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Hyper generalized quasi-Einstein manifolds and their applications

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Abstract. The object of the present paper is to study hyper generalized quasi-Einstein (*HGQE*) manifolds, focusing on their geometric and physical contributions. Among others things, it establishes that conharmonically flat Ricci semisymmetric *HGQE* manifolds enforce co-directionality among generators. Further, we prove that if the timelike vector field is a torse-forming vector field, then a *HGQE* spacetime with a Ricci soliton is a perfect fluid spacetime. We also show that a *HGQE* spacetime obeying Einstein's field equation with an energy-momentum tensor fulfilling the Codazzi condition is a Yang pure space. Ultimately, we construct an example to demonstrate the existence of *HGQE* spacetime.

1. Introduction

A Riemannian manifold (M^n, g) $(n \geq 3)$ is called an Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is in the following form

$$\mathrm{Ric} = \frac{r}{n} g, \tag{1.1}$$

where r denotes the scalar curvature. This proportionality condition (1.1) is termed the Einstein metric equation. Einstein manifolds derived their name from Albert Einstein due to the mathematical equivalence between the Einstein metric equation and a solution to the vacuum Einstein field equation. Thus, they serve as indispensable tools in general relativity.

Chaki and Maity [4] introduced a class of manifolds as a generalization of Einstein manifolds, termed quasi-Einstein manifolds. A Riemannian manifold (M^n, g) $(n \geq 3)$ is called a quasi-Einstein manifold if its Ricci tensor is not identically zero and satisfies the following condition

$$\mathrm{Ric}(X, Y) = \psi_1 g(X, Y) + \psi_2 A(X)A(Y) \quad \forall X, Y \in \mathfrak{X}(M). \tag{1.2}$$

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Here ψ_1 and ψ_2 are scalars of which $\psi_2 \neq 0$. A is a nonzero 1-form that satisfies

$$g(X, \theta_1) = A(X), \quad g(\theta_1, \theta_1) = A(\theta_1) = 1,$$

the vector field θ_1 is recognized as the generator of the manifold.

In general relativity, the Robertson-Walker spacetime, which we are familiar with, is a *QE* manifold. Further generalizations are also known as extended quasi-Einstein manifolds [18], generalized quasi-Einstein (*GQE*) manifolds [3, 8], mixed generalized quasi-Einstein manifolds[1] and many others.

In this paper, we will study a new type of generalization of quasi-Einstein manifold introduced by Shaikh, Özgür and Patra [27], called the hyper generalized quasi-Einstein manifold.

Definition 1.1. *A Riemannian manifold* (M^n, g) $(n \geq 3)$ *is said to be a hyper generalized quasi-Einstein manifold if its Ricci tensor is not identically zero with satisfying*

$$\begin{aligned}\mathrm{Ric}(X, Y) =&\psi_1 g(X, Y) + \psi_2 A(X)A(Y) + \psi_3\Big(A(X)B(Y) + B(X)A(Y)\Big)\\ &+ \psi_4\Big(A(X)D(Y) + D(X)A(Y)\Big),\end{aligned} \tag{1.3}$$

where ψ_1, ψ_2, ψ_3, ψ_4 *are scalars, called the associated scalars of this manifold, with* $\psi_2 \neq 0$, $\psi_3 \neq 0$, $\psi_4 \neq 0$. *A, B and D are nonzero 1-forms such that*

$$A(X) = g(X, \theta_1), \quad B(X) = g(X, \theta_2), \quad D(X) = g(X, \theta_3),$$

the vector fields θ_1, θ_2 *and* θ_3*, called generators of the manifold, are mutually orthogonal unit vector fields, i.e.*

$$g(\theta_1, \theta_2) = g(\theta_1, \theta_3) = g(\theta_2, \theta_3) = 0, \quad g(\theta_1, \theta_1) = g(\theta_2, \theta_2) = g(\theta_3, \theta_3) = 1.$$

If $\psi_4 = 0$ in (1.3), then the manifold reduces to a *GQE* manifold [3]. If $\psi_3 = \psi_4 = 0$ in (1.3), then the manifold turns into a *QE* manifold.

The meaning of *HGQE* manifold lies in the fact that such a connected semi-Riemannian manifold (M^4, g) endowed with a Lorentzian metric g of signature $(-, +, +, +)$ is relevant to study of general relativistic viscous fluid spacetime that admits heat flux. In [2], Bhunia, Pahan and Bhattacharyya gave an example of a physical model of the *HGQE* manifold.

Example 1.2. *If a connected 4-dimensional semi-Riemannian viscous fluid spacetime, which admits heat flux, obeys Einstein's field equation. Let a* $(0, 2)$*-type tensor field* T*, defined as the energy momentum tensor, denote the matter distribution of such a fluid. Thus, we have*

$$\begin{aligned}T(X, Y) =&\rho g(X, Y) + (\sigma + \rho)A(X)A(Y) + A(X)B(Y) + B(X)A(Y)\\ &+ A(X)D(Y) + D(X)A(Y)\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}&A(X) = g(X, \theta_1), \quad B(X) = g(X, \theta_2), \quad g(X, \theta_2) = g(X, \theta_3),\\ &A(\theta_1) = -1, \quad B(\theta_2) = D(\theta_3) = 1,\\ &g(\theta_1, \theta_2) = g(\theta_2, \theta_3) = g(\theta_1, \theta_3) = 0,\end{aligned}$$

where ρ *and* σ *represent the isotropic pressure and density, respectively. Also,* θ_1, θ_2 *and* θ_3 *are the timelike vector field (velocity vector field), the heat conduction vector field and the stress vector field, respectively.*

Next, the Einstein's field equation is given by [22]

$$\mathrm{Ric}(X, Y) - \frac{r}{2} g(X, Y) + \lambda g(X, Y) = \kappa T(X, Y), \tag{1.5}$$

where λ is known as the cosmological constant and κ is known as the gravitational constant. By virtue of (1.4), (1.5) takes the following form

$$\begin{aligned}\mathrm{Ric}(X,Y) =&(\kappa\rho + \frac{r}{2} - \lambda)g(X,Y) + \kappa(\sigma+\rho)A(X)A(Y)\\ &+\kappa\Big(A(X)B(Y) + B(X)A(Y)\Big)\\ &+\kappa\Big(A(X)D(Y) + D(X)A(Y)\Big),\end{aligned}$$

which implies that the spacetime is a HGQE manifold.

In 2018, Debnath [11] discussed the geometric properties of *HGQE* manifolds. In 2021, Bhunia, Pahan and Bhattacharyya [2] studied some physical applications of viscous-fluid *HGQE* spacetimes in general relativity. The same year, Chattopadhyay, Bhattacharyya and Debnath [5] investigated Ricci pseudosymmetry and projective pseudosymmtry on *HGQE* spacetimes.

