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Zero epi compact maps with an inclusion property

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Abstract. We present the notion of 0–epi for compact maps with a selection property. In particular we present coincidence, normalization and homotopy properties and a Leray–Schauder type alternative for these maps.

1. Introduction

In this paper we consider a general class of maps, which include the *DKT* [3], *HLPY* [9], Wu [15], Scalzo [14] and *KLU* [8] maps, and we present the notion of a 0–epi map. Note 0–epi maps were introduced by Furi, Martelli and Vignoli [4] and extended in the literature by other authors (see for example [5, 6, 11] and the references therein). The main goal of the paper is to present coincidence, normalization, Leray–Schauder type alternatives and homotopy properties for this class of maps. The paper will consist of two main sections. Section 2 considers Kakutani type and acyclic type maps (to be defined later) and most of the properties for Kakutani type maps carry over to acyclic type maps with one notable exception, namely the homotopy property. We overcome this in Section 3 by considering a more general class of maps, which include Kakutani and acyclic type maps, and we see that that the properties of Kakutani type maps carry over to admissible type maps (to be defined later).

Next we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

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Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [5]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

Next we consider a general class of maps, namely the PK maps of Park (which include Kak and Ad maps). Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class C of single valued continuous functions;

(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov\,(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that
(i). π_α and $i : K \to V$ are α–close;
(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES$(compact).

The following fixed point result can be found in [12].

Theorem 1.1. *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $T \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in T(x)$.*

Remark 1.2. *(i). Other variations of Theorem 1.1 can be found in [13].*
(ii). Every convex subset of a Hausdorff locally convex linear topological space is AES(compact) (see [2]).

Next we describe the maps due to Wu [15]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1) for Wu maps.

Theorem 1.3. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists a upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.4. *Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X, Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [10] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [9] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{\, int\, S^{-1}(w) : \; w \in W\}$; here $S^{-1}(w) = \{z \in Z : \; w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [9].

Theorem 1.5. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X, Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup \{\, int\, S^{-1}(w) : w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\, S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.6. *These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [3] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.*

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [8] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\, F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X, Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [7].

Remark 1.7. *These maps contain as a special case the Scalzo maps [14] in the literature (see [8 pg12]).*

Next we recall a selection theorem [8].

Theorem 1.8. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X, Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\, T(w)$ for all $w \in X$.*

Now we present a result needed in Section 3.

Theorem 1.9. *Let X, and Y be Hausdorff topological vector spaces, $\Phi \in Ad(X, Y)$ and $\Psi \in Ad(X, Y)$. Then $\Phi + \Psi \in Ad(X, Y)$.*

Proof. Let $X \stackrel{p}{\leftarrow} \Gamma \stackrel{q}{\to} Y$ and $X \stackrel{p_1}{\leftarrow} \Gamma_1 \stackrel{q_1}{\to} Y$ with $q\,(p^{-1}(x)) \subseteq \Phi(x)$ and $q_1\,(p_1^{-1}(x)) \subseteq \Psi(x)$ for any $x \in X$. Consider the fiber product of p and p_1, namely $\Gamma_2 = \{(u, v) \in \Gamma \times \Gamma_1 : \; p(u) = p_1(v)\}$, and let $p_2 : \Gamma_2 \to X$, $q_2 : \Gamma_2 \to Y$ be $p_2(u, v) = p(u)$ and $q_2(u, v) = q(u) + q_1(v)$. Note p_2 is a Vietoris map (note $p_2^{-1}(x) = p^{-1}(x) \times p_1^{-1}(x)$ and Künneth's theorem) and $q_2\,(p_2^{-1}(x)) \subseteq (\Phi + \Psi)(x)$ for $x \in X$. □

2. Convex valued and acyclic valued zero epi maps

In this section we present the notion of 0–epi maps for Kakutani and acyclic type maps and we present some properties of these maps. Let E be a Hausdorff topological vector space (so completely regular) and U an open subset of E. We begin with convex valued type maps (see Definition 2.3 below).

