Filomat 40:5 (2026), 1797–1808
https://doi.org/10.2298/FIL2605797S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pedal-Contrapedal curve pairs of spacelike frontals in null sphere

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Abstract. Due to the curves that exhibiting singularities in the null sphere can not be studied directly. Hence we aim to analyze these problems by using the duality theory. In addition, we define the pedal curves and contrapedal curves of spacelike frontals and investigate their geometric properties in null sphere. We also consider sufficient singular conditions for the contrapedal curves. Moreover, we give the particular relations between pedal-contrapedal curve pairs and Involute-evolute curve pairs.

1. Introduction

Pedal curves and contrapedal curves are fascinating classical topics in differential geometry, deeply rooted in the study of curves and their geometric companions [1–3]. Given a regular plane curve $\widetilde{\gamma}_p$ and a fixed point $\widetilde{p}$, the pedal curve traces the points $\widetilde{h}$ where the line $\widetilde{ph}$ meets the tangent to $\widetilde{\gamma}_p$ at right angles. Similarly, the contrapedal curve arises when $\widetilde{ph}$ is perpendicular to the normal lines of $\widetilde{\gamma}_p$ [5–7, 11, 12]. These constructions also have significant applications in the field of optics,(as the wave surface theory for describing the diffraction of light waves), mechanical engineering, and even celestial mechanics, revealing hidden symmetries in how curves interact with points in space [8, 9, 21, 22].

When curves have singularties, the classical definitions break down because the tangent lines are not be defined at singular points. To handel singular curve, the Legendre curves (frontals) and Legendre immersion (fronts) are introduced by T. Fukunaga and M. Takahashi [4]. By shifting perspective to the unit tangent bundle, they equipped singular curves with moving frames and curvatures that persist even at singular points [14–19]. Previous work by T. Nishimura has studied the singularities pedal curve for dual curve germs[13]. In [20], the authors obtained the condition for a pedal curve of a front to be a frontal, and considered some relationships of the evolute, the involute, and the offset of a front. Moreover, Li and Pei researched the pedal curve of front in the Euclidean plane and Euclidean sphere [10].

However as far as we konw, no literature exists regarding the pedal curves and contrapedal curves of spherical frontals in the null sphere. Inspired by the above work, we apparently need to establish the geometric property of pedal-contrapedal curve pairs of spacelike frontals in null sphere.

2020 *Mathematics Subject Classification*. Primary 53A04; Secondary 53A05, 57R45.
Keywords. Pedal curve, Contrapedal curve, Frontal, Singularities.
Received: 20 August 2025; Accepted: 07 December 2025
Communicated by Mića Stanković
Research supported by the Hainan Provincial Natural Science Foundation of China (125QN292), the Research Foundation for Talented Scholars of the Hainan Normal University (HSZK-KYQD-202402).
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In present paper, we firstly introduce the moving frame of Legendre curves (frontals) in null sphere, which is the most fundamental tool for studying the pedal-contrapedal curve pairs. Then we define the pedal curves and contrapedal curves of spherical frontals in null sphere. We recall the notions of Involute and evolute curves. Afterwards, we discuss the relationship between pedal-contrapedal curve pairs and Involute-evolute curve pairs in detail. Furthermore, we consider sufficient singular conditions for the contrapedal curves. Finally, we give some examples to show pedal-contrapedal curve pairs into 3-dimensional projection spaces.

2. Preliminaries

Let $\mathbb{R}^4$ be a four-dimensional Euclidean space, for any two vectors $\widetilde{\boldsymbol{a}} = (a^1, a^2, a^3, a^4) \in \mathbb{R}^4$ and $\widetilde{\boldsymbol{b}} = (b^1, b^2, b^3, b^4) \in \mathbb{R}^4$, the *pseudo inner product* of $\widetilde{\boldsymbol{a}}$ and $\widetilde{\boldsymbol{b}}$ is defined by $\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}} \rangle = -a^1b^1 + a^2b^2 + a^3b^3 + a^4b^4$. We call $(\mathbb{R}^4; \langle , \rangle)$ the Minkowski four-dimensional space and denote $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4; \langle , \rangle)$. We also define the pseudo-vector product of $\widetilde{\boldsymbol{a}} = (a^1, a^2, a^3, a^4)$, $\widetilde{\boldsymbol{b}} = (b^1, b^2, b^3, b^4)$ and $\widetilde{\boldsymbol{c}} = (c^1, c^2, c^3, c^4)$ by

$$\widetilde{\boldsymbol{a}} \wedge \widetilde{\boldsymbol{b}} \wedge \widetilde{\boldsymbol{c}} = \begin{vmatrix} -\boldsymbol{e}_a & \boldsymbol{e}_b & \boldsymbol{e}_c & \boldsymbol{e}_d \\ a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \\ c^1 & c^2 & c^3 & c^4 \end{vmatrix},$$

where $(\boldsymbol{e}_a, \mathbf{e}_b, \boldsymbol{e}_c, \boldsymbol{e}_d)$ is the canonical basis of $\mathbb{R}^4_1$. We call a non-zero vector $\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1$ is *spacelike, null* or *timelike* if $\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle$ is positive, zero or negative, respectively. The *norm* of a vector $\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1$ is defined as $\| \widetilde{\boldsymbol{a}} \| = \sqrt{|\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle|}$. Let $\boldsymbol{\gamma}_p : I \to \mathbb{R}^4_1$ be a regular curve (i.e., $\gamma_p'(t) \neq 0$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve γ_p is called *spacelike curve, null curve* or *timelike curve* if its velocity is $\langle \boldsymbol{\gamma}_p'(t), \boldsymbol{\gamma}_p'(t) \rangle > 0$, $\langle \boldsymbol{\gamma}_p'(t), \boldsymbol{\gamma}_p'(t) \rangle = 0$ or $\langle \boldsymbol{\gamma}_p'(t), \boldsymbol{\gamma}_p'(t) \rangle < 0$, respectively.

