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Toughness and A_α-spectral radius in graphs

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Abstract. Let $\alpha \in [0,1)$, and let G be a connected graph of order n with $n \geq f(\alpha)$, where $f(\alpha) = 6$ for $\alpha \in [0, \frac{2}{3}]$ and $f(\alpha) = \frac{4}{1-\alpha}$ for $\alpha \in (\frac{2}{3}, 1)$. A graph G is said to be t-tough if $|S| \geq tc(G-S)$ for each subset S of $V(G)$ with $c(G-S) \geq 2$, where $c(G-S)$ is the number of connected components in $G-S$. The A_α-spectral radius of G is denoted by $\rho_\alpha(G)$. In this paper, it is verified that G is a 1-tough graph unless $G = K_1 \vee (K_{n-2} \cup K_1)$ if $\rho_\alpha(G) \geq \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, where $\rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$ equals the largest root of $x^3 - ((\alpha+1)n + \alpha - 3)x^2 + (\alpha n^2 + (\alpha^2 - \alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (3\alpha^2 - \alpha + 1)n - 4\alpha^2 + 5\alpha - 3 = 0$. Further, we present an A_α-spectral radius condition for a graph to be a t-tough graph.

1. Introduction

Let G be an undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The order of G is the number $n = |V(G)|$ of its vertices and its size is the number $m = e(G)$ of its edges. A graph G is called trivial if $|V(G)| = 1$. For arbitrary $v \in V(G)$, the degree of v in G is defined as the number of edges which are adjacent to v and denoted by $d_G(v)$. For a given subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and write $G - S$ for $G[V(G) \setminus S]$. The complement of a graph G is a graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are nonadjacent in G. Let G_1 and G_2 be two vertex-disjoint graphs. We use $G_1 \cup G_2$ to denote the disjoint union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph formed from $G_1 \cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. We denote by K_n a complete graph of order n.

Let t be a nonnegative real number. A graph G is said to be t-tough if $|S| \geq tc(G-S)$ for each subset S of $V(G)$ with $c(G-S) \geq 2$, where $c(G-S)$ is the number of connected components in $G-S$. The toughness $t(G)$ of G is the largest real number t for which G is t-tough, or is ∞ if G is complete. This concept was first introduced by Chvátal [6] in 1973. Some results on toughness of graphs can be found in [4, 5, 7, 17, 21, 26, 28, 31–33].

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Given a graph G of order n, the adjacency matrix $A(G)$ of G is the $n \times n$ matrix in which entry a_{ij} is 1 or 0 according to whether v_i and v_j are adjacent or not, and $a_{ij} = 0$ if $i = j$. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be its eigenvalues in nonincreasing order. Note that the adjacency spectral radius of G is equal to $\lambda_1(G)$, written as $\rho(G)$.

Let $D(G)$ denote the diagonal matrix of vertex degree of G. Then $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and signless Laplacian matrix of G, respectively. For any $\alpha \in [0, 1]$, Nikiforov [18] introduced the A_α-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

It is easy to see that $A_\alpha(G) = A(G)$ if $\alpha = 0$, and $A_\alpha(G) = \frac{1}{2}Q(G)$ if $\alpha = \frac{1}{2}$. Note that $A_\alpha(G)$ is a real symmetric nonnegative matrix. Hence, the eigenvalues of $A_\alpha(G)$ are real, which can be indexed in nonincreasing order as $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$. The largest eigenvalue $\lambda_1(A_\alpha(G))$ is called the A_α-spectral radius of G, and denoted by $\rho_\alpha(G)$. Namely, $\rho_\alpha(G) = \lambda_1(A_\alpha(G))$. For some interesting spectral properties of $A_\alpha(G)$, we refer the reader to [2, 13–16, 18, 19, 25].

Many authors [8, 20, 22, 23, 27, 29, 30, 34, 35] obtained some results on spectral radius in graphs. Brouwer [3] investigated the relationship between toughness and eigenvalues and claimed that for any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$, and he further conjectured that the lower bound $\frac{d}{\lambda} - 2$ can be improved to $\frac{d}{\lambda} - 1$, where $\lambda = \max_{2 \leq i \leq n} |\lambda_i|$. Later, Gu [11] strengthened Brouwer's result and claimed the lower bound $\frac{d}{\lambda} - 2$ can be improved to $\frac{d}{\lambda} - \sqrt{2}$. Very recently, Gu [10] confirmed Brouwer's conjecture. Fan, Lin and Lu [9] presented two adjacency spectral radius conditions for the existence of a t-tough graph. It is natural and interesting to find a sufficient condition for a graph to be t-tough in view of its A_α-spectral radius. In this paper, we first show a A_α-spectral radius condition to guarantee that a graph is 1-tough.