Definition 2.1. *We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to C(E)$ is upper semicontinuous; here $\overline{U}$ denotes the closure of U in E and $C(E)$ denotes the family of nonempty closed subsets of E.*

Definition 2.2. *We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ and $0 \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.*

Definition 2.3. *We say $G \in BC(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$.*

Definition 2.4. *We say $G \in BC_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Now we present 0–epi maps.

Definition 2.5. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in BC_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Remark 2.6. *(i). In Definition 2.5 note $\emptyset \neq F(y) \cap \Phi(y) \subseteq F(y) \cap co\,(G(y))$ so there exists a $y \in U$ with $F(y) \cap co\,(G(y)) \neq \emptyset$.*
(ii). In Definition 2.5 (take $G = 0$) note there exists a $y \in U$ with $0 \in F(y)$.

Example 2.7. *Consider a $G \in CKLU(\overline{U}, E)$ (i.e. $G \in KLU(\overline{U}, E)$ with $co\,G$ a compact map and $co\,G(x) = \{0\}$ for $x \in \partial U$) and assume $\overline{U}$ is paracompact. Then Theorem 1.8 guarantees that there exists an upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ (note Φ is a compact map since $co\,G$ is a compact map) and $\Phi(x) = \{0\}$ for $x \in \partial U$, so Definition 2.4 makes sense if $G \in CKLU(\overline{U}, E)$.*

We begin with a coincidence result.

Theorem 2.8. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in BC(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi. In addition for any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} :\ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap co\,(G(x)) \neq \emptyset$.

Proof. Let $\Phi : \overline{U} \to CK(E)$ be an upper semicontinuous compact map with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and let

$$K = \left\{x \in \overline{U} :\ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $K \neq \emptyset$ (take $t = 0$ and from Remark 2.6 (ii) note there exists a $y \in U$ with $0 \in F(y)$), K is closed (note F and Φ are upper semicontinuous) and compact (note Φ is compact). By assumption $K \cap \partial U = \emptyset$ so since E is completely regular then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $J(x) = \mu(x)\,G(x)$ and $R(x) = \mu(x)\,\Phi(x)$ for $x \in \overline{U}$. Note $J \in BC(\overline{U}, E)$ since $R : \overline{U} \to CK(E)$ is an upper semicontinuous compact map with $R(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ (since for a fixed $x \in \overline{U}$ note $\mu(x)\,\Phi(x) \subseteq \mu(x)\,co\,(G(x)) = co\,(\mu(x)\,G(x))$ because $co(t\,A) = t\,co(A)$). In fact $J \in BC_0(\overline{U}, E)$ since if $x \in \partial U$ then $R(x) = \mu(x)\,\Phi(x) = \{0\}$ (note $\mu(\partial U) = 0$). Now since F is 0–epi then there exists a $y \in U$ with $F(y) \cap R(y) \neq \emptyset$. Note $y \in K$ so $\mu(y) = 1$ i.e. $F(y) \cap \Phi(y) \neq \emptyset$ so $F(y) \cap co\,(G(y)) \neq \emptyset$. □

Next we present a normalization result.

Theorem 2.9. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the identity mapping i is 0–epi.*

Proof. Note $i(x) = x$ for $x \in \overline{U}$ and $i \in A_{\partial U}(\overline{U}, E)$ (note $0 \notin i(x) = x$ for $x \in \partial U$ since $0 \in U$). Consider any $G \in BC_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$. Let

$$Q(x) = \begin{cases} \Phi(x), & x \in \overline{U} \\ \{0\}, & x \in E\backslash\overline{U}. \end{cases}$$

Note [16] that $Q : E \to CK(E)$ is a upper semicontinuous, compact map so Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in Q(x)$. If $x \in E\backslash\overline{U}$ then $Q(x) = \{0\}$, a contradiction since $0 \in U$. Thus $x \in U$ so $x \in \Phi(x)$ i.e. $i(x) \cap \Phi(x) \neq \emptyset$. Thus i is 0–epi. □

Now Theorem 2.8 and Theorem 2.9 gives us a Leray–Schauder type alternative.

