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On the Cauchy problem for Boussinesq-Love equation

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Abstract. We are devoted to the study of the Cauchy problem for Boussinesq- Love equation. We establish the regularity of the mild solution in the linear case. In the nonlinear case, we show the existence and unique on globally solution concerning the parameter. Additionally, in this paper, we also shown that the solution to the Boussinesq- Love equation converges to the solution of the Love equation.

1. Introduction

1.1. Statement of the problem

Let Ω be a simply connected and bounded domain in $\mathbb{R}$. Let T be a positive real number. The aim of this paper is to consider the initial value problem of the Boussinesq-Love equation

$$\begin{cases} u_{tt}(x,t) + (-\Delta)^q u(x,t) - \mu\Delta u_{tt}(x,t) - \eta u_{txx}(x,t) = Z(u(x,t)), & \text{in} \quad \Omega\times(0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad \text{in} \quad \Omega, \tag{2}$$

where Z is the source function, representing the effect of an external force, $u = u(x,t)$ describes the distribution at time t and space x, $(-\Delta)^q, q > 0$ is the fractional Laplacian operator and μ, η are positive constants. Here φ and ψ are the initial conditions which are defined later.

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1.2. Physical motivations and known results

Equation (1) is known as the Boussinesq-Love equation which describes longitudinal vibrations in a thin elastic rod, taking into account the inertia and under external load.
In [7],Nizameddin Iskenderov, Seriyye Allahverdiyeva consider the problem

$$u_{tt}(x,t) - u_{ttxx}(x,t) - \alpha u_{txx}(x,t) - \beta u_{xx}(x,t) = a(x,t)u(x,t) + f(x,t), \quad 0 < x < 1, 0 < t < T, \tag{3}$$

nonlocal integral condition

$$\int_0^t u(x,t)dx = 0, 0 < t < T, \tag{4}$$

and overdetermination condition

$$u(0,t) = h(t), 0 < t < T, \tag{5}$$

where $\alpha > 0, \beta > 0$ are known numbers, $f(x,t), \varphi(x), \psi(x), p(t)$, and $h(t)$ are given sufficiently smooth functions of $x \in [0,1]$ and $t \in [0,T]$. The authors prove existence and uniqueness of the classical solution to an inverse boundary value problem (3)-(4)-(5) .
For the case $q = 1, \eta = 0$ and $Z \equiv 0$, reduce equation (1) to Love waves or Love type waves equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2\omega^2 u_{xxtt} = 0, \tag{6}$$

presented by V. Radochova (see [4]), where u is the displacement, E is the Young modulus of the material and ρ is the mass density. Equation (6) plays an important role in numerous fields studied by mathematicians and physicists. In recent years a great attention has been devoted to the study this equation, see e.g.[1], [2], [3]. For example, in [5], Ngoc, Triet, Duy and Long considered an initial and boundary value problem for a nonlinear Love equation as follows

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - u_{xxtt} = f(x,t,u,u_t), & \text{in} \quad 0 < x < 1,\ 0 < t < T \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x), & 0 < x < 1. \end{cases} \tag{7}$$

Here some function $\widetilde{u}_0, \widetilde{u}_1, f$ are defined later. The authors applied the Faedo-Galerkin method to show the existence of a local weak solution of Problem (7) .
For equation (1) without term u_{txx}, (i.e., $\eta = 0$), in [6], the authors Nam, Nghia, Phuong study the initial value problem

$$\begin{cases} y_{tt}(x,t) + (-\Delta)^s y(x,t) - m\Delta y_{tt}(x,t) = G(x,t), & \text{in} \quad \Omega \times (0,T], \\ y|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{8}$$

with the initial conditions

$$y(x,0) = f(x), \ \ y_t(x,0) = g(x) \quad \text{in} \quad \Omega. \tag{9}$$

The authors are interested in studying a mild solution of the Love equation.They present the regularity of the mild solution and show the convergence of the solution of Love's equation to the solution of the wave equation.
The works investigated classical solution and weak solution.However, according to our knowledge, the mild solution of the Boussinesq-Love equation has not been investigated yet.

1.3. Contribution and Organization

The contributions of this paper are organized as follows. In Section 2, we give some preliminaries. In section (3) for the linear case, in subsection 3.1, we demonstrate an approach to present the formula of the mild solution and, based on it, we consider the regularity of the solution under three different assumptions in μ and η. In subsection 3.2, we show that the solution to the Boussinesq- Love equation converges to the solution of the Love equation as $\eta \to 0$. Next, in section 4, for the case nonlinear, we also consider under three different assumptions in μ and η and prove the global existence and uniqueness of the solution by using standard fixed point arguments. Furthermore, we study the regularity of the mild solution for the linear problem.

2. Preliminaries

In this section, we present the results needed for proof of the main result of this paper. We recall some basic properties of functional spaces. Throughout this paper, we consider the Laplace operator Δ defined on $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Denote by $\{\lambda_k\}_{k\geq 1}$ and $\{e_k(x)\}_{k\geq 1}$, the spectrum and sequence of eigenfunctions of Δ respectively, which satisfy $\Delta e_k(x) = -\lambda_k e_k(x)$, $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ...$, and $\lim\limits_{k\to\infty} \lambda_k = \infty$. The sequence $\{e_k(x)\}_{k\geq 1}$ forms an orthonormal basis of the space $L^2(\Omega)$. The fractional power Δ^q, $q > 0$, of the Laplacian operator Δ with fractional order $q > 0$ on Ω is defined by

$$\Delta^q v(x) := \sum_{k=1}^{\infty} (v, e_k) \lambda_k^q e_k(x). \tag{10}$$

We denote $f \lesssim_{M_1,M_2,...} g + h + ...$ instead $f \leq C_1 g + C_2 h + ...$ where $C_1, C_2, ..$ depend on $M_1, M_2, ..$
and the notation $f =_{M_1,M_2,...} g + h + ...$ will mean $f = C_1 g + C_2 h + ...$ where $C_1, C_2, ..$ depend on $M_1, M_2, ..$

Definition 2.1. *We recall the Hilbert scale space, which is given as follows*

$$\mathbb{H}^{\epsilon}(\Omega) := \left\{ f \in L^2(\Omega), \ \sum_{k=1}^{\infty} \lambda_k^{2\epsilon} \Big(\int_\Omega f(x) e_k(x) dx \Big)^2 < \infty \right\},$$

for any $\epsilon \geq 0$. And the norm is given by

$$\|f\|_{\mathbb{H}^{\epsilon}(\Omega)} := \left(\sum_{k=1}^{\infty} \lambda_k^{2\epsilon} \Big(\int_\Omega f(x) e_k(x) dx \Big)^2 \right)^{1/2}, \ \ f \in \mathbb{H}^{\epsilon}(\Omega).$$

Definition 2.2. *Let us denote $L^\infty(0, T; \mathbb{H}^{\gamma}(\Omega))$, the space of all functions $v : \Omega \times (0, T) \to \mathbb{R}$ such that*

$$\|v\|_{L^\infty(0,T;\mathbb{H}^{\gamma}(\Omega))} := \operatorname*{ess\,sup}_{t\in(0,T)} \|v(., t)\|_{\mathbb{H}^{\gamma}(\Omega)} < \infty.$$

Definition 2.3. *Let B be a Banach space. Let $O_{a,q}((0, T]; B)$ denote the weighted space of all functions $v \in C((0, T]; B)$ such that*

$$\|\psi\|_{O_{a,q}((0,T];B)} := \operatorname*{ess\,sup}_{t\in(0,T]} t^a e^{-qt} \|\psi(t, \cdot)\|_B < \infty, \tag{11}$$

where $a, q > 0$.

Definition 2.4. *Let B be a Banach space. For $\sigma \in (0, 1)$, we define the Hölder continuous space, denoted by*

$$C^{\sigma}([0, T]; B) = \left\{ v \in C([0, T]; B) \,\Big|\, \sup_{0\leq t<t'\leq T} \frac{|v(t') - v(t)|}{|t' - t|^{\sigma}} < \infty \right\},$$

corresponding to the norm

$$\|v\|_{C^\sigma([0,T];B)} := \sup_{0\le t<t'\le T} \frac{\|v(t') - v(t)\|_B}{|t'-t|^\sigma},$$

for all $v \in C^\sigma([0,T];B)$.

Lemma 2.5. *Let $k_1 > -1$, $k_2 > -1$ such that $k_1 + k_2 \ge -1$, $\rho > 0$ and $t \in [0,T]$. For $h > 0$, the following limit holds*

$$\lim_{\rho\to\infty} \left(\sup_{t\in[0,T]} t^h \int_0^1 \xi^{k_1}(1-\xi)^{k_2} e^{-\rho t(1-\xi)} \mathrm{d}\xi \right) = 0.$$

3. For the linear case

3.1. Regularity of mild solution

In this subsection, we investigate the regularity of the mild solution to problem (1), under the assumption that the forcing term is a prescribed function $Z(x,t)$ depending solely on the space and time variables. For this purpose, we first derive the mild solution formula. The main idea is to solve the differential equation using the Laplace transform. To illustrate this, we write the solution of the problem (1) as a Fourier series as follows.

$$u(x,t) = \sum_{k=1}^{\infty} u_k e_k(x), \tag{12}$$

where $u_k(t) = (u(.,t), e_k)$ satisfies

$$\begin{cases} \partial_t^2 u_k(t) + \frac{\lambda_k^q}{1+\mu\lambda_k} u_k(t) + \frac{\eta\lambda_k}{1+\mu\lambda_k} \partial_t u_k(t) = \frac{Z_k(t)}{1+\mu\lambda_k}, \\ u_k(0) = \varphi_k, \quad \frac{d}{dt} u_k(0) = \psi_k. \end{cases} \tag{13}$$

By applying the Laplace transform to the differential equation (13), we obtain the solution $u_k(t)$. Substituting this into the series representation (12), we arrive at the explicit expression for $u(x,t)$.

It is evident that the qualitative behavior of the solution to (13) is governed by the values of the parameters μ and η. In particular, the nature of the solution depends on the sign of the discriminant-like expression

$$\Delta_k := 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2.$$

Therefore, it is natural to distinguish between the following three cases:

i) **Oscillatory regime:** When

$$\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 > 0,$$

the system exhibits damped oscillatory behavior. This case corresponds to the underdamped regime.

ii) **Critical regime:** When

$$\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 = 0,$$

the system is at the critical damping threshold, where oscillations vanish, and the solution transitions to a purely exponential form.

iii) **Overdamped regime:** When

$$\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 < 0,$$

the solution exhibits exponential decay without oscillation, corresponding to the overdamped case.

We now proceed to analyze each of the three cases mentioned above in detail:

i) **Case 1:** Assume that $\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 > 0$. We introduce the following notations:

$$a = \frac{\eta\lambda_k}{2(1+\mu\lambda_k)}, \quad b = 2(1+\mu\lambda_k), \quad \text{and} \quad \xi = \sqrt{4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2}.$$

A direct computation yields the explicit representation

$$u_k(t) = e^{-at}\cos\left(\frac{\xi}{b}t\right)\varphi_k + \frac{b}{\xi}e^{-at}\sin\left(\frac{\xi}{b}t\right)\psi_k + \frac{2}{\xi}\int_0^t e^{-a(t-r)}\sin\left(\frac{\xi}{b}(t-r)\right)Z_k(r)\,dr.$$

Consequently, the solution $u(x,t)$ admits the Fourier expansion

$$u(x,t) = \sum_{k=1}^{\infty} e^{-at}\cos\left(\frac{\xi}{b}t\right)\varphi_k e_k(x) + \sum_{k=1}^{\infty}\frac{b}{\xi}e^{-at}\sin\left(\frac{\xi}{b}t\right)\psi_k e_k(x) + \sum_{k=1}^{\infty}\left[\frac{2}{\xi}\int_0^t e^{-a(t-r)}\sin\left(\frac{\xi}{b}(t-r)\right)Z_k(r)\,dr\right]e_k(x). \tag{14}$$

For $v \in L^2(\Omega)$ and $t \in [0,T]$, we define the operator families

- $$\mathcal{M}_q(t)v = \sum_{k=1}^{\infty} e^{-at}\cos\left(\frac{\xi}{b}t\right)(v,e_k)e_k(x), \tag{15}$$

- $$\mathcal{N}_q(t)v = \sum_{k=1}^{\infty} \frac{b}{\xi}e^{-at}\sin\left(\frac{\xi}{b}t\right)(v,e_k)e_k(x), \tag{16}$$

- $$\mathcal{O}_q(t)v = \sum_{k=1}^{\infty} \frac{2}{\xi}e^{-at}\sin\left(\frac{\xi}{b}t\right)(v,e_k)e_k(x). \tag{17}$$

Accordingly, the mild solution to problem (1)–(2) can be written as

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{O}_q(t-r)Z(r)\,dr. \tag{18}$$

ii) **Case 2:** Suppose that $\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 = 0$.
Set

$$\beta_1 = \sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}, \quad \beta_2 = \frac{\eta\lambda_k - \sqrt{\lambda_k^q(1+\mu\lambda_k)}}{1+\mu\lambda_k}.$$