We introduce the following definitions, the *de Sitter 3-space* by

$$S^3_1 = \{\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1 \mid \langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle = 1\}.$$

The *closed nullcone with the vertex* $\boldsymbol{p}$ by

$$\mathcal{NC}_p = \{\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1 \mid \langle \widetilde{\boldsymbol{a}} - \boldsymbol{p}, \widetilde{\boldsymbol{a}} - \boldsymbol{p} \rangle = 0\}.$$

The *open nullcone at the origin* by

$$\mathcal{LC}^* = \{\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1 \backslash \{\boldsymbol{0}\}, \langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle = 0\}.$$

The *unit sphere* in the open nullcone is defined as

$$S^2_+ = \{\widetilde{\boldsymbol{a}} \in \mathcal{LC}^* \mid \widetilde{a}^1 = 1\}.$$

The *canonical nullcone projection* $\theta : \mathcal{LC}^* \to S^2_+$ as

$$\theta(a^1, a^2, a^3, a^4) = (1, \frac{a^2}{a^1}, \frac{a^3}{a^1}, \frac{a^4}{a^1}).$$

And the *Euclidean unit 2-sphere* in $\mathbb{R}^3_0$ is denoted as

$$S^2_0 = \{\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1 \mid a_1 = 0,\ a_2^2 + a_3^2 + a_4^2 = 1\}.$$

Obviously, S^2_0 is a subspace of the *Euclidean 3-space* $\mathbb{R}^3_0 = \{\widetilde{\boldsymbol{a}} \in \mathbb{R}^4_1 \mid a^1 = 0\}$.

Let $\boldsymbol{\gamma}_f : I \to S^2_0$ be a smooth curve. We say that $(\boldsymbol{\gamma}_f, \boldsymbol{\nu}_f) : I \to \Delta \subset S^2_0 \times S^2_0$ is a Legendrian curve if

$$(\boldsymbol{\gamma}_f(t), \boldsymbol{\nu}_f(t))^* \theta = 0$$

for all $t \in I$, we call γ_f the frontal in S^2_0. In addition, the condition $(\gamma_f(t), \nu_f(t))^*\theta = 0$ is equivalent to

$$\langle \gamma_f{}'(t), \nu_f(t)\rangle = 0$$

for all $t \in I$.

Let $\gamma_p : I \to S^2_+$ be a spacelike front which may have singular points. We notice that the curve γ_p on the nullcone is degenerate, hence we can not study it directly. Then we can consider its dual curve $\widetilde{\gamma}_p$ by homeomorphic theory in the Euclidean space, so that $\widetilde{\gamma}_p$ have the same properties as the original curve γ_p. We consider the isometric mapping $\Gamma : S^2_+ \to S^2_0$ defined by $\Gamma(\eta) = \eta - e_p$, where $e_p = (1, 0, 0, 0)$. Then we obtain another spacelike front $\widetilde{\gamma}_p : I \to S^2_0$ defined by $\widetilde{\gamma}_p(t) = \Gamma(\gamma_p(t)) = \gamma_p(t) - e_p$, so that γ_p and $\widetilde{\gamma}_p$ have the same geometric properties as spherical curves. We define the unit principal normal vector $\widetilde{\nu}_p(t) = \widetilde{\gamma}_p(t) \wedge e_p \wedge \widetilde{\mu}_p(t)$. Then we have a moving frame $\{\widetilde{\gamma}_p(t), \widetilde{\mu}_p(t), \widetilde{\nu}_p(t)\}$ along the spacelike front $\widetilde{\gamma}_p$ and the Frenet-Serret type formula is given as follows:

$$\begin{pmatrix} \widetilde{\gamma}'_p(t) \\ \widetilde{\mu}'_p(t) \\ \widetilde{\nu}'_p(t) \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{\beta}_p(t) & 0 \\ -\widetilde{\beta}_p(t) & 0 & k_{\widetilde{\gamma}}(s) \\ 0 & -\widetilde{\ell}_p(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\gamma}_p(t) \\ \widetilde{\mu}_p(t) \\ \widetilde{\nu}_p(t) \end{pmatrix}, \tag{1}$$

where $\widetilde{\beta}_p(t) = \langle \widetilde{\gamma}'_p(t), \widetilde{\mu}_p(t)\rangle$ and $\widetilde{\ell}_p(t) = -\langle \widetilde{\mu}'_p(t), \widetilde{\nu}_p(t)\rangle$. Here, the pair $(\widetilde{\beta}_p(t), \widetilde{\ell}_p(t))$ is called the *geodesic curvature* of the spacelike frontal. We remark that $(\widetilde{\gamma}_p, \widetilde{\nu}_p) : I \to \Delta$ is the Legendrian immersion if the curvature $(\widetilde{\beta}_p, \widetilde{\ell}_p) \neq (0, 0)$.

