Filomat 40:5 (2026), 1865–1872
https://doi.org/10.2298/FIL2605865S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The maximum spectral radius of outerplanar 3-uniform hypergraphs with a given size

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Abstract. Given a 3-uniform hypergraph $\mathcal{G} = \big(V(\mathcal{G}), E(\mathcal{G})\big)$, the shadow of $\mathcal{G}$ is a graph $G = \big(V(G), E(G)\big)$ with $V(G) = V(\mathcal{G})$ and $E(G) = \{uv : uv \in e \text{ for some } e \in E(\mathcal{G})\}$. A graph is planar if it can be drawn on the plane such that its edges intersect only at their endpoints. A planar graph is outerplanar if it has a planar embedding such that all its vertices lie on the outer face. If the shadow of $\mathcal{G}$ is an (outer)planar graph, then $\mathcal{G}$ is a 3-uniform (outer)planar hypergraph. Let $P_1 + P_{n-1}$ be the join of P_1 and P_{n-1}, where P_n is a path with n vertices and $n \geqslant 3$. Ellingham et al. (2022) proved that for sufficiently large n, the n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the unique 3-uniform hypergraph whose shadow is $P_1 + P_{n-1}$. In this paper, we prove that for sufficiently large m, the m-edge outerplanar 3-uniform hypergraph of maximum spectral radius is also the unique 3-uniform hypergraph whose shadow is $P_1 + P_{m+1}$.

1. Introduction

Given a hypergraph $\mathcal{G}$, we denote the vertex set and edge set of $\mathcal{G}$ by $V(\mathcal{G})$ and $E(\mathcal{G})$, respectively. For any edge $e \in E(\mathcal{G})$, if $|e| = k$, then $\mathcal{G}$ is k-uniform, where $k \geqslant 2$. If $e = \{v_1, v_2, \cdots, v_k\}$, then we denote e by $v_1v_2\cdots v_k$. For an edge $e \in E(\mathcal{G})$, if $u \in e$, then u is incident with e. If e contains u and v, we say that u is adjacent to v.

A planar graph is a type of graph which can be drawn on the plane and its edges intersect only at their endpoints. When a planar graph is drawn on a plane without edges crossing, the edges and vertices of the planar graph divide the plane into some regions and we call each region a face. Let f be a face in a planar graph. The size of f is the number of the edges that form the boundary of f. If an edge borders the same face on both sides, it is counted twice. If the size of f is i, then we call f an i-face, where $i \geqslant 3$. We use $f(u_1u_2\cdots u_ku_1)$ to denote the k-face formed by the vertices $u_1, u_2, \cdots, u_k$, where $k \geqslant 3$. A planar graph is outerplanar if it has a planar embedding such that all its vertices lie on the outer face.

Given a graph G, let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. The 2-shadow of a hypergraph $\mathcal{G}$ is a graph G with $V(G) = V(\mathcal{G})$ and $E(G) = \{uv : uv \in e \text{ for some } e \in E(\mathcal{G})\}$. For simplicity,

2020 *Mathematics Subject Classification*. Primary 05C10; Secondary 05C35, 05C65.
Keywords. 3-uniform hypergraph, outerplanar, spectral radius.
Received: 17 March 2025; Revised: 18 November 2025; Accepted: 03 December 2025
Communicated by Paola Bonacini
Research supported by the National Natural Science Foundation of China under the grant number 12561064.
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we refer to the 2-shadow of $\mathcal{G}$ as the shadow of $\mathcal{G}$. We denote the shadow of $\mathcal{G}$ by $\partial(\mathcal{G})$. For a 3-uniform hypergraph $\mathcal{G}$, let the shadow of $\mathcal{G}$ be $\widetilde{G}$. If $\widetilde{G}$ is an outer(planar) graph, then we call $\mathcal{G}$ a 3-uniform outer(planar) hypergraph. If an edge xyz belongs to $E(\mathcal{G})$, then xyz forms a 3-face in $\widetilde{G}$. It is noted that if there is a 3-face (denoted by $f(u_1u_2u_3u_1)$) in $\widetilde{G}$, then u_1, u_2 and u_3 do not necessarily form an edge in $\mathcal{G}$.

Let C_n and K_n be a cycle and a complete graph with n vertices, respectively, where $n \geqslant 3$. Let S_n be a star with n vertices, where $n \geqslant 2$. The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained from G_1 and G_2 by adding all possible edges between G_1 and G_2.