In this case, the solution component $u_k(t)$ is given by

$$u_k(t) = \left[e^{-\beta_1 t} + \beta_2 t e^{-\beta_1 t}\right]\varphi_k + te^{-\beta_1 t}\psi_k + \int_0^t (t-r)e^{-\beta_1(t-r)}Z_k(r)\,dr.$$

Hence,

$$u(x,t) = \sum_{k=1}^{\infty}\left[e^{-\beta_1 t} + \beta_2 t e^{-\beta_1 t}\right]\varphi_k e_k(x) + \sum_{k=1}^{\infty} te^{-\beta_1 t}\psi_k e_k(x) + \sum_{k=1}^{\infty}\left[\int_0^t (t-r)e^{-\beta_1(t-r)}Z_k(r)\,dr\right]e_k(x). \tag{19}$$

We define the operators

- $$\mathcal{P}_q(t)v = \sum_{k=1}^{\infty} \left[e^{-\beta_1 t} + \beta_2 t e^{-\beta_1 t} \right] (v, e_k) e_k(x),$$

- $$\mathcal{Q}_q(t)v = \sum_{k=1}^{\infty} t e^{-\beta_1 t} (v, e_k) e_k(x). \tag{20}$$

Thus, the corresponding mild solution reads

$$u(t) = \mathcal{P}_q(t)\varphi + \mathcal{Q}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-r) Z(r)\, dr. \tag{21}$$

iii) **Case 3:** Finally, assume that $\Delta_k = 4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 < 0$
Define

$$a_1 = \frac{\eta\lambda_k}{\sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}}, \quad a_2 = \frac{1+\mu\lambda_k}{\sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}},$$

and

$$\xi_1 = \frac{\eta\lambda_k + \sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}}{2(1+\mu\lambda_k)}, \quad \xi_2 = \frac{\eta\lambda_k - \sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}}{2(1+\mu\lambda_k)}.$$

Then the solution $u_k(t)$ takes the form

$$\begin{aligned} u_k(t) = &\left[\left(\frac{1}{2} - a_1\right) e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right) e^{-\xi_2 t} \right] \varphi_k \\ &+ \left[a_2 e^{-\xi_2 t} - a_2 e^{-\xi_1 t} \right] \psi_k + \int_0^t \left[a_2 e^{-\xi_2(t-r)} - a_2 e^{-\xi_1(t-r)} \right] Z_k(r)\, dr. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) = &\sum_{k=1}^{\infty} \left[\left(\frac{1}{2} - a_1\right) e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right) e^{-\xi_2 t} \right] \varphi_k e_k(x) + \sum_{k=1}^{\infty} \left[a_2 e^{-\xi_2 t} - a_2 e^{-\xi_1 t} \right] \psi_k e_k(x) \\ &+ \sum_{k=1}^{\infty} \int_0^t \left[a_2 e^{-\xi_2(t-r)} - a_2 e^{-\xi_1(t-r)} \right] Z_k(r) e_k(x) dr. \end{aligned}$$

We now define the corresponding operators:

- $$\mathcal{R}_q(t)v = \sum_{k=1}^{\infty} \left[\left(\frac{1}{2} - a_1\right) e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right) e^{-\xi_2 t} \right] (v, e_k) e_k(x),$$

- $$\mathcal{S}_q(t)v = \sum_{k=1}^{\infty} \left[a_2 e^{-\xi_2 t} - a_2 e^{-\xi_1 t} \right] (v, e_k) e_k(x). \tag{22}$$

Hence, the mild solution in this case is given by

$$u(t) = \mathcal{R}_q(t)\varphi + \mathcal{S}_q(t)\psi + \int_0^t \mathcal{S}_q(t-r)Z(r)\,dr. \tag{23}$$

The main tool in our analysis is the use of mapping properties of the operators $\mathcal{M}_q(t)$, $\mathcal{N}_q(t)$, and $\mathcal{O}_q(t)$. We now establish several useful estimates for these operators.

Lemma 3.1. *Let $q > 1$, $1 < \gamma < \zeta$, and assume that $0 < \frac{2(\zeta-\gamma)}{q-1} \le 1$. Then the following estimates hold:*

a)

$$\left\|\mathcal{M}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\gamma(\Omega)} + t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}, \tag{24}$$

for any $v \in \mathbb{H}^\gamma(\Omega) \cap \mathbb{H}^\zeta(\Omega)$.

b)

$$\left\|\mathcal{N}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}, \tag{25}$$

for any $v \in \mathbb{H}^\zeta(\Omega)$

c)

$$\left\|\mathcal{O}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}, \tag{26}$$

for any $v \in \mathbb{H}^\zeta(\Omega)$.

Proof. We first prove part (a). By Parseval's identity, we obtain

$$\begin{aligned}\left\|\mathcal{M}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} &= \left\|\sum_{k=1}^\infty e^{-at}\cos\Big(\frac{\xi}{b}t\Big)\Big(v(.),e_k(.)\Big)e_k(x)\right\|^2_{\mathbb{H}^\gamma(\Omega)} \\ &= \sum_{k=1}^\infty \lambda_k^{2\gamma}e^{-at}\cos^2\Big(\frac{\xi}{b}t\Big)\Big|\Big(v(.),e_k(.)\Big)\Big|^2\end{aligned}$$

Recall that $a = \frac{\eta\lambda_k}{2(1+\mu\lambda_k)} \lesssim_{\mu,\eta} 1$ and $\frac{\xi}{b} \lesssim_{\mu,\eta,\lambda_1} \lambda_k^{\frac{(q-1)}{2}}$.
Using the elementary inequality $e^{-x} \le C_\epsilon x^\epsilon$ for any $0 < \epsilon \le 1$, we deduce

$$e^{-at} \lesssim_{\mu,\eta} t^\epsilon, \quad 0 < \epsilon \le 1. \tag{27}$$

Moreover, from $|\sin(x)| \le C_\epsilon x^\epsilon$, $0 < \epsilon \le 1$, it follows that

$$\begin{aligned}\left|\cos^2\Big(\frac{\xi}{b}t\Big)\right| &= \left|\cos^2\bigg(\frac{\sqrt{4\lambda_k^q(1+\mu\lambda_k)-\eta^2.\lambda_k^2}}{2(1+\mu\lambda_k)}t\bigg)\right| \\ &\le 1+\lambda_k^{(q-1)\epsilon}(\lambda_1+\mu)^\epsilon t^{2\epsilon} \lesssim_{\mu,\eta,\epsilon} 1+\lambda_k^{(q-1)\epsilon}t^{2\epsilon}.\end{aligned}$$

Consequently,

$$\left\|\mathcal{M}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^\epsilon \sum_{k=1}^{\infty} \lambda_k^{2\gamma}\left|\Big(v(.),e_k(.)\Big)\right|^2 + t^{3\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\lambda_k^{(q-1)\epsilon}\left|\Big(v(.),e_k(.)\Big)\right|^2 \tag{28}$$

$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} t^\epsilon \left\|v\right\|^2_{\mathbb{H}^\gamma(\Omega)} + t^{3\epsilon}\left\|v\right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)}, \tag{29}$$

that for any $0<\epsilon\le 1$. Choosing $0<\epsilon=\frac{2(\zeta-\gamma)}{q-1}\le 1$ and using the inequality $(a+b)^\nu \le a^\nu + b^\nu$ for $0<\nu<1$, we arrive at

$$\left\|\mathcal{M}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\gamma(\Omega)} + t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}.$$

We next prove part (b). By arguments similar to those above, one can verify that

$$\left|e^{-at}\sin^2\Big(\frac{\xi}{b}t\Big)\right| = \left|e^{\frac{-\eta\lambda_k}{(1+\mu\lambda_k)}t}\sin^2\left(\frac{\sqrt{4\lambda_k^q(1+\mu\lambda_k)-\eta^2.\lambda_k^2}}{2(1+\mu\lambda_k)}t\right)\right| \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{3\epsilon}\lambda_k^{(q-1)\epsilon}. \tag{30}$$

On the other hand,

$$\frac{b}{\xi} = \frac{2(1+\mu\lambda_k)}{\sqrt{4\lambda_k^q(1+\mu\lambda_k)-\eta^2\lambda_k^2}} \le \frac{2\lambda_k(\frac{1}{\lambda_k}+\mu)}{\lambda_k\sqrt{4\frac{1}{\lambda_k^{1-q}}\mu-\eta^2}} \lesssim_{\mu,\eta,\lambda_1} 1. \tag{31}$$

Combining (30) and (31), we find that

$$\begin{aligned}\left\|\mathcal{N}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} &= \left\|\sum_{k=1}^{\infty}\frac{b}{\xi}e^{-at}\sin\Big(\frac{\xi}{b}t\Big)\Big(v(.),e_k(.)\Big)e_k(x).\right\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &\lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{3\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+(q-1)\epsilon}\left|\Big(v(.),e_k(.)\Big)\right|^2\\ &=_{\mu,\eta,\epsilon,\lambda_1} t^{3\epsilon}\left\|v\right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)}.\end{aligned}$$

Taking again $0<\epsilon=\frac{2(\zeta-\gamma)}{q-1}\le 1$, we follows from the latter estimate that

$$\left\|\mathcal{N}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}. \tag{32}$$

Additionally, by the same techniques as in part b, one can see that $\left\|\mathcal{O}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}$, which implies Part c and completes the proof. □

Lemma 3.2. *Let $q>1$, $1<\gamma<\zeta$, and suppose that $0<\frac{2(\zeta-\gamma)}{q-1}\le 1$. Then the operators $\mathcal{P}_q(t)$ and $\mathcal{Q}_q(t)$ satisfy the following estimates:*

(a) *For any $v\in\mathbb{H}^\zeta(\Omega)$,*

$$\left\|\mathcal{P}_q(t)\,v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{2(\zeta-\gamma)}{q-1}+1}\,\|v\|_{\mathbb{H}^\zeta(\Omega)}\,. \tag{33}$$

(b) *For any $v\in\mathbb{H}^\zeta(\Omega)$,*

$$\left\|\mathcal{Q}_q(t)\,v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{\frac{\zeta-\gamma}{q-1}}\,\|v\|_{\mathbb{H}^\zeta(\Omega)}\,. \tag{34}$$

Proof. The proof follows by arguments analogous to those employed in Lemma 3.1.

First, we estimate the operator $\mathcal{P}_q(t)$. Using Parseval's identity and standard spectral techniques, we obtain

$$\left\|\mathcal{P}_q(t)\, v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{2\epsilon+2} \sum_{k=1}^{\infty} \lambda_k^{2\gamma+(q-1)\epsilon} \left|(v, e_k)\right|^2 . \tag{35}$$

Similarly, for the operator $Q_q(t)$, we have the estimate

$$\left\|Q_q(t)\, v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{\epsilon} \sum_{k=1}^{\infty} \lambda_k^{2\gamma+(q-1)\epsilon} \left|(v, e_k)\right|^2 . \tag{36}$$

Now, we choose $\epsilon = \frac{2(\zeta-\gamma)}{q-1}$, which satisfies $0 < \epsilon \leq 1$ by assumption. Substituting this value into (35) and using the embedding $\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega) \subset \mathbb{H}^\zeta(\Omega)$, we obtain

$$\left\|\mathcal{P}_q(t)\, v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{2\epsilon+2} \|v\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{2\epsilon+2} \|v\|^2_{\mathbb{H}^\zeta(\Omega)} .$$

In the same way, from (36), we deduce

$$\left\|Q_q(t)\, v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{\epsilon} \|v\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{\epsilon} \|v\|^2_{\mathbb{H}^\zeta(\Omega)} .$$

Taking square roots on both sides of the above inequalities completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $0 < \gamma$ and $\epsilon > 0$. Then the operators $\mathcal{R}_q(t)$ and $\mathcal{S}_q(t)$ satisfy the following estimates:*

(a)

$$\left\|\mathcal{R}_q(t)\, v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{2\epsilon} \|v\|_{\mathbb{H}^\gamma(\Omega)} , \tag{37}$$

(b)

$$\left\|\mathcal{S}_q(t)\, v\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} t^{2\epsilon} \|v\|_{\mathbb{H}^\gamma(\Omega)} , \tag{38}$$

for any $v \in \mathbb{H}^\gamma(\Omega)$.