3. Pedal Curves of Frontals on Null Sphere

In this section, firstly, we recall the pedal curves of regular curves in the sphere. For a point $H_p \in S^2_0 \setminus \{\pm\widetilde{n}_p(t) \mid t \in I\}$, the pedal curve $Pe_{\widetilde{\gamma}_p, H_p} : I \to S^2_0$ of a regular curve $\widetilde{\gamma}_p : I \to S^2_0$ is given by

$$Pe_{\widetilde{\gamma}_p, H_p}(t) = \frac{1}{\sqrt{1 - (H_p \cdot \widetilde{n}_p(t))^2}} \left(H_p - (H_p \cdot \widetilde{n}_p(t))\widetilde{n}_p(t)\right),$$

where, $\widetilde{n}_p(t)$ is the unit normal vector of $\widetilde{\gamma}_p$.

Let $(\widetilde{\gamma}_p, \widetilde{\nu}_p) : I \to \Delta_5 \subset S^2_0 \times S^2_0$ be a Legendrian curve with the geodesic curvature $(\widetilde{\beta}_p, \widetilde{\ell}_p)$ and $H_p \in S^2_0 \setminus \{\pm v_p(t) \mid t \in I\}$ be a fixed point, called a pedal point. We give the concept of a pedal curve of a frontal.

Definition 3.1. The pedal curve $Pe_{\widetilde{\gamma}_p, H_p} : I \to S^2_0$ of $\widetilde{\gamma}_p$ with respect to H_p is defined by

$$Pe_{\widetilde{\gamma}_p, H_p}(t) = \frac{1}{\sqrt{1 - \left(H_p \cdot \widetilde{v}_p(t)\right)^2}}(H_p - \left(H_p \cdot \widetilde{v}_p(t)\right)\widetilde{v}_p(t)).$$

Proposition 3.2. *Let $\widetilde{\gamma}_p : I \to S^2_0$ be a regular curve and a point $H_p \in S^2_0 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. Then the pedal curve of the regular curve coincides with the pedal curve of the frontal.*

Proof. Let $\widetilde{\gamma}_p : I \to S^2_0$ be a regular curve and a point $H_p \in S^2_0 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. Taking $\widetilde{\nu}_p(t) = \widetilde{n}_p(t)$, then $(\widetilde{\gamma}_p, \widetilde{n}_p)$ is a Legendre curve. By the definition of pedal curve of the regular curve, we have

$$\begin{aligned} Pe_{\widetilde{\gamma}_p, H_p}(t) &= \frac{1}{\sqrt{1 - (H_p \cdot \widetilde{n}_p(t))^2}} \left(H_p - (H_p \cdot \widetilde{n}_p(t))\widetilde{n}_p(t)\right) \\ &= \frac{1}{\sqrt{1 - (H_p \cdot \widetilde{\nu}_p(t))^2}} \left(H_p - (H_p \cdot \widetilde{\nu}_p(t))\widetilde{\nu}_p(t)\right) \\ &= Pe_{\widetilde{\gamma}_p, H_p}(t). \end{aligned}$$

□

Proposition 3.3. *Let* $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S_0^2 \times S_0^2$ *be a Legendrian curve with the geodesic curvature* $(\widetilde{\beta}_p, \widetilde{\ell}_p)$ *and a point* $H_p \in S_0^2 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. *The pedal curve* $Pe_{\widetilde{\gamma}_p, H_p}$ *of* $\widetilde{\boldsymbol{\gamma}}_p$ *is independent of the parametrization of* $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p)$.

Proof. Let $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S_0^2 \times S_0^2$ and $(\overline{\gamma}_p, \overline{\nu}_p) : \overline{I}_p \to \Delta_5 \subset S_0^2 \times S_0^2$ be parametrically equivalent via the change of parameter $t : \overline{I}_p \to I$. By the assumption, we have

$$(\overline{\gamma}_p(u), \overline{\nu}_p(u)) = (\widetilde{\boldsymbol{\gamma}}_p(t(u)), \widetilde{\boldsymbol{\nu}}_p(t(u))).$$

Then

$$\begin{aligned} Pe_{\overline{\gamma}_p, H_p}(u) &= \frac{1}{\sqrt{1-(H_p \cdot \overline{\nu}_p(u))^2}} \Big(H_p - (H_p \cdot \overline{\nu}_p(u))\overline{\nu}_p(u) \Big) \\ &= \frac{1}{\sqrt{1-(H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t(u)))^2}} \Big(H_p - (H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t(u)))\widetilde{\boldsymbol{\nu}}_p(t(u)) \Big) \\ &= Pe_{\widetilde{\gamma}_p, H_p}(t(u)). \end{aligned}$$

□

Proposition 3.4. *Let* $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S_0^2 \times S_0^2$ *be a Legendrian curve with the geodesic curvature* $(\widetilde{\beta}_p, \widetilde{\ell}_p)$ *and a point* $H_p \in S_0^2 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. *The pedal curve satisfies* $Pe'_{\widetilde{\gamma}_p, H_p}(t) = 0$ *if and only if* $\widetilde{\ell}_p(t) = 0$ *or* $H_p = \widetilde{\boldsymbol{\gamma}}_p(t)$.