The spectral radius of a graph G, denoted by $\rho(G)$, is the maximum modulus of all the eigenvalues of the adjacency matrix of G. Since the adjacency matrix of a graph G is symmetric and nonnegative, $\rho(G)$ must be a real number. When the order of a graph G is fixed, many interesting results on the extremal graph attaining the spectral radius have been obtained. It is easy to check that K_n is the unique graph having the maximum spectral radius among all graphs with n vertices, where $n \geqslant 3$. The spectral radius of planar graphs attracts the interest of researchers since it is a measure of the overall connectivity of planar graphs [6, 2]. In 1990, Cvetković and Rowlinson [6] conjectured that $K_1 + P_{n-1}$ attains the maximum spectral radius among all outerplanar graphs on n vertices. In 2017, Tait and Tobin [18] proved that for sufficiently large n, the conjecture is true. In 2021, Lin and Ning [11] determined that this conjecture holds for $n \geqslant 2$ except for $n = 6$. For planar graphs, Boots and Royle [2] and Cao and Vince [4] independently proposed the conjecture that $K_2 + P_{n-2}$ is the unique planar graph with the maximum spectral radius, where $n \geqslant 9$. In 2017, Tait and Tobin [18] also proved that this conjecture is correct when n is sufficiently large.

Which graph attains the maximum spectral radius among all graphs with a fixed number of edges? In 1985, Brualdi and Hoffman [3] proved that the union of K_k and some possible isolated vertices attains the maximum spectral radius among all e-edge graphs, where $e = \binom{k}{2}$ and $k \geqslant 2$. Brualdi and Hoffman [3] also conjectured that when $e = \binom{k}{2} + s < \binom{k+1}{2}$, $s \geqslant 1$ and $k \geqslant 2$, the extremal graph is obtained from a complete graph K_k and a vertex u by adding s edges between the vertex u and s vertices of K_k. Later, this conjecture was confirmed by Rowlinson [16].

In 2021, Zhai et al. [20] obtained the following Theorem 1.1.

Theorem 1.1. *[20] Let $r \geqslant 2$ and $m \geqslant 16r^2$. If G is a $K_{2,r+1}$-free graph with size m, then $\rho(G) \leqslant \sqrt{m}$, and the equality holds if and only if G is a star.*

We can check that S_m is the extremal graph having the maximum spectral radius among all graphs with a given size m without $K_{2,3}$, where $m \geqslant 64$. Since S_m is an outerplanar graph, we have Theorem 1.2.

Theorem 1.2. *Let $m \geqslant 64$. The m-edge outerplanar graph of maximum spectral radius is the star S_m.*

Given a positive integer m, let $G_m = K_2 + \frac{m-1}{2}K_1$ when m is odd and $G_m = K_1 + \left(S_{\frac{m-2}{2}} \cup K_1\right)$ when m is even. Fan et al. [8] determined the unique graph having the maximum spectral radius among all m-edge planar graphs when m is large enough.

Theorem 1.3. *[8] Let G be a planar graph with m edges. Suppose that m is sufficiently large. Then $\rho(G) \leqslant \rho(G_m)$, and the equality holds if and only if $G \cong G_m$.*

In the following, we introduce the spectral radius of a k-uniform hypergraph, where $k \geqslant 3$.

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. Let k and n be two positive integers. Let $[n] = \{1, 2, \cdots, n\}$. A real tensor (or hypermatrix) $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$ of order k and dimension n is a multi-dimensional array with entries $a_{i_1 i_2 \cdots i_k}$ such that $a_{i_1 i_2 \cdots i_k} \in \mathbb{R}$, where $i_1, i_2, \cdots, i_k \in [n]$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ be an n-dimensional complex column vector. Let $\boldsymbol{x}^{[k]} = (x_1^k, x_2^k, \cdots, x_n^k)^{\mathrm{T}}$. By using the product of tensors defined by Shao [17], $\mathcal{A}\boldsymbol{x}^{k-1}$ is simplified as $\mathcal{A}\boldsymbol{x}$. Then $\mathcal{A}\boldsymbol{x}$ is a vector in $\mathbb{C}^n$ whose i-th component is given by

$$(\mathcal{A}\boldsymbol{x}^{k-1})_i = (\mathcal{A}\boldsymbol{x})_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}, \text{ for each } i \in [n]. \tag{1}$$

We have

$$x^{\mathrm{T}}(\mathcal{A}x) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} \cdots x_{i_k}. \tag{2}$$

In 2005, Qi [13] and Lim [10] independently introduced the concept of tensor eigenvalues and the spectra of tensors as follows. If there exist a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ satisfying $(\mathcal{A}x)_i = \lambda x_i^{k-1}$ for each $i \in [n]$, then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ corresponding to λ. The spectral radius of $\mathcal{A}$ is defined to be the largest modulus of the eigenvalues of $\mathcal{A}$, i.e., $\rho(\mathcal{A}) = \max\{|\lambda|| \ \lambda$ is an eigenvalue of $\mathcal{A}\}$.

In 2012, Cooper and Dutle [5] introduced the definition of the adjacency tensor of a k-uniform hypergraph $\mathcal{H}$, where $k \geqslant 3$. Let $V(\mathcal{H}) = \{v_1, v_2, \cdots, v_n\}$, where $n \geqslant 3$. The adjacency tensor $\mathcal{A}(\mathcal{H}) = (a_{i_1 i_2 \cdots i_k})$ of $\mathcal{H}$ is an order k dimension n symmetric tensor, where $a_{i_1 i_2 \ldots i_k} = \frac{1}{(k-1)!}$ if $\{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\} \in E(\mathcal{H})$ and $a_{i_1 i_2 \ldots i_k} = 0$ otherwise.