Proof. We first consider the operator $\mathcal{R}_q(t)$. From the definitions of the coefficients in Case (iii), it is straightforward to verify that

$$a_1 = \frac{\eta\lambda_k}{\sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}} \lesssim_\eta 1, \quad a_2 = \frac{1+\mu\lambda_k}{\sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}} \lesssim_{\mu,\eta} 1. \tag{39}$$

Moreover, we have

$$\xi_1 = \frac{\eta\lambda_k + \sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}}{2(1+\mu\lambda_k)} \lesssim_{\mu,\eta} 1, \quad \xi_2 = \frac{\eta\lambda_k - \sqrt{\eta^2\lambda_k^2 - 4\lambda_k^q(1+\mu\lambda_k)}}{2(1+\mu\lambda_k)} \lesssim_{\mu,\eta} 1. \tag{40}$$

Using Parseval's identity, we compute

$$\begin{aligned}
\left\|\mathcal{R}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} &= \left\| \sum_{k=1}^{\infty} \left[\left(\frac{1}{2} - a_1\right) e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right) e^{-\xi_2 t} \right] (v, e_k) e_k \right\|^2_{\mathbb{H}^\gamma(\Omega)} \\
&= \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \left[\left(\frac{1}{2} - a_1\right) e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right) e^{-\xi_2 t} \right]^2 |(v, e_k)|^2.
\end{aligned}$$

Applying the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, together with the bound $e^{-x} \le C_\epsilon x^\epsilon$ for all $x > 0$, with $\epsilon > 0$, we obtain

$$\left[\left(\frac{1}{2} - a_1\right)e^{-\xi_1 t} + \left(\frac{1}{2} + a_1\right)e^{-\xi_2 t}\right]^2 \le 2\left(\frac{1}{2} - a_1\right)^2 e^{-2\xi_1 t} + 2\left(\frac{1}{2} + a_1\right)^2 e^{-2\xi_2 t} \lesssim_{\mu,\eta,\epsilon} t^{2\epsilon},$$

where we used the uniform boundedness of a_1, ξ_1, and ξ_2.
Consequently,

$$\left\|\mathcal{R}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} \sum_{k=1}^{\infty} \lambda_k^{2\gamma} t^{2\epsilon} |(v, e_k)|^2 = t^{2\epsilon} \|v\|^2_{\mathbb{H}^\gamma(\Omega)},$$

which yields the desired estimate (37).

We next turn to the operator $\mathcal{S}_q(t)$. Analogously, we estimate

$$\left[a_2 e^{-\xi_2 t} - a_2 e^{-\xi_1 t}\right]^2 = a_2^2 \left(e^{-\xi_2 t} - e^{-\xi_1 t}\right)^2 \lesssim_{\mu,\eta,\epsilon} t^{2\epsilon},$$

since the exponential function is smooth and Lipschitz on bounded intervals and $\xi_1, \xi_2 \lesssim 1$.
Therefore,

$$\left\|\mathcal{S}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \left[a_2 \left(e^{-\xi_2 t} - e^{-\xi_1 t}\right)\right]^2 |(v, e_k)|^2 \lesssim_{\mu,\eta,\epsilon} t^{2\epsilon} \|v\|^2_{\mathbb{H}^\gamma(\Omega)}, \tag{41}$$

which establishes (38) and completes the proof. □

We now establish the main results of the paper.

Theorem 3.4. *Let $q > 1$ and $\mu, \eta > 0$ be such that $4\lambda_1^q(1 + \mu\lambda_1) - \eta^2\lambda_1^2 > 0$.
Assume that $1 < \gamma < \zeta$, and $\frac{(\zeta-\gamma)}{q-1} \le \frac{1}{2}$. Suppose the initial data satisfy $\varphi \in \mathbb{H}^\gamma(\Omega), \psi \in \mathbb{H}^\zeta(\Omega)$, and the source term satisfies $Z \in L^2(0, T; \mathbb{H}^{\zeta-1}(\Omega))$.
Then the solution u to the problem (referenced elsewhere in the paper) satisfies the estimate:*

$$\|u\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\mu,\eta,\epsilon} T^{3\frac{\zeta-\gamma}{q-1}} \left\|\varphi\right\|_{\mathbb{H}^\gamma(\Omega)} + T^{3\frac{\zeta-\gamma}{q-1}} \left\|\varphi\right\|_{\mathbb{H}^\zeta(\Omega)} + T^{3\frac{\zeta-\gamma}{q-1}} \left\|\psi\right\|_{\mathbb{H}^\zeta(\Omega)} + T^{3\frac{\zeta-\gamma}{q-1}+\frac{1}{2}} \|Z\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

Moreover, the solution satisfies the Hölder continuity in time:

$$u \in C^{\frac{\epsilon}{2}}([0,T]; \mathbb{H}^\gamma(\Omega)), \quad \textit{where } \epsilon = \frac{2(\zeta-\gamma)}{q-1}.$$

Proof. By using triangle inequality, we have

$$\left\|u(x,t)\right\|_{\mathbb{H}^\gamma(\Omega)} \le \left\|\mathcal{M}_q(t)\varphi\right\|_{\mathbb{H}^\gamma(\Omega)} + \left\|\mathcal{N}_q(t)\psi\right\|_{\mathbb{H}^\gamma(\Omega)} + \left\|\int_0^t \mathcal{O}_q(t-r)Z(r)dr\right\|_{\mathbb{H}^\gamma(\Omega)}.$$

It is obvious that

$$\left\|\int_0^t \mathcal{O}_q(t-r)Z(r)dr\right\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \frac{4}{\xi^2}\left(\int_0^t e^{-a(t-r)} \sin\left(\frac{\xi}{b}(t-r)\right) Z_k(r)dr\right)^2.$$

Applying Hölder inequality, we get

$$\begin{aligned}
&\left(\int_0^t e^{-a(t-r)} \sin\left(\frac{\xi}{b}(t-r)\right) Z_k(r)dr\right)^2 \le \int_0^t e^{-2a(t-r)}dr. \int_0^t \sin^2\left(\frac{\xi}{b}(t-r)\right)\Big|Z(.), e_k(.)\Big)\Big|^2 dr \\
&\le \int_0^t C(\epsilon,\mu,\eta)(t-r)^\epsilon dr \int_0^t C(\lambda_1,\mu,\epsilon)\lambda_k^{(q-1)\epsilon}(t-r)^{2\epsilon}\Big|Z(.), e_k(.)\Big)\Big|^2 dr \\
&\lesssim_{\mu,\eta,\lambda_1,\epsilon} T^{3\epsilon+1} \int_0^t \lambda_k^{(q-1)\epsilon}\Big|Z(.), e_k(.)\Big)\Big|^2 dr.
\end{aligned}$$

and we conclude from (31) that $\frac{4}{\xi^2} \lesssim_{\mu,\eta,\lambda_1} \lambda_k^{-2}$, for this reason

$$\Big\| \int_0^t \mathcal{O}_q(t-r)Z(r)dr \Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\lambda_1,\epsilon} T^{3\epsilon+1} \int_0^t \Big\| Z(.,r)dr \Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}-1}(\Omega)} dr$$
$$=_{\mu,\eta,\lambda_1,\epsilon} T^{3\epsilon+1} \Big\| Z \Big\|^2_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

Hence, one has

$$\Big\| \int_0^t \mathcal{O}_q(t-r)Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\lambda_1,\epsilon} T^{\frac{3(\zeta-\gamma)}{q-1}+\frac{1}{2}} \Big\| Z \Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}. \tag{42}$$

Combining (24), (25) and (42) yields

$$\Big\| u(x,t) \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{\frac{(\zeta-\gamma)}{q-1}} \Big\| \varphi \Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{\frac{3(\zeta-\gamma)}{q-1}} \Big\| \varphi \Big\|_{\mathbb{H}^\zeta(\Omega)} + T^{\frac{3(\zeta-\gamma)}{q-1}} \Big\| \psi \Big\|_{\mathbb{H}^\zeta(\Omega)} + T^{\frac{3(\zeta-\gamma)}{q-1}+\frac{1}{2}} \Big\| Z \Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

It is a simple matter to

$$\Big\| u \Big\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{3\frac{(\zeta-\gamma)}{q-1}} \Big\| \varphi \Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{\frac{3(\zeta-\gamma)}{q-1}} \Big\| \varphi \Big\|_{\mathbb{H}^\zeta(\Omega)} + T^{\frac{(\zeta-\gamma)}{q-1}} \Big\| \psi \Big\|_{\mathbb{H}^\zeta(\Omega)} + T^{\frac{3(\zeta-\gamma)}{q-1}+\frac{1}{2}} \Big\| Z \Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

Our next claim is that $u \in C^{\frac{\epsilon}{2}}([0,T], \mathbb{H}^\gamma(\Omega))$
Observe that for any $0 \le t \le t+h \le T$ with $h > 0$. From (18), we get

$$u(t+h) - u(t) = \Big(\mathcal{M}_q(t+h) - \mathcal{M}_q(t)\Big)\varphi + \Big(\mathcal{N}_q(t+h) - \mathcal{N}_q(t)\Big)\psi$$
$$+ \int_0^t \Big(\mathcal{O}_q(t+h-r) - \mathcal{O}_q(t-r)\Big)Z(r)dr + \int_t^{t+h} \mathcal{O}_q(t+h-r)Z(r)dr.$$

We have divided the estimate into four step following

Step 1. Estimation of $\Big\| \Big(\mathcal{M}_q(t+h) - \mathcal{M}_q(t)\Big)\varphi \Big\|_{\mathbb{H}^\gamma(\Omega)}$.

We see at once that

$$\Big\| \Big(\mathcal{M}_q(t+h) - \mathcal{M}_q(t)\Big)\varphi \Big\|^2_{\mathbb{H}^\gamma(\Omega)} = \Big\| \sum_{k=1}^{\infty} \Big(e^{-a(t+h)} \cos\frac{\xi}{b}(t+h) - e^{-at}\cos\frac{\xi}{b}t\Big)\varphi_k e_k(x) \Big\|^2_{\mathbb{H}^\gamma(\Omega)}$$
$$= \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \Big(e^{-a(t+h)} \cos\frac{\xi}{b}(t+h) - e^{-at}\cos\frac{\xi}{b}t\Big)^2 |\varphi_k|^2.$$

It is immediate that

$$\Big| e^{-a(t+h)} \cos\frac{\xi}{b}(t+h) - e^{-at}\cos\frac{\xi}{b}t \Big| \le \Big| \Big(e^{-a(t+h)} - e^{-at}\Big)\cos\frac{\xi}{b}(t+h) \Big| + \Big| e^{-at}\Big(\cos\frac{\xi}{b}(t+h) - \cos\frac{\xi}{b}t\Big) \Big|$$
$$:= \mathbf{A_1(t)} + \mathbf{A_2(t)}.$$

We proceed now to estimate the term $\mathbf{A_1(t)}$. Using the inequality $|e^{-a} - e^{-b}| \le C_\gamma |a-b|^\gamma$, for any $0 < \gamma \le 1$, we know that

$$\Big| e^{-a(t+h)} - e^{-at} \Big| \lesssim_{\mu,\eta,\epsilon} h^\epsilon, \textit{ for any } \quad 0 < \epsilon \le 1. \tag{43}$$

By arguments similar to those used previously, we also have

$$\left|\cos\frac{\xi}{b}(t+h)\right| \le 1 + C(\mu,\eta,\epsilon,\lambda_1)T^{\epsilon}\lambda_k^{\frac{(q-1)\epsilon}{2}}. \tag{44}$$

Combining (43) with (44), we see that

$$\mathbf{A_1(t)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} h^{\epsilon} + T^{\epsilon}\lambda_k^{\frac{(q-1)\epsilon}{2}}h^{\epsilon}, \quad 0<\epsilon\le 1.$$

We next estimate the term $\mathbf{A_2(t)}$.
Since inequality $\left|\cos(m)-\cos(n)\right| \le C(\varepsilon)|m-n|^{\varepsilon}$, for $0<\epsilon\le 1$, it follows that

$$\left|\cos\frac{\xi}{b}(t+h) - \cos\frac{\xi}{b}t\right| \lesssim_{\mu,\eta,\epsilon,\lambda_1} \lambda_k^{\frac{(q-1)\epsilon}{2}}h^{\epsilon}, 0<\epsilon\le 1. \tag{45}$$

Combining (45) with (27), we see that $\mathbf{A_2(t)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \lambda_k^{\frac{(q-1)\epsilon}{2}}h^{2\epsilon}, 0<\epsilon\le 1$. Hence, we can assert that
$\left\|\Big(\mathcal{M}_q(t+h)-\mathcal{M}_q(t)\Big)\varphi\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma}|\varphi_k|^2 + h^{\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\varphi_k|^2$
$+ h^{2\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\varphi_k|^2 =_{\mu,\eta,\epsilon,\lambda_1,T} h^{\epsilon}\left\|\varphi\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} + h^{\epsilon}\left\|\varphi\right\|^2_{\mathbb{H}^{\zeta}(\Omega)} + h^{2\epsilon}\left\|\varphi\right\|^2_{\mathbb{H}^{\zeta}(\Omega)}$.
It is shown that

$$\left\|\Big(\mathcal{M}_q(t+h)-\mathcal{M}_q(t)\Big)\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\frac{\epsilon}{2}}\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + h^{\frac{\epsilon}{2}}\left\|\varphi\right\|_{\mathbb{H}^{\zeta}(\Omega)} + h^{\epsilon}\left\|\varphi\right\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