Proof. Differentiating $Pe_{\widetilde{\gamma}_p, H_p}$ and using the equation **??**, we have

$$\begin{aligned} Pe'_{\widetilde{\gamma}_p, H_p}(t) = &-\widetilde{\ell}_p(t) \frac{(H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t))(H_p \cdot \widetilde{\boldsymbol{\mu}}_p(t))}{(1-(H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t))^2)^{3/2}} \Big((H_p \cdot \widetilde{\boldsymbol{\gamma}}_p(t))\widetilde{\boldsymbol{\gamma}}_p(t) + (H_p \cdot \widetilde{\boldsymbol{\mu}}_p(t))\widetilde{\boldsymbol{\mu}}_p(t) \Big) \\ &- \widetilde{\ell}_p(t) \frac{1}{(1-(H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t))^2)^{1/2}} \Big((H_p \cdot \widetilde{\boldsymbol{\mu}}_p(t))\widetilde{\boldsymbol{\nu}}_p(t) + (H_p \cdot \widetilde{\boldsymbol{\nu}}_p(t))\widetilde{\boldsymbol{\mu}}_p(t) \Big). \end{aligned}$$

Since $\{\widetilde{\boldsymbol{\gamma}}_p(t), \widetilde{\boldsymbol{\nu}}_p(t), \widetilde{\boldsymbol{\mu}}_p(t)\}$ is an orthogonal frame, hence $Pe'_{\widetilde{\gamma}_p, H_p}(t) = 0$ if and only if $\widetilde{\ell}_p(t) = 0$ or $H_p = \widetilde{\boldsymbol{\gamma}}_p(t)$. □

4. Contrapedal Curve of Frontals in Null Sphere

Suppose that $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S_0^2 \times S_0^2$ is a Legendrian curve with the geodesic curvature $(\widetilde{\beta}_p, \widetilde{\ell}_p)$, For a point $H_p \in S_0^2 \setminus \{\pm\widetilde{\mu}_p(t) \mid t \in I\}$. We define a contrapedal curve of a frontal. Here we call H_p a contrapedal point.

Definition 4.1. The contrapedal curve $CPe_{\widetilde{\gamma}_p, H_p}(t) : I \to S_0^2$ of $\widetilde{\gamma}_p$ at H_p is defined by

$$CPe_{\widetilde{\gamma}_p, H_p}(t) = \frac{1}{\sqrt{1-(H_p \cdot \widetilde{\mu}_p(t))^2}} (H_p - (H_p \cdot \widetilde{\mu}_p(t))\widetilde{\mu}_p(t)).$$

Proposition 4.2. *If there exist three smooth maps* $\widetilde{f}_p : I \to \mathbb{R}$, $\widetilde{\varphi}_p : I \to \mathbb{R}$, *and* $\widetilde{\phi}_p : I \to \mathbb{R}$ *which satisfy*

$$H_p \cdot \widetilde{\mu}_p(t) = \widetilde{f}_p(t)\widetilde{\varphi}_p(t), \quad \widetilde{\beta}_p(t)(H_p \cdot \widetilde{\gamma}_p(t)) + \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\nu}_p(t)) = \widetilde{f}_p(t)\widetilde{\phi}_p(t)$$

and $(\widetilde{\varphi}_p(t), \widetilde{\phi}_p(t)) \neq (0, 0)$ *for all* $t \in I$, *then* $CPe_{\widetilde{\gamma}_p, H_p}$ *is a frontal. To be more precise,* $(CPe_{\widetilde{\gamma}_p, H_p}, \overline{v}_{H_p})$ *is a Legendre curve, where*

$$\overline{v}_{H_p}(t) = \widetilde{\kappa}_p(t)\widetilde{\mu}_p(t) + \widetilde{\tau}_p(t) \frac{H_p \cdot \widetilde{\gamma}_p(t)}{\sqrt{(H_p \cdot \widetilde{\gamma}_p(t))^2 + (H_p \cdot \widetilde{\nu}_p(t))^2}} \widetilde{\nu}_p(t)$$

$$- \widetilde{\tau}_p(t) \frac{H_p \cdot \widetilde{\nu}_p(t)}{\sqrt{(H_p \cdot \widetilde{\gamma}_p(t))^2 + (H_p \cdot \widetilde{\nu}_p(t))^2}} \widetilde{\gamma}_p(t),$$

and

$$\widetilde{\kappa}_p(t) = \sqrt{\frac{\widetilde{\varphi}_p^2(t)\left(\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\nu}_p(t)) + \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\gamma}_p(t))\right)^2}{\widetilde{\varphi}_p^2(t)\left(\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\nu}_p(t)) + \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\gamma}_p(t))\right)^2 + \widetilde{\phi}_p^2(t)\left((H_p \cdot \widetilde{\gamma}(t))_p^2 - (H_p \cdot \widetilde{\nu}_p(t))^2\right)}},$$

$$\widetilde{\tau}_p(t) = \frac{\widetilde{\phi}_p(t)}{\sqrt{\dfrac{\widetilde{\varphi}_p^2(t)\left(\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\nu}_p(t)) + \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\gamma}_p(t))\right)^2}{(H_p \cdot \widetilde{\gamma}_p(t))^2 - (H_p \cdot \widetilde{\nu}_p(t))^2} + \widetilde{\phi}_p^2(t)}}.$$

