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Directed graphs and int-graphic topology

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Abstract. In this work, we construct a topology $\mathcal{T}_G^{int}$ on the vertex set of a directed graph $G = (V, E)$. We investigate the fact that $(V, \mathcal{T}_G^{int})$ is an Alexandroff space to study some topological properties of the graph due to the existence of minimal basis for $\mathcal{T}_G^{int}$. Some continuity properties of functions are proved. Finally, some examples are given with connected and disconnected int-graphic topologies.

1. Introduction

Since the earlier work of L. Euler in resolving the problem of the Königsberg seven bridges [5], graph theory has attracted attention and becomes a fundamental branch of discrete mathematic structures.
This mathematical tool is simple to understand and to use for representing a lot of mathematical combinations. Nowadays, graph theory becomes a fundamental mathematical tool for a large of domain as chemistry, marketing and computers network. When we related graph theory and topology, we can use them to solve economic and the traffick flow problems [2, 9, 11]. Also, they are used in medical application and blood circulation [7, 10, 12, 13].
In a given topology, if any intersection of open sets is also an open set then the topology is called an Alexandroff topology [3, 14]. As topology is very interesting because we have a minimal bases and the characteristic properties can be studied by using minimal open sets.
In particlar, Jafarian Amiri *et al.* [8] introduced an Alexandroff topology on the set of vertices of a simple undirected graph called graphic topology. After that many topologies are introduced on undirected graphs but speaking about graphic topologies, in [15] the authors investigated the graphic topology and solved partially an open problem mentioned in [8] (Problem 2 page 658).
As an attempt to answer the first question (Problem 2 page 656 in [8]) and bypass its its problem, Zoman an Dammak in 2022, define a new topology in the paper [16].
Graphic topology was also be defined on fuzzy graph by Alzubaidi and Dammak in [4].
When the graph is directed, Abdu and Kilicman defined in [1] two topologies on the edges set. In this work, we investigate directed graph and graphic topology: we introduce the int-graphic topology on the vertices set.
This paper is composed of four sections in addition to the introduction and conclusion. Section 2 is devoted to some preliminaries in directed graph theory and topology. We give a set of subset of the vertices set

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V of a directed graph $G = (V, E)$ which is will be the subbases of our graphic topology. In section 3, we prove a lot of typical results as proving that the int-graphic topology is an Alexandroff topology, we prove some characterizations of minimal open sets and of the int-graphic topology results and we give some examples of open and closed sets. In Section 4, we study functions between digraphs and their relation with continuous and homeomorphism maps. Section 5 is devoted to int-graphic topology and connectedness.

2. Preliminaries

A directed graph, for short digraph, is a pair $G = (V, E)$ where V is a nonempty set called the vertices set and a set of ordered pairs E, subset of $V \times V$ named edges set. For each $e = ab \in E$, b is called the tail and b the head of the edge. Also, we say that b dominated a and a is dominated by b. To precise the direction, we say e is an edge from a to b and we write $a \to b$.
If the directed graph $G = (V, E)$ does not have loops ($(a, a) \notin E$, for all $a \in V$) and and no multiple edges in the same direction, it is called simple. We can also define an oriented graph and a tournament as particular types of directed graphs.

Definition 2.1. *Let $G = (V, E)$ be a directed graph. G is called an oriented graph if for all $a, b \in V$, at most one of ab and ba is in E. The directed graph G is called tournament if $\forall a, b \in V$, $ab \in E$ if and only if $ba \notin E$.*

Before giving the definition of the complement graph, let us recall the definition of complete directed simple graph.

Definition 2.2. *A directed graph $G = (V, E)$ is called complete if it is simple and if for any distinct vertices a and b there exists a unique edge $a \to b$ and a unique edge $b \to a$.*

Definition 2.3. *Suppose that $G = (V, E)$ is a simple directed graph. The simple directed graph $G^c = (V, E^c)$, where $E^c = \{ab;\ ab \notin E\}$, is called the complement of G.*

Definition 2.4. *In a directed graph $G = (V, E)$, a path from a vertex a_0 to a vertex a_n is a sequence of the form $P : a_0, a_1, \ldots, a_n$, where $a_{i-1}a_i \in E$ for $i = 1, \cdots, n$.*

Definition 2.5. *Let $G = (V, E)$ be a directed graph and let $a, b \in V$. We say that a and b are connected if there is a path from a to b and a path from b to a.*
We say that G is strongly connected if any two distinct vertices are connected.

