\ominus(\ominus a \oplus c)\} \\ &= \bigcup_{a \in A, c \in C} \{gyr[\ominus a, c](\ominus c \oplus a)\} \\ &\subset \bigcup_{a \in A, c \in C} \{gyr[\ominus a, c](V)\} \\ &= V.\end{aligned}$$

Since V is symmetric, then $\ominus A \oplus C \subset V$, and consequently, $C \subset A \oplus V$. It follows that

$$\begin{aligned}H &\subset C \oplus V \\ &\subset (A \oplus V) \oplus V \\ &= \bigcup_{a \in A, v \in V} \{(a \oplus v) \oplus V\} \\ &= \bigcup_{a \in A, v \in V} \{a \oplus (v \oplus gyr[v, a](V)\} \\ &= \bigcup_{a \in A, v \in V} \{a \oplus (v \oplus V)\} \\ &= A \oplus (V \oplus V).\end{aligned}$$

Thus, $H \subset A \oplus W$ which completes the proof. □

However, we do not know whether the corresponding conclusion in the above theorem also holds true for right ω-narrow strongly topological gyrogroups. Then we pose the following question.

Question 3.9. *Is each subgyrogroup H of a right ω-narrow strongly topological gyrogroup G also right ω-narrow?*

4. On compact subset of topological gyrogroups

It is known that for a topological group G, if K is a compact subset and F is a closed subset of G with $K \cap F = \emptyset$, then there exists an open neighborhood V of e in G such that $KV \cap F = \emptyset$ (see [1, Theorem 1.4.29]). But for topological gyrogroups, the following question was posed.

Question 4.1. ([7, Question 4.6]) *Let G be a topological gyrogroup, K a compact subset and F a closed subset of G such that $K \cap F = \emptyset$. Then is there an open neighborhood V of e in G such that $(K \oplus V) \cap F = \emptyset$, and $(V \oplus K) \cap F = \emptyset$? What if G is a strongly topological gyrogroup?*

In [5, Theorem 5.2], the author proved the above question in the case of strongly topological gyrogroups. In [19, Lemma 9], the authors gave a proof in the case of $(K \oplus V) \cap F = \emptyset$ (note topological gyrogroup is a rectifiable space [10]). In the following, a positive answer to this question in the other case of the topological gyrogroup is established.

Theorem 4.2. *Let G be a topological gyrogroup and F a compact subset of G. If a closed subset $P \subseteq G$ such that $F \cap P = \emptyset$, then there exists an open neighborhood V of e in G such that $(V \oplus F) \cap P = \emptyset$.*

Proof. Let G be a topological gyrogroup with a neighborhood base $\mathcal{U}$ of the identity element e. For every $y \in F$ there exists $U_y \in \mathcal{U}$ such that $(U_y \oplus y) \cap P = \emptyset$, since P is a closed subset of G. Take $V_y, V_y^{'} \in \mathcal{U}$, such that $V_y \oplus V_y \subseteq U_y$, and $V_y^{'} \oplus (V_y^{'} \oplus y) \subseteq (V_y \oplus V_y) \oplus y$. Since F is a compact subset and $F \subseteq \bigcup_{y \in F}\{V_y^{'} \oplus y\}$, there exists a finite subset $E \subseteq F$ such that $F \subseteq \bigcup_{y \in E}\{V_y^{'} \oplus y\}$. Then for every $t \in F$, there exists $y \in E$ such that $t \in V_y^{'} \oplus y$. Let $V = \bigcap_{y \in E} V_y^{'}$. It follows that

$$\begin{aligned} V \oplus t &\subseteq V \oplus (V_y^{'} \oplus y) \\ &\subseteq V_y^{'} \oplus (V_y^{'} \oplus y) \\ &\subseteq (V_y \oplus V_y) \oplus y \\ &\subseteq U_y \oplus y \\ &\subseteq G \setminus P. \end{aligned}$$

This implies $(V \oplus F) \cap P = \emptyset$. □

A subgyrogroup H of a topological gyrogroup G is *inner neutral (respectively, outer neutral)* if, for each open neighborhood U of the identity element e, there exists an open neighborhood V of e such that $V \oplus H \subseteq H \oplus U$ (respectively, $H \oplus V \subseteq U \oplus H$) [8, Definition 3.14]. If H is both inner neutral and outer neutral, then H is called a *neutral subgyrogroup*.