Theorem 1.1. Let $\alpha \in [0, 1)$, and let G be a connected graph of order n with $n \geq f(\alpha)$, where

$$f(\alpha) = \begin{cases} 6, & if\ \alpha \in [0, \frac{2}{3}]; \\ \frac{4}{1-\alpha}, & if\ \alpha \in (\frac{2}{3}, 1). \end{cases}$$

If $\rho_\alpha(G) \geq \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, then G is a 1-tough graph unless $G = K_1 \vee (K_{n-2} \cup K_1)$, where $\rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$ equals the largest root of $x^3 - ((\alpha+1)n + \alpha - 3)x^2 + (\alpha n^2 + (\alpha^2 - \alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (3\alpha^2 - \alpha + 1)n - 4\alpha^2 + 5\alpha - 3 = 0$.

Further, it is natural and interesting to find an A_α-spectral radius condition to ensure that a graph is a t-tough graph. Next, we present an A_α-spectral radius condition for a graph to be t-tough.

Theorem 1.2. Let $\alpha \in [\frac{1}{2}, \frac{3}{4})$, and let t be a positive integer. If G is a connected graph of order $n \geq \max\{5t^2 + 10t + 1, \frac{12t(1-\alpha)-2\alpha+1}{3-4\alpha}\}$ with $\rho_\alpha(G) \geq \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1))$, then G is a t-tough graph unless $G = K_{2t-1} \vee (K_{n-2t} \cup K_1)$.

2. Preliminaries

In this section, we put forward some necessary lemmas, which play a key role in proving our main results.

Lemma 2.1 (Nikiforov [18]). Let K_n be a complete graph of order n. Then

$$\rho_\alpha(K_n) = n - 1.$$

Lemma 2.2 (Nikiforov [18]). If G is a connected graph, and H is a proper subgraph of G, then

$$\rho_\alpha(G) > \rho_\alpha(H).$$

Lemma 2.3 (Zhao, Huang and Wang [25]). Let $\alpha \in [0, 1)$, and let $n_1 \geq n_2 \geq \cdots \geq n_t$ be positive integers with $n = \sum_{i=1}^{t} n_i + s$ and $n_1 \leq n - s - t + 1$. Then

$$\rho_\alpha(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})) \leq \rho_\alpha(K_s \vee (K_{n-s-t+1} \cup (t-1)K_1)),$$

where the equality holds if and only if $(n_1, n_2, \dots, n_t) = (n - s - t + 1, 1, \dots, 1)$.

Let M be a real matrix whose rows and columns are indexed by $V = \{1, 2, \dots, n\}$. Assume that M, with respect to the partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_t$, can be written as

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. Let b_{ij} denote the average row sum of M_{ij}, namely, b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then matrix $M_\pi = (b_{ij})$ is called the *quotient matrix* of M. If the row sum of every block M_{ij} is a constant, then the partition is *equitable*.

Lemma 2.4 (Haemers [12], You, Yang, So and Xi [24]). Let M be a real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then each eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the spectral radius of M_π equals the spectral radius of M.

Lemma 2.5 (Alhevaz, Baghipur, Ganie and Das [1]). Let G be a graph of order n with m edges, with no isolated vertices and let $\alpha \in [\frac{1}{2}, 1]$. Then

$$\rho_\alpha(G) \leq \frac{2m(1-\alpha)}{n-1} + \alpha n - 1.$$

If $\alpha \in (\frac{1}{2}, 1)$ and G is connected, equality holds if and only if $G = K_n$.

3. The proof of Theorem 1.1

In this section, we verify Theorem 1.1, which establishes an A_α-spectral radius condition for a graph to be 1-tough.

Proof of Theorem 1.1. Suppose to the contrary that G is not a 1-tough graph, there exists some nonempty subset $S \subseteq V(G)$ such that $c(G - S) \geq |S| + 1$. Let $|S| = s$. Then G is a spanning subgraph of $G_s^1 = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s+1}})$ for some positive integers $n_1 \geq n_2 \geq \cdots \geq n_{s+1}$ with $\sum_{i=1}^{s+1} n_i = n - s$. According to Lemma 2.2, we have

$$\rho_\alpha(G) \leq \rho_\alpha(G_s^1) \tag{1}$$

with equality if and only if $G = G_s^1$.