Proof. By differentiating $CPe_{\widetilde{\gamma}_p,H_p}$, we obtain

$$\begin{aligned} CPe_{\widetilde{\gamma}_p,H_p}{}'(t) = {} & \frac{\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\mu}_p(t))(H_p \cdot \widetilde{\nu}_p(t))^2 + \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\gamma}_p(t))(H_p \cdot \widetilde{\nu}_p(t))(H_p \cdot \widetilde{\mu}_p(t))}{(1 - (H_p \cdot \widetilde{\mu}_p(t))^2)^{3/2}} \widetilde{\gamma}_p(t) \\ & - \frac{\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\gamma}_p(t))(H_p \cdot \widetilde{\nu}_p(t))(H_p \cdot \widetilde{\mu}_p(t)) - \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\mu}_p(t))(H_p \cdot \widetilde{\gamma}_p(t))^2}{(1 - (H_p \cdot \widetilde{\mu}_p(t))^2)^{3/2}} \widetilde{\nu}_p(t) \\ & + \frac{(1 - (H_p \cdot \widetilde{\mu}_p(t))^2)\left(\widetilde{\beta}_p(t)(H_p \cdot \widetilde{\gamma}_p(t)) - \widetilde{\ell}_p(t)(H_p \cdot \widetilde{\nu}_p(t))\right)}{(1 - (H_p \cdot \widetilde{\mu}_p(t))^2)^{3/2}} \widetilde{\mu}_p(t). \end{aligned}$$

Then

$$CPe_{\widetilde{\gamma}_p,H_p}(t) \cdot \overline{v}_{H_p}(t) = 0, \quad CPe_{\widetilde{\gamma}_p,H_p}{}'(t) \cdot \overline{v}_{H_p}(t) = 0.$$

By the above assumptions, the unit vector field $\overline{v}_{H_p}$ is well-defined. We prove that $CPe_{\widetilde{\gamma}_p,H_p}$ is a frontal. □

In fact, if $\widetilde{\beta}_p(t_0)(H_p \cdot \widetilde{\gamma}_p(t_0)) - \widetilde{\ell}_p(t_0)(H_p \cdot \widetilde{\nu}_p(t_0)) \neq 0$ for some $t_0 \in I$, then t_0 is a regular point of $CPe_{\widetilde{\gamma}_p,H_p}$. We give the condition that $CPe_{\widetilde{\gamma}_p,H_p}$ is singular.

Theorem 4.3. *Suppose that $\widetilde{\beta}_p(t_0)(H_p \cdot \widetilde{\gamma}_p(t_0)) - \widetilde{\ell}_p(t_0)(H_p \cdot \widetilde{\nu}_p(t_0)) = 0$ for some $t_0 \in I$.*

(1) If $\widetilde{\beta}_p(t_0) = 0$, $\widetilde{\ell}_p(t_0) \neq 0$, then $CPe_{\widetilde{\gamma}_p,H_p}$ is singular at t_0 if and only if $H_p = \pm\widetilde{\gamma}_p(t_0)$.

(2) If $\widetilde{\beta}_p(t_0) \neq 0$, $\widetilde{\ell}_p(t_0) = 0$, then $CPe_{\widetilde{\gamma}_p,H_p}$ is singular at t_0 if and only if $H_p = \pm\widetilde{\nu}_p(t_0)$.

(3) If $\widetilde{\beta}_p(t_0) \neq 0$, $\widetilde{\ell}_p(t_0) \neq 0$, then $CPe_{\widetilde{\gamma}_p,H_p}$ is singular at t_0 if and only if $H_p \cdot \widetilde{\mu}_p(t_0) = 0$.

5. The Relationships Between Pedal-Contrapedal Curves Pairs and Involute-Evolute Curves Pairs

In this section, we will investigate the the relationships between pedal-contrapedal curves pairs and Involute-evolute curves pairs in S_0^2. Firstly, we recall the definition and geometric properties of the evolutes and involutes of frontals. We endow a Legendrian curve $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S_0^2 \times S_0^2$ with the geodesic curvature pair $(\widetilde{\beta}_p(t), \widetilde{\ell}_p(t)) \neq (0,0)$, the nullcone evolute $\widetilde{\boldsymbol{E}}(\widetilde{\boldsymbol{\gamma}}_p) : I \to S_0^2$ is defined by:

$$\widetilde{\boldsymbol{E}}(\widetilde{\boldsymbol{\gamma}}_p)(t) = \pm \frac{\widetilde{\ell}_p(t)}{\sqrt{\widetilde{\beta}_p^2(t) + \widetilde{\ell}_p^2(t)}} \widetilde{\boldsymbol{\gamma}}_p(t) \pm \frac{\widetilde{\beta}_p(t)}{\sqrt{\widetilde{\beta}_p^2(t) + \widetilde{\ell}_p^2(t)}} \widetilde{\boldsymbol{\nu}}_p(t).$$

Proposition 5.1. The nullcone evolute $\widetilde{\boldsymbol{E}}(\widetilde{\boldsymbol{\gamma}}_p) : I \to S^2_0$ of the Legendrian immersion $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p)$ is a spacelike front and $(\widetilde{\boldsymbol{E}}(\widetilde{\boldsymbol{\gamma}}_p), \widetilde{\boldsymbol{\nu}}_\varepsilon) : I \to \Delta \subset S^2_0 \times S^2_0$ is a spacelike Legendrian immersion with the geodesic curvature $(\widetilde{\beta}_\varepsilon, \widetilde{l}_\varepsilon)$, where $\widetilde{\boldsymbol{\nu}}_\varepsilon(t) = \widetilde{\boldsymbol{\mu}}_p(t)$ and

$$\widetilde{\beta}_\varepsilon(t) = \frac{\widetilde{\ell}'_p(t)\widetilde{\beta}_p(t)}{\widetilde{\beta}^2_p(t) + \widetilde{\ell}^2_p(t)} - \frac{\widetilde{\ell}_p(t)\widetilde{\beta}'_p(t)}{\widetilde{\beta}^2_p(t) + \widetilde{\ell}^2_p(t)},$$
$$\widetilde{\ell}_\varepsilon(t) = \pm\sqrt{\widetilde{\beta}^2_p(t) + \widetilde{\ell}^2_p(t)}.$$

The nullcone involute $\widetilde{\boldsymbol{I}}_p : I \to S^2_0$ of $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p)$ at t_0 is defined by

$$\widetilde{\boldsymbol{I}}_p(\widetilde{\boldsymbol{\gamma}}_p, t_0)(t) = -\sin\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\boldsymbol{\mu}}_p(t) + \cos\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\boldsymbol{\gamma}}_p(t),$$

where $t_0 \in I$.