The spectral radius of $\mathcal{H}$, denoted by $\rho(\mathcal{H})$, is defined as the spectral radius of $\mathcal{A}(\mathcal{H})$. The spectral radii of hypergraphs attract a lot of attention of researchers because of the fine properties of its corresponding eigenvector together with its popularity in graph counterpart [12].

In recent years, among the hypergraphs with a given number of vertices, many results regarding to the extremal hypergraphs with the maximum spectral radii have been obtained. For more details, one can refer to the monograph [15]. However, for the hypergraphs with a given number of edges, there are few results about the extremal hypergraphs attaining the maximum spectral radii. When the number of the edges of an r-uniform hypergraph is given, Bai and Lu [1] proved that the spectral radius of an r-uniform hypergraph with $e = \binom{k}{r}$ edges is at most $\binom{k-1}{r-1}$ and characterized the unique extremal hypergraph with the spectral radius $\binom{k-1}{r-1}$, where $k \geqslant r$ and $r \geqslant 3$.

Let $\mathcal{F}_n$ be a 3-uniform hypergraph with n vertices whose shadow is $K_1 + P_{n-1}$, where $n \geqslant 3$. It is obvious that $\mathcal{F}_n$ is unique. Ellingham et al. [7] obtained the following Theorem 1.4.

Theorem 1.4. *[7] For large enough n, the n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the fan hypergraph $\mathcal{F}_n$.*

Inspired by all the results as above, in this article, we consider the extremal hypergraph with the maximum spectral radius among all outerplanar 3-uniform hypergraphs with a given size m, where $m \geqslant 3$. The main result of this article is Theorem 1.5.

Theorem 1.5. *For large enough m, the m-edge outerplanar 3-uniform hypergraph of maximum spectral radius is the fan hypergraph $\mathcal{F}_{m+2}$.*

Remark: Among all outerplanar graphs with n vertices, Tait and Tobin [18] proved that $K_1 + P_{n-1}$ attains the maximum spectral radius. In Theorem 1.2, we obtain that the m-edge outerplanar graph of maximum spectral radius is the star S_m, where $m \geqslant 64$. But for outerplanar 3-uniform hypergraphs, whether the number of their vertices is given or the number of their edges is given, from Theorems 1.4 and 1.5, we get that the fan hypergraph is the unique extremal hypergraph attaining the maximum spectral radius.

This paper is organized as follows. In Section 2, some necessary lemmas are introduced. In Section 3, we prove that Theorem 1.5 holds and thus the hypergraph with the maximum spectral radius among all outerplanar 3-uniform hypergraphs with a given size is obtained.

2. Preliminary

In this section, we introduce some useful lemmas which are necessary for the proofs of our results.

Friedland et al. [9] introduced the definition of weak irreducibility of nonnegative tensors and proved that if a k-uniform hypergraph $\mathcal{G}$ is connected, then $\mathcal{A}(\mathcal{G})$ is weakly irreducible. Parts of the Perron-Frobenius theorem of nonnegative tensors are as follows.

Lemma 2.1. *[19] Let $\mathcal{A}$ be a nonnegative tensor of order k and dimension n, where $k \geqslant 2$. Then we have the following statements.*

(i). $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $x \in \mathbb{R}_+^n$ corresponding to it.

(ii). If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with the unique positive eigenvector $x \in \mathbb{R}_{++}^n$, up to a positive scaling coefficient.

Lemma 2.2. *[14] Let $\mathcal{A}$ be a nonnegative symmetric tensor of order k and dimension n. Then we have $\rho(\mathcal{A}) = \max\left\{x^T \mathcal{A} x \mid x \in \mathbb{R}_+^n, \|x\|_k = 1\right\}$. Furthermore, $x \in \mathbb{R}_+^n$ with $\|x\|_k = 1$ is an optimal solution of the above optimization problem if and only if it is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\rho(\mathcal{A})$.*

If a k-uniform hypergraph $\mathcal{G}$ is connected, then by Lemmas 2.1 and 2.2, there exists the unique positive x (called Perron vector) of the adjacency tensor $\mathcal{A}(\mathcal{G})$ corresponding to $\rho(\mathcal{G})$ with $\|x\|_k = 1$. By definition, for every component x_i of x, we have the following eigenequation

$$\rho(\mathcal{G}) x_i^{k-1} = \sum_{\{i, i_2, \cdots, i_k\} \in E(\mathcal{G})} x_{i_2} \cdots x_{i_k}. \tag{3}$$

Cooper and Dutle [5] obtained Lemma 2.3 as follows.

Lemma 2.3. *[5] Let $\mathcal{H}$ be a k-uniform hypergraph, where $k \geqslant 3$. If $\mathcal{G}$ is a sub-hypergraph of $\mathcal{H}$, then $\rho(\mathcal{G}) \leqslant \rho(\mathcal{H})$.*

3. The maximum spectral radius of outerplanar 3-uniform hypergraphs with a given size

In this section, to obtain the hypergraph with the maximum spectral radius among all outerplanar 3-uniform hypergraphs with a given size, we first introduce Lemmas 3.1–3.3.