Let $G_s^2 = K_s \vee (K_{n-2s} \cup sK_1)$. Using Lemma 2.3, we get

$$\rho_\alpha(G_s^1) \leq \rho_\alpha(G_s^2), \tag{2}$$

where the equality holds if and only if $(n_1, n_2, \dots, n_{s+1}) = (n - 2s, 1, \dots, 1)$. Let $\varphi(x) = x^3 - ((\alpha+1)n + \alpha - 3)x^2 + (\alpha n^2 + (\alpha^2 - \alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (3\alpha^2 - \alpha + 1)n - 4\alpha^2 + 5\alpha - 3$. Then $\rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$ equals the largest root of $\varphi(x) = 0$. We are to verify the following claim.

Claim 1. If $\alpha \in [0, 1)$ and $n \geq f(\alpha)$, then we obtain

$$\rho_\alpha(G_s^2) \leq \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$$

with equality if and only if $G_s^2 = K_1 \vee (K_{n-2} \cup K_1)$.

Proof. Recall that $G_s^2 = K_s \vee (K_{n-2s} \cup sK_1)$. Obviously, $1 \leq s \leq \frac{n-1}{2}$.

Consider the partition $V(G_s^2) = V(sK_1) \cup V(K_{n-2s}) \cup V(K_s)$. The corresponding quotient matrix of $A_\alpha(G_s^2)$ is equal to

$$B_1 = \begin{pmatrix} \alpha s & 0 & (1-\alpha)s \\ 0 & n+(\alpha-2)s-1 & (1-\alpha)s \\ (1-\alpha)s & (1-\alpha)(n-2s) & \alpha n - \alpha s + s - 1 \end{pmatrix}.$$

Then the characteristic polynomial of B_1 is

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - ((\alpha+1)n + (\alpha-1)s - 2)x^2 \\ &+ (\alpha n^2 + (\alpha^2 s - \alpha - 1)n - s^2 - (2\alpha - 1)s + 1)x \\ &- \alpha^2 sn^2 + (2\alpha^2 - 2\alpha + 1)s^2 n + (\alpha^2 + \alpha)sn \\ &- (3\alpha^2 - 5\alpha + 2)s^3 - (\alpha^2 - \alpha + 1)s^2 - \alpha s. \end{aligned} \quad (3)$$

Since the partition $V(G_s^2) = V(sK_1) \cup V(K_{n-2s}) \cup V(K_s)$ is equitable, it follows from Lemma 2.4 that $\rho_\alpha(G_s^2)$ is the largest root of $\varphi_{B_1}(x) = 0$. Namely, $\varphi_{B_1}(\rho_\alpha(G_s^2)) = 0$. Let $\eta_1 = \rho_\alpha(G_s^2) \geq \eta_2 \geq \eta_3$ be the three roots of $\varphi_{B_1}(x) = 0$ and $Q = \mathrm{diag}(s, n-2s, s)$. One checks that

$$Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}} = \begin{pmatrix} \alpha s & 0 & (1-\alpha)s \\ 0 & n+(\alpha-2)s-1 & (1-\alpha)s^{\frac{1}{2}}(n-2s)^{\frac{1}{2}} \\ (1-\alpha)s & (1-\alpha)s^{\frac{1}{2}}(n-2s)^{\frac{1}{2}} & \alpha n - \alpha s + s - 1 \end{pmatrix}$$

is symmetric, and also contains

$$\begin{pmatrix} \alpha s & 0 \\ 0 & n+(\alpha-2)s-1 \end{pmatrix}$$

as its submatrix. Since $Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}}$ and B_1 have the same eigenvalues, the Cauchy interlacing theorem (cf. [12]) implies that

$$\eta_2 \leq n + (\alpha - 2)s - 1 < n - 2. \quad (4)$$

If $s = 1$, then $G_s^2 = K_1 \vee (K_{n-2} \cup K_1)$ and $\varphi_{B_1}(x) = \varphi(x)$. And so $\rho_\alpha(G_s^2) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$. In what follows, we always assume $s \geq 2$.