Proposition 5.2. The nullcone involute $\widetilde{\boldsymbol{I}}_p(\widetilde{\boldsymbol{\gamma}}_p, t_0) : I \to S^2_0$ of the Legendrian immersion $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p)$ is a spacelike frontal and $(\widetilde{\boldsymbol{I}}_p(\widetilde{\boldsymbol{\gamma}}_p, t_0), \widetilde{\boldsymbol{\nu}}_I) : I \to \Delta \subset S^2_0 \times S^2_0$ is a spacelike Legendrian curve with the geodesic curvature $(\widetilde{\beta}_I, \widetilde{l}_I)$, where

$$\widetilde{\boldsymbol{\nu}}_I(t) = \cos\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\boldsymbol{\mu}}_p(t) + \sin\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\boldsymbol{\gamma}}_p(t),$$
$$\widetilde{\beta}_I(t) = \sin\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\ell}_p(t),$$
$$\widetilde{\ell}_I(t) = \cos\left(\int_{t_0}^{t} \widetilde{\beta}_p(t)dt\right)\widetilde{\ell}_p(t).$$

By the definitions of involute-evolute curve pairs and pedal-contrapedal curve pairs, we give the relationships between these curves.

Proposition 5.3. *For a Legendre curve $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S^2_0 \times S^2_0$ and a point $H_p \in S^2_0 \setminus \{\pm\widetilde{\mu}_p(t) \mid t \in I\}$, the pedal curve of $\widetilde{\varepsilon v}(\widetilde{\gamma}_p)$ at H_p coincides with the contrapedal curve of $(\widetilde{\gamma}_p, \widetilde{\nu}_p)$ at H_p, precisely,*

$$Pe_{\widetilde{\varepsilon v}(\widetilde{\gamma}_p),H_p}(t) = CPe_{\widetilde{\gamma}_p,H_p}(t).$$

Proof. By Proposition 5.1, $\widetilde{\nu_{\widetilde{\varepsilon}}}(t) = \widetilde{\mu}_p(t)$. Thus

$$\begin{aligned}
Pe_{\widetilde{\varepsilon v}(\widetilde{\gamma}_p),H_p}(t) &= \frac{1}{\sqrt{1 - (H_p \cdot \widetilde{\nu_{\widetilde{\varepsilon}}}(t))^2}}(H_p - (H_p \cdot \widetilde{\nu_{\widetilde{\varepsilon}}}(t))\widetilde{\nu_{\widetilde{\varepsilon}}}(t)) \\
&= \frac{1}{\sqrt{1 - (H_p \cdot \widetilde{\mu}_p(t)))^2}}(H_p - (H_p \cdot \widetilde{\mu}(t))\widetilde{\mu}_p(t)) \\
&= CPe_{\widetilde{\gamma}_p,H_p}(t).
\end{aligned}$$

□

By the dual definitions of these curves, we have the following.

Proposition 5.4. *For a Legendre curve $(\widetilde{\boldsymbol{\gamma}}_p, \widetilde{\boldsymbol{\nu}}_p) : I \to \Delta_5 \subset S^2_0 \times S^2_0$ and a point $H_p \in S^2_0 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$, the contrapedal curve of $Inv(\widetilde{\boldsymbol{\gamma}}_p, t_0)$ at H_p coincides with the pedal curve of $\widetilde{\boldsymbol{\gamma}}_p$ at H_p. Precisely,*

$$CPe_{Inv(\widetilde{\gamma}_p,t_0),H_p}(t) = Pe_{\widetilde{\gamma}_p,H_p}(t).$$

Proof. By Proposition 5.2, $\widetilde{\mu}_I(t) = -\widetilde{\nu}_p(t)$. Thus

$$\begin{aligned} CPe_{Inv(\widetilde{\gamma}_p,t_0),H_p}(t) &= \frac{1}{\sqrt{1-(H_p\cdot\widetilde{\mu}_I(t))^2}}(H_p-(H_p\cdot\widetilde{\mu}_I(t))\widetilde{\mu}_I(t)) \\ &= \frac{1}{\sqrt{1-(H_p\cdot(-\widetilde{\nu}_p(t)))^2}}(H_p-(H_p\cdot(-\widetilde{\nu}_p(t)))(-\widetilde{\nu}_p(t))) \\ &= \frac{1}{\sqrt{1-(H_p\cdot\widetilde{\nu}_p(t))^2}}(H_p-(H_p\cdot\widetilde{\nu}_p(t))\widetilde{\nu}_p(t)) \\ &= Pe_{\widetilde{\gamma}_p,H_p}(t). \end{aligned}$$

□

Example 5.5. Suppose that $\gamma_p : I \to S^2_+$ is a spacelike front given by

$$\gamma_p(t) = \left\{1,\ A,\ At^2,\ At^3\right\}$$

and

$$\widetilde{\gamma}_p(t) = \left\{0,\ A,\ At^2,\ At^3\right\},$$

where $A = \dfrac{1}{\sqrt{1+t^6+t^4}}$. Take $\widetilde{\boldsymbol{\nu}}_p(t) : I \to S^2_0$:

$$\widetilde{\boldsymbol{\nu}}_p(t) = \left\{0, -\frac{t^3}{\sqrt{t^6+9t^2+4}}, \frac{3t}{\sqrt{t^6+9t^2+4}}, -\frac{2}{\sqrt{t^6+9t^2+4}}\right\},$$