Next, we introduce the int-graphic topology. For this, let $a \in V$ be a vertex of a simple directed graph $G = (V, E)$. Consider the set of all int-neighbors of a,

$$\mathcal{M}_a = \{b \in V;\ \ ba \in E\}$$

and the set of the out-neighbors of a,

$$\mathcal{N}_a = \{b \in V;\ \ ab \in E\}.$$

For a vertex in directed graph we define two degrees: the out-degree and the int-degree.

Definition 2.6. *Let $G = (V, E)$ be a directed graph. For $a \in V$, we define the out-degree of a as*

$$d^+(a) = card(\mathcal{N}_a),$$

the cardinal of the set of all vertices that dominated a and the int-degree of a by

$$d^-(a) = card(\mathcal{M}_a),$$

the cardinal of the set of all vertices that are dominated by a.

Now, the minimum out-degree and the minimum int-degree of a digraph $G = (V, E)$ are given by

$$\delta^+(G) = \min\{d^+(a),\ a \in V\},$$

and

$$\delta^-(G) = \min\{d^-(a),\ a \in V\}.$$

Also, the maximum out-degree and maximum int-degree of G are defined as

$$\Delta^+(G) = \max\{d^+(a),\ a \in V\},$$

and

$$\Delta^-(G) = \max\{d^-(a),\ a \in V\}.$$

We are going to give the definition of a topology and some related notations and properties.

Definition 2.7. *Let V be a non empty set. A topology $\mathcal{T}$ for V is a set of subsets of V satisfying*

(1) $\emptyset,\ V \in \mathcal{T}$.
(2) $\forall\ U,\ V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
(3) $\forall\ \{U_i\}_{i \in I}$ a family of elements in $\mathcal{T}$, the union $\cup_{i \in I} U_i \in \mathcal{T}$.

An element of $\mathcal{T}$ is called an open set of V and we say that $(V, \mathcal{T})$ is a topological space.

Definition 2.8. *Suppose that $(V, \mathcal{T})$ is a topological space.*

(1) A subset A of V is called closed set if its complement $A^c = V \setminus A$ is an open set of V.
(2) Let A be a subset of V. The closure of A in V, $\overline{A}$, is the smallest closed set of V containing A.

Let V be a non empty set. Suppose that $\mathcal{S} \subset 2^V$ a family of subsets of V. Assume that $\cup_{M \in \mathcal{S}} M = V$, and define $\mathcal{B}$ as the set of all possible intersections of elements of $\mathcal{S}$. Now, let $\mathcal{T}$ be the set of all possible unions of elements of $\mathcal{B}$, Then $(V, \mathcal{T})$ is a topological space [6].
The set $\mathcal{B}$ is called a basis for the topology $\mathcal{T}$ and $\mathcal{S}$ a subbases for $\mathcal{T}$. In order to introduce our topology on the vertices set of a digraph $G = (V, E)$, consider the set

$$\mathcal{S}_G^{int} = \{\mathcal{M}_a, a \in V\}. \tag{1}$$

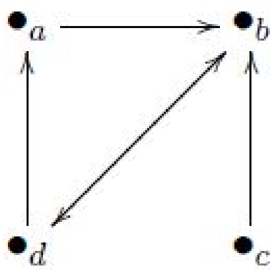
Our first result is the following.

Theorem 2.9. *Suppose that $G = (V, E)$ is a simple directed graph such that for all $a \in V,\ \mathcal{N}_a \neq \emptyset$ (i.e without isolated point). Then, $\mathcal{S}_G^{int}$ is a subbases of a topology for the set V.*

Proof. Since G has no isolated vertices (i.e., $N_a \neq \emptyset$ for all $a \in V$), for every $b \in V$, there exists some $a \in V$ such that $b \in M_a$, ensuring $V = \bigcup_{a \in V} M_a$. and the result follows. □

The topology for V introduced above is called int-graphic topology of the graph G. It is denoted by $\mathcal{T}_G^{int}$.