Note that K_{n-1} is a proper subgraph of $K_1 \vee (K_{n-2} \cup K_1)$. In terms of (4), Lemmas 2.1 and 2.2, we get

$$\rho_\alpha(K_1 \vee (K_{n-2} \cup K_1)) > \rho_\alpha(K_{n-1}) = n - 2 > \eta_2. \quad (5)$$

Write $\theta(n) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$. Note that $\varphi(\theta(n)) = 0$. A simple calculation yields that

$$\begin{aligned}\varphi_{B_1}(\theta(n)) =& \varphi_{B_1}(\theta(n)) - \varphi(\theta(n)) \\ =& (s-1)((1-\alpha)(\theta(n))^2 + (\alpha^2 n - 2\alpha - s)\theta(n) - \alpha^2 n^2 \\ &+ (2\alpha^2 - 2\alpha + 1)sn + (3\alpha^2 - \alpha + 1)n - (3\alpha^2 - 5\alpha + 2)s^2 \\ &- (4\alpha^2 - 6\alpha + 3)s - 4\alpha^2 + 5\alpha - 3). \end{aligned} \quad (6)$$

The following proof will be divided into two cases.

Case 1. $0 \leq \alpha \leq \frac{2}{3}$.

Let $h(x) = (1-\alpha)x^2 + (\alpha^2 n - 2\alpha - s)x - \alpha^2 n^2 + (2\alpha^2 - 2\alpha + 1)sn + (3\alpha^2 - \alpha + 1)n - (3\alpha^2 - 5\alpha + 2)s^2 - (4\alpha^2 - 6\alpha + 3)s - 4\alpha^2 + 5\alpha - 3$ be a real function in x. Then the symmetry axis of $h(x)$ is $x = \frac{-\alpha^2 n + 2\alpha + s}{2(1-\alpha)}$. Obviously, $h(x)$ is increasing when $x \geq \frac{-\alpha^2 n + 2\alpha + s}{2(1-\alpha)}$. By plugging the value $\theta(n)$ into x of $h(x)$, we obtain

$$\begin{aligned}h(\theta(n)) =& (1-\alpha)(\theta(n))^2 + (\alpha^2 n - 2\alpha - s)\theta(n) - \alpha^2 n^2 \\ &+ (2\alpha^2 - 2\alpha + 1)sn + (3\alpha^2 - \alpha + 1)n - (3\alpha^2 - 5\alpha + 2)s^2 \\ &- (4\alpha^2 - 6\alpha + 3)s - 4\alpha^2 + 5\alpha - 3 \end{aligned}$$

and it follows from (6) that

$$\varphi_{B_1}(\theta(n)) = (s-1)h(\theta(n)). \tag{7}$$

Note that $n-2s \geq 1$. We have $n \geq 2s+1$. Together with $s \geq 2$ and (5), we conclude

$$\frac{-\alpha^2 n + 2\alpha + s}{2(1-\alpha)} < n-2 < \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1)) = \theta(n),$$

and so

$$\begin{aligned}
h(\theta(n)) >& h(n-2) \\
=&(1-\alpha)(n-2)^2 + (\alpha^2 n - 2\alpha - s)(n-2) - \alpha^2 n^2 \\
&+ (2\alpha^2 - 2\alpha + 1)sn + (3\alpha^2 - \alpha + 1)n - (3\alpha^2 - 5\alpha + 2)s^2 \\
&- (4\alpha^2 - 6\alpha + 3)s - 4\alpha^2 + 5\alpha - 3 \\
=&(1-\alpha)n^2 + ((2\alpha^2 - 2\alpha)s + \alpha^2 + \alpha - 3)n - (3\alpha^2 - 5\alpha + 2)s^2 \\
&- (4\alpha^2 - 6\alpha + 1)s - 4\alpha^2 + 5\alpha + 1.
\end{aligned} \tag{8}$$

Let $p(x,s) = (1-\alpha)x^2 + ((2\alpha^2 - 2\alpha)s + \alpha^2 + \alpha - 3)x - (3\alpha^2 - 5\alpha + 2)s^2 - (4\alpha^2 - 6\alpha + 1)s - 4\alpha^2 + 5\alpha + 1$ be a real function in x. According to (8), we get

$$h(\theta(n)) > p(n,s). \tag{9}$$

Clearly, $p(x,s)$ is increasing when $x \geq -\frac{(2\alpha^2-2\alpha)s+\alpha^2+\alpha-3}{2(1-\alpha)}$. If $s \geq 3$, then we can verify that