Step 2. Estimation of $\left\|\Big(\mathcal{N}_q(t+h)-\mathcal{N}_q(t)\Big)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)}$.
From (16), we have

$$\left\|\Big(\mathcal{N}_q(t+h)-\mathcal{N}_q(t)\Big)\psi\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} = \left\|\sum_{k=1}^{\infty}\frac{b}{\xi}\Big(e^{-a(t+h)}\sin\frac{\xi}{b}(t+h) - e^{-at}\sin\frac{\xi}{b}t\Big)\psi_k e_k(x)\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} \tag{46}$$

$$= \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\frac{b^2}{\xi^2}\Big(e^{-a(t+h)}\sin\frac{\xi}{b}(t+h) - e^{-at}\sin\frac{\xi}{b}t\Big)^2 |\psi_k|^2. \tag{47}$$

By an argument analogous to the previous one. We get that

$$\Big[e^{-a(t+h)}\sin\frac{\xi}{b}(t+h) - e^{-at}\sin\frac{\xi}{b}t\Big] \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{\epsilon}\lambda_k^{\frac{(q-1)\epsilon}{2}}h^{\epsilon} + \lambda_k^{\frac{(q-1)\epsilon}{2}}h^{2\epsilon},\ 0<\epsilon\le 1. \tag{48}$$

Combining (47),(31) and (48), we have the following bound

$$\left\|\Big(\mathcal{N}_q(t+h)-\mathcal{N}_q(t)\Big)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} h^{\frac{\epsilon}{2}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)} + h^{\epsilon}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

Step 3. Estimation of $\left\|\int_0^t\Big(O_q(t+h-r)-O_q(t-r)\Big)Z(r)dr\right\|_{\mathbb{H}^{\gamma}(\Omega)}$.
It is clear that

$$\left\|\int_0^t\Big(O_q(t+h-r)-O_q(t-r)\Big)Z(r)dr\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} =$$

$$\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\frac{4}{\xi^2}\bigg(\int_0^t\Big[e^{-a(t+h-r)}\sin\frac{\xi}{b}(t+h-r) - e^{-a(t-r)}\sin\frac{\xi}{b}(t-r)\Big]|Z_k(r)|dr\bigg)^2.$$

By using Holder inequality, we find that

$$\left(\int_0^t \left[e^{-a(t+h-r)}\sin\left(\frac{\xi}{b}(t+h-r)\right)-e^{-a(t-r)}\sin\left(\frac{\xi}{b}(t-r)\right)\right]\left|Z_k(r)\right|dr\right)^2$$
$$\leq T\int_0^t \left[e^{-a(t+h-r)}\sin\left(\frac{\xi}{b}(t+h-r)\right)-e^{-a(t-r)}\sin\left(\frac{\xi}{b}(t-r)\right)\right]^2\left|Z_k(r)\right|^2 dr.$$

Combining (48) with inequality $(a+b)^2 \leq 2a^2+2b^2$, we have

$$\int_0^t \left[e^{-a(t+h-r)}\sin\left(\frac{\xi}{b}(t+h-r)\right)-e^{-a(t-r)}\sin\left(\frac{\xi}{b}(t-r)\right)\right]^2\left|Z_k(r)\right|^2 dr \tag{49}$$
$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} \int_0^t \left[T^{2\epsilon}\lambda_k^{(q-1)\epsilon}h^{2\epsilon}+\lambda_k^{(q-1)\epsilon}h^{4\epsilon}\right]\left|Z_k(r)\right|^2 dr. \tag{50}$$

Thus, from (50) and (31) one can estimate as follows

$$\left\|\int_0^t \left(O_q(t+h-r)-O_q(t-r)\right)Z(r)dr\right\|^2_{\mathbb{H}^\gamma(\Omega)}$$
$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{2\epsilon}h^{2\epsilon}\int_0^t\sum_{k=1}^\infty \lambda_k^{2\gamma+(q-1)\epsilon-2}\left|Z_k(r)\right|^2 dr + h^{4\epsilon}\int_0^t\sum_{k=1}^\infty \lambda_k^{2\gamma+(q-1)\epsilon-2}\left|Z_k(r)\right|^2 dr$$
$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{2\epsilon}h^{2\epsilon}\int_0^T\left\|Z(r)\right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}-1}(\Omega)}dr + h^{4\epsilon}\int_0^T\left\|Z(r)\right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}-1}(\Omega)}dr.$$

Let $\epsilon = \frac{2(\zeta-\gamma)}{q-1}$, we have thus proved that

$$\left\|\int_0^t \left(O_q(t+h-r)-O_q(t-r)\right)Z(r)dr\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^\epsilon h^\epsilon\left\|Z\right\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))} + \qquad h^{2\epsilon}\left\|Z\right\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

Step 4. Estimation of $\left\|\int_t^{t+h} O_q(t+h-r)Z(r)dr\right\|_{\mathbb{H}^\gamma(\Omega)}$.
Arguing as above we can see that

$$\left\|\int_t^{t+h} O_q(t+h-r)Z(r)dr\right\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^\infty \lambda_k^{2\gamma}\frac{4}{\xi^2}\left(\int_t^{t+h} e^{-a(t+h-r)}\sin\frac{\xi}{b}(t+h-r)\,|Z_k(r)\,|dr\right)^2$$
$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} \sum_{k=1}^\infty \lambda_k^{2\gamma-2}\int_t^{t+h}(t+h-r)^\epsilon dr\int_t^{t+h}\lambda_k^{(q-1)\epsilon}T^{2\epsilon}\left|Z_k(r)\right|^2 dr$$
$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{2\epsilon}h^{\epsilon+1}\sum_{k=1}^\infty\int_0^T \lambda_k^{2\gamma+(q-1)\epsilon-2}\left|Z_k(r)\right|^2 dr$$
$$=_{\mu,\eta,\epsilon,\lambda_1} T^{2\epsilon}h^{\epsilon+1}\int_0^T\left\|Z(r)\right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}-1}(\Omega)}dr, 0<\epsilon\leq 1.$$

By choose $\epsilon = \frac{2(\zeta-\gamma)}{q-1}$, we have immediately that

$$\left\|\int_t^{t+h} O_q(t+h-r)Z(r)dr\right\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^\epsilon h^{\frac{\epsilon+1}{2}}\|Z\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

Using the results just obtained we get

$$\Big\|u(t+h)-u(t)\Big\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} h^{\frac{\epsilon}{2}}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^{\gamma}(\Omega)}+\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}\Big)$$
$$+h^{\epsilon}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}\Big)+h^{2\epsilon}\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}+h^{\frac{\epsilon+1}{2}}\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta-1}(\Omega))}.$$

We have thus proved that $u \in C^{\frac{\epsilon}{2}}([0,T],\mathbb{H}^{\gamma}(\Omega))$. The proof is complete. □

Theorem 3.5. *Let $q>1$, and $\mu,\eta>0$ such that $4\lambda_1^q(1+\mu\lambda_1)-\eta^2\lambda_1^2=0$. Let $\varphi\in\mathbb{H}^{\gamma}(\Omega)$ for any $\gamma\geq 0$. Let us assume that $\psi\in\mathbb{H}^{\zeta}(\Omega)$ and $Z\in L^2(0,T;\mathbb{H}^{\zeta}(\Omega))$ with $1<\gamma<\zeta$ and $0<\frac{(\zeta-\gamma)}{q-1}\leq\frac{1}{2}$. Let ϵ such that $0<\epsilon=\frac{2(\zeta-\gamma)}{q-1}$. Then we have*

$$\Big\|u\Big\|_{L^\infty(0,T;\mathbb{H}^{\gamma}(\Omega))}\leq C(\epsilon,\mu,\eta,\lambda_1)T^{\epsilon+1}\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+C(\epsilon,\mu,\eta)T^{\frac{\epsilon}{2}}\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+C(\mu,\eta,\epsilon)T^{\epsilon+\frac{3}{2}}\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta}(\Omega))}.$$

Furthermore, we show that $u\in C^{\theta}([0,T],\mathbb{H}^{\gamma}(\Omega))$, where $\theta=min\{\frac{\epsilon}{2},\frac{1}{2}\}$.

Proof. Applying Hölder inequality and an easy computation shows that

$$\Big\|\int_0^t \mathcal{Q}_q(t-r)Z(r)dr\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}\lesssim_{\mu,\eta,\epsilon} T^{2\epsilon+3}\int_0^t\Big\|Z(.,r)dr\Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)}dr\lesssim_{\mu,\eta,\epsilon} T^{2\epsilon+3}\Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^{\zeta}(\Omega))}. \tag{51}$$

We can now combine the results of Lemma (3.2) with (51) and obtain the following result

$$\Big\|u(x,t)\Big\|_{\mathbb{H}^{\gamma}(\Omega)}\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{\epsilon+1}\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+T^{\frac{\epsilon}{2}}\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+T^{\epsilon+\frac{3}{2}}\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta}(\Omega))}.$$

It follows immediately that

$$\Big\|u\Big\|_{L^\infty(0,T;\mathbb{H}^{\gamma}(\Omega))}\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{\epsilon+1}\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+T^{\frac{\epsilon}{2}}\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}+T^{\epsilon+\frac{3}{2}}\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\zeta}(\Omega))}.$$

In the next part, we prove $u\in C^{\theta}([0,T],\mathbb{H}^{\gamma}(\Omega))$. Let $0\leq t\leq t+h\leq T$, with $h>0$.
From (19), we get

$$u(t+h)-u(t)=\Big(\mathcal{P}_q(t+h)-\mathcal{P}_q(t)\Big)\varphi+\Big(\mathcal{Q}_q(t+h)-\mathcal{Q}_q(t)\Big)\psi$$
$$+\int_0^t\Big(\mathcal{Q}_q(t+h-r)-\mathcal{Q}_q(t-r)\Big)Z(r)dr+\int_t^{t+h}\mathcal{Q}_q(t+h-r)Z(r)dr.$$

We have divided the estimate into four step following
Step 1. Estimation of $\Big\|\Big(\mathcal{P}_q(t+h)-\mathcal{P}_q(t)\Big)\varphi\Big\|_{\mathbb{H}^{\gamma}(\Omega)}$.
Our estimation starts with the observation that

$$\Big\|\Big(\mathcal{P}_q(t+h)-\mathcal{P}_q(t)\Big)\varphi\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}=\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big[\Big(e^{-\beta_1(t+h)}-e^{-\beta_1 t}\Big)+\beta_2\Big((t+h)e^{-\beta_1(t+h)}-te^{-\beta_1 t}\Big)\Big]^2|\varphi_k|^2.$$

On the other hand, we have that

$$\Big|e^{-\beta_1(t+h)}-e^{-\beta_1 t}\Big|\lesssim_{\mu,\eta,\epsilon} h^{\epsilon}\lambda_k^{\frac{(q-1)\epsilon}{2}},$$

and

$$\Big((t+h)e^{-\beta_1(t+h)} - te^{-\beta_1 t}\Big) = t\Big(e^{-\beta_1(t+h)} - e^{-\beta_2 t}\Big) + he^{-\beta_1(t+h)} \tag{52}$$

$$\lesssim_{\mu,\eta,\epsilon} h^{\epsilon} T \lambda_k^{\frac{(q-1)\epsilon}{2}} + hT^{\epsilon}\lambda_k^{\frac{(q-1)\epsilon}{2}}, \text{ for any } \quad 0 < \epsilon \le 1. \tag{53}$$

Hence, we can assert that

$$\Big\|\Big(\mathcal{P}_q(t+h) - \mathcal{P}_q(t)\Big)\varphi\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,T} h^{\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\varphi_k|^2 + h^{\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\varphi_k|^2 + h\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\varphi_k|^2$$
$$=_{\mu,\eta,\epsilon,T} h^{\epsilon}\Big\|\varphi\Big\|^2_{\mathbb{H}^{\zeta}(\Omega)} + h^{\epsilon}\Big\|\varphi\Big\|^2_{\mathbb{H}^{\zeta}(\Omega)} + h\Big\|\varphi\Big\|^2_{\mathbb{H}^{\zeta}(\Omega)}.$$

It follows that

$$\Big\|\Big(\mathcal{P}_q(t+h) - \mathcal{P}_q(t)\Big)\varphi\Big\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\frac{\epsilon}{2}}\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)} + h^{\frac{1}{2}}\Big\|\varphi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

Step 2. Estimation of $\Big\|\Big(\mathcal{Q}_q(t+h) - \mathcal{Q}_q(t)\Big)\psi\Big\|_{\mathbb{H}^{\gamma}(\Omega)}$.
By virtue of (20) one can

$$\Big\|\Big(\mathcal{Q}_q(t+h) - \mathcal{Q}_q(t)\Big)\psi\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)} = \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big[(t+h)e^{-\beta_1(t+h)} - te^{-\beta_1 t}\Big]^2|\psi_k|^2. \tag{54}$$