Example 2.10. *Consider the directed simple graph G:*



We have

x	a	b	c	d
$\mathcal{M}_x$	$\{d\}$	$\{a,c,d\}$	$\emptyset$	$\{b\}$

$\mathcal{S}_G^{int} = \Big\{\emptyset, \{b\}, \{d\}, \{a,c,d\}\Big\}$.
Then, the basis of the topology $\mathcal{T}_G^{int}$ *is*
$\mathcal{B} = \Big\{\{b\}, \{d\}, \{a,c,d\}\Big\}$.
We get $\mathcal{T}_G^{int} = \Big\{\emptyset, \{b\}, \{d\}, \{b,d\}, \{a,c,d\}, \{a,b,c,d\}\Big\}$.

For the int-graphic topology, we introduce the following definition for locally finite directed graph that we will use is to prove that this topology is an Alexandroff topology.

Definition 2.11. *We call a directed graph $G = (V, E)$ locally finite if $\forall\ a \in V$ the set of its dominating vertices $\mathcal{N}_a$ is a finite set.*

In the sequel, We will consider $G = (V, E)$ a digraph by meaning a simple locally finite directed graph without isolated vertex ($\mathcal{N}_a \neq \emptyset$ and $\mathcal{N}_a$ is finite).

3. Typical study of the int-graphic topology

One of fundamental property of topological space is being an Alexandroff space: this means any intersection of open sets is an open set. We prove the following result.

Theorem 3.1. *If $G = (V, E)$ is a digraph, then its int-graphic topology is an Alexandroff topology.*

Proof. Since the topology $\mathcal{T}_G^{int}$ has a subbases $\mathcal{S}_G^{int}$, it is sufficient to prove that any intersection of elements of $\mathcal{S}_G^{int}$ is an open set. Let A a subset of V and consider $\cap_{a\in A}\mathcal{M}_a$. We have two cases.
• *Case 1.* $\cap_{a\in A}\mathcal{M}_a = \emptyset$, then $\cap_{a\in A}\mathcal{M}_a$ is an open set by the definition of a topology.
• *Case 2.* $\cap_{a\in A}\mathcal{M}_a \neq \emptyset$.
Let $b \in \cap_{a\in A}\mathcal{M}_a$, then for all $a \in A$, we have $b \in \mathcal{M}_a$ and so $a \in \mathcal{N}_b$.
Since G is locally finite (Definition 2.9), N_b is finite, and thus $A \subset N_b$ implies A is finite. Hence, $\cap_{a\in A}M_a$ is an open set.

□

As consequence of the last theorem, we can deduce that the int-graphic topology $\mathcal{T}_G^{int}$ has a minimal basis $\mathcal{U}_G = \{U_a;\ \ a \in V\}$, where U_a is the intersection of all open sets containing a, it is the smallest open set containing the vertex a.
We have the following important characterisation of the minimal open sets.

Theorem 3.2. *Let $G = (V, E)$ be a digraph and let $a \in V$ a vertex of G. The minimal open set containing a, U_a, satisfies*

$$U_a = \bigcap_{b\in\mathcal{N}_a} \mathcal{M}_b$$

and U_a is a finite set.