Based on these studies, we study certain geometric properties of *HGQE* manifolds and apply some of them to physical research. To extend the results of Chattopadhyay, Bhattacharyya and Debnath [5] under curvature conditions, we consider a conharmonically flat Ricci semisymmetric *HGQE* manifold and prove that if zero is not an eigenvalue of the Ricci operator, then the vector fields θ_1 and $\theta_2+\theta_3$ are co-directional. Moreover, generalizing the work of Huang, Zhou and Lu [19] on some geometric properties of mixed quasi-Einstein manifolds admitting Ricci solitons, we investigate some physical applications of *HGQE* manifolds with Ricci solitons. Our results show that if the timelike vector field θ_1 is a torse-forming vector field, then a *HGQE* spacetime endowed with a Ricci soliton (g, θ_1, Λ) is a perfect fluid spacetime.

The structure of the paper is as follows.

In Section 2, we discuss *HGQE* manifolds with the generators θ_1, θ_2 and θ_3 serving as Killing vector fields and recurrent vector fields. Subsequently, we analyze *HGQE* manifolds admitting Codazzi type of Ricci tensor. In Section 3, we investigate some curvature properties of *HGQE* manifolds, namely Ricci semisymmetry and conharmonically flat Ricci semisymmetry. In Section 4, we study *HGQE* manifolds with Ricci solitons. In Section 5, we consider *HGQE* spacetimes under Einstein's field equation. Section 6 deals with physical applications of *HGQE* spacetimes. Finally, a non-trivial *HGQE* spacetime's example is presented.

2. Some Special Vector Fields

Now, setting $X = Y = e_i$ in (1.3), where $\{e_i : i = 1, 2, ..., n\}$ is an orthonormal basis of the tangent space at each point of the manifold, and summing over i with $1 \le i \le n$, we find

$$r = n\psi_1 + \psi_2. \tag{2.1}$$

In the section, we discuss some geometric properties of *HGQE* manifolds with special vector fields.

2.1. *HGQE manifolds with the generators θ_1, θ_2 and θ_3 serving as Killing vector fields*

Let the generators θ_1, θ_2 and θ_3 of *HGQE* manifolds be Killing vector fields. Then, one has

$$(\mathcal{L}_{\theta_1}g)(X,Y) = 0, \quad (\mathcal{L}_{\theta_2}g)(X,Y) = 0, \quad (\mathcal{L}_{\theta_3}g)(X,Y) = 0, \tag{2.2}$$

where $\mathcal{L}$ denotes the Lie derivative.

In this subsection, we consider a *HGQE* manifold with Killing vector fields θ_1, θ_2 and θ_3. So, we have the following theorem.

Theorem 2.1. *If the generators of a HGQE manifold are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor.*

Proof. Eq. (2.2) can be written as

$$g(\nabla_X\theta_1, Y) + g(X, \nabla_Y\theta_1) = 0,$$
$$g(\nabla_X\theta_2, Y) + g(X, \nabla_Y\theta_2) = 0,$$
$$g(\nabla_X\theta_3, Y) + g(X, \nabla_Y\theta_3) = 0.$$

It is clear that

$$g(\nabla_X\theta_1, Y) = (\nabla_X A)(Y), \quad g(\nabla_X\theta_2, Y) = (\nabla_X B)(Y)$$

and

$$g(\nabla_X\theta_3, Y) = (\nabla_X D)(Y).$$

Thus

$$\begin{aligned}
&(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0,\\
&(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0,\\
&(\nabla_X D)(Y) + (\nabla_Y D)(X) = 0 \quad \forall X, Y \in \mathfrak{X}(M).
\end{aligned} \tag{2.3}$$

Since $\psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, it follows from (1.3) that

$$\begin{aligned}
(\nabla_Z \mathrm{Ric})(X, Y) =&\psi_2\Big((\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)\Big)\\
&+ \psi_3\Big((\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y)\\
&+ (\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z A)(Y)\Big)\\
&+ \psi_4\Big((\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y)\\
&+ (\nabla_Z D)(X)A(Y) + D(X)(\nabla_Z A)(Y)\Big).
\end{aligned} \tag{2.4}$$

Finally, combining (2.3) and (2.4), we obtain

$$(\nabla_X \mathrm{Ric})(Y, Z) + (\nabla_Y \mathrm{Ric})(Z, X) + (\nabla_Z \mathrm{Ric})(X, Y) = 0.$$

This implies that the manifold has cyclic parallel Ricci tensor [21]. □

2.2. HGQE manifolds with the generators θ_1, θ_2 and θ_3 serving as recurrent vector fields

A vector field θ is called a recurrent vector field if it satisfies [26]

$$\nabla_X\theta = \omega(X)\theta,$$

where ω is a nonzero 1-form.

A Riemannian manifold (M^n, g) ($n \geq 3$) is referred to as a generalized Ricci recurrent manifold if its Ricci tensor satisfies [9]

$$(\nabla_Z \mathrm{Ric})(X, Y) = \alpha(Z)\mathrm{Ric}(X, Y) + \beta(Z)g(X, Y),$$

where α and β are nonzero 1-forms. If $\beta = 0$, then (M^n, g) becomes a Ricci recurrent manifold [23].

Let the generators of a *HGQE* manifold be recurrent vector fields. Therefore, we obtain the following theorem.

Theorem 2.2. *If the generators of a HGQE manifold are recurrent vector fields admitting the same vector of recurrence. Suppose that the associated scalars are constants. Then the manifold is a generalized Ricci recurrent manifold.*

Proof. Since the generators are recurrent vector fields with the same vector of recurrence, it follows that

$$(\nabla_Z A)(X) = \pi(Z)A(X), \tag{2.5}$$
$$(\nabla_Z B)(X) = \pi(Z)B(X), \tag{2.6}$$
$$(\nabla_Z D)(X) = \pi(Z)D(X) \quad \forall X, Z \in \mathfrak{X}(M), \tag{2.7}$$

where π is a nonzero 1-form. It is noted that $\psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{R}$. Thus, making use of (2.5), (2.6) and (2.7) in (2.4), we obtain

$$\begin{aligned}(\nabla_Z \mathrm{Ric})(X, Y) =& 2\pi(Z)\psi_2 A(X)A(Y)\\ &+ 2\pi(Z)\psi_3\big(A(X)B(Y) + B(X)A(Y)\big)\\ &+ 2\pi(Z)\psi_4\big(A(X)D(Y) + D(X)A(Y)\big),\end{aligned}$$

i.e.

$$(\nabla_Z \mathrm{Ric})(X, Y) = \eta_1(Z)\mathrm{Ric}(X, Y) + \eta_2(Z)g(X, Y),$$

where $\eta_1(Z) = 2\pi(Z)$ and $\eta_2(Z) = -2\psi_1\pi(Z)$. □

2.3. *HGQE manifolds admitting Codazzi type of Ricci tensor*

If the Ricci tensor of a Riemannian manifold (M^n, g) $(n \geq 3)$ satisfies

$$(\nabla_Z \mathrm{Ric})(X, Y) = (\nabla_X \mathrm{Ric})(Z, Y), \tag{2.8}$$

then (M^n, g) is said to admit Codazzi type of Ricci tensor [14].