From (53) it follows that

$$\Big\|\Big(\mathcal{Q}_q(t+h) - \mathcal{Q}_q(t)\Big)\psi\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\psi_k|^2 + h\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\frac{(q-1)\epsilon}{2}}|\psi_k|^2$$
$$=_{\mu,\eta,\epsilon,\lambda_1,T} h^{\epsilon}\Big\|\psi\Big\|^2_{\mathbb{H}^{\zeta}(\Omega)} + h\Big\|\psi\Big\|^2_{\mathbb{H}^{\zeta}(\Omega)}.$$

Therefore,

$$\Big\|\Big(\mathcal{Q}_q(t+h) - \mathcal{Q}_q(t)\Big)\psi\Big\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\frac{\epsilon}{2}}\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)} + h^{\frac{1}{2}}\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

Step 3. Estimation of $\Big\|\int_0^t\Big(\mathcal{Q}_q(t+h-r) - \mathcal{Q}_q(t-r)\Big)Z(r)dr\Big\|_{\mathbb{H}^{\gamma}(\Omega)}$.
We first observe that

$$\Big\|\int_0^t\Big(\mathcal{Q}_q(t+h-r) - \mathcal{Q}_q(t-r)\Big)Z(r)dr\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}$$
$$= \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\bigg(\int_0^t\Big[(t+h-r)e^{-\beta_1(t+h-r)} - (t-r)e^{-\beta_1(t-r)}\Big]|Z_k(r)|dr\bigg)^2.$$

By an argument similar to the previous one and using bounded in (53). We get

$$\Big\|\int_0^t\Big(\mathcal{Q}_q(t+h-r) - \mathcal{Q}_q(t-r)\Big)Z(r)dr\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\epsilon} T^3h^{2\epsilon}\int_0^t\sum_{k=1}^{\infty}\lambda_k^{2\gamma+(q-1)\epsilon}\Big|Z_k(r)\Big|^2dr$$
$$+T^{2\epsilon+1}h^2\int_0^t\sum_{k=1}^{\infty}\lambda_k^{2\gamma+(q-1)\epsilon}\Big|Z_k(r)\Big|^2dr$$
$$\lesssim_{\mu,\epsilon} T^3h^{2\epsilon}\int_0^T\Big\|Z(r)\Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)}dr + T^{2\epsilon+1}h^2\int_0^T\Big\|Z(r)\Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)}dr.$$

By choose $\epsilon = \frac{2(\zeta-\gamma)}{q-1}$, we have that

$$\Big\| \int_0^t \Big(Q_q(t+h-r) - Q_q(t-r)\Big) Z(r)dr \Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\epsilon} T^3 h^{2\epsilon} \Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\zeta(\Omega))} + T^{2\epsilon+1} h^2 \|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\zeta(\Omega))}.$$

This give

$$\Big\| \int_0^t \Big(Q_q(t+h-r) - Q_q(t-r)\Big) Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\epsilon} T^{\frac{3}{2}} h^{\epsilon} \Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))} + T^{\epsilon+\frac{1}{2}} h \|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))}.$$

Step 4. Estimation of $\Big\| \int_t^{t+h} Q_q(t+h-r)Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)}$.
It is easy to verify that

$$\begin{aligned}
\Big\| \int_t^{t+h} Q_q(t+h-r)Z(r)dr \Big\|^2_{\mathbb{H}^\gamma(\Omega)} &= \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \Big(\int_t^{t+h} (t+h-r) e^{-\beta_1(t+h-r)} \, |Z_k(r)\, | dr \Big)^2 \\
&\lesssim_{\mu,\epsilon} \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \int_t^{t+h} (t+h-r)^{\epsilon+2} dr \int_t^{t+h} \lambda_k^{(q-1)\epsilon} \Big|Z_k(r)\Big|^2 dr \\
&\lesssim_{\mu,\epsilon} h^{\epsilon+3} \sum_{k=1}^{\infty} \int_0^T \lambda_k^{2\gamma+(q-1)\epsilon} \Big|Z_k(r)\Big|^2 dr =_{\mu,\epsilon} h^{\epsilon+3} \int_0^T \Big\|Z(r)\Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon}{2}}(\Omega)} dr \\
&=_{\mu,\epsilon} h^{\epsilon+3} \|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\zeta(\Omega))}.
\end{aligned}$$

We contend that

$$\Big\| \int_t^{t+h} Q_q(t+h-r)Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\epsilon} h^{\frac{\epsilon+3}{2}} \|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))}.$$

Thus, we have immediately that

$$\begin{aligned}
\Big\|u(t+h) - u(t)\Big\|_{\mathbb{H}^\gamma(\Omega)} &\lesssim_{\mu,\eta,\epsilon,\lambda_1,T} h^{\frac{\epsilon}{2}} \Big(\Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)} \Big) + h^{\frac{1}{2}} \Big(\Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} \Big) \\
&\quad + T h^{\epsilon} \Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))} + T^{\epsilon+\frac{1}{2}} h \|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))} + h^{\frac{\epsilon+3}{2}} \|Z\Big\|_{L^2(0,T;\mathbb{H}^\zeta(\Omega))}.
\end{aligned}$$

The result above shows that $u \in C^\theta([0,T], \mathbb{H}^\gamma(\Omega))$ where $\theta = min\{\frac{\epsilon}{2}, \frac{1}{2}\}$. □

Theorem 3.6. *Let* $q > 1$, *and* $\mu, \eta > 0$ *such that* $4\lambda_1^q(1+\mu\lambda_1) - \eta^2\lambda_1^2 < 0$. *Let* $\varphi \in \mathbb{H}^\gamma(\Omega)$ *for any* $\gamma \geq 0$. *Let us assume that* $\psi \in \mathbb{H}^\gamma(\Omega)$ *and* $Z \in L^2(0,T;\mathbb{H}^\gamma(\Omega))$ *with* $0 < \gamma$. *Let* $0 < \epsilon \leq 1$. *Then we have*

$$\Big\|u\Big\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{2\epsilon} \Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{2\epsilon} \Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{4\epsilon+1} \Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

Moreover, we obtain that $u \in C^\epsilon([0,T], \mathbb{H}^\gamma(\Omega))$.

Proof. Our proof starts with the observation that

$$\Big\|u(x,t)\Big\|_{\mathbb{H}^\gamma(\Omega)} \leq \Big\|\mathcal{R}_q(t)\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + \Big\|\mathcal{S}_q(t)\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} + \Big\| \int_0^t \mathcal{S}_q(t-r)Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)}.$$

The application of Holder inequality and (41), yields

$$\Big\| \int_0^t \mathcal{S}_q(t-r)Z(r)dr \Big\|^2_{\mathbb{H}^\gamma(\Omega)} \le \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \int_0^t C(\epsilon,\mu,\eta)(t-r)^{2\epsilon}dr \int_0^t C(\lambda_1,\mu,\epsilon) \tag{55}$$

$$\times (t-r)^{2\epsilon}\Big| Z(.), e_k(.)\Big)\Big|^2 dr \le C(\lambda_1,\epsilon,\mu,\eta)T^{4\epsilon+1}\Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}. \tag{56}$$

We can now combine the results of lemma (3.3) with (56) and obtain the following estimate

$$\Big\|u(x,t)\Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\lambda_1,\epsilon,\mu,\eta} T^{2\epsilon}\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{2\epsilon}\Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{4\epsilon+1}\Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

It follows immediately that

$$\Big\|u\Big\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\lambda_1,\epsilon,\mu,\eta} T^{2\epsilon}\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{2\epsilon}\Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{4\epsilon+1}\Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

To complete the proof, it just remains to prove $u \in C^{\frac{\epsilon}{2}}([0,T],\mathbb{H}^\gamma(\Omega))$. We let $0 \le t \le t+h \le T$ with $h > 0$. It holds that

$$u(t+h) - u(t) = \Big(\mathcal{R}_q(t+h) - \mathcal{R}_q(t)\Big)\varphi + \Big(\mathcal{S}_q(t+h) - \mathcal{S}_q(t)\Big)\psi$$
$$+ \int_0^t \Big(\mathcal{S}_q(t+h-r) - \mathcal{S}_q(t-r)\Big)Z(r)dr + \int_t^{t+h} \mathcal{S}_q(t+h-r)Z(r)dr.$$

We have divided the estimate into four step following

Step 1. Estimation of $\Big\|\Big(\mathcal{R}_q(t+h) - \mathcal{R}_q(t)\Big)\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)}$.

In view of the above, there is

$$\Big\|\Big(\mathcal{R}_q(t+h) - \mathcal{R}_q(t)\Big)\varphi\Big\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big[\Big(\frac{1}{2} - a_1\Big)\Big(e^{-\xi_1(t+h)} - e^{-\xi_1 t}\Big) + \Big(\frac{1}{2} + a_1\Big)\Big(e^{-\xi_2(t+h)} - e^{-\xi_2 t}\Big)\Big]^2 |\varphi_k|^2.$$

Using the inequality $|e^{-a} - e^{-b}| \le C_\epsilon |a-b|^\epsilon$ for any $0 < \epsilon \le 1$, we know that

$$\Big[\Big(\frac{1}{2} - a_1\Big)\Big(e^{-\xi_1(t+h)} - e^{-\xi_1 t}\Big) + \Big(\frac{1}{2} + a_1\Big)\Big(e^{-\xi_2(t+h)} - e^{-\xi_2 t}\Big)\Big] \lesssim_\epsilon (\xi_1 h)^\epsilon + \Big(\frac{1}{2} + a_1\Big)(\xi_2 h)^\epsilon.$$

From the estimates in (39) and (40), we get that

$$\Big\|\Big(\mathcal{R}_q(t+h) - \mathcal{R}_q(t)\Big)\varphi\Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^{2\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma}|\varphi_k|^2 =_{\mu,\eta,\epsilon} h^{2\epsilon}\Big\|\varphi\Big\|^2_{\mathbb{H}^\gamma(\Omega)}.$$

It follows that

$$\Big\|\Big(\mathcal{R}_q(t+h) - \mathcal{R}_q(t)\Big)\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^{\epsilon}\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)}.$$

Step 2. Estimation of $\Big\|\Big(\mathcal{S}_q(t+h) - \mathcal{S}_q(t)\Big)\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)}$.

From (22), we have

$$\Big\|\Big(\mathcal{S}_q(t+h) - \mathcal{S}_q(t)\Big)\psi\Big\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big[a_2\Big(e^{-\xi_1 t} - e^{-\xi_1(t+h)}\Big) + a_2\Big(e^{-\xi_2(t+h)} - e^{-\xi_2 t}\Big)\Big]^2 |\psi_k|^2. \tag{57}$$

By an argument analogous to the previous one. We get

$$\Big\|\Big(\mathcal{S}_q(t+h)-\mathcal{S}_q(t)\Big)\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^\epsilon \Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)}.$$

Step 3. Estimation of $\Big\| \int_0^t \Big(\mathcal{S}_q(t+h-r)-\mathcal{S}_q(t-r)\Big)Z(r)dr\Big\|_{\mathbb{H}^\gamma(\Omega)}$.
It follows immediately that

$$\Big\| \int_0^t \Big(\mathcal{S}_q(t+h-r)-\mathcal{S}_q(t-r)\Big)Z(r)dr\Big\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^\infty \lambda_k^{2\gamma}\bigg(\int_0^t a_2\bigg[\Big(e^{-\xi_2(t+h-r)}-e^{-\xi_2(t-r)}\Big) + \Big(e^{-\xi_1(t-r)}-e^{-\xi_1(t+h-r)}\Big)\bigg]\,|Z_k(r)\,|dr\bigg)^2.$$

The proof is standard, we have estimate

$$\bigg|\Big(e^{-\xi_2(t+h-r)}-e^{-\xi_2(t-r)}\Big)+\Big(e^{-\xi_1(t-r)}-e^{-\xi_1(t+h-r)}\Big)\bigg| \lesssim_{\mu,\eta,\epsilon} h^\epsilon.$$

We thus get

$$\Big\| \int_0^t \Big(\mathcal{S}_q(t+h-r)-\mathcal{S}_q(t-r)\Big)Z(r)dr\Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^{2\epsilon}\int_0^t \sum_{k=1}^\infty \lambda_k^{2\gamma}\Big|Z_k(r)\Big|^2 dr$$
$$\lesssim_{\mu,\eta,\epsilon} h^{2\epsilon}\int_0^T \Big\|Z(r)\Big\|^2_{\mathbb{H}^\gamma(\Omega)}dr =_{\mu,\eta,\epsilon} h^{2\epsilon}\Big\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

Consequently, one derives

$$\Big\| \int_0^t \Big(\mathcal{S}_q(t+h-r)-\mathcal{S}_q(t-r)\Big)Z(r)dr\Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^{\epsilon}\Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