For a given vertex u in $\mathcal{G}$, let $E_{\mathcal{G}}(u)$ denote the set of the edges which contain u. We use $N_{\mathcal{G}}(u)$ to denote the set of neighbors of u in $V(\mathcal{G})$. Without confusion, $N_{\mathcal{G}}(u)$ is written as $N(u)$.

We introduce Lemma 3.1 as follows. Lemma 3.1 will give us a useful tool to compare the spectral radii of two (hyper)graphs.

Lemma 3.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two connected k-uniform hypergraphs with the same vertex set V, where $k \geqslant 2$. Furthermore, we suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same Perron vector (denoted by x) with $\|x\|_k = 1$ and $x \in \mathbb{R}_{++}^n$. If there is a vertex v in V satisfying $E_{\mathcal{G}_1}(v) \subsetneq E_{\mathcal{G}_2}(v)$, then $\rho(\mathcal{G}_1) < \rho(\mathcal{G}_2)$.*

Proof. By using eigenequation (3) of $\mathcal{G}_1$ and $\mathcal{G}_2$ at vertex v, we have

$$\rho(\mathcal{G}_2) x_v^2 - \rho(\mathcal{G}_1) x_v^2 = \sum_{\{v, i_2, \cdots, i_k\} \in E_{\mathcal{G}_2}(v) \setminus E_{\mathcal{G}_1}(v)} x_{i_2} \cdots x_{i_k} > 0, \tag{4}$$

where the inequality in (4) is obtained from Lemma 2.1 (ii) and the fact that $E_{\mathcal{G}_1}(v) \subsetneq E_{\mathcal{G}_2}(v)$. Thus, we get $\rho(\mathcal{G}_1) < \rho(\mathcal{G}_2)$ □

Based on Lemma 2.3, we introduce Lemma 3.2 as follows.

Lemma 3.2. *Let $\mathcal{H}$ be a k-uniform connected hypergraph, where $k \geqslant 3$. If $\mathcal{G}$ is a proper sub-hypergraph of $\mathcal{H}$, then $\rho(\mathcal{G}) < \rho(\mathcal{H})$.*

Proof. By Lemma 2.3, we get that $\rho(\mathcal{G}) \leqslant \rho(\mathcal{H})$. We only need to prove that $\rho(\mathcal{G}) = \rho(\mathcal{H})$ is impossible. We suppose that $\rho(\mathcal{G}) = \rho(\mathcal{H})$. If $V(\mathcal{G}) = V(\mathcal{H})$, since $\mathcal{G}$ is a proper sub-hypergraph of $\mathcal{H}$, it is easy to check that $\rho(\mathcal{G}) < \rho(\mathcal{H})$. If $V(\mathcal{G}) \subsetneq V(\mathcal{H})$, let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the Perron vector of $\mathcal{G}$, where $\|x\|_k = 1$ and $x \in \mathbb{R}_{++}^n$. Let y be the $|V(\mathcal{H})|$-dimensional vector of $\mathcal{A}(\mathcal{H})$ and $y = (x_1, x_2, \cdots, x_n, 0, \cdots, 0)^{\mathrm{T}}$. Let z be a $|V(\mathcal{H})|$-dimensional vector with $\|z\|_k = 1$. By Lemmas 2.1 and 2.2, we obtain

$$\rho(\mathcal{G}) = \rho(\mathcal{H}) = \max_{z} z^{\mathrm{T}} \mathcal{A}(\mathcal{H}) z > y^{\mathrm{T}} \mathcal{A}(\mathcal{H}) y = x^{\mathrm{T}} \mathcal{A}(\mathcal{G}) x = \rho(\mathcal{G}),$$

a contradiction. Thus, we get Lemma 3.2. □

Let $\mathcal{H}^*$ be the hypergraph attaining the maximum spectral radius among all outerplanar 3-uniform hypergraphs with m edges, where $m \geqslant 3$. Let $f\big(\partial(\mathcal{H}^*)\big)$ denote the face set of $\partial(\mathcal{H}^*)$. Let ϕ be a mapping and we define ϕ as follows. Let $\phi : E(\mathcal{H}^*) \to f\big(\partial(\mathcal{H}^*)\big)$ and $\phi(xyz) \to f(xyzx)$, where xyz is an edge in $\mathcal{H}^*$ and $f(xyzx)$ is a face in $f\big(\partial(\mathcal{H}^*)\big)$. It is noted that $\partial(\mathcal{H}^*)$ is a planar graph and the mapping ϕ from $E(\mathcal{H}^*)$ to $f\big(\partial(\mathcal{H}^*)\big)$ is unique. Let $\overline{f}$ be the outer face of $\partial(\mathcal{H}^*)$. We classify $f\big(\partial(\mathcal{H}^*)\big)\backslash\{\overline{f}\}$ into two subsets and denote the two subsets by $f_1(\mathcal{H}^*)$ and $f_2(\mathcal{H}^*)$, where $f_1(\mathcal{H}^*)$ (respectively, $f_2(\mathcal{H}^*)$) is the face set in which each face has (respectively, does not have) a preimage in $E(\mathcal{H}^*)$. If a face belongs to $f_1(\mathcal{H}^*)$ (respectively, $f_2(\mathcal{H}^*)$), then we call it a nonempty face (respectively, an empty face).