Theorem 2.10. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Suppose $G \in BC(\overline{U}, E)$ and $x \notin t\,co\,G(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in \overline{U}$ with $x \in co\,G(x)$.*

Proof. Suppose $x \notin co\,G(x)$ for $x \in \partial U$ (otherwise we are finished). Then

$$x \notin t\,co\,G(x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad t \in [0, 1]. \tag{2.1}$$

Consider any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$. Now (2.1) guarantees that $\left\{x \in \overline{U} : \; i(x) \cap t\,\Phi(x) \neq \emptyset \;\text{ for some }\; t \in [0, 1]\right\}$ does not intersect ∂U since if there exists a $x \in \partial U$ with $i(x)\,(= x) \in t\Phi(x)$ for some $t \in [0, 1]$ then $x \in t\,co\,G(x)$, which contradicts (2.1). Now i is 0–epi (Theorem 2.9) so Theorem 2.8 (with $F = i$) guarantees a $x \in U$ with $i(x) \cap co\,G(x) \neq \emptyset$ i.e. $x \in co\,G(x)$. □

Next we present a homotopy type result.

Theorem 2.11. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is 0–epi and there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\,.\,, 1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - H(x, 1)$ for $x \in \partial U$. Also suppose there exists a upper semicontinuous compact map $h : \overline{U} \times [0, 1] \to CK(E)$ with $h(x, t) \subseteq co\,H(x, t)$ for $(x, t) \in \overline{U} \times [0, 1]$, $h(x, 0) = \{0\}$ for $x \in \partial U$ and $h(y, 1) \subseteq H(y, 1)$ for $y \in U$. In addition for any $G \in BC_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ suppose*

$$\left\{x \in \overline{U} : \; F(x) \cap \; [\Phi(x) + h(x, t)] \neq \emptyset \;\; \textit{for some} \;\; t \in [0, 1]\right\}$$

does not intersect ∂U. Then $F(\,.\,) - H(\,.\,, 1) : \overline{U} \to C(E)$ is 0–epi.

Remark 2.12. *Note from the above assumptions that $F(\,.\,) - H(\,.\,, 1) \in A_{\partial U}(\overline{U}, E)$.*

Proof. Consider any $G \in BC_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$. We must show there exists a $x \in U$ with $[F(x) - H(x, 1)] \cap \Phi(x) \neq \emptyset$. Let

$$K = \left\{x \in \overline{U} : \; F(x) \cap \; [\Phi(x) + h(x, t)] \neq \emptyset \;\; \text{for some} \;\; t \in [0, 1]\right\}.$$

Note $K \neq \emptyset$. To see this take $t = 0$ and note $G(\,.\,) + H(\,.\,, 0) \in BC_0(\overline{U}, E)$ since $\Phi(\,.\,) + h(\,.\,, 0) : \overline{U} \to CK(E)$ is a upper semicontinuous compact map with $\Phi(x) + h(x, 0) \subseteq co\,(G(x)) + co\,(H(x, 0)) = co\,(G(x) + H(x, 0))$ for $x \in \overline{U}$ (note $co\,(A) + co\,(B) = co\,(A + B)$) and finally note for $x \in \partial U$ that $\Phi(x) + h(x, 0) = \{0\}$. Now since F is 0–epi then there exists a $x \in U$ with $F(x) \cap [\Phi(x) + h(x,t)] \neq \emptyset$, so $K \neq \emptyset$. Note also that K is closed, compact and by assumption $K \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $J(x) = G(x) + H(x, \mu(x))$ and $R(x) = \Phi(x) + h(x, \mu(x))$ for $x \in \overline{U}$. Note $R : \overline{U} \to CK(E)$ is a upper semicontinuous compact map and for $x \in \overline{U}$ we have $R(x) = \Phi(x) + h(x, \mu(x)) \subseteq co\,(G(x)) + co\,(H(x, \mu(x))) = co\,(G(x) + H(x, \mu(x))) = co\,(J(x))$, so $J \in BC(\overline{U}, E)$. In fact $J \in BC_0(\overline{U}, E)$ since if $x \in \partial U$ then $R(x) = \Phi(x) + h(x,0) = \{0\}$ (note $\mu(\partial U) = 0$). Now since F is 0–epi then there exists a $x \in U$ with $F(x) \cap [\Phi(x) + h(x, \mu(x))] \neq \emptyset$. Thus $x \in K$ so $\mu(x) = 1$, and so $F(x) \cap [\Phi(x) + h(x,1)] \neq \emptyset$. Also since we assumed $h(y,1) \subseteq H(y,1)$ for $y \in U$ then $F(x) \cap [\Phi(x) + H(x,1)] \neq \emptyset$, so $[F(x) - H(x,1)] \cap \Phi(x) \neq \emptyset$. □

