Filomat 40:5 (2026), 1915–1924
https://doi.org/10.2298/FIL2605915H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Mathematical modeling of the discrete time investment strategy of insurance companies

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Abstract. The target of the insurance companies is to manage the risk of losses (in the form of wealth or health, etc.) of their clients. The clients pay premiums to the companies to buy a policy, while the companies invest the collected premiums in markets (risky/non-risky) to handle the risks. Continuous time trading investment strategies are modeled by several authors, but no body can trade continuously. A discrete time investment mathematical model to deal with this kind of business has been introduced. For this purpose, first, an upper bound of the claim size has been calculated. Secondly, explicit form of the probability function of ruin has been studied. It has been observed that this probability of ruin is less than one. The established results can be used for the identification of the optimal claim size, financial markets, optimal investment strategies, and optimal investment time to reduce the ruin of the companies.

1. Introduction

Insurance is a contract which covers loss of an insurer in return of regular small payments known as premiums. The purpose of buying an insurance policy is to reduce risks of losses. A company which writes these contracts is called insurance company. These companies collect premiums, invest their wealth in clever way and pay the claims for losses to their clients. It is shown by Benckert [2], and Zuanetti, Diniz and Leite [25] that these claims are usually log-normally distributed. Wealth of these companies is defined as: initial surplus plus premiums collected plus earnings from investments minus claims from their clients. If size of claims become greater than the total wealth, that is wealth goes negative, then the company is said to bankrupted. In other words, these companies face high risk to bankrupt. The probability of bankruptcy of an insurance company is called ruin probability. To avoid ruin, these companies need

2020 *Mathematics Subject Classification*. Primary 03B48; Secondary 37A50, 47N30.
Keywords. Wealth function; Ruin probability; Investments modeling; Discrete time investments; Regularity properties; Partial differential equations.
Received: 31 March 2023; Revised: 22 January 2026; Accepted: 24 January 2026
Communicated by Marija Milošević
Funding Statement: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2603).
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to invest cleverly. In this regard, Lundberg [17] established a risk model in insurance theory based on Poisson claims processes. In his work the author has been shown that if insurance claims have exponential moments, then the ruin probability function is bounded above by exponentially decreasing function of the initial amount at hand. Dufresne and Gerber [6] proposed three techniques to study ruin probability. In the first method the authors used a relation between ruin probability and maximal aggregate loss random variable, in the second method the claim amount distribution is assumed to be the combination of translated exponential distributions whereas in the third scheme, the ruin has been linked with stationary distribution of certain associated process. Kalashnikov [11] calculated bounds for rare events with application. In this study the author took initial surplus securing prescribed risk level to derive bounds of ruin probability in cases where claims size has light and heavy tails. Moreover, the author investigated continuity estimates for the probability with respect to governing parameters of wealth process. Browne [3] studied the ruin probabilities when investor of a company is allowed to invest in risky assets modeled using geometric Brownian motion. The author also considered the case when the investor can invest funds in bond market. Browne [3] discussed optimal financing strategies for a company facing uncontrollable cash flow and having opportunity to invests in one risky asset. He also showed that minimum ruin probability is given by an exponential function of the initial wealth if only constant amount is invested in a risky asset. Promislow and Young [20] applied two controls to minimize ruins, that is, investing in risky assets and purchasing quota-share reinsurance. In this work, the authors obtained analytic expression for minimum probability of ruin. Schmidli [21] considered a classical risk model where a company is allowed to invest into risky assets as well as reinsurance. Using Hamilton-Jacobi-Bellman approach, he found an optimal strategy which gives an increasing solution to HJB equation. Kasozi, Mahera and Mayambala [14] studied ruin under which insurance companies enter into quota-share reinsurance arrangements with a reinsurer. They obtained a second order Volterra integro-differential equation, solution of which shows that quota-share reinsurance increases survival of the insurer. Li, Zhou and Yin [15] investigated an optimal reinsurance model under both proportional and fixed transaction costs which lead them to a mixed regular control and optimal stopping problem. Moller and Max [19] used general approach for investigating differential equations to evaluate probabilities of ruin both for finite and infinite times. Mishura and Ragulina [24] worked on optimal control which reduces ruin probability for the case when an insurance company is able to adjust a franchise amount continuously. Li, Li and Young [16] considered an insurer's re-insurance–investment case under mean-variance criterion and obtained Hamilton-Jacobi-Bellman type equation solution of which gives explicit equilibrium re-insurance–investment strategy. Burnecki, Teuerle and Wilkowska [5] considered two-dimensional risk process and derived an infinite-time ruin probability formula. Hussain and Aqsa [22] studied several investment strategies, obtained solvency condition under every investment strategy and initiated some expressions for ruin probability function. Recently, Kasumo [13] proposed diffusion-perturbed risk model comprising an investment return and surplus generating processes. Through the HJB approach, he derived second-order Volterra integro differential equation solution which shows that reinsurance and investments play an enormous role in the survival of insurance companies. Bulinskaya and Shigida [4] studied periodic-review insurance model under the assumption that all claims are independent and some ratio of the surplus is invested in bank account. The condition which leads to decide the premium size is called solvency condition. For more details on the investment process, solvency conditions and ruin probabilities, we refer the reader to Asmussen [1], Elamer et al. [7], Gerber [8], Grandell [9], Grimmett [10] and Tao Jiang [23] and some references therein.