When we consider that a *HGQE* manifold admits Eq. (2.8), we obtain the following theorem.

Theorem 2.3. *Let (M^n, g) be a HGQE manifold satisfying Codazzi type of Ricci tensor. Suppose that the associated scalars are constants.*

(i) If the generators θ_2 and θ_3 are parallel vector fields, then the 1-form A is closed.
(ii) If the generators θ_1 and θ_3 are parallel vector fields, then the 1-form B is closed.
(iii) If the generators θ_1 and θ_2 are parallel vector fields, then the 1-form D is closed.

Proof. As $\psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{R}$, taking $Y = \theta_1$ in (2.8) gives

$$\begin{aligned}\psi_2\big((\nabla_Z A)(X) - (\nabla_X A)(Z)\big) &+ \psi_3\big(A(X)(\nabla_Z B)(\theta_1) - A(Z)(\nabla_X B)(\theta_1)\\ &+ (\nabla_Z B)(X) - (\nabla_X B)(Z)\big)\\ &+ \psi_4\big(A(X)(\nabla_Z D)(\theta_1) - A(Z)(\nabla_X D)(\theta_1)\\ &+ (\nabla_Z D)(X) - (\nabla_X D)(Z)\big) = 0.\end{aligned} \tag{2.9}$$

In virtue of $Z(g(\theta_1, \theta_2)) = Z(g(\theta_1, \theta_3)) = 0$, which indicates that

$$\begin{aligned}&(\nabla_Z B)(\theta_1) = -(\nabla_Z A)(\theta_2),\\ &(\nabla_Z D)(\theta_1) = -(\nabla_Z A)(\theta_3) \quad \forall Z \in \mathfrak{X}(M).\end{aligned} \tag{2.10}$$

Therefore, (2.9) can be expressed as

$$\begin{aligned}\psi_2\big((\nabla_Z A)(X) - (\nabla_X A)(Z)\big) &+ \psi_3\big(A(Z)(\nabla_X A)(\theta_2) - A(X)(\nabla_Z A)(\theta_2)\\ &+ (\nabla_Z B)(X) - (\nabla_X B)(Z)\big)\\ &+ \psi_4\big(A(Z)(\nabla_X A)(\theta_3) - A(X)(\nabla_Z A)(\theta_3)\\ &+ (\nabla_Z D)(X) - (\nabla_X D)(Z)\big) = 0.\end{aligned} \tag{2.11}$$

Assuming that the generators θ_1, θ_2 and θ_3 are parallel vector fields. Then

$$(\nabla_Z A)(X) = 0, \tag{2.12}$$
$$(\nabla_Z B)(X) = 0, \tag{2.13}$$
$$(\nabla_Z D)(X) = 0 \quad \forall X, Z \in \mathfrak{X}(M). \tag{2.14}$$

Substituting (2.13) and (2.14) in (2.9) yields

$$\psi_2\Big((\nabla_Z A)(X) - (\nabla_X A)(Z)\Big) = 0.$$

Since $\psi_2 \neq 0$, the above gives

$$(\nabla_Z A)(X) - (\nabla_X A)(Z) = 0,$$

i.e.

$$dA(X, Z) = 0.$$

Similarly, using (2.12) and (2.14) in (2.11) gives

$$\psi_3\Big((\nabla_Z B)(X) - (\nabla_X B)(Z)\Big) = 0.$$

As $\psi_3 \neq 0$, this shows that

$$dB(X, Z) = 0.$$

Finally, inserting (2.12) and (2.13) in (2.11), we get

$$\psi_4\Big((\nabla_Z D)(X) - (\nabla_X D)(Z)\Big) = 0.$$

As $\psi_4 \neq 0$, this establishes that $dD(X, Z) = 0$. □

3. Some Curvature Properties

If the Ricci tensor of a Riemannian manifold (M^n, g) $(n \geq 3)$ satisfies

$$\nabla \mathrm{Ric} = 0, \tag{3.1}$$

then the manifold is said to be Ricci symmetric.

If the Riemannian curvature tensor and Ricci tensor of a Riemannian manifold (M^n, g) $(n \geq 3)$ satisfies

$$\mathrm{R}(X, Y) \cdot \mathrm{Ric} = 0, \tag{3.2}$$

then the manifold is said to be Ricci semisymmetric [12].

In a Riemannian manifold (M^n, g) $(n \geq 3)$, the conharmonic curvature tensor C of type $(1, 3)$ is given by [20]

$$\mathrm{C}(X, Y)Z = \mathrm{R}(X, Y)Z - \frac{1}{n-2}\Big(\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y + g(Y, Z)LX - g(X, Z)LY\Big), \tag{3.3}$$

where the Ricci operator L is defined by $g(LX, Y) = \mathrm{Ric}(X, Y)$. It is clear that for $\mathrm{C} = 0$, then the manifold becomes a conharmonically flat manifold.

In this section, we consider the curvature-related properties of *HGQE* manifolds.

3.1. Ricci symmetric HGQE manifolds

When *HGQE* manifold is Ricci symmetric, we obtain the following theorem.