Remark 2.13. *In the proof above one can see that in the statement of Theorem 2.11 that if we replace $H(\,.\,,1) : \overline{U} \to C(E)$ upper semicontinuous, $0 \notin F(x) - H(x,1)$ for $x \in \partial U$ and $h(y,1) \subseteq H(y,1)$ for $y \in U$ with $co\,H(\,.\,,1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - co\,H(x,1)$ for $x \in \partial U$, then the argument guarantees that $F(\,.\,) - co\,H(\,.\,,1) : \overline{U} \to C(E)$ is 0–epi.*

From the ideas in this section note we could also consider the following class of maps.

Definition 2.14. *We say $G \in B(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$.*

Definition 2.15. *We say $G \in B_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 2.16. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in B_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Basically the same proofs (with minor adjustments) in Theorem's 2.8–2.11 establish the following.

Theorem 2.17. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in B(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 2.16). In addition for any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \; F(x) \cap t\,\Phi(x) \neq \emptyset \;\; \text{for some} \;\; t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 2.18. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the identity mapping i is 0–epi (in the sense of Definition 2.16).*

Theorem 2.19. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Suppose $G \in B(\overline{U}, E)$ and $x \notin t\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then there exists a $x \in \overline{U}$ with $x \in G(x)$.*

Theorem 2.20. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is 0–epi (in the sense of Definition 2.16) and there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,,1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - H(x,1)$ for $x \in \partial U$. Also suppose there exists a upper semicontinuous compact map $h : \overline{U} \times [0,1] \to CK(E)$ with $h(x,t) \subseteq H(x,t)$ for $(x,t) \in \overline{U} \times [0,1]$ and $h(x,0) = \{0\}$ for $x \in \partial U$. In addition for any $G \in B_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to CK(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ suppose*

$$\left\{x \in \overline{U} : \; F(x) \cap \; [\Phi(x) + h(x,t)] \neq \emptyset \;\; \text{for some} \;\; t \in [0,1]\right\}$$

does not intersect ∂U. Then $F(\,.\,) - H(\,.\,,1) : \overline{U} \to C(E)$ is 0–epi (in the sense of Definition 2.16).

Now we show how a very slight modification of the reasonng above enables us to obtain coincidence, normalization and Leray–Schauder type results for acyclic–valued type maps (see Definition 2.21 below).

Definition 2.21. *We say $G \in ABC(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$.*

Definition 2.22. *We say $G \in ABC_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 2.23. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in ABC_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Theorem 2.24. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in ABC(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 2.23). In addition for any upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0, 1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap co\,(G(x)) \neq \emptyset$.

Proof. The proof follows the same reasoning (with minor adjustments) as in Theorem 2.8 (we only need to note that here $R : \overline{U} \to Ac(E)$ since homeomorphic spaces have isomorphic homology groups). □

Basically the same proofs (with minor adjustments) as in Theorem 2.9 and Theorem 2.10 establish the following results.

Theorem 2.25. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the identity mapping i is 0–epi (in the sense of Definition 2.23).*

Theorem 2.26. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Suppose $G \in ABC(\overline{U}, E)$ and $x \notin t\,co\,G(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in \overline{U}$ with $x \in co\,G(x)$.*

Remark 2.27. *One could also follow the reasoning in Theorem 2.11 for acyclic maps provided one puts conditions on Φ and h so that in the adjusted Theorem 2.11 we have $\Phi(\,.\,) + h(\,.\,, 0) : \overline{U} \to Ac(E)$ and $R : \overline{U} \to Ac(E)$ (note the addition of two acyclic maps might not be an acyclic map). Instead of presenting this we will consider a more general class of maps in Section 3 (which include acyclic type maps) where the homotopy property will follow immediately from the reasoning in Theorem 2.11.*

From the ideas in this section note we could also consider the following class of maps.

