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The chromatic number of $\{P_6$, banner, diamond$\}$-free graphs

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Abstract. Let P_6 denote a 6-vertex path, a *banner* denote a 4-cycle with a pendant vertex, and a *diamond* denote a K_4 minus an edge. We demonstrate that for any graph G in the class of $\{P_6, \text{banner}, \text{diamond}\}$-free graphs, the chromatic number is always bounded above by the maximum of 3 and the clique number. This strictly strengthens a result of Lan, Zhou and Liu [21] by replacing their C_4-free condition with the weaker banner-free requirement.

1. Introduction

Throughout this paper, all graphs are finite, simple, and undirected. A graph G is said to *contain* a graph H as an induced subgraph if there exists an induced subgraph of G that is isomorphic to H. The graph G is called *H-free* if no induced subgraph of G is isomorphic to H. More generally, for a family of graphs $\mathcal{H}$, we say G is *$\mathcal{H}$-free* if G does not contain any graph from $\mathcal{H}$ as an induced subgraph.

For a graph G and a vertex subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. The set X is called:

- a *clique* if $G[X]$ is complete,
- an *independent set* if $G[X]$ has no edges.

The *clique number* $\omega(G)$ is the maximum size of a clique in G (we sometimes write $\omega(X)$ for $\omega(G[X])$). The *chromatic number* $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. Note that $\chi(G) \geq \omega(G)$ holds for any graph G.

For a vertex $v \in V(G)$, its neighborhood $N_G(v)$ is the set of vertices adjacent to v. For a vertex subset $X \subseteq V(G)$, the *neighborhood* of X in G, denoted $N_G(X)$, is defined as the set of all vertices in $V(G) \setminus X$ that are adjacent to at least one vertex in X. The subscript G may be omitted when the graph is clear from context.

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In his seminal work [6], Erdős demonstrated the existence of triangle-free graphs with arbitrarily large chromatic number. Specifically, given any integer $g \geq 1$, there exists a graph containing no triangles but requiring at least g colors for a proper vertex coloring. This shows that no universal function of $\omega(G)$ can bound $\chi(G)$ for all graphs G.

A graph G is called *perfect* if and only if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$, otherwise it is *imperfect*. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the positive integers. A family of graphs $\mathcal{G}$ is said to be *χ-bounded* when there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ (referred to as a *bounding function*) satisfying $\chi(G) \leq f(\omega(G))$ for every graph G belonging to $\mathcal{G}$. If f can be chosen to be a polynomial, we say $\mathcal{G}$ is *polynomially χ-bounded*.

A major research direction in χ-boundedness involves identifying graph families $\mathcal{H}$ for which the family of $\mathcal{H}$-free graphs is χ-bounded, and determining optimal χ-binding functions for such classes. Gyárfás [11] and Sumner [26] independently proposed:

Conjecture 1.1 ([11, 26]). *For any forest F, the family of all graphs excluding F as an induced subgraph is χ-bounded.*

While the general case remains unresolved, this conjecture has been confirmed for various specific tree classes (see e.g. [5, 11, 13, 18–20, 22, 23]).

- A *hole* is defined as an induced cycle on four or more vertices.
- An *anti-hole* is the graph complement of a hole.
- A hole or anti-hole is *odd* if its vertex count is an odd number, and *even* otherwise.
- A *q-hole* refers specifically to a hole comprising exactly q vertices.

The celebrated Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas [4] characterizes perfect graphs:

Theorem 1.2 ([4]). *The following conditions are equivalent for any graph G:*

- *G is perfect.*
- *G contains neither odd holes nor odd anti-holes as induced subgraphs.*

This establishes that the class of {odd hole, odd antihole}-free graphs is χ-bounded with identity binding function. Building upon this foundation, researchers have systematically examined multiple graph classes characterized by some forbidden induced subgraphs, identifying several families that admit linear χ-binding functions.

For an arbitrary positive integer t, we use the following standard notation:

- P_t - the path graph on t vertices.
- C_t - the cycle graph on t vertices.
- K_t - the complete graph on t vertices.

A *banner* is the graph obtained by adding a pendant vertex to a 4-cycle, while a *diamond* is K_4 minus one edge (see Figure 1). Notably, all graph families discussed here forbid odd holes and odd anti-holes of length ≥ 7, so by Theorem 1.2, their members are either perfect or contain C_5.