Theorem 3.1. *Let (M^n, g) be a HGQE manifold with $\psi_3 = \psi_4$. If (M^n, g) is Ricci symmetric, then the associated scalars are constants and the vector fields θ_1 and $\theta_2 + \theta_3$ are parallel vector fields.*

Proof. Taking the covariant derivative of (1.3) along the vector field Z and using (3.1) gives

$$\begin{aligned}(\nabla_Z \mathrm{Ric})(X,Y) =& Z(\psi_1)g(X,Y) + Z(\psi_2)A(X)A(Y)\\ &+ \psi_2\Big((\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)\Big)\\ &+ Z(\psi_3)\Big(A(X)B(Y) + B(X)A(Y)\Big)\\ &+ \psi_3\Big((\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y)\\ &+ (\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z A)(Y)\Big)\\ &+ Z(\psi_4)\Big(A(X)D(Y) + D(X)A(Y)\Big)\\ &+ \psi_4\Big((\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y)\\ &+ (\nabla_Z D)(X)A(Y) + D(X)(\nabla_Z A)(Y)\Big) = 0.\end{aligned} \tag{3.4}$$

Taking $X = Y = \theta_1$ in (3.4) gives

$$Z(\psi_1) + Z(\psi_2) + 2\psi_3(\nabla_Z B)(\theta_1) + 2\psi_4(\nabla_Z D)(\theta_1) = 0. \tag{3.5}$$

Setting $X = Y = \theta_2$ in (3.4) leads to

$$Z(\psi_1) + 2\psi_3(\nabla_Z A)(\theta_2) = 0. \tag{3.6}$$

Again, letting $X = Y = \theta_3$ in (3.4) yields

$$Z(\psi_1) + 2\psi_4(\nabla_Z A)(\theta_3) = 0. \tag{3.7}$$

In view of (2.10), (3.6) and (3.7), (3.5) turns into

$$Z(3\psi_1 + \psi_2) = 0. \tag{3.8}$$

Next, contracting (3.4) over X, Y and using (2.10), one gets

$$Z(n\psi_1 + \psi_2) = 0. \tag{3.9}$$

Combining (3.8) and (3.9)($n \neq 3$), we conclude $\psi_1 \in \mathbb{R}$ and $\psi_2 \in \mathbb{R}$. Thus, (3.4) reduces to

$$\begin{aligned}(\nabla_Z \mathrm{Ric})(X,Y) =& \psi_2\Big((\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)\Big)\\ &+ Z(\psi_3)\Big(A(X)B(Y) + B(X)A(Y)\Big)\\ &+ \psi_3\Big((\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y)\\ &+ (\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z A)(Y)\Big)\\ &+ Z(\psi_4)\Big(A(X)D(Y) + D(X)A(Y)\Big)\\ &+ \psi_4\Big((\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y)\\ &+ (\nabla_Z D)(X)A(Y) + D(X)(\nabla_Z A)(Y)\Big) = 0.\end{aligned} \tag{3.10}$$

Since $\psi_3 \neq 0$ and $\psi_4 \neq 0$, it follows from (3.6) and (3.7) that

$$(\nabla_Z A)(\theta_2) = (\nabla_Z B)(\theta_1) = 0$$

and

$$(\nabla_Z A)(\theta_3) = (\nabla_Z D)(\theta_1) = 0.$$

Substituting $Y = \theta_1$ in (3.10) yields

$$\psi_2(\nabla_Z A)(X) + Z(\psi_3)B(X) + \psi_3(\nabla_Z B)(X) + Z(\psi_4)D(X) + \psi_4(\nabla_Z D)(X) = 0. \tag{3.11}$$

Setting $Y = \theta_2$ in (3.10) leads to

$$Z(\psi_3)A(X) + \psi_3(\nabla_Z A)(X) + \psi_4 A(X)(\nabla_Z D)(\theta_2) = 0. \tag{3.12}$$

Taking $X = \theta_2$ in (3.11) gives

$$Z(\psi_3) + \psi_4(\nabla_Z D)(\theta_2) = 0. \tag{3.13}$$

Similarly, putting $X = \theta_3$ in (3.11), we get

$$Z(\psi_4) + \psi_3(\nabla_Z B)(\theta_3) = 0. \tag{3.14}$$

Since $Z(g(\theta_2, \theta_3)) = 0$, it follows that

$$(\nabla_Z B)(\theta_3) + (\nabla_Z D)(\theta_2) = 0.$$

Noticing that $\psi_3 = \psi_4$, from (3.13) and (3.14), we establish $\psi_3, \psi_4 \in \mathbb{R}$ and $(\nabla_Z D)(\theta_2) = 0$. Consequently, (3.12) simplifies to

$$\psi_3(\nabla_Z A)(Y) = 0 \Rightarrow \quad g(\nabla_Z \theta_1, Y) = 0 \quad (as \quad \psi_3 \neq 0),$$

which implies that θ_1 is a parallel vector field.

Based on all the above results, (3.11) becomes

$$(\nabla_Z B)(X) + (\nabla_Z D)(X) = 0 \Rightarrow g\Big(\nabla_Z(\theta_2 + \theta_3), Y\Big) = 0,$$

which means that $\theta_2 + \theta_3$ is also a parallel vector field. □

3.2. Ricci semisymmetric HGQE manifolds

Now, we turn to study a Ricci semisymmetric *HGQE* manifold and obtain the following theorem.

Theorem 3.2. *If a HGQE manifold is Ricci semisymmetric, then the curvature tensor satisfies*

$$\mathrm{R}(X, Y, \theta_i, \theta_j) = 0, \quad (i, j \in \{1, 2, 3\}).$$

Proof. In view of (1.3), (3.2) can be written as

$$\begin{aligned}
&\psi_1\Big(g(\mathrm{R}(X,Y)Z, W) + g(Z, \mathrm{R}(X,Y)W)\Big) \\
&\quad + \psi_2\Big(A(\mathrm{R}(X,Y)Z)A(W) + A(\mathrm{R}(X,Y)W)A(Z)\Big) \\
&\quad + \psi_3\Big(A(\mathrm{R}(X,Y)Z)B(W) + B(\mathrm{R}(X,Y)Z)A(W) \\
&\quad + A(\mathrm{R}(X,Y)W)B(Z) + B(\mathrm{R}(X,Y)W)A(Z)\Big) \\
&\quad + \psi_4\Big(A(\mathrm{R}(X,Y)Z)D(W) + D(\mathrm{R}(X,Y)Z)A(W) \\
&\quad + A(\mathrm{R}(X,Y)W)D(Z) + D(\mathrm{R}(X,Y)W)A(Z)\Big) = 0.
\end{aligned}$$

Since $g(\mathrm{R}(X,Y)Z,W) = \mathrm{R}(X,Y,Z,W)$, the above gives

$$\begin{aligned}&\psi_2\Big(A(\mathrm{R}(X,Y)Z)A(W) + A(\mathrm{R}(X,Y)W)A(Z)\Big)\\&\quad + \psi_3\Big(A(\mathrm{R}(X,Y)Z)B(W) + B(\mathrm{R}(X,Y)Z)A(W)\\&\quad + A(\mathrm{R}(X,Y)W)B(Z) + B(\mathrm{R}(X,Y)W)A(Z)\Big)\\&\quad + \psi_4\Big(A(\mathrm{R}(X,Y)Z)D(W) + D(\mathrm{R}(X,Y)Z)A(W)\\&\quad + A(\mathrm{R}(X,Y)W)D(Z) + D(\mathrm{R}(X,Y)W)A(Z)\Big) = 0.\end{aligned} \tag{3.15}$$