Definition 2.28. *We say $G \in AB(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$.*

Definition 2.29. *We say $G \in AB_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists an upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 2.30. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in AB_0(\overline{U}, E)$ and any upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Basically the same proofs (with minor adjustments) establish the following results.

Theorem 2.31. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in AB(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 2.30). In addition for any upper semicontinuous compact map $\Phi : \overline{U} \to Ac(E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 2.32. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the identity mapping i is 0–epi (in the sense of Definition 2.16).*

Theorem 2.33. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E and $0 \in U$. Suppose $G \in AB(\overline{U}, E)$ and $x \notin t\,G(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in \overline{U}$ with $x \in G(x)$.*

3. Admissible type zero epi maps

It is possible to consider many of the ideas of the last section for a more general class of maps, namely admissible type maps. Again E will be a Hausdorff topological vector space and U an open subset of E.

Definition 3.1. *We say $G \in AdBC(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$.*

Definition 3.2. *We say $G \in AdBC_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 3.3. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in AdBC_0(\overline{U}, E)$ and any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Theorem 3.4. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in AdBC(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 3.3). In addition for any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap co\,(G(x)) \neq \emptyset$.

Proof. The proof follows the same reasoning (with minor adjustments) as in Theorem 2.8 (we only need to note that here $R \in Ad(\overline{U}, E)$ (see for example [6])). □

For a normalization type result we need to present a different type argument for admissible maps.

Theorem 3.5. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $\overline{U}$ a retract of E (i.e. there exists a retraction $r : E \to \overline{U}$). Then the identity mapping i is 0–epi (in the sense of Definition 3.3).*

Proof. Consider any $G \in AdBC_0(\overline{U}, E)$ and any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$. Let

$$\Omega = \left\{x \in \overline{U} : \ x \in \lambda\,\Phi(x) \ \text{ for some } \ \lambda \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ (note $0 \in U$) and Ω is closed and compact. Note $\Omega \cap \partial U = \emptyset$ since if there exists a $x \in \partial U$ and $x \in \Omega$ then for some $\lambda \in [0,1]$ we have $x \in \lambda\,\Phi(x) = \{0\}$, so $x = 0$ which is a contradiction since $0 \in U$. Now there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \backslash U) = 0$ and $\mu(\Omega) = 1$. Let $J(x) = \mu(x)\,\Phi(r(x))$. Note [5, 6] that $J \in Ad(E, E)$ is a compact map. Now Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in J(x)$. If $x \in E \backslash U$ then $\mu(x) = 0$, so $x \in \{0\}$ i.e. $x = 0$, a contradiction since $0 \in U$. Thus $x \in U$ so $x \in \mu(x)\,\Phi(r(x)) = \mu(x)\,\Phi(x)$, so $x \in \Omega$ and so $\mu(x) = 1$. Thus $x \in \Phi(x)$ i.e. $i(x) \cap \Phi(x) \neq \emptyset$. Thus i is 0–epi (in the sense of Definition 3.3). □

Remark 3.6. *If E is a locally convex topological vector space and U is an open convex subset of E then $\overline{U}$ is a retract of E. Either quote Dugundji's extension theorem or alternatively let $r : E \to \overline{U}$ be given by*

$$r(x) = \frac{x}{\max\{1, \eta(x)\}} \quad \textit{for} \quad x \in E$$

where η is the Minkowski functional on $\overline{U}$ i.e. $\eta(x) = \inf\{\alpha : \; x \in \alpha\, \overline{U}\}$.

Now Theorem 3.4 and Theorem 3.5 gives us a Leray–Schauder type alternative.