A maximum outerplanar graph is an outerplanar graph which satisfies that no additional edge can be added while maintaining the outerplanarity. By Euler's formula, an n-vertex outerplanar graph is maximum if and only if it has exactly $2n-3$ edges (equivalently, $n-2$ interior faces) with $n \geqslant 3$. An outerplanar 3-uniform hypergraph is maximum if its shadow is maximum.

Lemma 3.3. *$\partial(\mathcal{H}^*)$ is a maximum outerplanar graph and every 3-face in $\partial(\mathcal{H}^*)$ is a nonempty face, i.e., $\mathcal{H}^*$ is a maximum outerplanar 3-uniform hypergraph.*

Proof. To obtain Lemma 3.3, we need to prove that Claims 1–3 hold.

Claim 1. $\partial(\mathcal{H}^*)$ is connected.

Proof. If $\partial(\mathcal{H}^*)$ is not connected, then $\mathcal{H}^*$ is also not connected and $\mathcal{H}^*$ must contain at least 2 connected components. We assume that $\mathcal{H}^* = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_s$, where $\rho(\mathcal{H}^*) = \rho(\mathcal{H}_1) \geqslant \rho(\mathcal{H}_2) \geqslant \cdots \geqslant \rho(\mathcal{H}_s)$ and $s \geqslant 2$. Let $f(x_1x_2\cdots x_px_1)$ and $f(y_1y_2\cdots y_qy_1)$ be the outer faces of $\partial(\mathcal{H}_1)$ and $\partial(\mathcal{H}_2)$, respectively, where $p, q \geqslant 3$. We contract x_1 and y_1 into a vertex in $\mathcal{H}^*$ and denote the new vertex by x_1'. We denote the resulting hypergraph by $\mathcal{H}'$. It is easy to check that $\mathcal{H}' = \mathcal{H}_1' \cup \mathcal{H}_3 \cup \cdots \cup \mathcal{H}_s$, where $\mathcal{H}_1'$ is the connected component which is composed of $\mathcal{H}_1$ and $\mathcal{H}_2$. By Lemma 3.2, we get $\rho(\mathcal{H}') = \rho(\mathcal{H}_1') > \rho(\mathcal{H}_1)$.

Now we prove that $\mathcal{H}'$ is an outerplanar 3-uniform hypergraph. It suffices to show that $\partial(\mathcal{H}_1')$ is an outerplanar graph. It is obvious that $\mathcal{H}'$ has a planar drawing and $f(x_1'x_2\cdots x_px_1'y_2\cdots y_qx_1')$ is an outer face in $\partial(\mathcal{H}_1')$ since $\partial(\mathcal{H}_1)$ and $\partial(\mathcal{H}_2)$ are two outerplanar graphs, where $p, q \geqslant 3$. Thus, we obtain an outerplanar 3-uniform hypergraph $\mathcal{H}'$ which satisfies $\rho(\mathcal{H}') > \rho(\mathcal{H}^*)$ and $e(\mathcal{H}') = e(\mathcal{H}^*)$. By the definition of $\mathcal{H}^*$, we get a contradiction. □

By Claim 1 in Lemma 3.3, $\mathcal{H}^*$ is connected. We assume that $\boldsymbol{x}$ is the Perron vector of $\mathcal{A}(\mathcal{H}^*)$ corresponding to $\rho(\mathcal{H}^*)$ with $\|\boldsymbol{x}\|_3 = 1$.

Claim 2. $\partial(\mathcal{H}^*)$ does not contain a cut vertex.

Proof. If $\partial(\mathcal{H}^*)$ contains a cut vertex, then we denote a cut vertex of $\partial(\mathcal{H}^*)$ by u. Therefore, $\partial(\mathcal{H}^*) - u$ contains at least two connected components. If $\partial(\mathcal{H}^*) - u$ contains at least three connected components, then we assume that $d_{\partial(\mathcal{H}^*)}(u) = d$ and let $N_{\partial(\mathcal{H}^*)}(u) = \{u_1, u_2, \cdots, u_i, \cdots, u_j, \cdots, u_{d-1}, u_d\}$ with $j - i > 1$. If $\partial(\mathcal{H}^*) - u$ exactly contains two connected components, we still assume that $d_{\partial(\mathcal{H}^*)}(u) = d$ and let $N_{\partial(\mathcal{H}^*)}(u) = \{u_1, u_2, \cdots, u_i, u_j, \cdots, u_{d-1}, u_d\}$ with $j - i = 1$. The vertices in $N_{\partial(\mathcal{H}^*)}(u)$ are labeled according to the anticlockwise of the planar drawing of $\partial(\mathcal{H}^*)$. $\partial(\mathcal{H}^*)$ and its vertex u are shown in Figure 1 (a). It is noted that the vertices between u_i and u_j do not exist in Figure 1 (a) if $\partial(\mathcal{H}^*) - u$ exactly contains two connected components.