Proof. (i) The intersection

$$\bigcap_{b\in\mathcal{N}_a} \mathcal{M}_b$$

is an open set since $\mathcal{N}_a$ is finite and $\mathcal{M}_b$ is an open set ($\mathcal{M}_b \in \mathcal{S}_G^{int}$).
$b \in \mathcal{N}_a$ is equivalent to $a \in \mathcal{M}_b$ and so

$$a \in \bigcap_{b\in\mathcal{N}_a} \mathcal{M}_b$$

and then

$$U_a \subset \bigcap_{b \in \mathcal{N}_a} \mathcal{M}_b.$$

(ii) Conversely, since $\mathcal{T}_G^{int}$ is a topology constructed from the subbases $\mathcal{S}_G^{int}$, there exists $A \subset V$ such that

$$U_a = \bigcap_{b \in A} \mathcal{M}_b$$

When $b \in A$, $a \in \mathcal{M}_b$ and then $b \in \mathcal{N}_a$. We get $A \subset \mathcal{N}_a$ and hence

$$\bigcap_{b \in \mathcal{N}_a} \mathcal{M}_b \subset \bigcap_{b \in A} \mathcal{M}_b$$

Therefore

$$\bigcap_{b \in \mathcal{N}_a} \mathcal{M}_b \subset U_a.$$

□

We have also the following characterisation of U_a in particular cases.

Corollary 3.3. *Let $G = (V, E)$ be a digraph and let $a, b \in V$ two distinct vertices.*

(Case 1.) *If $\mathcal{N}_a = \{b\}$, then $U_a = \mathcal{M}_b$.*
(Case 2.) *If $b \in \mathcal{N}_a$, then $U_a \subset \mathcal{M}_b$.*

Proof.

(Case 1.) $U_a = \bigcap_{c \in \mathcal{N}_a} \mathcal{M}_c = \mathcal{M}_b$ since $\mathcal{N}_a = \{b\}$.
(Case 2.) From the fact that

$$U_a = \bigcap_{a \in \mathcal{N}_c} \mathcal{M}_c$$

we have $U_a \subset \mathcal{M}_c$, for all $c \in \mathcal{N}_a$.
If $b \in \mathcal{N}_a$, we get $U_a \subset \mathcal{M}_b$.

□

Corollary 3.4. *Let a, b be two vertices of a digraph $G = (V, E)$. Then, $b \in U_a$ if, and only if, $\mathcal{N}_a \subset \mathcal{N}_b$. That is,*

$$U_a = \{b \in V;\ \ \mathcal{N}_a \subset \mathcal{N}_b\}.$$

Proof. Using Theorem 3.2, we get

$$b \in U_a \Leftrightarrow b \in \mathcal{M}_c, \forall c \in \mathcal{N}_a$$

so,

$$b \in U_a \Leftrightarrow \forall c \in \mathcal{N}_a, c \in \mathcal{N}_b$$

which equivalent to

$$\mathcal{N}_a \subset \mathcal{N}_b.$$

□

Corollary 3.5. *Consider $G = (V, E)$ a digraph and $a \in V$. We have*

(i) $U_a \cap \mathcal{N}_a = \emptyset$.
(ii) *If $U_b \subset \mathcal{N}_a$, then $U_a \cap U_b = \emptyset$.*

Proof.

(i) By contradiction. Suppose that there exists $b \in U_a \cap \mathcal{N}_a$. We obtain $\mathcal{N}_a \subset \mathcal{N}_b$ since $b \in U_a$. But also $b \in \mathcal{N}_a$ and so $b \in \mathcal{N}_b$ but this is impossible in a simple graph.

(ii) If $U_b \subset \mathcal{N}_a$, then $U_b \cap U_a \subset \mathcal{N}_a \cap U_a$. From (i), we obtain $U_a \cap U_b = \emptyset$.

□

Corollary 3.6. *Consider $G = (V, E)$ a digraph and $a \in V$. We have $\overline{\{a\}} \subset \overline{U_a} \subset \overline{\mathcal{N}_a^c}$ and $\overline{\mathcal{N}_a} \subset U_a^c$. When G is a tournament, $\mathcal{M}_a^c = \mathcal{N}_a^c \cup \{a\}$.*

Theorem 3.7. *Let G be a directed graph and $a \in V$. We have $b \in \overline{\{a\}}$ if and only if $\mathcal{N}_b \subset \mathcal{N}_a$. That is,*

$$\overline{\{a\}} = \{b \in V;\ \ \mathcal{N}_b \subset \mathcal{N}_a\}.$$

Proof. We know that $b \in \overline{\{a\}}$ if and only if $O \cap \{a\} \neq \emptyset$ for all open set O containing b. But, this is equivalent to $U_b \cap \{a\} \neq \emptyset$. So, $b \in \overline{\{a\}}$ if and only if $a \in U_b$ and we conclude by Corollary 3.4.