Taking $Z = W = \theta_1$ in (3.15) gives

$$\psi_3\mathrm{R}(X,Y,\theta_1,\theta_2) + \psi_4\mathrm{R}(X,Y,\theta_1,\theta_3) = 0. \tag{3.16}$$

Setting $Z = W = \theta_2$ in (3.15), we obtain

$$\psi_3\mathrm{R}(X,Y,\theta_2,\theta_1) = 0.$$

Since $\psi_3 \neq 0$, this shows that

$$\mathrm{R}(X,Y,\theta_1,\theta_2) = 0. \tag{3.17}$$

Applying (3.17) in (3.16) reduces it to

$$\psi_4\mathrm{R}(X,Y,\theta_1,\theta_3) = 0.$$

As $\psi_4 \neq 0$, we conclude

$$\mathrm{R}(X,Y,\theta_1,\theta_3) = 0. \tag{3.18}$$

Putting $Z = \theta_1$ and $W = \theta_3$ in (3.15), we get

$$\psi_2\mathrm{R}(X,Y,\theta_1,\theta_3) + \psi_3\mathrm{R}(X,Y,\theta_2,\theta_3) = 0. \tag{3.19}$$

In view of (3.18) and (3.19), there holds

$$\psi_3\mathrm{R}(X,Y,\theta_2,\theta_3) = 0.$$

Since $\psi_3 \neq 0$, it follows that

$$\mathrm{R}(X,Y,\theta_2,\theta_3) = 0.$$

Based on the above results, the remaining cases of $\mathrm{R}(X,Y,\theta_i,\theta_j) = 0, \quad (i,j \in \{1,2,3\})$ are hold and trivial. □

3.3. *Conharmonically flat Ricci semisymmetric HGQE manifolds*

Next, we consider a conharmonically flat Ricci semisymmetric *HGQE* manifold and obtain the following theorem.

Theorem 3.3. *Let (M^n, g) be a conharmonically flat HGQE manifold. If (M^n, g) is Ricci semisymmetric. Suppose that zero is not an eigenvalue of the Ricci operator L. Then*

- *the vector fields θ_1 and θ_2 are co-directional,*
- *the vector fields θ_1 and θ_3 are co-directional,*
- *the vector fields θ_1 and $\theta_2 + \theta_3$ are co-directional.*

Proof. Since the *HGQE* manifold is conharmonically flat, (3.3) yields

$$\mathrm{R}(X,Y)Z = \frac{1}{n-2}\Big(\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + g(Y,Z)LX - g(X,Z)LY\Big).$$

Ricci semisymmetric condition $\mathrm{R}(X,Y)\cdot\mathrm{Ric} = 0$ now gives

$$\begin{aligned} g(Y,Z)\mathrm{Ric}(LX,W) - g(X,Z)\mathrm{Ric}(LY,W) + g(Y,W)\mathrm{Ric}(LX,Z) \\ - g(X,W)\mathrm{Ric}(LY,Z) = 0. \end{aligned} \tag{3.20}$$

Let b be an eigenvalue of L with corresponding eigenvectors X and Y. Then

$$LX = bX, \quad LY = bY.$$

Thus

$$\mathrm{Ric}(LX,W) = b\mathrm{Ric}(X,W), \quad \mathrm{Ric}(LY,W) = b\mathrm{Ric}(Y,W) \quad \forall W \in \mathfrak{X}(M). \tag{3.21}$$

Substituting (3.21) into (3.20) and cancelling the nonzero scalar b leads to

$$\begin{aligned} g(Y,Z)\mathrm{Ric}(X,W) - g(X,Z)\mathrm{Ric}(Y,W) + g(Y,W)\mathrm{Ric}(X,Z) \\ - g(X,W)\mathrm{Ric}(Y,Z) = 0. \end{aligned} \tag{3.22}$$

Next, taking $Z = W = \theta_2$ in (3.22) gives

$$\psi_3\Big(A(X)B(Y) - B(X)A(Y)\Big) = 0.$$

As $\psi_3 \neq 0$, the above gives

$$A(X)B(Y) - B(X)A(Y) = 0. \tag{3.23}$$

This implies that θ_1 and θ_2 are co-directional [6].

Likewise, taking $Z = W = \theta_3$ in (3.22) and using $\psi_4 \neq 0$ gives

$$A(X)D(Y) - D(X)A(Y) = 0. \tag{3.24}$$

This implies that θ_1 and θ_3 are co-directional.

In consequence of (3.23) and (3.24), we obtain

$$A(X)\Big((B(Y) + D(Y)\Big) - \Big(B(X) + D(X)\Big)A(Y) = 0,$$

i.e.

$$A(X)E(Y) - E(X)A(Y) = 0,$$

where

$$E(X) = g(X,\nu) = g(X,\theta_2 + \theta_3).$$

This implies that θ_1 and $\theta_2 + \theta_3$ are also co-directional. □

4. Ricci solitons on the HGQE manifolds

Ricci soliton [16] is a self-similar solution of the Ricci flow and plays a crucial role in understanding the formation of singularity. Its definition is as follows.

Definition 4.1. *A triplet (g, θ, Λ) on a Riemannian manifold (M^n, g) $(n \geq 3)$ is called a Ricci soliton if the vector field θ satisfies*

$$\frac{1}{2}(\mathcal{L}_\theta g)(X, Y) + \mathrm{Ric}(X, Y) = \Lambda g(X, Y) \quad \forall X, Y \in \mathfrak{X}(M). \tag{4.1}$$

Here $\mathcal{L}_\theta$ represents the Lie derivative with respect to θ, and $\Lambda \in \mathbb{R}$.

A Ricci soliton is said to be expanding, steady, or shrinking if Λ is positive, zero, or negative, respectively.

A vector field θ is said to be a θ(Ric)- vector field if it satisfies [17]

$$\nabla_X \theta = cLX,$$

where c is constant and L denotes the Ricci operator. For $c \neq 0$, then the vector field θ is called a proper θ(Ric)- vector field.

A vector field θ is said to be a torse-forming vector field if it satisfies [32]

$$\nabla_X \theta = hX + \phi(X)\theta,$$

where h is a scalar, ϕ is a 1-form.

In this section, we study *HGQE* manifolds with Ricci solitons and prove the following theorems.