Combining with the fact that $\partial(\mathcal{H}^*)$ is an outerplanar graph, we can check that $f(uu_1u_2\cdots u_iu\cdots uu_j\cdots u_{d-1}u_du)$ with $j - i > 1$ or $f(uu_1u_2\cdots u_iuu_j\cdots u_{d-1}u_du)$ with $j - i = 1$ is an outer face of $\partial(\mathcal{H}^*)$. Let $\mathcal{H}''$ be the hypergraph obtained from $\mathcal{H}^*$ by contracting u_1 and u_d into a vertex and we denote the new vertex by u_1'. It is noted that $f(u_1'u_2\cdots u_iu\cdots uu_j\cdots u_{d-1}u_1')$ with $j - i > 1$ or $f(u_1'u_2\cdots u_iuu_j\cdots u_{d-1}u_1')$ with $j - i = 1$ is an outer face of $\partial(\mathcal{H}'')$ and $\partial(\mathcal{H}'')$ is an outerplanar graph, where $\partial(\mathcal{H}'')$ is shown in Figure 1 (b) and the vertices between u_i and u_j do not exist if $j - i = 1$. Furthermore, we obtain that $\mathcal{H}''$ is an outerplanar 3-uniform hypergraph.

It is easy to check that $e(\mathcal{H}'') = e(\mathcal{H}^*)$. To find a contradiction, it suffices to prove that $\rho(\mathcal{H}'') > \rho(\mathcal{H}^*)$. Recall that $\boldsymbol{x}$ is the Perron vector of $\mathcal{A}(\mathcal{H}^*)$ corresponding to $\rho(\mathcal{H}^*)$. Let $\boldsymbol{y}$ be a vector of $\mathcal{A}(\mathcal{H}'')$ with

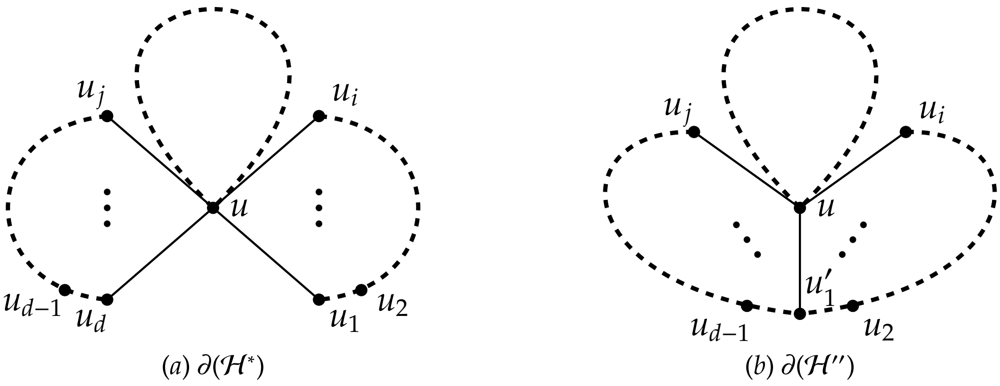

Figure 1: $\partial(\mathcal{H}^*)$, $\partial(\mathcal{H}'')$ and their outer faces

dimension $n-1$ which satisfies $y_v = x_v$ for $v \neq u_1'$ and $y_v = (x_{u_d}^3 + x_{u_1}^3)^{\frac{1}{3}}$ for $v = u_1'$. It is obvious that $y_{u_1'} > \max\{x_{u_d}, x_{u_1}\}$. Let z be an $(n-1)$-dimensional vector with $\|z\|_3 = 1$. By Lemmas 2.1 and 2.2, we have

$$\rho(\mathcal{H}'') = \max_z z^{\mathrm{T}}\mathcal{A}(\mathcal{H}'')z \geqslant y^{\mathrm{T}}\mathcal{A}(\mathcal{H}'')y > x^{\mathrm{T}}\mathcal{A}(\mathcal{H}^*)x = \rho(\mathcal{H}^*). \tag{5}$$

By (5), we obtain $\rho(\mathcal{H}'') > \rho(\mathcal{H}^*)$. By the definition of $\mathcal{H}^*$, we get a contradiction. □

By Claims 1 and 2 in Lemma 3.3, we can conclude that $\partial(\mathcal{H}^*)$ is an outerplanar graph without a cut vertex, which illustrates that there is a Hamilton cycle in $\partial(\mathcal{H}^*)$. To completely prove Lemma 3.3, we need to prove that every inner face of $\partial(\mathcal{H}^*)$ is a nonempty face.

Claim 3. $f_2(\mathcal{H}^*)$ is empty.

Proof. To prove this claim, we first prove that Claims 3.1 and 3.2 hold.

Claim 3.1. $f_2(\mathcal{H}^*)$ does not contain a k-face, where $k \geqslant 4$.