The P_3-free graphs are precisely the graphs whose connected components are all complete graphs. (Given two vertex-disjoint graphs G and H: Their *union* $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.) For P_t-free graphs ($t \geq 4$), we summarize some known χ-bounding results:

- P_4-free graphs are perfect [25]
- For P_5-free graphs:

  - $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$ when $\omega(G) \geq 3$ [7, 10]
  - $\chi(G) \leq \omega(G)^{\log_2 \omega(G)}$ [24]
  - General bound: $\chi(G) \leq (t-2)^{\omega(G)-1}$ for P_t-free graphs [10]

- For specific forbidden pairs:
  - $\{P_6, C_4\}$-free: $\chi(G) \leq \frac{5}{4}\omega(G)$ [16] (improving [8])
  - $\{P_5, \text{gem}\}$-free: $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$ [3] (improving [1])
  - $\{P_5, \text{gem}, \text{co-gem}\}$-free: $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$ [15]
  - $\{P_5, \text{diamond}\}$-free: $\chi(G) \leq \omega(G) + 1$ [14]
  - $\{P_6, \text{diamond}\}$-free:
    - $\chi(G) \leq \omega(G) + 3$ [2] (improving [17])
    - $\chi(G) \leq \max\{6, \omega(G)\}$ [9]
  - $\{P_6, \text{diamond}, K_4\}$-free: $\chi(G) \leq 6$ [17]

In this work, we establish the following chromatic bounding result for an extended family of graphs:

Theorem 1.3. *Every $\{P_6, \text{banner}, \text{diamond}\}$-free graph G satisfies $\chi(G) \leq \max\{3, \omega(G)\}$.*

Our result significantly improves the work of Lan, Zhou, and Liu [21] by extending the analysis from C_4-free to the broader class of banner-free graphs, which includes both C_4-free and claw-free graphs as special cases. The proof leverages Theorem 1.2 to reduce the problem to imperfect graphs, then systematically analyzes their structure through a case decomposition.

Paper Organization

- Section 2: Structural analysis of an imperfect $\{P_6, \text{banner}, \text{diamond}\}$-free graph.
- Section 3: Proof of the chromatic bound (Theorem 1.3).

Notation

For any positive integer q, define $[q] := \{1, \ldots, q\}$. Given a graph G:

- For $X \subseteq V(G)$, $G[X]$ denotes the induced subgraph on X, and $G \setminus X := G[V(G) \setminus X]$.
- For $x \in V(G)$, $N(x)$ denotes its neighborhood.
- A *q-clique* is a complete subgraph on q vertices.
- Two non-adjacent vertices u and v are comparable if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.
- Edges between two disjoint subsets $A, B \subseteq V(G)$ form a *matching* if:

$$\max_{a \in A} |N(a) \cap B| \leq 1 \quad \text{and} \quad \max_{b \in B} |N(b) \cap A| \leq 1.$$

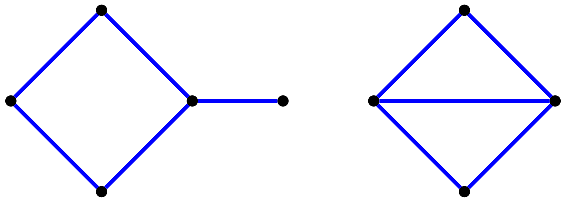

Figure 1: The forbidden configurations: banner (left) and diamond (right).

2. Structure analysis

From Theorem 1.2, any imperfect $\{P_6, \text{banner}, \text{diamond}\}$-free graph G must contain a 5-hole, since:

- P_6-freeness excludes holes of length ≥ 7;
- Diamond-freeness excludes anti-holes of length ≥ 7.

Consider a 5-hole in G with vertex set $C = \{u_1, \ldots, u_5\}$ and edges $u_i u_{i+1}$ (with indices modulo 5). We define a partition of $V(G)$ into:

$$\mathcal{N} := \{v \in V(G) \setminus C \mid N_C(v) \neq \emptyset\} \quad \text{(neighboring vertices)}$$
$$\mathcal{R} := V(G) \setminus (C \cup \mathcal{N}) \quad \text{(remaining vertices)}$$

Thus, $V(G) = C \cup \mathcal{N} \cup \mathcal{R}$.