$\widetilde{\boldsymbol{\mu}}_p(t) : I \to S^2_0$:

$$\widetilde{\boldsymbol{\mu}}_p(t) = \left\{0,\ N\left(3t^4+2t^2\right),\ N\left(t^6-2\right),\ -N\left(t^5+3t\right)\right\},$$

where

$$N = \frac{1}{\sqrt{4+t^6+9t^2}\sqrt{1+t^6+t^4}}.$$

Here the curvature pair

$$(\widetilde{\beta}_p(t), \widetilde{\ell}_p(t)) = \left(-\frac{t}{(1+t^6+t^4)^{3/2}}, \frac{6}{4+t^6+9t^2}\sqrt{1+t^6+t^4}\right).$$

Moreover,

$$\left\langle \widetilde{\gamma}_p, \widetilde{\boldsymbol{\nu}}_p \right\rangle = 0 \ and\ \left\langle \widetilde{\gamma}'_p, \widetilde{\boldsymbol{\nu}}_p \right\rangle = 0.$$

Thus $(\widetilde{\gamma}_p, \widetilde{\boldsymbol{\nu}}_p)$ is a spacelike Legendrian curve.

Let $K_p = \widetilde{\gamma}_p(0) = (0,0,1,0)$, then $K_p \in S^2_0 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. Hence, the pedal curve

$$Pe_{\widetilde{\gamma}_p,K_p}(t) = \left(0, \frac{PK_1}{G}, \frac{PK_2}{G}, \frac{PK_3}{G}\right),$$

where

$$PK_1 = \frac{3t^4}{t^6+9t^2+4},$$

$$PK_2 = 1 - \frac{9t^2}{t^6+9t^2+4},$$

$$PK_3 = \frac{6t}{t^6 + 9t^2 + 4},$$

$$G = \sqrt{1 - \frac{9t^2}{t^6 + 9t^2 + 4}}.$$

Then we draw the picture of $Pe_{\widetilde{\gamma}_p,K_p}$ at the point K_p into three dimensional projection spaces (see Fig. 1).

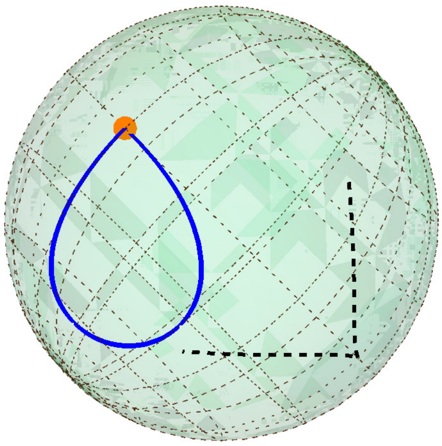

Fig. 1. The projections of the images of frontal $\widetilde{\gamma}$ (black dashed line) and its pedal curve (blue line), the pedal point K_p (orange).

Let $K_p = \widetilde{\gamma}_p(0) = (0, 0, 1, 0)$, then $K_p \in S^2_0 \setminus \{\pm\widetilde{\mu}_p(t) \mid t \in I\}$ is a singular point of $\widetilde{\gamma}_p$. Hence, the contrapedal curve

$$CPe_{\widetilde{\gamma}_p,K_p}(t) = (0, \frac{CK_1}{A}, \frac{CK_2}{A}, \frac{CK_3}{A}),$$

where

$$CK_1 = \frac{t^2\left(3t^2 + 2\right)\left(2 - t^6\right)}{(t^6 + 9t^2 + 4)\,(t^6 + t^4 + 1)},$$

$$CK_2 = 1 - \frac{\left(t^6 - 2\right)^2}{(t^6 + 9t^2 + 4)\,(t^6 + t^4 + 1)},$$

$$Ck_3 = \frac{t\left(t^4 + 3\right)\left(t^6 - 2\right)}{(t^6 + 9t^2 + 4)\,(t^6 + t^4 + 1)},$$

$$A = \sqrt{1 - \frac{(t^6 - 2)^2}{(t^6 + 9t^2 + 4)(t^6 + t^4 + 1)}}.$$

Then we draw the picture of $CPe_{\widetilde{\gamma}_p, K_p}$ at the point K_p into three dimensional projection spaces (see Fig. 2).

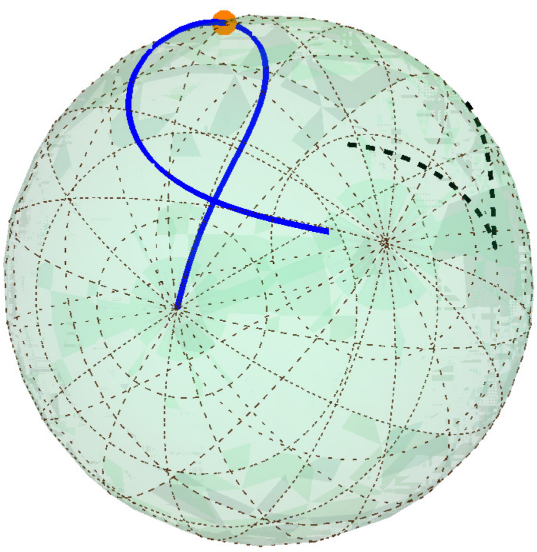

Fig. 2. The projections of the images of frontal $\widetilde{\gamma}$ (black dashed line) and its pedal curve (blue line), the pedal point K_p (orange).