Practically, the insurance companies do not pay any claim to the policyholders up-to a certain fixed time, called reserving time. Sometimes, this extra (delay) time is due to actions like claim negotiator, claim manager, etc., before the claim payment day. In this reserving time, these companies charge the premiums and invest all their wealth in some financial markets. This rule is common in insurance companies. An example of such claims is IBNR (a claim which has occurred but is either not yet known to the company or known but the claim manager needs time for documentations). Moreover, no body can trade continuously in risky assets, only discrete time investments are possible. The literature shows that no one has considered delay payments and discrete time investments to study and analyze the ruin probability function.

In this work, we introduce a model on wealth function of insurance companies who charge premiums at fixed rates, make investments under risky and risk free markets in discrete times, and pay claims in the future specified (non-random) times. In the first result, we investigate largest claim size which does not produce ruin when investments are done. In the second result, we obtain explicit form of ruin probability function which shows that ruin probability is not equal to one a.s. if the claims size becomes greater than the premiums collected. In the onward results, we show that this function is decreasing and smooth with respect to the premium parameter. Moreover, it satisfies second order partial differential equation and its second derivative with respect to initial wealth is bounded.

The insurance companies can use the results to identify the optimal claim size, premium rate, suitable time interval, and financial markets for investments to minimize ruin and set the future goals.

2. Preliminary Results

Consider a probability space (Ω, F, P), where Ω represents sample space, $P : \Omega \to [0,1]$ probability measure and F denotes sigma-algebra on Ω with filtration $F_t, t \geq 0$. Further, on the filtered probability space let us take a risky asset $S_t, t \geq 0$, a Poisson process N_t, claims of sizes $X_i, i = 1, 2, ...$, and wealth function $w(t)$.

Assume investors of an insurance company start with initial amount $w(0)$, collect premiums at a fixed rate c, invest all their wealth in a risky asset S_t that follows geometric Brownian motion with constant volatility σ and constant drift a, i.e.,

$$dS_t = S_t(adt + \sigma dB_t),\ t \geq 0, \tag{1}$$

where B_t, is standard Brownian motion. In case a claim $X_i, i = 1, 2, ...$, occurs, the company pays from the wealth $w(t)$.

The first time when the wealth $w(t), t \geq 0$, of the portfolio drops below zero is refer to be ruin time. Mathematically, it is defined as

$$\mathcal{T}(w(0)) = \inf_{0 \leq t}\{w(t) < 0 \mid w(0) > 0\}. \tag{2}$$

The over-all surplus $w(t)$ of the portfolio is (see Hussain and Aqsa [22]) given as

$$w(t) = w(0) + ct + a\int_0^t w(s)ds + \sigma\int_0^t w(s)dB_s - \sum_{i=1}^{N_t} X_i, \tag{3}$$

where N_t represents the number of claims appeared up to time t. If N_t is a Poisson process then one has the classical jump-diffusion processes. Note that (Grimmett and Stirzaker [10]) a random process N_t, is called Poisson with intensity λ if it takes values from $\{0, 1, 2, ...\}$ and has distribution

$$P(N_t = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t},\ k = 0, 1, 2, ...$$

Moreover, the random variables X_i are log-normally distributed (see Benckert [2] and Zuanetti, Diniz and Leite [25]).