Structural Properties

- **Neighborhood Restrictions**: Since G is {banner, diamond}-free, each vertex in $\mathcal{N}$ satisfies:
  - Either adjacent to exactly one vertex in C, or
  - Adjacent to exactly two consecutive vertices in C.
- **Partition of $\mathcal{N}$**: For $i \in \{1, 2, \ldots, 5\}$ (with $u_6 := u_1$), define:

$$A_i := \{u \in \mathcal{N} \mid N(u) \cap C = \{u_i\}\},$$
$$B_{i,i+1} := \{u \in \mathcal{N} \mid N(u) \cap C = \{u_i, u_{i+1}\}\}.$$

  Let $A = \bigcup_{i=1}^{5} A_i$ and $B = \bigcup_{i=1}^{5} B_{i,i+1}$. Then:

$$V(G) = C \cup A \cup B \cup \mathcal{R}.$$

In what follows, we establish several properties of G, each accompanied by a concise proof.

(1) *$V(G) = C \cup A \cup B$.*

If $\mathcal{R}$ were non-empty, then a vertex from $\mathcal{R}$ and a vertex from $\mathcal{N}$, together with the 5-hole C, would induce a P_6, a contradiction.

(2) *$B = B_{i,i+1} \cup B_{i+2,i+3}$ for some index $i \in \{1, 2, \ldots, 5\}$.*

Suppose for contradiction that for some i, both $B_{i-1,i}$ and $B_{i,i+1}$ are nonempty. Let $b_1 \in B_{i-1,i}$ and $b_2 \in B_{i,i+1}$. Then:

- If $b_1 b_2 \notin E(G)$, then $\{b_1, u_{i-1}, u_{i-2}, u_{i+2}, u_{i+1}, b_2\}$ induces a P_6;
- If $b_1 b_2 \in E(G)$, then $\{u_i, b_1, b_2, u_{i+1}\}$ induces a diamond.

In both cases, we obtain a contradiction to our assumptions, proving the claim.

(3) *Each connected component of A_i forms a clique.*

This is an immediate consequence of G being diamond-free.

(4) *No edges exist between A_i and A_{i+1}.*

Assume for contradiction that there exist adjacent vertices $a_1 \in A_i$ and $a_2 \in A_{i+1}$. Then the vertex set $\{a_1, a_2, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$ would induce a P_6 in G, contradicting our assumptions.

(5) *Every vertex in A_i is adjacent to every vertex in A_{i+2}.*

Suppose for contradiction that there exist nonadjacent vertices $a_1 \in A_i$ and $a_2 \in A_{i+2}$. Then the vertex set $\{a_1, u_i, u_{i-1}, u_{i-2}, u_{i+2}, a_2\}$ induces a P_6 in G, contradicting the assumptions on G.

(6) *If both A_i and A_{i+2} are nonempty, then $|A_i| = |A_{i+2}| = 1$. As a consequence, if $|A_i| \geq 2$ for some i, then $A_{i+2} = \emptyset$ and $A_{i+3} = \emptyset$.*

By symmetry, suppose for contradiction that for some i, both A_i and A_{i+2} are nonempty and $|A_{i+2}| \geq 2$. Let $a_1 \in A_i$ and $a_2, a_3 \in A_{i+2}$. Then $a_1a_2, a_1a_3 \in E(G)$ by (5). This implies:

- If $a_2a_3 \notin E(G)$, then $\{u_i, a_1, a_2, a_3, u_{i+2}\}$ induces a banner;
- If $a_2a_3 \in E(G)$, then $\{a_1, a_2, a_3, u_{i+2}\}$ induces a diamond.

In both cases, we obtain a contradiction to our assumptions, proving the claim.

(7) *The set $B_{i,i+1}$ induces a clique in G.*

Suppose there exist nonadjacent vertices $b_1, b_2 \in B_{i,i+1}$. Then the subgraph induced by $\{b_1, u_i, u_{i+1}, b_2\}$ forms a diamond, contradicting the diamond-freeness of G.