□

Recall that a topological space V is called compact if for any open cover of V, we can restrict a finite cover for V.
For the int-graphic topology of a directed graph G we prove the following result.

Theorem 3.8. *Let $G = (V, E)$ be a digraph. Then, $(V, \mathcal{T}_G^{int})$ is a compact topological space if and only if V is a finite set.*

Proof. (i) Suppose that $\mathcal{T}_G^{int}$ is a compact topology. The minimal basis $\{U_a, a \in V\}$ is an open cover of V, so there exists a finite subcover $\{U_a, a \in V'\}$, where V' a finite subset of V. Since we deal with minimal basis, we get $V = V'$. Therefore, V is finite .
(ii) If V is a finite set , from any open cover we have a finite subcover and then $(V, \mathcal{T}_G^{int})$ is a compact topological space.

□

Proposition 3.9. *Let $G = (V, E)$ be a digraph. Then, $A = \{x \in V,\ \ d^+(x) = \Delta^+(G)\}$ is an open set for the int-graphic topology on G.*

Proof. Let $a \in A$, we are going to prove that the minimal open set containing a, U_a, is a subset of A.
Let $x \in U_a$, then from Corollary 3.4, $\mathcal{N}_a \subset \mathcal{N}_x$. We get

$$\Delta^+(G) = d^+(a) \leq d^+(x) \leq \Delta^+(G).$$

So, $x \in A$ and hence $a \in U_a \subset A$. Therefore A is an open set.

□

Proposition 3.10. *Let $G = (V, E)$ be a digraph. Then $B = \{x \in V,\ \ d^+(x) = \delta^+(G)\}$ is a closed set for the int-graphic topology of the graph G.*

Proof. Our idea is to prove that $\overline{B} \subset B$. For this, let $x \in \overline{B}$. For any open set O containing x, we have $O \cap B \neq \emptyset$. In particular $U_x \cap B \neq \emptyset$.
Let $b \in U_x \cap B$. Since $b \in U_x$, we have $\mathcal{N}_x \subset \mathcal{N}_b$.
As $b \in B$,we get $d^+(b) = \delta^+(G)$ and so,

$$\delta^+(G) \leq d^+(x) \leq d^+(b) = \delta^+(G).$$

Then $d^+(x) = \delta^+(G)$ and so $x \in B$. We have proved that for all $x \in \overline{B}$, $x \in B$. Therefore B is a closet set.

□

Proposition 3.11. *Suppose that $G = (V, E)$ is a finite digraph. Then we have the following result.*

(1) The family $T^c_G = \{A;\ A^c \in \mathcal{T}^{int}_G\}$ is a topology for V.

(2) When G is an oriented graph and $\mathcal{T}^{int}_G = T^c_G$, the topology $\mathcal{T}^{int}_G$ is the discrete topology for V.

Proof. (1) *First*, $\emptyset = V^c$ and $V = \emptyset^c$ and $V, \emptyset \in \mathcal{T}^{int}_G$ and so $\emptyset, V \in T^c_G$.
Second, let U_1 and U_2 in T^c_G, then $(U_1 \cap U_2)^c = U_1^c \cup U_2^c \in \mathcal{T}^{int}_G$. Therefore

$$U_1 \cap U_2 \in T^c_G.$$

Finally, for any family $\{U_i\}$ in T^c_G, we have $(\cup_i U_i)^c = \cap_i (U_i^c)$. Since $\mathcal{T}^{int}_G$ is an Alexandroff topology, $\cup_i U_i \in T^c_G$.