$$-\frac{(2\alpha^2 - 2\alpha)s + \alpha^2 + \alpha - 3}{2(1-\alpha)} < 2s+1 \leq n,$$

and so

$$\begin{aligned}
p(n,s) \geq& p(2s+1,s) \\
=&(1-\alpha)(2s+1)^2 + ((2\alpha^2 - 2\alpha)s + \alpha^2 + \alpha - 3)(2s+1) \\
&- (3\alpha^2 - 5\alpha + 2)s^2 - (4\alpha^2 - 6\alpha + 1)s - 4\alpha^2 + 5\alpha + 1 \\
=&(s^2-3)\alpha^2 - (3s^2 - 2s - 5)\alpha + 2s^2 - 3s - 1 \\
\geq&\frac{4}{9}(s^2-3) - \frac{2}{3}(3s^2 - 2s - 5) + 2s^2 - 3s - 1 \\
=&\frac{4s^2 - 15s + 9}{9} \\
\geq&0,
\end{aligned}$$

where the last two inequalities hold from $\frac{3s^2-2s-5}{2(s^2-3)} > \frac{2}{3} \geq \alpha$ and $s \geq 3$, respectively.

If $s = 2$, then $-\frac{(2\alpha^2-2\alpha)s+\alpha^2+\alpha-3}{2(1-\alpha)} = -\frac{5\alpha^2-3\alpha-3}{2(1-\alpha)} < 6 = f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{2}{3}$. Hence, for $0 \leq \alpha \leq \frac{2}{3}$ and $n \geq f(\alpha) = 6$, we get

$$p(n,2) \geq p(6,2) = 6\alpha^2 - 17\alpha + 9 > 0.$$

Thus, we obtain $p(n,s) > 0$ for $2 \leq s \leq \frac{n-1}{2}$. Together with (7), (9) and $s \geq 2$, we have

$$\varphi_{B_1}(\theta(n)) = (s-1)h(\theta(n)) > (s-1)p(n,s) \geq 0.$$

Recall that $\rho_\alpha(G_s^2)$ is the largest root of $\varphi_{B_1}(x) = 0$. From (6), the symmetry axis of $\varphi_{B_1}(\theta(n))$ is $\theta(n) = \frac{-\alpha^2 n+2\alpha+s}{2(1-\alpha)}$. Thus, we see that $\varphi_{B_1}(\theta(n))$ is increasing in the interval $[\frac{-\alpha^2 n+2\alpha+s}{2(1-\alpha)}, +\infty)$. Note that $\frac{-\alpha^2 n+2\alpha+s}{2(1-\alpha)} < n-2$. Together with (5), we have $\frac{-\alpha^2 n+2\alpha+s}{2(1-\alpha)} < n-2 < \theta(n)$. Combining these with $\varphi_{B_1}(\theta(n)) > 0$, we deduce

$$\rho_\alpha(G_s^2) < \theta(n) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1))$$

for $2 \le s \le \frac{n-1}{2}$.

Case 2. $\frac{2}{3} < \alpha < 1$.

In terms of (3), we obtain

$$\begin{aligned}\varphi_{B_1}(n-2) =&(n-2)^3 - ((\alpha+1)n + (\alpha-1)s - 2)(n-2)^2\\ &+ (\alpha n^2 + (\alpha^2 s - \alpha - 1)n - s^2 - (2\alpha-1)s + 1)(n-2)\\ &- \alpha^2 s n^2 + (2\alpha^2 - 2\alpha + 1)s^2 n + (\alpha^2+\alpha)sn\\ &- (3\alpha^2 - 5\alpha + 2)s^3 - (\alpha^2 - \alpha + 1)s^2 - \alpha s\\ =&(3\alpha-2)(1-\alpha)s^3 + ((2\alpha^2 - 2\alpha)n - \alpha^2 + \alpha + 1)s^2\\ &+ ((1-\alpha)n^2 - (\alpha^2 - 3\alpha + 3)n - \alpha + 2)s\\ &+ (\alpha-1)n^2 - (2\alpha-3)n - 2\\ &:= \Phi(s,n).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{\partial\Phi(s,n)}{\partial s} =&3(3\alpha-2)(1-\alpha)s^2 + 2((2\alpha^2 - 2\alpha)n - \alpha^2 + \alpha + 1)s\\ &+ (1-\alpha)n^2 - (\alpha^2 - 3\alpha + 3)n - \alpha + 2.\end{aligned}$$