Step 4. Estimation of $\Big\| \int_t^{t+h} \mathcal{S}_q(t+h-r)Z(r)dr\Big\|_{\mathbb{H}^\gamma(\Omega)}$.
By a similar argument,

$$\Big\| \int_t^{t+h} \mathcal{S}_q(t+h-r)Z(r)dr\Big\|^2_{\mathbb{H}^\gamma(\Omega)} = \sum_{k=1}^\infty \lambda_k^{2\gamma}\bigg(\int_t^{t+h} a_2\Big[e^{-\xi_2(t+h-r)}-e^{-\xi_1(t+h-r)}\Big]\,|Z_k(r)\,|dr\bigg)^2.$$

It holds that

$$\Big|e^{-\xi_2(t+h-r)}-e^{-\xi_1(t+h-r)}\Big| \lesssim_\epsilon (\xi_2-\xi_1)^\epsilon (t+h-r)^\epsilon,$$

and

$$(\xi_2-\xi_1)^\epsilon \lesssim_{\mu,\eta,\epsilon} 1.$$

From the above it follows that

$$\Big\| \int_t^{t+h} \mathcal{S}_q(t+h-r)Z(r)dr\Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} \sum_{k=1}^\infty \lambda_k^{2\gamma}\int_t^{t+h}(\xi_2-\xi_1)^{2\epsilon}(t+h-r)^{2\epsilon}dr\int_t^{t+h}\Big|Z_k(r)\Big|^2 dr$$
$$\lesssim_{\mu,\eta,\epsilon} h^{2\epsilon+1}\sum_{k=1}^\infty\int_0^T \lambda_k^{2\gamma}\Big|Z_k(r)\Big|^2 dr =_{\mu,\eta,\epsilon} h^{2\epsilon+1}\int_0^T\Big\|Z(r)\Big\|^2_{\mathbb{H}^\gamma(\Omega)}dr =_{\mu,\eta,\epsilon} h^{2\epsilon+1}\|Z\Big\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

Thus, we have immediately that

$$\Big\| \int_t^{t+h} \mathcal{S}_q(t+h-r)Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^{\epsilon+\frac{1}{2}} \|Z\|_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

Using the results just obtained we get

$$\Big\| u(t+h) - u(t) \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon} h^\epsilon \Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + h^\epsilon \Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)} + h^\epsilon \Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^\gamma(\Omega))} + h^{2\epsilon+1} \|Z\|^2_{L^2(0,T;\mathbb{H}^\gamma(\Omega))}.$$

We have thus proved that $u \in C^\epsilon([0,T], \mathbb{H}^\gamma(\Omega))$. □

3.2. *Convergence of the mild solution when $\eta \to 0^+$*

In this section, we show that the mild solution the Boussinesq-Love equation convergences to the mild solution the Love equation . Let u_* be the mild solution to problem

$$\begin{cases} u_{tt}(x,t) + (-\Delta)^q u(x,t) - \mu \Delta u_{tt}(x,t) = Z(x,t)), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{58}$$

with the initial conditions

$$u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x) \quad \text{in} \quad \Omega. \tag{59}$$

Our goal is to prove that u converges to u^* when $\eta \to 0^+$. We can now formulate our main results.

Theorem 3.7. *For any $q > 1$, and $\mu, \eta > 0$ such that $4\lambda_1^q(1+\mu\lambda_1) - \eta^2\lambda_1^2 > 0$. Let $\varphi \in \mathbb{H}^\zeta(\Omega)$ and $\psi \in \mathbb{H}^\zeta(\Omega)$ for any $\max\{1, \frac{(q+1)-(q-1)\epsilon}{2}\} < \gamma < \zeta, \epsilon = \frac{2(\zeta-\gamma)}{q-1} \le 1$. Let us assume that $Z \in L^2(0,T;\mathbb{H}^\zeta(\Omega)) \cap L^2(0,T;\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega))$. Then, we have*

$$\Big\| u - u^* \Big\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\epsilon,\mu,\lambda_1,T} \eta^\epsilon \Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \eta^{\epsilon+2} \Big\|\psi\Big\|_{\mathbb{H}^{\gamma+\frac{3+q(\epsilon-1)-\epsilon}{2}}(\Omega)} + \eta^{\epsilon+2} \Big\|Z\Big\|_{L^2(0,T;\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega))}.$$

Proof. We denote $b^* = (1+\mu\lambda_k)$ and $\xi^* = \sqrt{\lambda_k^q(1+\mu\lambda_k)}$. An easy computation shows that the mild solution of (58)-(59) is given by

$$u^*(t) = \mathcal{M}_q^*(t)\varphi + \mathcal{N}_q^*(t)\psi + \int_0^t \mathcal{O}_q^*(t-r)Z(r)dr, \tag{60}$$

where

$$\mathcal{M}_q^*(t)v = \sum_{k=1}^\infty \cos\Big(\frac{\xi^*}{b^*}t\Big)\Big(v(.), e_k(.)\Big)e_k(x),$$

$$\mathcal{N}_q^*(t)v = \sum_{k=1}^\infty \frac{b^*}{\xi^*}\sin\Big(\frac{\xi^*}{b^*}t\Big)\Big(v(.), e_k(.)\Big)e_k(x),$$

and

$$\mathcal{O}_q^*(t)v = \sum_{k=1}^\infty \frac{1}{\xi^*}\sin\Big(\frac{\xi^*}{b^*}t\Big)\Big(v(.), e_k(.)\Big)e_k(x).$$

Our proof starts with the estimating $\left\|u(t)-u^*(t)\right\|_{\mathbb{H}^\gamma(\Omega)}$.
From (18) and (60) we get that

$$u(t)-u^*(t)=\Big(\mathcal{M}_q(t)-\mathcal{M}_q^*(t)\Big)\varphi+\Big(\mathcal{N}_q(t)-\mathcal{N}_q^*(t)\Big)\psi+\int_0^t\Big(O_q(t-r)-O_q^*(t-r)\Big)Z(r)dr.$$

The the estimating $\left\|u(t)-u^*(t)\right\|_{\mathbb{H}^\gamma(\Omega)}$ falls naturally into three parts.

Part 1. Estimation of $\left\|\Big(\mathcal{M}_q(t)-\mathcal{M}_q^*(t)\Big)\varphi\right\|_{\mathbb{H}^\gamma(\Omega)}$.
From Parseval's equality, we find that

$$\left\|\Big(\mathcal{M}_q(t)-\mathcal{M}_q^*(t)\Big)\varphi\right\|^2_{\mathbb{H}^\gamma(\Omega)}=\sum_{k=1}^\infty\lambda_k^{2\gamma}\left[e^{-at}\cos\left(\frac{\xi}{b}t\right)-\cos\left(\frac{\xi^*}{b^*}t\right)\right]^2|\varphi_k|^2.$$

It is worth pointing out that

$$\left|e^{-at}\cos\left(\frac{\xi}{b}t\right)-\cos\left(\frac{\xi^*}{b^*}t\right)\right|=e^{-at}\left|\cos\left(\frac{\xi}{b}t\right)-\cos\left(\frac{\xi^*}{b^*}t\right)\right|+\left|\Big(e^{-at}-1\Big)\cos\left(\frac{\xi^*}{b^*}t\right)\right|.$$

It is straightforward to verify that

$$\frac{\xi^*}{b^*}\lesssim_\mu\lambda_k^{\frac{q-1}{2}}. \tag{61}$$

Using the inequality $|\cos x|\le 1+C_\epsilon x^\epsilon$, we infer

$$|\cos\left(\frac{\xi^*}{b^*}t\right)|\le 1+C(\mu,\epsilon)\lambda_k^{\frac{q-1}{2}\epsilon}t^\epsilon.$$

We continue to apply the inequality $\Big|\cos(m)-\cos(n)\Big|\le C(\varepsilon)|m-n|^\varepsilon$, for $0<\epsilon\le 1$, and a direct computation yields

$$\left|\cos\left(\frac{\xi}{b}t\right)-\cos\left(\frac{\xi^*}{b^*}t\right)\right|\lesssim_\epsilon\left|\left(\frac{\sqrt{4\lambda_k^q(1+\mu\lambda_k)}}{2(1+\mu\lambda_k)}t\right)-\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\right|^\epsilon=0. \tag{62}$$

Finally, noting that

$$|e^{-at}-1|\le\ |e^{-at}|\lesssim_\epsilon a^\epsilon t^\epsilon\lesssim_{\mu,\epsilon}\eta^\epsilon t^\epsilon.$$

We conclude,

$$\left\|\Big(\mathcal{M}_q(t)-\mathcal{M}_q^*(t)\Big)\varphi\right\|^2_{\mathbb{H}^\gamma(\Omega)}\lesssim_{\epsilon,\mu}\eta^{2\epsilon}t^{2\epsilon}\sum_{k=1}^\infty\lambda_k^{2\gamma}|\varphi_k|^2+\eta^{2\epsilon}t^{4\epsilon}\sum_{k=1}^\infty\lambda_k^{2\gamma+(q-1)\epsilon}|\varphi_k|^2.$$

We thus get

$$\left\|\Big(\mathcal{M}_q(t)-\mathcal{M}_q^*(t)\Big)\varphi\right\|_{\mathbb{H}^\gamma(\Omega)}\lesssim_{\mu,\epsilon,T}\eta^\epsilon\left\|\varphi\right\|_{\mathbb{H}^\zeta(\Omega)}, \tag{63}$$

where we have used embedding $H^{\gamma}(\Omega) \hookrightarrow H^{\zeta}(\Omega)$.

Step 2. Estimation of $\left\|\left(\mathcal{N}_q(t) - \mathcal{N}_q^*(t)\right)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)}$.

We first observe that

$$\left\|\left(\mathcal{N}_q(t) - \mathcal{N}_q^*(t)\right)\psi\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \left[\frac{b}{\xi} e^{-at} \sin\frac{\xi}{b} t - \frac{b^*}{\xi^*} \sin\frac{\xi^*}{b^*} t\right]^2 |\psi_k|^2.$$

It is plain that

$$\left[\frac{b}{\xi} e^{-at} \sin\frac{\xi}{b} t - \frac{b^*}{\xi^*} \sin\frac{\xi^*}{b^*} t\right] = \frac{b}{\xi} e^{-at}\left[\sin\frac{\xi}{b} t - \sin\frac{\xi^*}{b^*} t\right] + \left[\frac{b}{\xi} - \frac{b^*}{\xi^*}\right] e^{-at} \sin\frac{\xi^*}{b^*} t$$
$$+ \left[e^{-at} - 1\right]\frac{\xi^*}{b^*} \sin\frac{\xi^*}{b^*} t := \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3}.$$

A trivial verification and combining with (62),we obtain

$$|\mathbf{B_1}| \lesssim_{\epsilon} \frac{b}{\xi}(at)^{\epsilon}\left|\frac{\xi}{b} t - \frac{\xi^*}{b^*} t\right|^{\epsilon} \lesssim_{\epsilon,\mu,\lambda_1} 0.$$

An easy computation shows that

$$\left|\frac{b}{\xi} - \frac{b^*}{\xi^*}\right| \leq \frac{\eta^2 \lambda_k^2 (1 + \mu\lambda_k)}{\lambda_k^q (4\lambda_k^q (1 + \mu\lambda_k) - \eta^2 \lambda_k^2)} \frac{1}{\frac{2(1+\mu\lambda_k)}{\sqrt{4\lambda_k^q (1+\mu\lambda_k) + \eta^2 \lambda_k^2}} + \sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}} \tag{64}$$

$$\leq \frac{\eta^2 \lambda_k^2 (1 + \mu\lambda_k)^2 \lambda_k^{\frac{-q}{2}}}{4\lambda_k^q (1 + \mu\lambda_k) - \eta^2 \lambda_k^2)\sqrt{1 + \mu\lambda_k}}. \tag{65}$$

Combining (65) with (31), we obtain that the following estimate

$$\left|\frac{b}{\xi} - \frac{b^*}{\xi^*}\right| \lesssim_{\epsilon,\mu,\lambda_1} \eta^2 \lambda_k^{\frac{3-q}{2}}. \tag{66}$$

Combining (27),(61) and (66) gives $|\mathbf{B_2}| \lesssim_{\epsilon,\mu,\lambda_1} \eta^{\epsilon+2} t^{2\epsilon} \lambda_k^{\frac{3+q(\epsilon-1)-\epsilon}{2}}$.
We proceed now estimate the term $|B_3|$. We have

$$|\mathbf{B_3}| \lesssim_{\epsilon,\mu} \eta^{\epsilon} t^{2\epsilon} \sqrt{\frac{1 + \mu\lambda_k}{\lambda_k^q}} \left(\sqrt{\frac{\lambda_k^q}{1 + \mu\lambda_k}}\right)^{\epsilon} \lesssim_{\epsilon,\mu,\lambda_1} \eta^{\epsilon} t^{2\epsilon} \lambda_k^{\frac{1-\epsilon+q\epsilon-q}{2}}.$$

Hence,

$$\left\|\left(\mathcal{N}_q(t) - \mathcal{N}_q^*(t)\right)\psi\right\|^2_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\epsilon} \eta^{\epsilon+4} T^{4\epsilon} \|\psi\|^2_{\mathbb{H}^{\gamma+\frac{3+q(\epsilon-1)-\epsilon}{2}}(\Omega)} + \eta^{2\epsilon} t^{4\epsilon} \|\psi\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)(\epsilon-1)}{2}}(\Omega)}. \tag{67}$$

Since $\max\{1, \frac{(q+1)-(q-1)\epsilon}{2}\} < \gamma < \zeta$, we have $\gamma + \frac{(q-1)(\epsilon-1)}{2} > 0$ and it is easy to see $\gamma + \frac{(q-1)(\epsilon-1)}{2} < \gamma + \frac{3+q(\epsilon-1)-\epsilon}{2}$. Hence, by using the properties concerning embedding in Hilbert Scale spaces, we can deduce that

$$\left\|\left(\mathcal{N}_q(t) - \mathcal{N}_q^*(t)\right)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\epsilon} \eta^{\epsilon+2} T^{2\epsilon} \|\psi\|_{\mathbb{H}^{\gamma+\frac{3+q(\epsilon-1)-\epsilon}{2}}(\Omega)}. \tag{68}$$

Step 3. Estimation of $\left\| \int_0^t \left(O_q(t-r) - O_q^*(t-r)\right) Z(r)dr \right\|_{\mathbb{H}^\gamma(\Omega)}$.