The probability of ruin is defined as

$$\psi(a, c, \sigma; w(0)) = P(\mathcal{T}(w) < \infty \mid w(0) > 0). \tag{4}$$

In the classical risk theory, claims occur at random times $\mathcal{T}_i, i = 1, 2, \ldots$, where the time between claims is denoted by $\tau_1 = \mathcal{T}_1, \tau_2 = \mathcal{T}_2 - \mathcal{T}_1, \ldots$. When the random variables $\{\tau_i\}_{i\geq 1}$ are identically distributed and independent, and N_t is a Poisson process then the aggregate claims process $\sum_{i=1}^{N_t} X_i$ is called compound Poisson process, where $\{N_t\}_{t=0}^{\infty}$ and X_i's are independent.

A more realistic approach for the insurance companies, is to allow delays on the claims payment. In more details, the insurance firm does not pay any claim to the policyholders until reserving time. Common practice in insurance industries is: for the same (reserving) time duration, the firm collects premiums and make invests in several risky and non-risky markets. Therefore, for simplicity, we fix the natural number k, the claims payment time T_k, $k = 1, 2, \dots$, and discuss the wealth function and one step ruin probability.

Assume at current time T_{k-1}, the investors of an insurance company invest a ratio $qw(T_{k-1}), 0 \le q \le 1$, of the wealth $w(T_{k-1})$ in a bank account and $(1-q)w(T_{k-1})$ in risky assets satisfying equation (1) during the time interval (T_{k-1}, T_k). Moreover, they are non active in stock market however collect premiums at a constant rate c which go to bank account. At time T_k, they pay all the claims (or some of the claims) $\sum_{i=N_{T_{k-1}-}}^{N_{T_k-}} X_i$ collected during $[T_{k-1}, T_k)$, where T_k- is left to T_k. The over-all surplus $w(t), T_{k-1} \le t < T_k$, of the portfolio can be expressed as ([22])

$$w(t) = \left\{c(t - T_{k-1}) + qw(T_{k-1})\right\}e^{r(t-T_{k-1})} + (1-q)\left(a\int_{T_{k-1}}^{t} w(s)ds + \sigma\int_{T_{k-1}}^{t} w(s)dB_s\right) - \sum_{i=N_{T_{k-1}-}}^{N_t} X_i \tag{5}$$

where N_t represents the number of claims occurred in time $[T_{k-1}, t)$, while the part $(1-q)w(T_{k-1})$ of the portfolio evolves as (1). The solvency condition under (5) is $c > \lambda t$.

3. Formulation of the Model and the Main Results

In practice, insurance companies need real world trading strategies. For this purpose, we model the wealth function of an insurance company who collect premiums at a constant rate, pays claims at non-random predetermined time and invests surplus in risky and non-risky markets. We study ruin probability function under this model and discuss its properties.

The Proposed Model: *Let us fix a natural number k, assume the investors of an insurance company have positive wealth $w(T_{k-1})$ at current time $T_{k-1}, k = 1, 2, \dots$ with $T_0 = 0$. They invest some fixed ratio $q \in [0, 1]$, of their wealth in the risky asset S_t, satisfies (1), by buying $\Delta(T_{k-1})$ number of shares at time T_{k-1}, while the remaining amount $(1-q)w(T_{k-1})$ in a bank account paying interest rate r. Investors are non-active on risky market during the time interval (T_{k-1}, T_k) however charges premiums at a fixed rate c which add to bank account. At negotiated time $T_k, k = 1, 2, \dots$, they pay no/some/all claims (if occurred) $Y_k = \sum_{i=N_{T_{k-1}}}^{N_{T_k-}} X_i$ collected during $[T_{k-1}, T_k)$.*

A possible sample path of the wealth function under our proposed investment strategy is given below:

Note: Note that sum of the claims occurred during time interval $[T_{k-1}, T_k)$ are paid at time T_k, that is, there is no claim payments at time T_0 and hence no ruin at this time.