Let $H_p = \widetilde{\gamma}_p(0) = (0, 1, 0, 0)$, then $H_p \in S_0^2 \setminus \{\pm\widetilde{\nu}_p(t) \mid t \in I\}$. Hence, the pedal curve $Pe_{\widetilde{\gamma}_p, H_p}(t) = (0, PH_1, PH_2, PH_3)$, where

$$PH_1 = \sqrt{\frac{9t^2 + 4}{t^6 + 9t^2 + 4}},$$

$$PH_2 = \frac{3t^4}{9t^2 + 4}\sqrt{\frac{9t^2 + 4}{t^6 + 9t^2 + 4}},$$

$$PH_3 = -\frac{2t^3}{9t^2 + 4}\sqrt{\frac{9t^2 + 4}{t^6 + 9t^2 + 4}}.$$

Then we draw the picture of $Pe_{\widetilde{\gamma}_p, H_p}$ at the point H_p into three dimensional projection spaces (see Fig. 3).

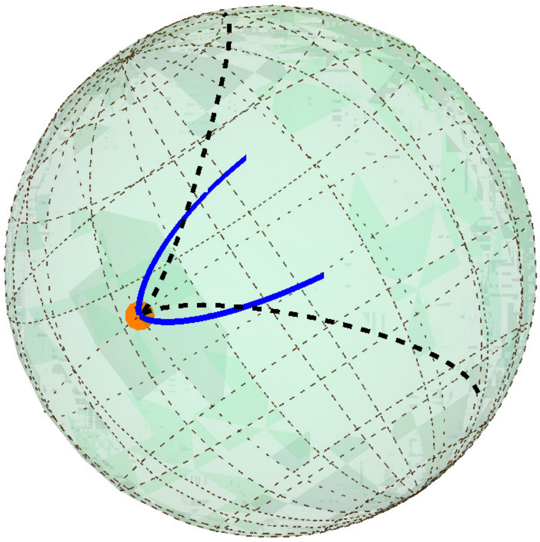

Fig. 3. The projections of the images of frontal $\widetilde{\gamma}$ (black dashed line) and its pedal curve (blue line), the pedal point H_p (orange).

Let $H_p = \widetilde{\gamma}_p(0) = (0, 1, 0, 0)$, then $H_p \in S_0^2 \setminus \{\pm\widetilde{\mu}_p(t) \mid t \in I\}$ is a singular point of $\widetilde{\gamma}_p$. Hence, the contrapedal curve

$$CPe_{\widetilde{\gamma}_p, H_p}(t) = (0, \frac{CH_1}{M}, \frac{CH_2}{M}, \frac{CH_3}{M}),$$

where

$$CH_1 = 1 - \frac{t^4\left(3t^2+2\right)^2}{(t^6+9t^2+4)\,(t^6+t^4+1)},$$

$$CH_2 = -\frac{t^2\left(3t^2+2\right)\left(t^6-2\right)}{(t^6+9t^2+4)\,(t^6+t^4+1)},$$

$$CH_3 = \frac{t^3\left(3t^2+2\right)\left(t^4+3\right)}{(t^6+9t^2+4)\,(t^6+t^4+1)},$$

$$M = \frac{1}{\sqrt{1 - \frac{t^4(3t^2+2)^2}{(t^6+9t^2+4)\left(t^6+t^4+1\right)}}}.$$

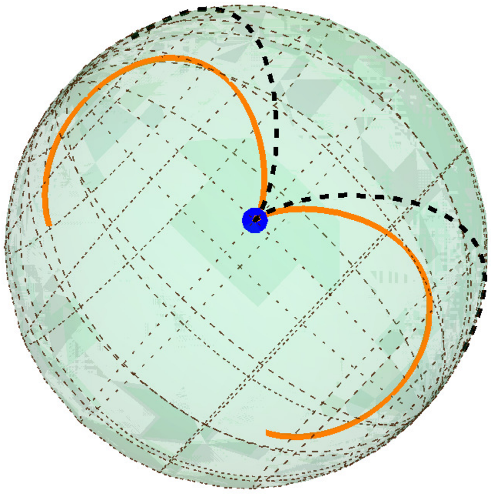

Fig. 4. The projections of the images of frontal $\widetilde{\gamma}$ (black dashed line) and its contrapedal curve (orange line), the contrapedal point H_p (blue).

Noticing that $\widetilde{\beta}_p(0)(H_p \cdot \widetilde{\gamma}_p(0)) - \widetilde{\ell}_p(0)(H_p \cdot \widetilde{\nu}_p(0)) = 0, \widetilde{\beta}_p(0) = 0, \widetilde{\ell}_p(0) \neq 0, H_p = \widetilde{\gamma}_p(0)$. Hence, $CPe_{\widetilde{\gamma}_p,H_p}$ is singular at $t = 0$. Then we draw the picture of $CPe_{\widetilde{\gamma}_p,H_p}$ at the point H_p into projection 3-spaces (see Fig. 4).

Acknowledgment. The authors would like to thank the reviewers for helpful comments to improve the original manuscript. This work was supported by Hainan Provincial Natural Science Foundation of China (125QN292), the Research Foundation for Talented Scholars of the Hainan Normal University (HSZK-KYQD-202402).

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