Note that $\frac{2}{3} < \alpha < 1$ and $n \ge f(\alpha) = \frac{4}{1-\alpha}$. By a simple calculation, we have

$$\begin{aligned}\frac{\partial\Phi(s,n)}{\partial s}\Big|_{s=2} =&(1-\alpha)n^2 + (7\alpha^2 - 5\alpha - 3)n - 40\alpha^2 + 63\alpha - 18\\ \ge&(1-\alpha)\left(\frac{4}{1-\alpha}\right)^2 + (7\alpha^2 - 5\alpha - 3)\left(\frac{4}{1-\alpha}\right) - 40\alpha^2 + 63\alpha - 18\\ =&\frac{1}{1-\alpha}(40\alpha^3 - 75\alpha^2 + 61\alpha - 14)\\ >&0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\Phi(s,n)}{\partial s}\Big|_{s=\frac{n-1}{2}} =&\frac{1}{4}((1-\alpha)(\alpha-2)n^2 + (2\alpha^2 - 6\alpha - 4)n - 5\alpha^2 + 7\alpha + 6)\\ \le&\frac{1}{4}\left((1-\alpha)(\alpha-2)\left(\frac{4}{1-\alpha}\right)^2 + (2\alpha^2 - 6\alpha - 4)\left(\frac{4}{1-\alpha}\right) - 5\alpha^2 + 7\alpha + 6\right)\\ =&\frac{1}{4(1-\alpha)}(5\alpha^3 - 4\alpha^2 - 7\alpha - 42)\\ <&0.\end{aligned}$$

This implies that $\varphi_{B_1}(n-2) = \Phi(s,n) \ge \min\left\{\Phi(2,n), \Phi\left(\frac{n-1}{2}, n\right)\right\}$ because the leading coefficient of $\Phi(s,n)$ (view as a cubic polynomial of s) is positive, and $2 \le s \le \frac{n-1}{2}$. According to $\frac{2}{3} < \alpha < 1$ and $n \ge f(\alpha) = \frac{4}{1-\alpha}$, we obtain

$$\begin{aligned}\Phi(2,n) =&(1-\alpha)n^2+(6\alpha^2-4\alpha-3)n-28\alpha^2+42\alpha-10\\ \geq&(1-\alpha)\left(\frac{4}{1-\alpha}\right)^2+(6\alpha^2-4\alpha-3)\left(\frac{4}{1-\alpha}\right)-28\alpha^2+42\alpha-10\\ =&\frac{2}{1-\alpha}(14\alpha^3-23\alpha^2+18\alpha-3)\\ >&0,\end{aligned}$$

and

$$\begin{aligned}\Phi\left(\frac{n-1}{2},n\right) =&\frac{1}{8}((\alpha^2-3\alpha+2)n^3-(5\alpha^2-19\alpha+16)n^2\\ &+(3\alpha^2-25\alpha+34)n+\alpha^2+\alpha-20)\\ \geq&\frac{1}{8}((\alpha^2-3\alpha+2)\left(\frac{4}{1-\alpha}\right)^3-(5\alpha^2-19\alpha+16)\left(\frac{4}{1-\alpha}\right)^2\\ &+(3\alpha^2-25\alpha+34)\left(\frac{4}{1-\alpha}\right)+\alpha^2+\alpha-20)\\ =&\frac{1}{8(1-\alpha)^2}(\alpha^4-13\alpha^3+11\alpha^2+45\alpha-12)\\ >&0.\end{aligned}$$

Consequently, we conclude $\varphi_{B_1}(n-2)\geq\min\left\{\Phi(2,n),\Phi\left(\frac{n-1}{2},n\right)\right\}>0$ for $2\leq s\leq\frac{n-1}{2}$. As $\eta_2<n-2<\rho_\alpha(K_1\vee(K_{n-2}\cup K_1))$ (see (5)), we have

$$\rho_\alpha(G_s^2)<n-2<\rho_\alpha(K_1\vee(K_{n-2}\cup K_1))$$

for $2\leq s\leq\frac{n-1}{2}$. This verifies Claim 1. □

In terms of (1), (2) and Claim 1, we obtain

$$\rho_\alpha(G)\leq\rho_\alpha(K_1\vee(K_{n-2}\cup K_1))$$

with equality if and only if $G=K_1\vee(K_{n-2}\cup K_1)$. Which is a contradiction to the A_α-spectral condition of Theorem 1.1. This completes the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

In this section, we prove Theorem 1.2, which provides an A_α-spectral radius condition for a graph to be t-tough.