(ii) Let $a \in V$. If the digraph $G = (V, E)$ is an oriented graph, then $\left(\mathcal{N}_a \cup \{x\}\right)^c = \mathcal{M}_a$. That is, $\left(\mathcal{N}_a \cup \{a\}\right)^c \in \mathcal{T}^{int}_G$. Since $\mathcal{T}^{int}_G = \mathcal{T}^c_G$, we have $\mathcal{N}_a \cup \{a\} \in \mathcal{T}^{int}_G$. Therefore, $U_a \subset \mathcal{N}_a \cup \{a\}$. From Corollary 3.5, we have get $U_a \subset \{a\}$ and so $U_a = \{a\}$ and the result follows.

□

4. On Functions Between Digraphs

Two topological spaces are said homeomorphic if there exists a continuous bijective function h between them and h^{-1} is also continuous. The function h is called homeomorphism.
For directed graphs, we can define an isomorphism between them as follows.

Definition 4.1. *Let $G = (V, E)$ and $G' = (V', E')$ be two digraphs. We say that the two graphs are isomorphic if there exists a bijection $h : V \to V'$ such that*

$$(a, b) \in E \ \text{ if and only if } \ (h(a), h(b)) \in E'.$$

We have the following result relating the int-graphic topologies and being isomorphic.

Theorem 4.2. *Let $G = (V, E)$ and $G' = (V', E')$ be two digraphs. Suppose that these diagraphs are isomorphic and $h : V \to V'$ is an isomorphism. Then,*

$$h : (V, \mathcal{T}^{int}_G) \to (V', \mathcal{T}^{int}_{G'})$$

is an homeomorphism.

Proof. Let A be an open set of V', we have to prove that$h^{-1}(A)$ is an open set of V. Since the topology $\mathcal{T}^{int}_{G'}$ has a subbases $\mathcal{S}^{int}_{G'}$, it is sufficient to prove the result for $A \in \mathcal{S}^{int}_{G'}$. So, suppose that $A = \mathcal{M}_b$, where $b \in V'$ and $b = h(a)$. We get

$$\begin{aligned}
h^{-1}(A) &= \{x \in V,\ \ h(x) \in A\} \\
h^{-1}(A) &= \{x \in V,\ \ h(x) \in \mathcal{M}_b\} \\
&= \{x \in V,\ \ (h(x), b) \in E'\} \\
&= \{x \in V,\ \ (h(x), h(a)) \in E'\} \\
&= \{x \in V,\ \ (x, a) \in E\} \\
&= \mathcal{M}_a \in \mathcal{T}^{int}_G.
\end{aligned}$$

So, the function h is continuous.
$h : V \to V'$ is bijective and we have $h^{-1} : V' \to V$. Let now $A = \mathcal{M}_a \in \mathcal{T}^{int}_G$ and set $b = h(a) \in V'$. We obtain

$$\begin{aligned}
\left(h^{-1}\right)^{-1}(A) &= \{y \in V',\ \ h^{-1}(y) \in \mathcal{M}_a\} \\
&= \{y \in V',\ \ (h^{-1}(y), a) \in E\} \\
&= \{y \in V',\ \ (h^{-1}(y), h^{-1}(b)) \in E\} \\
&= \{y \in V',\ \ (y, b) \in E'\} \\
&= \mathcal{M}_b.
\end{aligned}$$

Then, the function h is an homeomorphism.

□

The converse of the above result is not true in general. As an example, the two following digraphs have the same int-graphic topology (the discrete topology) but they are not isomorphic.

Next, we are going to prove two results relating continuity with the subbases.

Theorem 4.3. *Let $G = (V, E)$ and $G' = (V', E')$ be two digraphs and let $h : V \to V'$ be a given function. The function h is continuous for the graphic topologies if and only if*

$$\mathcal{N}_a \subset \mathcal{N}_b \Longrightarrow \mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}, \ \ \forall\, a, b \in V.$$

Proof. For the beginning, suppose that h is continuous and $\mathcal{N}_a \subset \mathcal{N}_b$, this means $b \in U_a$ from the Corollary 3.4.
We have also

$$a \in U_a \subset h^{-1}\big(U_{h(a)}\big)$$

so,

$$b \in h^{-1}\big(U_{h(a)}\big).$$

Therefore $h(b) \in U_{h(a)}$. By using again the Corollary 3.4, we find that $\mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}$.