Theorem 4.2. *If a HGQE manifold is endowed with (g, θ_1, Λ), then*

- $\Lambda = \psi_1 + \psi_2$,
- *the integral curves of θ_1 being geodesic is equivalent to the manifold being a QE manifold.*

Proof. Since (g, θ_1, Λ) is a Ricci soliton on a *HGQE* manifold, from (4.1), it follows that

$$\begin{aligned}\frac{1}{2}(\mathcal{L}_{\theta_1} g)(X, Y) + \Big(\psi_1 g(X, Y) + \psi_2 A(X)A(Y) + \psi_3\big(A(X)B(Y) + B(X)A(Y)\big)\\ + \psi_4\big(A(X)D(Y) + D(X)A(Y)\big)\Big) = \Lambda g(X, Y).\end{aligned} \tag{4.2}$$

Setting $X = \theta_1$ into (4.2) leads to

$$g(\nabla_{\theta_1}\theta_1, Y) + 2\Big((\psi_1 + \psi_2)A(Y) + \psi_3 B(Y) + \psi_4 D(Y)\Big) = 2\Lambda A(Y). \tag{4.3}$$

Taking $Y = \theta_1$ into (4.3) gives

$$\Lambda = \psi_1 + \psi_2. \tag{4.4}$$

In view of (4.4), (4.3) simplifies to

$$g(\nabla_{\theta_1}\theta_1, Y) = -2\psi_3 B(Y) - 2\psi_4 D(Y). \tag{4.5}$$

Given that the integral curves of θ_1 are geodesic on a *HGQE* manifold. In such a case we have

$$2\psi_3 B(Y) + 2\psi_4 D(Y) = 0. \tag{4.6}$$

Putting $Y = \theta_2$ into (4.6) provides $\psi_3 = 0$. Again, taking $Y = \theta_3$ into (4.6) gives $\psi_4 = 0$. This shows that the manifold is a *QE* manifold.

Conversely, given that the manifold is a *QE* manifold, we get $\psi_3 = \psi_4 = 0$. Thus (4.5) reduces to

$$g(\nabla_{\theta_1}\theta_1, Y) = 0.$$

This shows that $\nabla_{\theta_1}\theta_1 = 0$. □

Theorem 4.3. *If a HGQE manifold is endowed with (g, θ_3, Λ), then*

- $\Lambda = \psi_1$,
- *the integral curves of θ_3 being geodesic is equivalent to the manifold being a GQE manifold.*

Proof. The proof is similar to Theorem 4.2. □

Theorem 4.4. *If a HGQE manifold endowed with (g, θ_1, Λ) is conharmonically flat, then*

- $n\psi_1 + \psi_2 = 0$,
- *the soliton being steady is equivalent to the generator θ_1 being divergence-free.*

Proof. Since a *HGQE* manifold endowed with (g, θ_1, Λ) is conharmonically flat, then

$$\begin{aligned}\mathrm{R}(X, Y, Z, W) = &\frac{1}{n-2}\Big(\mathrm{Ric}(Y, Z)g(X, W) - \mathrm{Ric}(X, Z)g(Y, W) + g(Y, Z)\mathrm{Ric}(X, W) \\ &- g(X, Z)\mathrm{Ric}(Y, W)\Big).\end{aligned} \tag{4.7}$$

Contracting X and W in (4.7) yields

$$r = 0. \tag{4.8}$$

From (2.1) and (4.8), one obtains

$$n\psi_1 + \psi_2 = 0.$$

In view of (4.1) and (4.8), then contracting X and Y in (4.1) leads to

$$\mathrm{div}\theta_1 = n\Lambda.$$

This implies that $\mathrm{div}\theta_1 = 0 \Leftrightarrow \Lambda = 0$. □

Theorem 4.5. *If the generator θ_1 of a HGQE manifold endowed with (g, θ_1, Λ) is a $\theta_1(\mathrm{Ric})$- vector field, then θ_1 is a proper $\theta_1(\mathrm{Ric})$- vector field and the soliton is steady.*

Proof. Let θ_1 be a $\theta_1(\mathrm{Ric})$- vector field. Then

$$(\mathcal{L}_{\theta_1} g)(X, Y) = g(\nabla_X \theta_1, Y) + g(X, \nabla_Y \theta_1) = 2c\mathrm{Ric}(X, Y). \tag{4.9}$$

Since (g, θ_1, Λ) is a Ricci soliton on a *HGQE* manifold, by virtue of (4.9), (4.1) turns into

$$\begin{aligned}(c+1)\Big[&\psi_1 g(X, Y) + \psi_2 A(X)A(Y) + \psi_3\big(A(X)B(Y) + B(X)A(Y)\big) \\ &+ \psi_4\big(A(X)D(Y) + D(X)A(Y)\big)\Big] = \Lambda g(X, Y).\end{aligned} \tag{4.10}$$

Taking $X = Y = \theta_1$ in (4.10) gives

$$\Lambda = (\psi_1 + \psi_2)(c + 1). \tag{4.11}$$

Substituting $X = \theta_1$ and $Y = \theta_2$ in (4.10) yields

$$\psi_3(c + 1) = 0.$$

Since $\psi_3 \neq 0$, we get

$$c = -1. \tag{4.12}$$

This implies that θ_1 is a proper $\theta_1(\mathrm{Ric})$- vector field.

In view of (4.11) and (4.12), we conclude

$$\Lambda = 0.$$

This implies that the soliton is steady. □

Theorem 4.6. *If the generator θ_1 of a HGQE manifold endowed with (g, θ_1, Λ) is a torse-forming vector field, then the manifold reduces to a QE manifold and $h = \psi_2$.*

Proof. Since θ_1 is torse-forming, then

$$\nabla_X \theta_1 = hX + \phi(X)\theta_1. \tag{4.13}$$

Taking the inner product of (4.13) with θ_1, we get

$$\phi(X) = -hA(X). \tag{4.14}$$

Combining (4.13) and (4.14), one has

$$\nabla_X \theta_1 = h\Big(X - A(X)\theta_1\Big). \tag{4.15}$$

Taking $X = \theta_1$ in (4.15) gives

$$\nabla_{\theta_1} \theta_1 = 0. \tag{4.16}$$

A triplet (g, θ_1, Λ) can be expressed as

$$\begin{aligned}&\frac{1}{2}\Big(g(\nabla_X \theta_1, Y) + g(X, \nabla_Y \theta_1)\Big) + \psi_1 g(X, Y) + \psi_2 A(X)A(Y)\\ &\quad + \psi_3\Big(A(X)B(Y) + B(X)A(Y)\Big) + \psi_4\Big(A(X)D(Y) + D(X)A(Y)\Big)\\ &=\Lambda g(X, Y).\end{aligned} \tag{4.17}$$