(8) *No edges exist between $B_{i,i+1}$ and $A_i \cup A_{i+1}$.*

Without loss of generality, consider $a \in A_i$ and $b \in B_{i,i+1}$. If a and b were adjacent, the vertex set $\{a, u_i, u_{i+1}, b\}$ would induce a diamond in G, contradicting the diamond-freeness assumption.

(9) *Either $B_{i,i+1} = \emptyset$ or $A_{i-1} \cup A_{i+2} = \emptyset$.*

By symmetry, suppose for contradiction that for some i, both $B_{i,i+1}$ and $A_{i-1} \cup A_{i+2}$ are nonempty. Let $a \in A_{i+2}$ and $b \in B_{i,i+1}$. Then:

- If $ab \notin E(G)$, then the set $\{a, u_{i+2}, u_{i+3}, u_{i+4}, u_i, b\}$ induces a P_6;
- If $ab \in E(G)$, then the set $\{a, b, u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a banner.

In both cases, we obtain a contradiction to our assumptions, proving the claim.

(10) *If $B_{i,i+1} \neq \emptyset$, then either $A_i \cup A_{i+1} = \emptyset$ or $A_{i+3} = \emptyset$.*

Suppose for contradiction that for some i, both A_i and A_{i+3} are nonempty. Let $a_1 \in A_i$, $a_2 \in A_{i+3}$, and $b \in B_{i,i+1}$. Then $a_1a_2 \in E(G)$ by (4) and $a_1b \notin E(G)$ by (8). Furthermore:

- If $a_2b \notin E(G)$, then the set $\{a_1, a_2, u_{i+3}, u_{i+2}, u_{i+1}, b\}$ induces a P_6;
- If $a_2b \in E(G)$, then the set $\{b, u_i, a_1, a_2, u_{i+3}\}$ induces a banner.

In both cases, we obtain a contradiction to our assumptions, proving the claim.

3. Proof of Theorem 1.3

The aim of this section is to prove Theorem 1.3.
To prove Theorem 1.3, we use Theorems 3.1 and 3.2 below.

Theorem 3.1 ([27]). *For any $\{P_6, K_3\}$-free graph G without comparable vertices:*

- *G admits a proper 4-coloring*
- *G fails to be 3-colorable if and only if:*
  - *G contains an induced Grötzsch graph, and*
  - *G is isomorphic to an induced subgraph of the 16-vertex Clebsch graph. (Visualizations appear in Figure 2.)*

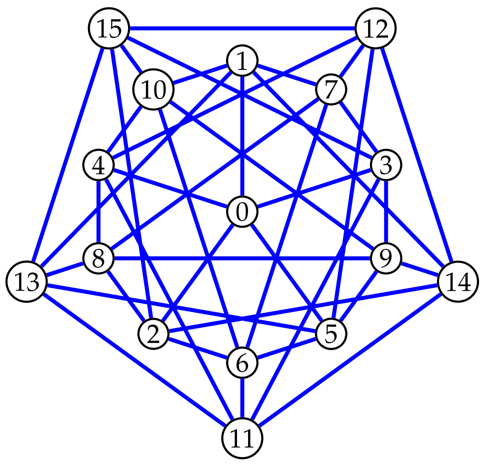

Figure 2: The Clebsch graph. The induced subgraphs on $\{0, 1, \ldots, 10\}$ and $\{0, 1, 2, 3, 7\}$ yield the Grötzsch graph and a banner, respectively.

Theorem 3.2. *Suppose G is a connected graph satisfying the following conditions:*

- *G is imperfect;*
- *G contains no induced subgraph isomorphic to P_6, banner, or diamond;*
- *The clique number $\omega(G)$ is at least 3;*
- *G has no clique cutsets;*
- *G contains no comparable vertices.*

Then $\chi(G) = \omega(G)$.