Next, we suppose that $\forall a,\ b \in V$,

$$\mathcal{N}_a \subset \mathcal{N}_b \ \text{ gives } \ \mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}.$$

Let $O \in \mathcal{T}^{int}_{G'}$, we will prove that $h^{-1}(O)$ is an open set of V. For this, let $a \in h^{-1}(O)$ and $x \in U_a$.
We have $\mathcal{N}_a \subset \mathcal{N}_x$ and so $\mathcal{N}_{h(a)} \subset \mathcal{N}_{h(x)}$. Then, $h(x) \in U_{h(a)}$ and we know that $U_{h(a)} \subset O$. Therefore, $x \in h^{-1}(O)$ and hence $\forall a \in h^{-1}(O), U_a \subset h^{-1}(O)$. The theorem follows.

□

Theorem 4.4. *Let $G = (V, E)$ and $G' = (V', E')$ be two digraphs and let $h : V \to V'$ be a given function. The function h is an homeomorphism for the graphic topologies if and only if*

$$\mathcal{N}_a \subset \mathcal{N}_b \Longleftrightarrow \mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}, \ \ \forall\, a, b \in V. \quad (2)$$

Proof. Suppose that h is an homeomorphism and let $a, b \in V$ such that $\mathcal{N}_a \subset \mathcal{N}_b$, from the Theorem 4.3 we get

$$\mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}.$$

Also, if $\mathcal{N}_{h(a)} \subset \mathcal{N}_{h(b)}$, we can use the Theorem 4.3 for the function h^{-1}, we get

$$\mathcal{N}_a \subset \mathcal{N}_b.$$

Conversely, suppose that equivalence (2) is satisfied. By Theorem 4.3, one implication gives h is continuous and the other implies that h^{-1} is continuous.

□

5. Int-graphic topology and connectedness

In the section 2, we have recalled the definition of connected (in fact strongly connected) directed graph. For a topological space, we give the following definition.

Definition 5.1. *A topological space $(V, \mathcal{T})$ is called connected if V is not be the union of two disjoint proper open sets, that is if A and B are two open sets such that*

$$A \cap B = \emptyset \text{ and } A \cup B = V$$

then, $A = \emptyset$ or $B = \emptyset$.

When the graph is undirected, if the graph is not connected, the graphic topology on V is not connected also, see [8]. The proof is based in the fact that each connected component of the graph is an open set. If we consider a directed graph, the result is not true. First, we give the following definition about connected components.

Definition 5.2. *Let $G = (V, E)$ be a digraph. The connected components of G are the elements of the family $A_1, A_2, \cdots$ of subsets of V satisfying*

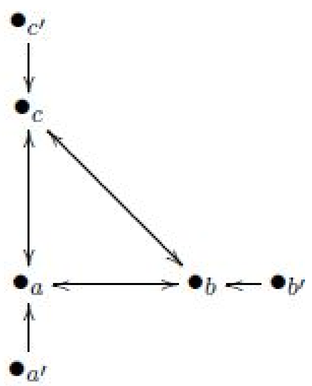
1. *$V = \cup_i A_i$.*
2. *$A_i \cap A_j = \emptyset, \forall\, i \neq j$.*
3. *If $a, b \in A_i$, $i = 1, 2, \cdots$, there exist at least two paths joining a and b, one path from a to b and another path from b to a.*
4. *If $a \in A_i$, $b \in A_j$ and $i \neq j$, then there is no pair of paths: one from a to b and one from b to a.*

Then, each subset V_i is called connected component of the digraph G.

It is clear that any strongly connected digraph has one connected component.
A finite digraph has a finite connected components.

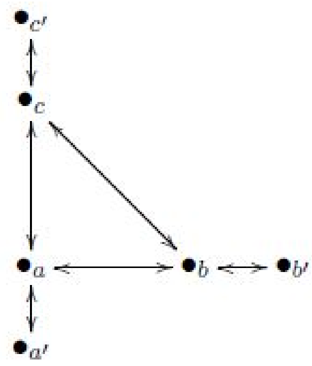
In the following example, the directed graph is not strongly connected but the int-graphic topology is connected.