Substituting $X = \theta_1$ in (4.17) yields

$$\frac{1}{2} g(\nabla_{\theta_1} \theta_1, Y) + (\psi_1 + \psi_2)A(Y) + \psi_3 B(Y) + \psi_4 D(Y) = \Lambda A(Y). \tag{4.18}$$

In virtue of (4.16), (4.18) reduces to

$$(\psi_1 + \psi_2)A(Y) + \psi_3 B(Y) + \psi_4 D(Y) = \Lambda A(Y). \tag{4.19}$$

Taking $Y = \theta_1$ into (4.19) gives

$$\Lambda = \psi_1 + \psi_2. \tag{4.20}$$

Putting $Y = \theta_2$ into (4.19) provides

$$\psi_3 = 0. \tag{4.21}$$

Again, setting $Y = \theta_3$ into (4.19) leads to

$$\psi_4 = 0, \tag{4.22}$$

which means that the manifold becomes a *QE* manifold.

In view of (4.15), (4.21) and (4.22), (4.17) reads as

$$(\psi_1 + h)g(X, Y) + (\psi_2 - h)A(X)A(Y) = \Lambda g(X, Y). \tag{4.23}$$

Substituting $X = Y = \theta_2$ into (4.23) yields

$$\Lambda = \psi_1 + h. \tag{4.24}$$

Comparing (4.20) and (4.24), we get $h = \psi_2$. □

5. HGQE spacetimes obeying Einstein's field equation

First of all, we give the definition of *HGQE* spacetime.

Definition 5.1. *A connected Lorentzian manifold* (M^4, g) *is called a hyper generalized quasi-Einstein spacetime if its Ricci tensor satisfies Eq. (1.3).*

Different from Definition 1.1, where θ_1 *is a unit timelike vector field, that is,* $g(\theta_1, \theta_1) = -1$.

Next, we give some definitions required for the proof of subsequent theorems in this section.

Definition 5.2 ([10]). *The energy momentum tensor* T *of type (0,2) is said to be semisymmetric if it follows the condition*

$$R(X, Y) \cdot T = 0.$$

Definition 5.3 ([22]). *The curl of a vector field* θ *is defined by*

$$(curl\theta)(X, Y) = g(\nabla_X \theta, Y) - g(X, \nabla_Y \theta).$$

If $curl\theta = 0$, *then the vector field is said to be irrotational.*

In the section, we study a *HGQE* spacetime that admits Einstein's field equation and prove the following theorems.

Theorem 5.4. *Let* (M^4, g) *be a HGQE spacetime obeying Einstein's field equation. Then the energy-momentum tensor is of Codazzi type if and only if its Ricci tensor is of Codazzi type.*

Proof. Using [28, Theorem 4.1] to obtain the theorem. □

Theorem 5.5. *In a HGQE spacetime obeying Einstein's field equation, if the energy-momentum tensor is semisymmetric, then the spacetime is Ricci semisymmetric.*

Proof. Using [10, Theorem 2.1] to obtain the theorem. □

Notably, the fundamental features of *HGQE* manifolds are also preserved in connected Lorentzian manifolds. It is merely because in the spacetime that the associated vector field θ_1 corresponding to the 1-form A is assumed to be a unit timelike vector. Hence, all the theorems of *HGQE* manifolds are correct in *HGQE* spacetimes.

Theorem 5.6. *Let* (M^4, g) *be a HGQE spacetime obeying Einstein's field equation. Let the energy-momentum tensor be of Codazzi type. Suppose that the associated scalars are constants.*

(i) If the generators θ_2 *and* θ_3 *are parallel vector fields, then the timelike vector field* θ_1 *is irrotational.*
(ii) If the generators θ_1 *and* θ_3 *are parallel vector fields, then the vector field* θ_2 *is irrotational.*
(iii) If the generators θ_1 *and* θ_2 *are parallel vector fields, then the vector field* θ_3 *is irrotational.*

Proof. By combining Theorem 2.3, Definition 5.3 and Theorem 5.4, we obtain the theorem. □

Theorem 5.7. *Let* (M^4, g) *be a HGQE spacetime obeying Einstein's field equation. If the energy-momentum tensor is semisymmetric, then the curvature tensor satisfies*

$$R(X, Y, \theta_i, \theta_j) = 0, \quad (i, j \in \{1, 2, 3\}).$$

Proof. In view of Theorem 3.2 and Theorem 5.5, we obtain the theorem. □

6. Physical applications of HGQE spacetimes

A Lorentzian manifold (M^4, g) is referred to as a perfect fluid spacetime [7] when its Ricci tensor satisfies Eq. (1.2). Here θ_1 is a unit timelike vector field, satisfying $g(\theta_1, \theta_1) = A(\theta_1) = -1$.

The conformal curvature tensor $\overline{\mathrm{C}}$ of type $(1,3)$ of a Lorentzian manifold (M^4, g) is given by [25]

$$\begin{aligned}\overline{\mathrm{C}}(X,Y)Z =& \mathrm{R}(X,Y)Z - \frac{1}{n-2}\Big(\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + g(Y,Z)LX - g(X,Z)LY\Big) \\ &+ \frac{r}{(n-1)(n-2)}\Big(g(Y,Z)X - g(X,Z)Y\Big),\end{aligned} \tag{6.1}$$

where L denotes Ricci operator and r is scalar curvature. It is clear that for $\overline{\mathrm{C}} = 0$, then the manifold is called a conformally flat manifold.

We suppose that the vector fields U and V correspond respectively to the 1-forms η_1 and η_2. It is easy to see from Theorem 2.2 that $V = -\psi_1 U$, which means that U and V are co-directional. We take U as a unit timelike vector field. In [13], Dey proved the following proposition.

Proposition 6.1. *[13, Proposition 3.2] A conformally flat generalized Ricci recurrent spacetime is a perfect fluid spacetime, provided the vector fields U and V are co-directional.*

Since we also consider that θ_1 is a unit timelike vector field, from Theorem 2.2, we get the following theorem.

Theorem 6.2. *If the generators of a HGQE spacetime are recurrent vector fields admitting the same vector of recurrence. Suppose that the associated scalars are constants. Then the conformally flat spacetime is a perfect fluid spacetime, provided U is a unit timelike vector field.*

The divergence of the conformal curvature tensor $\overline{\mathrm{C}}$ in a spacetime can be expressed as [28]

$$\mathrm{div}\overline{\mathrm{C}}(X,Y,Z) = \frac{1}{2}\Big((\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z)\Big) - \frac{1}{6}\Big(X(r)g(Y,Z) - Y(r)g(X,Z)\Big).$$

Petrov [24] stated that spacetime can be classified into six categories: I, II, III, D, N, and O. In [29], Sharma proved the following theorem.