Theorem 3.7. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $\overline{U}$ a retract of E. Suppose $G \in AdBC(\overline{U}, E)$ and $x \notin t\, co\, G(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in \overline{U}$ with $x \in co\, G(x)$.*

Proof. Suppose $x \notin co\, G(x)$ for $x \in \partial U$ (otherwise we are finished). Then $x \notin t\, co\, G(x)$ for $x \in \partial U$ and $t \in [0, 1]$. Consider any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$. Note (see the reasoning in Theorem 2.10) $\left\{x \in \overline{U} : \; i(x) \cap t\, \Phi(x) \neq \emptyset \;\text{ for some }\; t \in [0, 1]\right\}$ does not intersect ∂U. Now i is 0–epi (in the sense of Definition 3.3) from Theorem 3.5. Thus Theorem 3.4 (with $F = i$) guarantees a $x \in U$ with $i(x) \cap co\, G(x) \neq \emptyset$ i.e. $x \in co\, G(x)$ □

Theorem 3.8. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is 0–epi (in the sense of Definition 3.3) and there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\,.\,, 1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - H(x, 1)$ for $x \in \partial U$. Also suppose there exists a compact map $h \in Ad(\overline{U} \times [0, 1], E)$ with $h(x, t) \subseteq co\, H(x, t)$ for $(x, t) \in \overline{U} \times [0, 1]$, $h(x, 0) = \{0\}$ for $x \in \partial U$ and $h(y, 1) \subseteq H(y, 1)$ for $y \in U$. In addition for any $G \in AdBC_0(\overline{U}, E)$ and any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ suppose*

$$\left\{x \in \overline{U} : \; F(x) \cap \, [\Phi(x) + h(x, t)] \neq \emptyset \;\; \textit{for some} \;\; t \in [0, 1]\right\}$$

does not intersect ∂U. Then $F(\,.\,) - H(\,.\,, 1) : \overline{U} \to C(E)$ is 0–epi (in the sense of Definition 3.3).

Proof. The proof follows the same reasoning (with minor adjustments) as in Theorem 2.11 (we only need to note that the sum of two admissible maps is admissible (see Theorem 1.9)). □

Remark 3.9. *There is also an analogue of Remark 2.13 here. In the statement of Theorem 3.8 if we replace $H(\,.\,, 1) : \overline{U} \to C(E)$ upper semicontinuous, $0 \notin F(x) - H(x, 1)$ for $x \in \partial U$ and $h(y, 1) \subseteq H(y, 1)$ for $y \in U$ with $co\, H(\,.\,, 1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - co\, H(x, 1)$ for $x \in \partial U$, then the argument guarantees that $F(\,.\,) - co\, H(\,.\,, 1) : \overline{U} \to C(E)$ is 0–epi i (in the sense of Definition 3.3).*

From the ideas in this section note we could also consider the following class of maps.

Definition 3.10. *We say $G \in AdB(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$.*

Definition 3.11. *We say $G \in AdB_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 3.12. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in AdB_0(\overline{U}, E)$ and any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $y \in U$ with $F(y) \cap \Phi(y) \neq \emptyset$.*

Basically the same proofs (with minor adjustments) in Theorem's 3.4–3.5, Theorem 3.7 and Theorem 3.8 establish the following.

Theorem 3.13. *Let E be a Hausdorff topological vector space, U an open subset of E, $G \in AdB(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 3.12). In addition for any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \ \textit{for some} \ \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 3.14. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $\overline{U}$ a retract of E. Then the identity mapping i is 0–epi (in the sense of Definition 3.12).*

Theorem 3.15. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $\overline{U}$ a retract of E. Suppose $G \in AdB(\overline{U}, E)$ and $x \notin t\,G(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in \overline{U}$ with $x \in G(x)$.*

Theorem 3.16. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is 0–epi (in the sense of Definition 3.12) and there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\,.\,, 1) : \overline{U} \to C(E)$ upper semicontinuous and $0 \notin F(x) - H(x, 1)$ for $x \in \partial U$. Also suppose there exists a compact map $h \in Ad(\overline{U} \times [0, 1], E)$ with $h(x, t) \subseteq H(x, t)$ for $(x, t) \in \overline{U} \times [0, 1]$ and $h(x, 0) = \{0\}$ for $x \in \partial U$. In addition for any $G \in AdB_0(\overline{U}, E)$ and any compact map $\Phi \in Ad(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap \ [\Phi(x) + h(x,t)] \neq \emptyset \ \ \textit{for some} \ \ t \in [0,1]\right\}$$

does not intersect ∂U. Then $F(\,.\,) - H(\,.\,, 1) : \overline{U} \to C(E)$ is 0–epi (in the sense of Definition 3.12).

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