We claim that Indeed,

$$\left\| \int_0^t \left(O_q(t-r) - O_q^*(t-r)\right) Z(r)dr \right\|^2_{\mathbb{H}^\gamma(\Omega)} \leq \sum_{k=1}^{\infty} \lambda_k^{2\gamma} \left[\int_0^t \left| \left(\frac{2}{\xi} e^{-a(t-r)} \sin \frac{\xi}{b}(t-r) - \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) \right) Z_k(r) \right| dr \right]^2.$$

By using Holder inequality

$$\left[\int_0^t \left| \left(\frac{2}{\xi} e^{-a(t-r)} \sin \frac{\xi}{b}(t-r) - \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) \right) Z_k(r) \right| dr \right]^2 \tag{69}$$

$$\leq \int_0^t \left| \frac{2}{\xi} e^{-a(t-r)} \sin \frac{\xi}{b}(t-r) - \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) \right|^2 dr \int_0^t |Z_k(r)|^2 dr. \tag{70}$$

In next section we estimate the integral in (70). By adding and subtracting terms we get

$\left[\frac{2}{\xi} e^{-a(t-r)} \sin \frac{\xi}{b}(t-r) - \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) \right] = \frac{2}{\xi} e^{-a(t-r)} \left[\sin \frac{\xi}{b}(t-r) - \sin \frac{\xi^*}{b^*}(t-r) \right]$

$+ \left[\frac{2}{\xi} - \frac{1}{\xi^*} \right] e^{-a(t-r)} \sin \frac{\xi^*}{b^*}(t-r) + [e^{-a(t-r)} - 1] \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) := \mathbf{C_1} + \mathbf{C_2} + \mathbf{C_3}.$

We can now proceed analogously to the estimates of $\mathbf{B_1}$, one has $|\mathbf{C_1}| \lesssim_{\epsilon,\mu,\lambda_1} 0$.

It is easy to verify that the following bounds

$$\left| \frac{2}{\xi} - \frac{1}{\xi^*} \right| \leq \left| \frac{\eta^2 \lambda_k^2}{\left(4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2\right)(1+\mu\lambda_k)\lambda_k^q} \frac{1}{\frac{2}{\sqrt{4\lambda_k^q(1+\mu\lambda_k)-\eta^2\lambda_k^2}} + \frac{1}{\sqrt{(1+\mu\lambda_k)\lambda_k^q}}} \right| \tag{71}$$

$$\leq \left| \frac{\eta^2 \lambda_k^2}{\left(4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2\right)} \right| \lesssim_{\epsilon,\mu,\lambda_1} \eta^2 \lambda_k^2. \tag{72}$$

On the other hand, we have

$$|e^{-a(t-r)}| \lesssim_{\epsilon,\mu} \eta^\epsilon (t-r)^\epsilon. \tag{73}$$

Combining (72), (73), and (61), we obtain

$$|\mathbf{C_2}| \lesssim_{\epsilon,\mu,\lambda_1} \eta^{\epsilon+2}(t-r)^{2\epsilon} \lambda_k^{\frac{(q-1)\epsilon+4}{2}}.$$

This follows by the same method as in estimate $|\mathbf{B_2}|$, we get

$$|\mathbf{C_3}| \lesssim_{\epsilon,\mu} \eta^\epsilon (t-r)^\epsilon \lambda_k^{\frac{(q-1)\epsilon-(q+1)}{2}}.$$

Hence,

$$\left| \frac{2}{\xi} e^{-a(t-r)} \sin \frac{\xi}{b}(t-r) - \frac{1}{\xi^*} \sin \frac{\xi^*}{b^*}(t-r) \right|^2 \lesssim_{\epsilon,\mu,\lambda_1} \eta^{2\epsilon+4}(t-r)^{4\epsilon} \lambda_k^{(q-1)\epsilon+4} + \eta^{2\epsilon}(t-r)^{2\epsilon} \lambda_k^{(q-1)\epsilon-(q+1)}.$$

From the above observation, we can deduce that

$$\left\| \int_0^t \left(O_q(t-r) - O_q^*(t-r)\right) Z(r)dr \right\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\epsilon,\mu,\lambda_1} \eta^{2\epsilon+4} T^{4\epsilon+1} \int_0^t \sum_{k=1}^{\infty} \lambda_k^{2\gamma+(q-1)\epsilon+4}$$

$$\times |Z_k(r)|^2 dr + \eta^{2\epsilon} T^{2\epsilon+1} \int_0^t \sum_{k=1}^{\infty} \lambda_k^{2\gamma+(q-1)\epsilon-(q+1)} |Z_k(r)|^2 dr$$

$$\lesssim_{\epsilon,\mu,\lambda_1} \eta^{2\epsilon+4} T^{4\epsilon+1} \int_0^T \left\| Z(r) \right\|^2_{\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega)} dr + \eta^{2\epsilon} T^{2\epsilon+1} \int_0^T \left\| Z(r) \right\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon-(q+1)}{2}}(\Omega)} dr.$$

Again using the the properties concerning embedding in Hilbert Scale spaces together with the fact that $0 < \gamma + \frac{(q-1)\epsilon-(q+1)}{2} < \gamma + 2 + \frac{(q-1)\epsilon}{2}$, we infer that

$$\Big\| \int_0^t \Big(O_q(t-r) - O_q^*(t-r)\Big) Z(r)dr \Big\|^2_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\epsilon,\mu,\lambda_1} \eta^{2\epsilon+4} T^{4\epsilon+1} \int_0^T \Big\| Z(r) \Big\|^2_{\mathbb{H}^{\gamma+\frac{(q-1)\epsilon-(q+1)}{2}}(\Omega)} dr$$

$$=_{\epsilon,\mu,\lambda_1} \eta^{2\epsilon+4} T^{4\epsilon+1} \Big\| Z \Big\|^2_{L^2(0,T;\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega))}.$$

We therefore conclude that

$$\Big\| \int_0^t \Big(O_q(t-r) - O_q^*(t-r)\Big) Z(r)dr \Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\epsilon,\mu,\lambda_1,T} \eta^{\epsilon+2} \Big\| Z \Big\|_{L^2(0,T;\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega))}. \tag{74}$$

Combining (63), (68) and (74), we have that

$$\Big\| u - u^* \Big\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim_{\epsilon,\mu,\lambda_1,T} \eta^\epsilon \Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \eta^{\epsilon+2} \Big\|\psi\Big\|_{\mathbb{H}^{\gamma+\frac{3+q(\epsilon-1)-\epsilon}{2}}(\Omega)} + \eta^{\epsilon+2} \Big\| Z \Big\|_{L^2(0,T;\mathbb{H}^{\gamma+2+\frac{(q-1)\epsilon}{2}}(\Omega))}.$$

This concludes the proof. □

Remark 3.8. *The result of this theorem is also hold for the case* $4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 = 0$ *and* $4\lambda_k^q(1+\mu\lambda_k) - \eta^2\lambda_k^2 < 0$. *The results of these cases and the proof are left to the reader.*

4. For the nonlinear case

In this section, we study nonlinear problem

$$\begin{cases} u_{tt}(x,t) + (-\Delta)^q u(x,t) - \mu\Delta u_{tt}(x,t) - \eta u_{txx}(x,t) = Z(u(x,t)), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{75}$$

with the initial conditions

$$u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x) \quad \text{in} \quad \Omega. \tag{76}$$

We also present three results corresponding to the three cases mentioned, where we prove the existence, uniqueness and regularity of the mild solution.

Theorem 4.1. *For any* $q > 1$, *and* $\mu, \eta > 0$ *such that* $4\lambda_1^q(1+\mu\lambda_1) - \eta^2\lambda_1^2 > 0$. *Let* $\varphi \in \mathbb{H}^\gamma(\Omega)$ *and* $\psi \in \mathbb{H}^\zeta(\Omega)$ *for any* $1 < \gamma < \zeta, \epsilon = \frac{2(\zeta-\gamma)}{q-1} \le 1$. *Let us assume that and* $Z : \mathbb{H}^\gamma(\Omega) \to \mathbb{H}^\zeta(\Omega)$ *such that* $Z(\mathbf{0}) = \mathbf{0}$ *and*

$$\|Z(\theta_1) - Z(\theta_2)\|_{\mathbb{H}^\zeta(\Omega)} \le \mathcal{L}\|\theta_1 - \theta_2\|_{\mathbb{H}^\gamma(\Omega)}, \tag{77}$$

$\mathcal{L}$ *is a postive constant. Then problem* (1)-(2) *has a unique solution* $u \in O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))$ *for* $a, b_0 > 0$, b_0 *large enough. In addition, we obtain*

$$\|u(.,t)\|_{\mathbb{H}^\gamma(\Omega)} \le \frac{3}{2} C(\mu,\eta,\epsilon,\lambda_1) T^{a+\frac{3(\zeta-\gamma)}{q-1}} t^{-a} \Big(\Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)} \Big), 0 \le t \le T.$$

Proof. This proof will be based on the contraction mapping theorem. Firstly, we define the function $\Phi : O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega)) \to O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))$

$$\Phi u(t) := \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t O_q(t-r) Z(u(r))dr. \tag{78}$$

Since the property $Z(\mathbf{0}) = \mathbf{0}$ and combine with Lemma (3.1), we infer that

$$\left\|\Phi\left(u(t)=\mathbf{0}\right)\right\|_{\mathbb{H}^{\gamma}(\Omega)} = \left\|\mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} t^{\frac{(\zeta-\gamma)}{q-1}}\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|\varphi\right\|_{\mathbb{H}^{\zeta}(\Omega)} + t^{\frac{3(\zeta-\gamma)}{q-1}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

It is worth noticing that $e^{-bt} < 1$, and using the embedding $H^{\zeta}(\Omega) \hookrightarrow H^{\gamma}(\Omega)$, we see that

$$t^a e^{-bt}\left\|\Phi\left(u(t)=\mathbf{0}\right)\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{a+\frac{(\zeta-\gamma)}{q-1}}\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + T^{a+\frac{3(\zeta-\gamma)}{q-1}}\left\|\varphi\right\|_{\mathbb{H}^{\zeta}(\Omega)} + T^{a+\frac{3(\zeta-\gamma)}{q-1}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)} \tag{79}$$

$$\lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{a+\frac{3(\zeta-\gamma)}{q-1}}\left(\left\|\varphi\right\|_{\mathbb{H}^{\zeta}(\Omega)} + \left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}\right), \tag{80}$$

which implies that $\Phi(w=0) \in \mathcal{O}_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))$. Hence, we contend that the mapping Φ is well defined. Next, we proceed to show that Φ is a contraction mapping. Let us arbitrarily take u_1, u_2. Then

$$\Phi u_1(t) - \Phi u_2(t) = \int_0^t \mathcal{O}_q(t-r)\Big(Z(u_1(r)) - Z(u_2(r))\Big)dr.$$

By an argument analogous to the the the lemma (3.1), we get

$$\left\|\int_0^t \mathcal{O}_q(t-r)\Big(Z(u_1(r)) - Z(u_2(r))\Big)dr\right\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \int_0^t (t-r)^{\frac{3(\zeta-\gamma)}{q-1}}\left\|\Big(Z(u_1(r)) - Z(u_2(r))\Big)\right\|_{\mathbb{H}^{\zeta}(\Omega)} dr.$$

Since the globally Lipschitz property of Z yields to

$$\|\Phi u_1(t) - \mathcal{P}u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{L}\int_0^t (t-r)^{\frac{3(\zeta-\gamma)}{q-1}}\left\|u_1(r) - u_2(r)\right\|_{\mathbb{H}^{\gamma}(\Omega)} dr.$$