It is known a compact subgroup of a topological group is neutral (see [1, Corollary 1.8.20]). For strongly topological gyrogroups, a compact strong subgyrogroup of a strongly topological gyrogroup is neutral in [8, Proposition 3.16]. In the following, we show that a compact subgyrogroup of a topological gyrogroup is neutral also.

Theorem 4.3. *Let G be a topological gyrogroup and H a compact subgyrogroup of G. Then for each open neighborhood U of the identity element e, there exists an open neighborhood V of e such that $V \oplus H \subseteq H \oplus U$ and $H \oplus V \subseteq U \oplus H$, i.e., H is a neutral subgyrogroup.*

Proof. Let G be a topological gyrogroup with a symmetric neighborhood base $\mathcal{U}$ of the identity element e and take $U \in \mathcal{U}$. For each $y \in H$, there exists $U_y, V_y \in \mathcal{U}$ such that $y \oplus U_y \subseteq y \oplus U$ and $V_y \oplus (V_y \oplus y) \subseteq y \oplus U_y$. It is clear that $H \subseteq \bigcup_{y \in H}(V_y \oplus y)$. Since H is a compact subset, there exists a finite subset $F \subseteq H$ such that $H \subseteq \bigcup_{y \in F}(V_y \oplus y)$. Put $V_1 = \bigcap_{y \in F} V_y$. Then V_1 is an open neighborhood of e.

For each $x \in H$, there exists $y \in F$ such that $x \in V_y \oplus y$. Then

$$\begin{aligned} V_1 \oplus x &\subseteq V_1 \oplus (V_y \oplus y) \\ &\subseteq V_y \oplus (V_y \oplus y) \\ &\subseteq y \oplus U_y \\ &\subseteq y \oplus U \\ &\subseteq H \oplus U. \end{aligned}$$

Therefore, we have $V_1 \oplus H \subseteq H \oplus U$.

For each $b \in H$, there exists $U_b, V_b \in \mathcal{U}$ such that $U_b \oplus b \subseteq U \oplus b$ and $(b \oplus V_b) \oplus V_b \subseteq U_b \oplus b$. Clearly, $H \subseteq \bigcup_{b \in H}(b \oplus V_b)$. Since H is a compact subset, there exists a finite subset $E \subseteq H$ such that $H \subseteq \bigcup_{b \in E}(b \oplus V_b)$. Put $V_2 = \bigcap_{b \in E} V_b$. Then V_2 is an open neighborhood of e.

For each $a \in H$, there exists $b \in E$ such that $a \in b \oplus V_b$. Then

$$\begin{aligned} a \oplus V_2 &\subseteq (b \oplus V_b) \oplus V_2 \\ &\subseteq (b \oplus V_b) \oplus V_b \\ &\subseteq U_b \oplus b \\ &\subseteq U \oplus b \\ &\subseteq U \oplus H. \end{aligned}$$

Thus, we have $H \oplus V_2 \subseteq U \oplus H$.

Let $V = V_1 \cap V_2$. Then $V \oplus H \subseteq H \oplus U$, and $H \oplus V \subseteq U \oplus H$. Therefore, H is a neutral subgyrogroup. □

It is known that every Hausdorff first-countable paratopological group has a regular G_δ-diagonal in [16, Theorem 2.1]. At the end of the article, we discuss the diagonal property of a strongly paratopological gyrogroup.