Proof of Theorem 1.2. Suppose to the contrary that G is not a t-tough graph, there exists some nonempty subset $S\subseteq V(G)$ such that $tc(G-S)-1\geq|S|$. Let $|S|=s$ and $c(G-S)=c$, then $tc-1\geq s$. If $n\geq(t+1)c-1$, then G is a spanning subgraph of $G_1=K_{tc-1}\vee(K_{n_1}\cup K_{n_2}\cup\cdots\cup K_{n_c})$ for some positive integers $n_1\geq n_2\geq\cdots\geq n_c$ with $\sum_{i=1}^{c}n_i=n-tc+1$. In terms of Lemma 2.2, we conclude

$$\rho_\alpha(G)\leq\rho_\alpha(G_1) \tag{10}$$

with equality if and only if $G=G_1$.

Let $G_2=K_{tc-1}\vee(K_{n-(t+1)c+2}\cup(c-1)K_1)$. By virtue of Lemma 2.3, we obtain

$$\rho_\alpha(G_1)\leq\rho_\alpha(G_2), \tag{11}$$

where the equality holds if and only if $(n_1, n_2, \dots, n_c) = (n - (t+1)c + 2, 1, \dots, 1)$.

If $c = 2$, then we have $G_2 = K_{2t-1} \vee (K_{n-2t} \cup K_1)$. According to (10) and (11), we get

$$\rho_\alpha(G) \le \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1)),$$

where the equality holds if and only if $G = K_{2t-1} \vee (K_{n-2t} \cup K_1)$. In what follows, we consider $c \ge 3$.

Recall that $G_2 = K_{tc-1} \vee (K_{n-(t+1)c+2} \cup (c-1)K_1)$. According to $\alpha \in [\frac{1}{2}, \frac{3}{4})$ and Lemma 2.5, we obtain

$$\begin{aligned}\rho_\alpha(G_2) \le& \frac{2e(G_2)(1-\alpha)}{n-1} + \alpha n - 1\\ =& \frac{(1-\alpha)((n-c+1)(n-c) + 2(tc-1)(c-1))}{n-1} + \alpha n - 1\\ =& \frac{1-\alpha}{n-1}((2t+1)c^2 - (2n+2t+3)c + n^2 + n + 2) + \alpha n - 1. \end{aligned} \tag{12}$$

Let $\varphi(c) = (2t+1)c^2 - (2n+2t+3)c + n^2 + n + 2$. Note that $n \ge (t+1)c - 1$. We conclude $3 \le c \le \frac{n+1}{t+1}$. By a direct computation, it follows from $n \ge \max\{5t^2 + 10t + 1, \frac{12t(1-\alpha)-2\alpha+1}{3-4\alpha}\} \ge 5t^2 + 10t + 1 > 4t^2 + 6t + 1$ that

$$\varphi(3) - \varphi\left(\frac{n+1}{t+1}\right) = \frac{(n-3t-2)(n-4t^2-6t-1)}{(t+1)^2} > 0,$$

which implies that $\varphi(c)$ attains its maximum value at $c = 3$ when $3 \le c \le \frac{n+1}{t+1}$. Combining this with (12), $\alpha \in [\frac{1}{2}, \frac{3}{4})$ and $n \ge \max\{5t^2 + 10t + 1, \frac{12t(1-\alpha)-2\alpha+1}{3-4\alpha}\} \ge \frac{12t(1-\alpha)-2\alpha+1}{3-4\alpha}$, we get

$$\begin{aligned}\rho_\alpha(G_2) \le& \frac{(1-\alpha)\varphi(3)}{n-1} + \alpha n - 1\\ =& \frac{(1-\alpha)(n^2 - 5n + 12t + 2)}{n-1} + \alpha n - 1\\ =& n - 2 + \frac{-(3-4\alpha)n + 12t(1-\alpha) - 2\alpha + 1}{n-1}\\ \le& n - 2 + \frac{-(3-4\alpha)\cdot\frac{12t(1-\alpha)-2\alpha+1}{3-4\alpha} + 12t(1-\alpha) - 2\alpha + 1}{n-1}\\ =& n-2. \end{aligned} \tag{13}$$

Since K_{n-1} is a proper subgraph of $K_{2t-1} \vee (K_{n-2t} \cup K_1)$, we conclude

$$\rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1)) > \rho_\alpha(K_{n-1}) = n - 2 \tag{14}$$

by Lemmas 2.1 and 2.2. It follows from (10), (11), (13) and (14) that

$$\rho_\alpha(G) \le \rho_\alpha(G_1) \le \rho_\alpha(G_2) \le n - 2 < \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1)).$$

In conclusion, we have

$$\rho_\alpha(G) \le \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1))$$

with equality if and only if $G = K_{2t-1} \vee (K_{n-2t} \cup K_1)$. Which is a contradiction to the A_α-spectral radius condition of Theorem 1.2.