By multiplying both sides of the above estimate by $t^a e^{-bt}$, we obtain

$$t^a e^{-bt}\|\Phi u_1(t) - \Phi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{L}\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{\frac{3(\zeta-\gamma)}{q-1}} r^a e^{-br}\left\|u_1(r) - u_2(r)\right\|_{\mathbb{H}^{\gamma}(\Omega)} dr.$$

Noting that $\operatorname{ess\,sup}_{0\le r\le T} r^a e^{-br}\left\|u_1(r) - u_2(r)\right\|_{\mathbb{H}^{\gamma}(\Omega)} = \left\|u_1 - u_2\right\|_{\mathcal{O}_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))}$, it follows that

$$t^a e^{-bt}\|\Phi u_1(t) - \Phi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{L}\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{\frac{3(\zeta-\gamma)}{q-1}} dr\left\|u_1 - u_2\right\|_{\mathcal{O}_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))}.$$

If we take $r = t\xi$, then $dr = td\xi$ and we get

$$\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{\frac{3(\zeta-\gamma)}{q-1}} dr = t^{1+3\frac{(\zeta-\gamma)}{q-1}}\int_0^1 \xi^{-a} e^{bt(1-\xi)}(1-\xi)^{\frac{3(\zeta-\gamma)}{q-1}} d\xi.$$

Since $1 < q$, $1 < \gamma < \zeta$, we see that $1 + \frac{(\zeta-\gamma)}{q-1} > 0$. Moreover $a > -1$ and $\frac{3(\zeta-\gamma)}{q-1} > -1$ by virtue of Lemma 2.5 one can see

$$\lim_{b\to+\infty}\sup_{0\le t\le T}\left(t^{1+\frac{3(\zeta-\gamma)}{q-1}}\int_0^1 \xi^{-a} e^{bt(1-\xi)}(1-\xi)^{\frac{3(\zeta-\gamma)}{q-1}} d\xi\right) = 0.$$

Hence, there exists a positive $b_0 > 0$ such that

$$C(\lambda_1,\mu,\eta,\epsilon)\mathcal{L}\int_0^t t^a r^{-a} e^{b_0(r-t)}(t-r)^{\frac{3(\zeta-\gamma)}{q-1}} dr \le \frac{1}{3}.$$

this implies immediately that

$$t^a e^{-b_0 t}\|\Phi u_1(t) - \Phi u_2(t)\|_{\mathbb{H}^\gamma(\Omega)} \leq \frac{1}{3}\Big\|u_1 - u_2\Big\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))}.$$

It is to be noticed that the right above is independent of t. So, by taking esssupremum with respect to t, we obtain

$$\|\Phi u_1 - \Phi u_2\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} \leq \frac{1}{3}\Big\|u_1 - u_2\Big\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))}.$$

Therefore, we can assert that Φ is a contraction in space $O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))$. By applying Banach fixed point theorem, we conclde that (1)-(2) has a unique mild solution $u \in O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))$
Since $\Phi u = u$, it follows that

$$\|u\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} \leq \|\Phi u - \Phi\,(u=0)\,\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} + \|\Phi\,(u=0)\,\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} \tag{81}$$

$$\leq \frac{1}{3}\Big\|u\Big\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} + \operatorname*{ess\,sup}_{0\leq t\leq T} t^a e^{-bt}\Big\|\Phi(u=0)(t)\Big\|_{\mathbb{H}^\gamma(\Omega)}. \tag{82}$$

From (80), we get

$$\operatorname*{ess\,sup}_{0\leq t\leq T} t^a e^{-bt}\Big\|\Phi(u=0)(t)\Big\|_{\mathbb{H}^\gamma(\Omega)} \leq C(\mu,\eta,\epsilon,\lambda_1)T^{a+\frac{3(\zeta-\gamma)}{q-1}}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}\Big). \tag{83}$$

Combining (82) and (83), we derive that

$$\|u\|_{O_{a,b_0}((0,T];\mathbb{H}^\gamma(\Omega))} \leq \frac{3}{2}C(\mu,\eta,\epsilon,\lambda_1)T^{a+\frac{3(\zeta-\gamma)}{q-1}}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}\Big).$$

This clearly forces

$$\|u(.,t)\|_{\mathbb{H}^\gamma(\Omega)} \leq \frac{3}{2}C(\mu,\eta,\epsilon,\lambda_1)T^{a+\frac{3(\zeta-\gamma)}{q-1}}t^{-a}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^\zeta(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}\Big), 0 \leq t \leq T.$$

This concludes the proof. □

Theorem 4.2. *For any $q > 1$, and $\mu, \eta > 0$ such that $4\lambda_1^q(1+\mu\lambda_1) - \eta^2\lambda_1^2 = 0$. Let $\varphi \in \mathbb{H}^\gamma(\Omega)$ and $\psi \in \mathbb{H}^\zeta(\Omega)$ for any $1 < \gamma < \zeta, \epsilon = \frac{2(\zeta-\gamma)}{q-1} \leq 1$. Let us assume that $Z : \mathbb{H}^\gamma(\Omega) \to \mathbb{H}^\zeta(\Omega)$ such that $Z(\mathbf{0}) = \mathbf{0}$ and*

$$\|Z(\theta_1) - Z(\theta_2)\|_{\mathbb{H}^\zeta(\Omega)} \leq \mathcal{K}\|\theta_1 - \theta_2\|_{\mathbb{H}^\gamma(\Omega)}, \tag{84}$$

$\mathcal{K}$ is a postive constant. Then problem (1)-(2) *has a unique solution $u \in \Xi_{d,\rho_0}((0,T];\mathbb{H}^\gamma(\Omega))$ for $d, \rho_0 > 0, \rho_0$ large enough and we have*

$$\|u(.,t)\|_{\mathbb{H}^\gamma(\Omega)} \leq C(\mu,\eta,\epsilon,\lambda_1)T^{d+1+\frac{2(\zeta-\gamma)}{q-1}}t^{-d}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}\Big), 0 \leq t \leq T.$$

Proof. Let us consider the operator Ψ: $\Xi_{d,\rho}((0,T];\mathbb{H}^\gamma(\Omega)) \to \Xi_{d,\rho}((0,T];\mathbb{H}^\gamma(\Omega))$

$$\Psi u(t) := \mathcal{P}_q(t)\varphi + \mathcal{Q}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-r)Z(u(r))dr. \tag{85}$$

Lemma (3.2) now shows that

$$t^d e^{-\rho t}\Big\|\Psi\,(u(t)=\mathbf{0})\,\Big\|_{\mathbb{H}^\gamma(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} T^{d+1+\frac{2(\zeta-\gamma)}{q-1}}\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + T^{d+\frac{(\zeta-\gamma)}{q-1}}\Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}.$$

Consequently, operator Ψ map $\Xi_{d,\rho}((0,T];\mathbb{H}^{\gamma}(\Omega))$ into itself.
Analysis similar to that in the proof of Theorem (4.1) one has

$$\|\Psi u_1(t)-\Psi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{K}\int_0^t (t-r)^{\frac{(\zeta-\gamma)}{q-1}}\Big\|u_1(r)-u_2(r)\Big\|_{\mathbb{H}^{\gamma}(\Omega)}dr.$$

By multiplying both sides of the above estimate by $t^d e^{-\rho t}$, we obtain

$$t^d e^{-\rho t}\|\Psi u_1(t)-\Psi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{K}\int_0^t t^d r^{-d} e^{\rho(r-t)}(t-r)^{\frac{(\zeta-\gamma)}{q-1}} r^d e^{-\rho r}\Big\|u_1(r)-u_2(r)\Big\|_{\mathbb{H}^{\gamma}(\Omega)}dr.$$

It follows readily from $\operatorname{ess\,sup}_{0\le r\le T} r^d e^{-\rho r}\Big\|u_1(r)-u_2(r)\Big\|_{\mathbb{H}^{\gamma}(\Omega)} = \Big\|u_1-u_2\Big\|_{O_{d,\rho}((0,T];\mathbb{H}^{\gamma}(\Omega))}$
that

$$t^d e^{-\rho t}\|\Psi u_1(t)-\Psi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \lesssim_{\mu,\eta,\epsilon,\lambda_1} \mathcal{L}\int_0^t t^d r^{-d} e^{\rho(r-t)}(t-r)^{\frac{(\zeta-\gamma)}{q-1}} dr\Big\|u_1-u_2\Big\|_{O_{d,\rho}((0,T];\mathbb{H}^{\gamma}(\Omega))}.$$

By the change of variables $r = t\xi$, we have

$\int_0^t t^d r^{-a} e^{\rho(r-t)}(t-r)^{\frac{(\zeta-\gamma)}{q-1}} dr = t^{1+\frac{(\zeta-\gamma)}{q-1}}\int_0^1 \xi^{-d} e^{\rho t(1-\xi)}(1-\xi)^{\frac{(\zeta-\gamma)}{q-1}} d\xi.$
Since $1<q$, $1<\gamma<\zeta$, we see that $1+\frac{(\zeta-\gamma)}{q-1}>0$. Furthermore, $d>-1$ and $\frac{(\zeta-\gamma)}{q-1}>-1$, we invoke Lemma (2.5) to deduce that

$$\lim_{b\to+\infty}\sup_{0\le t\le T}\left(t^{1+\frac{(\zeta-\gamma)}{q-1}}\int_0^1 \xi^{-d} e^{\rho t(1-\xi)}(1-\xi)^{\frac{(\zeta-\gamma)}{q-1}} d\xi\right)=0.$$

For this reason, there exists a positive $\rho_0>0$ such that

$$C(\lambda_1,\mu,\eta,\epsilon)\mathcal{L}\int_0^t t^a r^{-d} e^{\rho(r-t)}(t-r)^{\frac{(\zeta-\gamma)}{q-1}} dr \le \frac{1}{2}.$$

This gives

$$t^d e^{-\rho_0 t}\|\Psi u_1(t)-\Psi u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \le \frac{1}{2}\Big\|u_1-u_2\Big\|_{O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))}. \tag{86}$$

We see that the right of (86) is independent of t. So, by taking esssupremum with respect to t, we have

$$\|\Psi u_1-\Psi u_2\|_{O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))} \le \frac{1}{2}\Big\|u_1-u_2\Big\|_{O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))}.$$

We deduce that Ψ is a contraction mapping. Hence, by Banach fixed point theorem, Ψ has a unique fixed point in space $O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))$. Thus, it follows that the (1)-(2) has a unique mild solution $u\in O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))$
By an argument analogous to that used for the proof of Theorem (4.1) we can see that

$$\|u\|_{O_{d,\rho_0}((0,T];\mathbb{H}^{\gamma}(\Omega))} \le 2C(\mu,\eta,\epsilon,\lambda_1)T^{d+1+\frac{2(\zeta-\gamma)}{q-1}}\left(\Big\|\varphi\Big\|_{\mathbb{H}^{\gamma}(\Omega)}+\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}\right).$$

The result above shows

$$\|u(.,t)\|_{\mathbb{H}^{\gamma}(\Omega)} \le C(\mu,\eta,\epsilon,\lambda_1)T^{d+1+\frac{2(\zeta-\gamma)}{q-1}}t^{-d}\left(\Big\|\varphi\Big\|_{\mathbb{H}^{\gamma}(\Omega)}+\Big\|\psi\Big\|_{\mathbb{H}^{\zeta}(\Omega)}\right), 0\le t\le T.$$

□

Theorem 4.3. *For any* $q > 1$, *and* $\mu, \eta > 0$ *such that* $4\lambda_1^q(1 + \mu\lambda_1) - \eta^2\lambda_1^2 < 0$. *Let* $\varphi \in \mathbb{H}^\gamma(\Omega)$ *and* $\psi \in \mathbb{H}^\gamma(\Omega)$ *for any* $0 < \gamma, \quad 0 < \epsilon \leq 1$. *Let us assume that and* $Z : \mathbb{H}^\gamma(\Omega) \to \mathbb{H}^\gamma(\Omega)$ *such that* $Z(\mathbf{0}) = \mathbf{0}$ *and*

$$\|Z(\theta_1) - Z(\theta_2)\|_{\mathbb{H}^\gamma(\Omega)} \leq \mathcal{M}\|\theta_1 - \theta_2\|_{\mathbb{H}^\gamma(\Omega)}, \tag{87}$$

$\mathcal{M}$ *is a positive constant. Then problem 1-2 has a unique solution* $u \in \Xi_{d,\rho_0}((0,T];\mathbb{H}^\gamma(\Omega))$ *for* $d, \rho_0 > 0$, ρ_0 *large enough and we obtain*

$$\|u(.,t)\|_{\mathbb{H}^\gamma(\Omega)} \leq C(\mu,\eta,\epsilon,\lambda_1,T)t^{-d}\Big(\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + \Big\|\psi\Big\|_{\mathbb{H}^\gamma(\Omega)}\Big), 0 \leq t \leq T,$$

here C *depend on* $\mu, \eta, \epsilon, \lambda_1, T$.

This follows by the same method as in theorem (4.1) and (4.2), so we omit it.

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