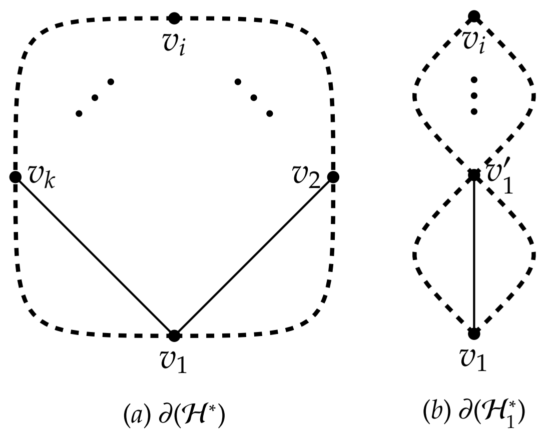

Figure 2: $\partial(\mathcal{H}^*)$, $\partial(\mathcal{H}_1^*)$ and their outer faces

We suppose that $f_2(\mathcal{H}^*)$ contains a k-face, where $k \geqslant 4$. Let $f(v_1v_2\cdots v_kv_1)$ be a k-face in $\partial(\mathcal{H}^*)$, where $f(v_1v_2\cdots v_kv_1)$ is not an outer face and $k \geqslant 4$. In Figure 2 (a), the dashed lines represent the outer face of $\partial(\mathcal{H}^*)$ and the k-face forms the unique cycle $C_k = v_1v_2\cdots v_i\cdots v_kv_1$ in $\partial(\mathcal{H}^*)$, where $k \geqslant 4$ and $3 \leqslant i \leqslant k-1$. Let $\mathcal{H}_1^*$ be the hypergraph obtained from $\mathcal{H}^*$ by contracting v_2 and v_k into a vertex and we denote the new vertex by v_1'. Then $\partial(\mathcal{H}_1^*)$ is shown in Figure 2 (b). We can easily check that $\mathcal{H}_1^*$ is an outerplanar 3-uniform hypergraph since there is an outer face formed by all the vertices in $V(\partial(\mathcal{H}_1^*))$ of $\partial(\mathcal{H}_1^*)$, and the outer face of $\partial(\mathcal{H}_1^*)$ is precisely formed by the dashed lines in Figure 2 (b).

It is easy to check that $e(\mathcal{H}^*) = e(\mathcal{H}_1^*)$. To find a contradiction, it suffices to prove that $\rho(\mathcal{H}_1^*) > \rho(\mathcal{H}^*)$. Let $\boldsymbol{y}$ be a vector of $\mathcal{A}(\mathcal{H}_1^*)$ with dimension $n-1$ which satisfies $y_w = x_w$ for $w \neq v_1'$ and $y_w = (x_{v_2}^3 + x_{v_k}^3)^{\frac{1}{3}}$ for $w = v_1'$. We get that $y_{v_1'} > \max\{x_{v_2}, x_{v_k}\}$. Let $\boldsymbol{z}$ be an $(n-1)$-dimensional vector with $\|\boldsymbol{z}\|_3 = 1$. By Lemmas 2.1 and 2.2, we have

$$\rho(\mathcal{H}_1^*) = \max_{\boldsymbol{z}} \boldsymbol{z}^{\mathrm{T}}\mathcal{A}(\mathcal{H}_1^*)\boldsymbol{z} \geqslant \boldsymbol{y}^{\mathrm{T}}\mathcal{A}(\mathcal{H}_1^*)\boldsymbol{y} > \boldsymbol{x}^{\mathrm{T}}\mathcal{A}(\mathcal{H}^*)\boldsymbol{x} = \rho(\mathcal{H}^*). \tag{6}$$

By (6), we obtain $\rho(\mathcal{H}_1^*) > \rho(\mathcal{H}^*)$. By the definition of $\mathcal{H}^*$, we get a contradiction. Thus, we obtain Claim 3.1.

Claim 3.2. $f_2(\mathcal{H}^*)$ does not contain a 3-face.

We suppose that $f_2(\mathcal{H}^*)$ contains a 3-face. Let $f(abca)$ be a 3-face in $\partial(\mathcal{H}^*)$. In Figure 3 (a), the shaded regions denote the nonempty faces in $\partial(\mathcal{H}^*)$ and the unshaded regions denote the empty faces in $\partial(\mathcal{H}^*)$. According to the relationship between $\mathcal{H}^*$ and $\partial(\mathcal{H}^*)$, for any two faces sharing a common edge in $\partial(\mathcal{H}^*)$, at least one of them is a nonempty face. It follows that $f(ab'ca)$, $f(abc'a)$ and $f(a'bca')$ are nonempty faces.