Proof. **[Proof of Theorem 1.3, assuming Theorem 3.2.]** We consider five cases:
Case 1. If G is perfect, then $\chi(G) = \omega(G)$ immediately holds.
Case 2. If G is disconnected with components $\{Q_1, \ldots, Q_t\}$,

$$\chi(G) = \max\{\chi(Q_1), \ldots, \chi(Q_t)\} \quad \text{and} \quad \omega(G) = \max\{\omega(Q_1), \ldots, \omega(Q_t)\}.$$

Case 3. If x and y are two non-adjacent vertices in G satisfying the neighborhood inclusion $N(x) \subseteq N(y)$, then

$$\chi(G) = \chi(G \setminus x) \quad \text{and} \quad \omega(G) = \omega(G \setminus x).$$

Case 4. If G has a clique T such that $G \setminus T$ decomposes into $\{Q_1, \dots, Q_t\}$, then

$$\chi(G) = \max\{\chi(G[V(Q_i) \cup T]) \mid 1 \leq i \leq t\}$$

and

$$\omega(G) = \max\{\omega(G[V(Q_i) \cup T]) \mid 1 \leq i \leq t\}.$$

Case 5. If $\omega(G) \leq 2$, then G contains no Grötzsch graph as an induced subgraph (since the vertices with indices $0, 1, 2, 3, 7$ in Figure 2 induce a banner). So, G is 3-colorable by Theorem 3.1.

In all cases, the chromatic number is bounded above by $\max\{3, \omega(G)\}$. Thus, the desired result follows by applying Theorem 3.2. □

Proof. **[Proof of Theorem 3.2.]** Let G be a graph satisfying all hypotheses of Theorem 3.2. By Theorem 1.2, any such graph G must contain a 5-hole, since:

- P_6-freeness excludes holes of length ≥ 7;
- Diamond-freeness excludes anti-holes of length ≥ 7.

Consider a 5-hole in G with vertex set $C = \{u_1, \dots, u_5\}$ and edges $u_i u_{i+1}$ (with indices modulo 5). We adopt the vertex partition described in Section 2.

We now construct an $\omega(G)$-coloring of G. Two vertex sets are said to *not conflict* if no edge connects vertices of the same color between them.

By (2) and by symmetry of the 5-cycle, we may assume

$$B = B_{2,3} \cup B_{4,5}.$$

Claim 3.3. *Either $B_{2,3} \neq \emptyset$ or $B_{4,5} \neq \emptyset$.*

Proof. Suppose for contradiction that both $B_{2,3}$ and $B_{4,5}$ are empty. Then, $V(G) = A \cup C$ by (1). From the assumption $\omega(G) \geq 3$, it follows that there exists an index $i \in \{1, \dots, 5\}$ such that A_i is not an independent set; hence, there exist adjacent vertices $x_1, x_2 \in A_i$. It follows from (6) that $A_3 = A_4 = \emptyset$. Moreover, (4) implies that A_1 is anti-complete to both A_2 and A_5. Therefore, $\{u_1\}$ is a clique cutset of G separating A_1 and $\{u_2, u_3, u_4, u_5\}$, contradicting our initial assumption. This proves Claim 3.3. □

Claim 3.4. *If exactly one of $B_{2,3}$ or $B_{4,5}$ is nonempty, then Theorem 3.2 holds.*

Proof. Without loss of generality, assume $B_{2,3} \neq \emptyset$. By (9), we have $A_1 = A_4 = \emptyset$. Thus, $V(G) = A_2 \cup A_3 \cup A_5 \cup B_{2,3} \cup C$ by (1). By (10), either $A_2 \cup A_3 = \emptyset$ or $A_5 = \emptyset$.

If $A_5 = \emptyset$, then $\{u_2, u_3\}$ is a clique cutset of G separating $A_2 \cup A_3 \cup B_{2,3}$ and $\{u_1, u_4, u_5\}$, contradicting the initial assumption. Therefore, $A_5 \neq \emptyset$ and $A_2 \cup A_3 = \emptyset$.

We now construct an $\omega(G)$-coloring of G as follows:

- Coloring the vertices u_1, u_2, u_3, u_4, u_5 with colors $1, 3, 2, 1, 2$, respectively.
- Coloring $B_{2,3}$ with colors from $\{1, 4, 5, \dots, \omega(G)\}$.
- **Coloring** A_5:
  - Each component K of A_5 is a clique (by (3)) with $|K| \leq \omega(G) - 1$.
  - The subgraph formed by edges between $B_{2,3}$ and K is a matching; otherwise, for $a \in K$ and $b_1, b_2 \in B_{2,3}$, $\{b_1, b_2, a, u_2\}$ would induce a diamond.