Example 5.3.

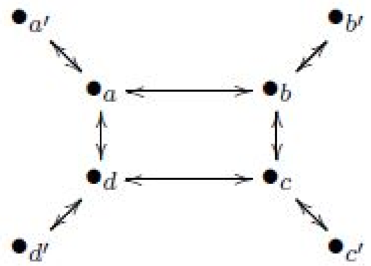


x	$\mathcal{M}_x$	$\mathcal{N}_x$	U_x
a	$\{a', b, c\}$	$\{b, c\}$	$\{a\}$
a'	$\emptyset$	$\{a\}$	$\{a', b, c\}$
b	$\{a, b', c\}$	$\{a, c\}$	$\{b\}$
b'	$\emptyset$	$\{b\}$	$\{b', a, c\}$
c	$\{a, b, c'\}$	$\{a, b\}$	$\{c\}$
c'	$\emptyset$	$\{c\}$	$\{c', a, b\}$

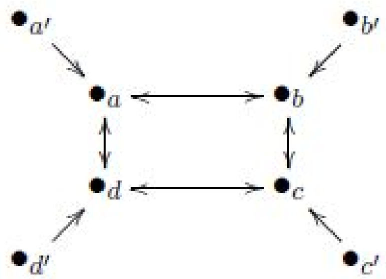
Example 5.4. *In this example, the digraph is strongly connected and the int-graphic topology is also connected.*



Example 5.5. *Here, the digraph is strongly connected but the int-graphic topology is not connected.*



Example 5.6. *This is an example of non strongly connected digraph with disconnected int-graphic topology.*



We see that, the four possibilities exist, not like graphic topology on directed graphs where we have: if the graph is not connected then the graphic topology on V is disconnected [**?**]. Next, we give some elementary results about connectedness of the int-graphic topology.

Theorem 5.7. *Suppose that $G = (V, E)$ is a bipartite digraph, then $\mathcal{T}_G^{int}$ is a disconnected topology.*

Proof. Since $G = (V, E)$ is a bipartite digraph, we can write $V = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$.
In addition, edges only exist between A_1 and A_2, so $M_a \subset A_2$ for $a \in A_1$ and $M_b \subset A_1$ for $b \in A_2$. This ensures $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = V$, as every vertex is either in A_1 or A_2.

□

Theorem 5.8. *Suppose that $G = (V, E)$ is a strongly connected bipartite digraph, then the topological space $(V, \mathcal{T}_G^{int})$ is disconnected.*

Proof. The same proof as the Theorem 5.7.

Theorem 5.9. *Let $G = (V, E)$ be a directed graph with $V = \{x_1, \cdots, x_n\}$ and $E = \{(x_i, x_{i+1}), i = 1 \ldots, n\} \cup \{(x_n, x_1)\}$, Gis a one sense directed cycle. Then $(V, \mathcal{T}_G^{int})$ is disconnected topological space.*

Proof. Since $\mathcal{M}_{x_i} = \{x_{i-1}\}$, for $i = 2, \ldots, n$ and $\mathcal{M}_{x_1} = \{x_n\}$, the $\mathcal{T}_G^{int}$ is the discrete topology. □

Conclusions

In this research , we define the int-graphic topology for the vertices set of a directed graph $G = (V, E)$ by using the interior neighbors. We use a subbases formed by the interior neighborhoods by supposing that the each vertices has at least an outer neighbor. We prove that this topology $\mathcal{T}_G^{int}$ is an Alexandroff topology and we give some characterizations of the minimal bases' elements. Functions between topological graphic spaces are investigated, Some examples are given to study the connectivity of the graph and of the int-graphic topology and the open question sets: are there some necessary and sufficient conditions for the connectivity of $(V, \mathcal{T}_G^{int})$?

Availability of data or material

all data used in this work is included in the paper.

Competing interests

The author mention that they have no competing interests.

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