Theorem 6.3. *If a spacetime with* $\mathrm{div}\overline{\mathrm{C}} = 0$ *admits a conformal Killing vector field, then the spacetime is either of Petrov type N or Petrov type O.*

We suppose that θ_1 is a unit timelike vector field. From Theorem 3.1, we can deduce that $r \in \mathbb{R}$ and then $\mathrm{div}\overline{\mathrm{C}} = 0$. Therefore, we can get the following theorem.

Theorem 6.4. *If a Ricci symmetric HGQE spacetime with $\psi_3 = \psi_4$ admits a conformal Killing vector field, then the spacetime is either of Petrov type N or Petrov type O.*

When we consider θ_1 as a unit timelike vector field. In [30], Suh, Majhi and De stated that $\mathrm{div}\theta_1$ represents the expansion scalar. Thus, from Theorem 4.4, we can obtain the following theorem.

Theorem 6.5. *Let (M^4, g) be a conharmonically flat HGQE spacetime endowed with (g, θ_1, Λ). If the soliton is steady, then the expansion scalar vanishes.*

It is easy to see that in a *HGQE* spacetime, when $\psi_3 = \psi_4 = 0$, then it can be reduced to a perfect fluid spacetime. So, let θ_1 be a unit timelike vector field. From Theorem 4.6, we can state the following theorem.

Theorem 6.6. *If the generator θ_1 of a HGQE spacetime endowed with (g, θ_1, Λ) is a torse-forming vector field, then the spacetime is a perfect fluid spacetime.*

To generalize Einstein's vacuum equations while taking the Yang-Mills equations of gauge theory as their underlying structure, Yang [31] introduced a system of equations called Yang's equations. A sapcetime is said to be a Yang Pure Space [15], if it satisfies Yang's equation (2.8). Thus, from Theorem 5.4, we get the following theorem.

Theorem 6.7. *Let (M^4, g) be a HGQE spacetime obeying Einstein's field equation. If the energy-momentum tensor is of Codazzi type, then the spacetime is a Yang Pure Space.*

7. Example of a HGQE spacetime

In this section, we construct a non-trivial example of hyper generalized quasi-Einstein spacetime to demonstrate its existence.

Example 7.1. *We consider a Lorentzian manifold (M^4, g) with a Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = -\frac{1}{x^2}(dx^1)^2 + \frac{1}{\frac{1}{x^2}-4}(dx^2)^2 + (x^2)^2(dx^3)^2 + (x^2 sinx^3)^2(dx^4)^2,$$

where $i, j = 1, 2, 3, 4$. The non-vanishing components of Christoffel symbols are

$$\Gamma^1_{12} = -\frac{1}{2x^2}, \quad \Gamma^2_{22} = \frac{1}{2x^2(1-4x^2)}, \quad \Gamma^3_{32} = \Gamma^4_{42} = \frac{1}{x^2}, \quad \Gamma^2_{33} = 4x^2 - 1,$$

$$\Gamma^4_{43} = cotx^3, \quad \Gamma^2_{44} = (4x^2-1)(sinx^3)^2, \quad \Gamma^3_{44} = -\frac{sin(2x^3)}{2}.$$

Also, the nonzero curvature tensors are

$$\mathrm{R}_{1221} = -\frac{(1-3x^2)}{(x^2)^3(1-4x^2)}, \quad \mathrm{R}_{1331} = \frac{(1-4x^2)}{2(x^2)^2}, \quad \mathrm{R}_{1441} = \frac{(1-4x^2)(sinx^3)^2}{2(x^2)^2},$$

$$\mathrm{R}_{2332} = \frac{1}{2(4x^2-1)}, \quad \mathrm{R}_{2442} = \frac{(sinx^3)^2}{2(4x^2-1)}, \quad \mathrm{R}_{3443} = x^2(1-5x^2)(sinx^3)^2.$$

Thus, the nonzero Ricci tensors are given by

$$\mathrm{Ric}_{11} = -\frac{1}{(x^2)^3}, \quad \mathrm{Ric}_{22} = -\frac{3}{x^2(1-4x^2)}, \quad \mathrm{Ric}_{33} = -3, \quad \mathrm{Ric}_{44} = -3(sinx^3)^2.$$

Now, we consider the associated scalars as follows.

$$\psi_1 = -\frac{3}{(x^2)^2}, \quad \psi_2 = -\frac{4}{(x^2)^2}, \quad \psi_3 = \frac{1}{(x^2)^2}, \quad \psi_4 = \frac{2}{(x^2)^2}.$$

We take the 1-forms as follow.

$$A_i(x) = \begin{cases} \sqrt{\frac{1}{x^2}}, & if\, i = 1 \\ 0, & if\, i = 2,3,4 \end{cases}, \quad B_i(x) = \begin{cases} x^2, & if\, i = 3 \\ 0, & if\, i = 1,2,4 \end{cases}$$

and

$$D_i(x) = \begin{cases} x^2 sinx^3, & if\, i = 4 \\ 0, & if\, i = 1,2,3. \end{cases}$$

Then

$$\mathrm{Ric}_{11} = \psi_1 g_{11} + \psi_2 A_1 A_1 + \psi_3(A_1B_1 + B_1A_1) + \psi_4(A_1D_1 + D_1A_1), \tag{7.1}$$
$$\mathrm{Ric}_{22} = \psi_1 g_{22} + \psi_2 A_2 A_2 + \psi_3(A_2B_2 + B_2A_2) + \psi_4(A_2D_2 + D_2A_2), \tag{7.2}$$
$$\mathrm{Ric}_{33} = \psi_1 g_{33} + \psi_2 A_3 A_3 + \psi_3(A_3B_3 + B_3A_3) + \psi_4(A_3D_3 + D_3A_3), \tag{7.3}$$
$$\mathrm{Ric}_{44} = \psi_1 g_{44} + \psi_2 A_4 A_4 + \psi_3(A_4B_4 + B_4A_4) + \psi_4(A_4D_4 + D_4A_4). \tag{7.4}$$

It can be easily proved that (7.1),(7.2), (7.3) and (7.4) are true. We shall show that the 1-forms are unit and orthogonal,

$$g^{ij}A_iA_j = -1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}D_iD_j = 1,$$
$$g^{ij}A_iB_j = 0, \quad g^{ij}A_iD_j = 0, \quad g^{ij}B_iD_j = 0.$$

So, the manifold is a *HGQE* spacetime.

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