Let $n \le (t+1)c - 2$, then G is a spanning subgraph of $K_{n-c} \vee cK_1$ with $c \ge \lceil \frac{n+2}{t+1} \rceil$. Let $G_3 = K_{n-\lceil \frac{n+2}{t+1} \rceil} \vee \lceil \frac{n+2}{t+1} \rceil K_1$. By virtue of Lemma 2.2, we conclude

$$\rho_\alpha(G) \le \rho_\alpha(K_{n-c} \vee cK_1) \le \rho_\alpha(G_3), \tag{15}$$

with equalities if and only if $G = G_3$. Note that $2e(G_3) = (n - \lceil \frac{n+2}{t+1} \rceil)(n - \lceil \frac{n+2}{t+1} \rceil - 1) + 2\lceil \frac{n+2}{t+1} \rceil(n - \lceil \frac{n+2}{t+1} \rceil) = (n - \lceil \frac{n+2}{t+1} \rceil)(n + \lceil \frac{n+2}{t+1} \rceil - 1) < (n - \frac{n+2}{t+1})(n + \frac{n+2}{t+1})$. In view of Lemma 2.5, we possess

$$\begin{aligned}\rho_\alpha(G_3) \leq & \frac{2e(G_3)(1-\alpha)}{n-1} + \alpha n - 1 \\ < & \frac{(1-\alpha)(n - \frac{n+2}{t+1})(n + \frac{n+2}{t+1})}{n-1} + \alpha n - 1 \\ = & n - 2 + \frac{\psi(n)}{(t+1)^2(n-1)},\end{aligned} \tag{16}$$

where $\psi(n) = -(1-\alpha)n^2 + ((2-\alpha)t^2 + (4-2\alpha)t + 3\alpha - 2)n - t^2 - 2t - 5 + 4\alpha$. Notice that

$$\frac{(2-\alpha)t^2 + (4-2\alpha)t + 3\alpha - 2}{2(1-\alpha)} < 5t^2 + 10t + 1 \leq n$$

by $\alpha \in [\frac{1}{2}, \frac{3}{4})$, $t \geq 1$ and $n \geq \max\{5t^2 + 10t + 1, \frac{12t(1-\alpha) - 2\alpha + 1}{3 - 4\alpha}\}$. For $t \geq 2$, we have

$$\begin{aligned}\psi(n) \leq & \psi(5t^2 + 10t + 1) \\ = & -(1-\alpha)(5t^2 + 10t + 1)^2 + ((2-\alpha)t^2 + (4-2\alpha)t + 3\alpha - 2)(5t^2 + 10t + 1) \\ & - t^2 - 2t - 5 + 4\alpha \\ = & (5t^2 + 10t + 1)((4\alpha - 3)t^2 + (8\alpha - 6)t + 4\alpha - 3) - t^2 - 2t - 5 + 4\alpha \\ \leq & (5t^2 + 10t + 1)(4(4\alpha - 3) + 2(8\alpha - 6) + 4\alpha - 3) \\ & - t^2 - 2t - 5 + 4\alpha \qquad \left(\text{since } t \geq 2 \text{ and } \alpha < \frac{3}{4}\right) \\ = & 9(4\alpha - 3)(5t^2 + 10t + 1) - t^2 - 2t - 5 + 4\alpha \\ < & 0 \qquad \left(\text{since } t \geq 2 \text{ and } \alpha < \frac{3}{4}\right).\end{aligned}$$

We check that $\psi(n) < 0$ also holds for $t = 1$. Combining these with (14), (15) and (16), we conclude

$$\rho_\alpha(G) \leq \rho_\alpha(G_3) < n - 2 < \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1)),$$

which contradicts $\rho_\alpha(G) \geq \rho_\alpha(K_{2t-1} \vee (K_{n-2t} \cup K_1))$. This completes the proof of Theorem 1.2. □

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

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