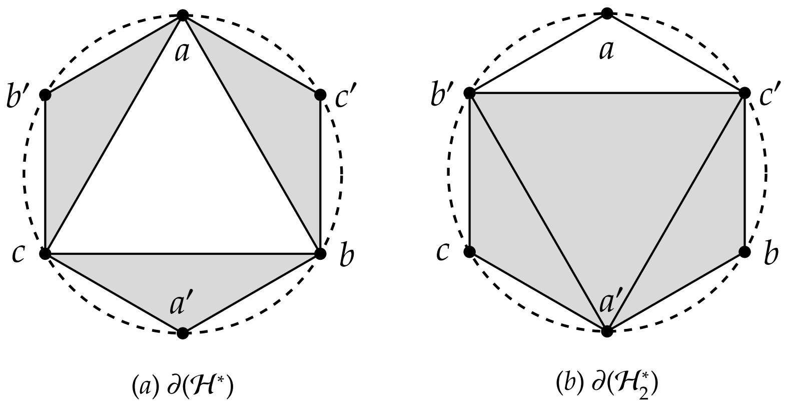

Figure 3: $\partial(\mathcal{H}^*)$, $\partial(\mathcal{H}_2^*)$ and their outer faces

Next, we prove that $x_{a'} > x_a$, $x_{b'} > x_b$ and $x_{c'} > x_c$. By symmetry of a and a', b and b', and c and c', it suffices to prove that $x_{a'} > x_a$. We suppose that $x_{a'} \leqslant x_a$. Let $\overline{\mathcal{H}} = \mathcal{H}^* - a'bc + abc$. It is obvious that $e(\overline{\mathcal{H}}) = e(\mathcal{H}^*)$. By Lemma 2.2, we have $\rho(\overline{\mathcal{H}}) - \rho(\mathcal{H}^*) \geqslant 3(x_a x_b x_c - x_{a'} x_b x_c) \geqslant 0$. If $\rho(\overline{\mathcal{H}}) = \rho(\mathcal{H}^*)$, we can check that $\boldsymbol{x}$ is also the Perron vector of $\mathcal{A}(\overline{\mathcal{H}})$ corresponding to $\rho(\overline{\mathcal{H}})$. By Lemma 3.1 and combining with the fact $E_{\mathcal{H}^*}(a) \subsetneq E_{\overline{\mathcal{H}}}(a)$, we have $\rho(\overline{\mathcal{H}}) > \rho(\mathcal{H}^*)$. By the definition of $\mathcal{H}^*$, we get a contradiction. So we have $x_{a'} > x_a$, $x_{b'} > x_b$ and $x_{c'} > x_c$.

Let $\mathcal{H}_2^* = \mathcal{H}^* + a'b'c' + a'bc' + a'b'c - a'bc - abc' - ab'c$, where $\partial(\mathcal{H}_2^*)$ is shown in Figure 3 (b). It is obvious that $e(\mathcal{H}_2^*) = e(\mathcal{H}^*)$. By the analyses as above, we have $x_{a'}x_{b'}x_{c'} > x_{a'}x_b x_c$, $x_{a'}x_b x_{c'} > x_a x_b x_{c'}$ and $x_{a'}x_{b'}x_c > x_a x_{b'} x_c$. By Lemma 2.2, we get

$$\rho(\mathcal{H}_2^*) - \rho(\mathcal{H}^*) \geqslant 3(x_{a'}x_{b'}x_{c'} + x_{a'}x_b x_{c'} + x_{a'}x_{b'}x_c - x_{a'}x_b x_c - x_a x_b x_{c'} - x_a x_{b'} x_c) > 0. \tag{7}$$

Thus, we obtain $\rho(\mathcal{H}_2^*) > \rho(\mathcal{H}^*)$. By the definition of $\mathcal{H}^*$, we get a contradiction. Thus, we obtain Claim 3.2.

By the proofs of Claims 3.1 and 3.2, we get Claim 3.

By the proofs of Claims 1, 2 and 3.1, we can check that $\partial(\mathcal{H}^*)$ is a maximum outerplanar graph. By the proof of Claim 3.2, we obtain that every 3-face in $\partial(\mathcal{H}^*)$ is a nonempty face. Thus we get that $\mathcal{H}^*$ is a maximum outerplanar 3-uniform hypergraph and Lemma 3.3 holds. □

Now we prove that Theorem 1.5 holds.

The proof of Theorem 1.5. By Lemma 3.3, we know that $\mathcal{H}^*$ is a maximum outerplanar 3-uniform hypergraph with m edges, where $m \geqslant 3$. Since a maximum n-vertex outerplanar 3-uniform hypergraph exactly has $n-2$ edges, where $n \geqslant 3$, we obtain that $\mathcal{H}^*$ is a maximum outerplanar 3-uniform hypergraph with $m+2$ vertices. By the definition of $\mathcal{H}^*$, it is obvious that $\mathcal{H}^*$ is the hypergraph that attains the maximum spectral radius among all maximum outerplanar 3-uniform hypergraphs with $m+2$ vertices, where $m \geqslant 3$. By Theorem 1.4, we can deduce that when m is large enough, among all maximum outerplanar 3-uniform hypergraphs with $m+2$ vertices, the fan hypergraph $\mathcal{F}_{m+2}$ attains the maximum spectral radius. Thus, we have Theorem 1.5. □

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