For a space X, by Δ_X we denote the set $\{(x, x) : x \in X\}$. A space X has a regular G_δ-diagonal if there exists a collection $\{W_n\}_n$ of open sets in $X \times X$ such that $\Delta_X = \bigcap_n W_n = \bigcap_n \overline{W_n}$ (see [12]). In [22, Proposition 1], Zenor showed that space X has a regular G_δ-diagonal if and only if there is a sequence $\{\gamma_n : n \in \omega\}$ of open covers of X such that if x and y are distinct points of X, then there exist $n \in \omega$ and open neighborhoods U and V of x and y, respectively, such that no member of γ_n intersects both U and V.

Lemma 4.4. ([13, Lemma 2.21]) *Let the neighborhood base $\mathcal{U}$ at e of G witness that G is a strongly paratopological gyrogroup. Then we have $(a \oplus O) \oplus W = a \oplus (O \oplus W)$ for each $a \in G$ and $O, W \in \mathcal{U}$.*

Theorem 4.5. *Let G be a Hausdorff strongly paratopological gyrogroup. If G has countable π-character, then G has a regular G_δ-diagonal.*

Proof. Let G be a paratopological gyrogroup with an open neighborhood basis $\mathcal{U}$ of the identity element e. Take two distinct points $y, z \in G$. Choose $U \in \mathcal{U}$ satisfying $(y \oplus (U \oplus (U \oplus U))) \cap (z \oplus (U \oplus (U \oplus U)) = \emptyset$. We can find $V \in \mathcal{U}$ such that $V \subseteq U$, $\ominus y \oplus (V \oplus y) \subseteq U$ and $\ominus z \oplus (V \oplus z) \subseteq U$, that is $V \oplus y \subseteq y \oplus U$ and $V \oplus z \subseteq z \oplus U$.

Take a local π-base $\{V_n : n \in \omega\}$ at the identity e in G. Then there exists $n \in \omega$ such that $V_n \subseteq V$. Put

$$\gamma_n = \{((x \oplus V_n) \oplus (\ominus V_n)) \cap (\ominus V_n \oplus (V_n \oplus x)) : x \in G\}.$$

Clearly, the family $\{\gamma_n : n \in \omega\}$ is a sequence of open covers of G.

We claim that no member of γ_n intersects both $y \oplus U$ and $z \oplus U$. Suppose the contrary, then there exist $a, b \in U$, $c, d, f, g \in V_n$ and $x \in G$ such that $y \oplus a = (x \oplus c) \oplus (\ominus f)$ and $z \oplus b = \ominus g \oplus (d \oplus x)$. This implies that $x = \ominus d \oplus (g \oplus (z \oplus b))$. It follows that

$$\begin{aligned} (y \oplus a) \oplus gyr[x \oplus c, \ominus f](f) &= ((x \oplus c) \oplus (\ominus f)) \oplus gyr[x \oplus c, \ominus f](f) \\ &= x \oplus c \\ &= (\ominus d \oplus (g \oplus (z \oplus b))) \oplus c \\ &= \ominus d \oplus ((g \oplus (z \oplus b)) \oplus gyr[(g \oplus (z \oplus b)), \ominus d](c)). \end{aligned}$$

Therefore, we have

$$d \oplus ((y \oplus a) \oplus gyr[x \oplus c, \ominus f](f)) = (g \oplus (z \oplus b)) \oplus gyr[(g \oplus (z \oplus b)), \ominus d](c).$$

Now by Lemma 4.4, we have

$$\begin{aligned}
d \oplus ((y \oplus a) \oplus gyr[x \oplus c, \ominus f](f)) &\in V_n \oplus ((y \oplus U) \oplus V_n) \\
&\subseteq V \oplus ((y \oplus U) \oplus V) \\
&= V \oplus (y \oplus (U \oplus V)) \\
&= (V \oplus y) \oplus gyr[V, y](U \oplus V) \\
&\subseteq (y \oplus U) \oplus (U \oplus V) \\
&= y \oplus (U \oplus (U \oplus V)) \\
&\subseteq y \oplus (U \oplus (U \oplus U)).
\end{aligned}$$