– For each component K, order its vertices as follows: For every vertex $v \in K$, let $c(v)$ be the color of its unique neighbor in $B_{2,3}$ (if such a neighbor exists). Sort the vertices of K in ascending order of $c(v)$, placing vertices with no neighbor in $B_{2,3}$ at the end. Let the ordered sequence be $a_1, a_2, \ldots, a_\ell$, where $\ell \leq \omega(G) - 1$.

– Coloring the vertices $a_1, a_2, \ldots, a_\ell$ with colors $3, 1, 4, 5, \ldots, \omega(G)$, respectively.

This yields a proper $\omega(G)$-coloring, proving Claim 3.4. □

By Claims 3.3 and 3.4, both $B_{2,3}$ and $B_{4,5}$ are nonempty, and by (9), we have $A_1 = A_3 = A_4 = \emptyset$. That is, $V(G) = A_2 \cup A_5 \cup B_{2,3} \cup B_{4,5} \cup C$ by (1). Furthermore, (10) implies that either $A_2 = \emptyset$ or $A_5 = \emptyset$. By symmetry, we assume $A_5 = \emptyset$.

We now construct a proper coloring of G by sequentially coloring C, A, and B, ensuring at each step that the new coloring does not conflict with previously colored vertices.

1. **Coloring** C: Coloring the vertices u_1, u_2, u_3, u_4, u_5 with colors $1, 3, 2, 1, 2$ respectively.
2. **Coloring** B:
   - By (7), both $B_{2,3}$ and $B_{4,5}$ are cliques with size at most $\omega(G) - 2$.
   - First color $B_{4,5}$ using colors from $\{3, 4, \ldots, \omega(G)\}$.
   - The diamond-freeness implies that the edges between $B_{2,3}$ and $B_{4,5}$ form a matching. Suppose otherwise: if $b_1 \in B_{2,3}$ has two neighbors $b_2, b_3 \in B_{4,5}$, then $\{b_1, b_2, b_3, u_4\}$ would induce a diamond.
   - Order the vertices in $B_{2,3}$ as follows: For every vertex $v \in B_{2,3}$, let $c(v)$ be the color of its unique neighbor in $B_{4,5}$ (if such a neighbor exists). Sort the vertices of $B_{2,3}$ in ascending order of $c(v)$, placing vertices with no neighbor in $B_{4,5}$ at the end. Let the ordered sequence be $b_1, b_2, \ldots, b_\ell$, where $\ell \leq \omega(G) - 2$.
   - If $\omega(G) = 3$, then color $B_{2,3}$ with color 1. If $\omega(G) \geq 4$, then Color the vertices $b_1, b_2, \ldots, b_\ell$ with colors $\omega(G), 1, 4, 5, \ldots, \omega(G) - 1$, respectively.
3. **Coloring** A_2:
   - Each component K of A_2 is a clique (by (3)) with $|K| \leq \omega(G) - 1$.
   - The subgraph formed by edges between $B_{4,5}$ and K is a matching; otherwise, for $a \in K$ and $b_1, b_2 \in B_{4,5}$, $\{b_1, b_2, a, u_4\}$ would induce a diamond.
   - For each component K, order its vertices as follows: For every vertex $v \in K$, let $c(v)$ be the color of its unique neighbor in $B_{4,5}$ (if such a neighbor exists). Sort the vertices of K in ascending order of $c(v)$, placing vertices with no neighbor in $B_{4,5}$ at the end. Let the ordered sequence be $a_1, a_2, \ldots, a_\ell$, where $\ell \leq \omega(G) - 1$.
   - Coloring the vertices $a_1, a_2, \ldots, a_\ell$ with colors $1, 2, 4, 5, \ldots, \omega(G)$, respectively.

This coloring ensures:

- No conflicts between A_2 and $B_{4,5}$ (by construction).
- No conflicts between A_2 and $B_{2,3}$ (by (8)).

Thus, we obtain a proper $\omega(G)$-coloring of G, proving Theorem 3.2. □

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Data Availibility

Data sharing is not applicable to this article because no datasets were generated or analyzed during the current study.

Declarations/Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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