and

$$\begin{aligned}
(g \oplus (z \oplus b)) \oplus gyr[(g \oplus (z \oplus b)), \ominus d](c) &\in (V_n \oplus (z \oplus U)) \oplus V_n \\
&\subseteq (V \oplus (z \oplus U)) \oplus V \\
&= ((V \oplus z) \oplus gyr[V, z](U)) \oplus V \\
&= ((V \oplus z) \oplus U) \oplus V \\
&\subseteq ((z \oplus U) \oplus U) \oplus V \\
&= (z \oplus U) \oplus (U \oplus V) \\
&= z \oplus (U \oplus (U \oplus V)) \\
&\subseteq z \oplus (U \oplus (U \oplus U)).
\end{aligned}$$

Which contradicts the choice of U. Therefore, no member of γ_n intersects both $y \oplus V$ and $z \oplus V$. We can conclude that G has a regular G_δ-diagonal. □

Acknowledgements

The authors would like to express their sincere appreciation to the referee for his/her careful reading of the paper and valuable comments.

References

[1] A. V. Arhangel'skiĭ, M. Tkachenko, *Topological Groups and Related Structures,* Atlantics Press, World Sci., 2008.
[2] W. Atiponra, *Topological gyrogroups: generalization of topological groups*. Topol. Appl. **224** (2017), 73-82.
[3] W. Atiponrat, R. Maungchang, *Complete regularity of paratopological gyrogroups*, Topol. Appl. **270** (2020): 106951.
[4] T. Banakh, A. Ravsky, *Each regular paratopological group is completely regular*, Proc. Am. Math. Soc. **145** (2017), 1373-1382.
[5] M. Bao, *On strongly topological gyrogroups*. Filomat **38** (2024), 2821-2833.
[6] M. Bao, F. Lin, *Feathered gyrogroups and gyrogroups with countable pseudocharacter*, Filomat **33** (2019), 5113-5124.
[7] M. Bao, R. Shen, X. Xu, *A class of quotient spaces in strongly topological gyrogroups*, Houston J. Math. **48** (2022), 655-676.
[8] M. Bao, X. Xu, *A note on (strongly) topological gyrogroups*, Topol. Appl. **307** (2022): 107950.
[9] M. Bao, X. Zhang, X. Xu, *Separability in (strongly) topological gyrogroups*, Filomat **35** (2021), 4381-4390.
[10] Z. Cai, S. Lin, W. He, *A note on Paratopological Loops*, Bulletin of the Malaysian Math. Sci. Soc. **42** (2019), 2535-2547.
[11] R. Engelking, *General Topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[12] R. Engelking , *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed., 1989.
[13] Y. Jin, L. Xie, *On the continuity of the inverse in (strongly) paratopological gyrogroups*, Filomat **38** (2024), 5449-5462.
[14] Y. Jin, L. Xie, P. Yan, *Three-space properties in paratopological gyrogroups*, Filomat **39** (2025), 1181-1196.
[15] P. Li, R. Shen, *On topological gyrogroups*, Filomat **37** (2023), 5087-5093.
[16] C. Liu, *A note on paratopological groups*, Comment. Math. Univ. Carolin. **47** (2006), 633-640.
[17] T. Suksumran, K. Wiboonton, *Isomorphism theorems for gyrogroups and L-subgyrogroups*, J. Geom. Symmetry Phys. **37** (2015), 67-83.
[18] A.A. Ungar, *Beyond the Einstein Addition Law and Its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, Fundamental Theories of Physics, vol. 117, Springer, Netherlands, 2002.
[19] L. Q. Tuyen, O. V. Tuyen, *Some properties of rectifiable spaces,* Fasciculi Mathematici **60** (2018), 181-190.
[20] A.A. Ungar, *Analytic hyperbolic geometry: Mathematical foundations and applications*, World Scientific, Hackensack, 2005.
[21] A. A. Ungar, *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific, Hackensack, New Jersey, 2008.
[22] P. Zenor, *On spaces with regular G_δ-diagonals*, Pacific J. Math. **40** (1972), 759-763.