At time t, wealth of an insurance company under the above model can be expressed as

$$w(t) = \Delta(T_{k-1})S_t + ((1-q)w(T_{k-1}) + c(t - T_k))e^{r(t-T_{k-1})}, \tag{6}$$

$t \in [T_{k-1}, T_k)$ and $k = 1, 2, \dots$ where S_t, solution to (1), is given as

$$S_t = S_{T_{k-1}} e^{\left(a - \frac{\sigma^2}{2}\right)(t-T_{k-1}) - \sigma(B_t - B_{T_{k-1}})}, \ t \in [T_{k-1}, T_k), k = 1, 2, \dots \tag{7}$$

At time T_k, we have the following forms of the wealth function:

$$w(T_k) = w(T_k-) - \sum_{i=N_{T_{k-1}}}^{N_{T_k-}} X_i \tag{8}$$

from the Figure 1 while

$$\begin{aligned} w(T_k) &= qw(T_k) + (1-q)w(T_k) \\ &= \Delta(T_k)S_{T_k} + (1-q)w(T_k) \end{aligned} \tag{9}$$

holds from the model.
From (6), we write the left limit value $w(T_k-)$, in terms of $\mathfrak{T}_k = T_k - T_{k-1}$, as follows:

$$\begin{aligned} w(T_k-) &= S_{T_k}\Delta(T_{k-1}) + ((1-q)w(T_{k-1}) + c\mathfrak{T}_k)e^{r\mathfrak{T}_k} \\ &= \Delta(T_{k-1})S_{T_{k-1}}e^{\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k+\sigma(B_{T_k}-B_{T_{k-1}})} + ((1-q)w(T_{k-1}) + c\mathfrak{T}_k)e^{r\mathfrak{T}_k}, \end{aligned} \tag{10}$$

where we have used solution S_{T_k} of (1) and its continuity on time interval $[T_k, T_{k-1})$. Using this, expression (8) takes the form

$$w(T_k) = \Delta(T_{k-1})S_{T_{k-1}}e^{\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k+\sigma(B_{T_k}-B_{T_{k-1}})} + ((1-q)w(T_{k-1}) + c\mathfrak{T}_k)e^{r\mathfrak{T}_k} - \sum_{i=N_{T_{k-1}}}^{N_{T_k-}} X_i. \tag{11}$$

From the latter expressions, we observe that $w(t)$ is right continuous with left hand limit. Moreover, $w(T_k)$ is a Markov process'therefore, if T_{k-1} is current time with $w(T_{k-1}) > 0$ then expression for the ruin probability at future time T_k can be written as

$$\psi\left(a, c, \sigma; w(T_{k-1}), T_k\right) = P\left(w(T_k) < 0 | w(T_{k-1}) > 0\right). \tag{12}$$

In the following result, we have identified largest size of a claim which does not produce ruin almost sure under the proposed model.

Theorem 3.1. *For the proposed model with $w(T_{k-1}) > 0$, $q \in [0, 1]$, ruin cannot occur almost sure at time $T_k, k = 1, 2, \dots$ if the claim size Y_k satisfies*

$$Y_k = \sum_{i=N_{T_{k-1}}}^{N_{T_k-}} X_i < \left((1-q)w(T_{k-1}) + c\mathfrak{T}_k\right)e^{r\mathfrak{T}_k}. \tag{13}$$

Proof. Using (11), we can write

$$\begin{aligned} & P\left(w(T_k) < 0 \middle| w(T_{k-1})\right) \\ =\ & P\left(\Delta(T_{k-1})S_{T_{k-1}}e^{\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k+\sigma(B_{T_k}-B_{T_{k-1}})} + ((1-q)w(T_{k-1}) + c\mathfrak{T}_k)e^{r\mathfrak{T}_k} < Y_k \middle| w(T_{k-1})\right) \\ =\ & P\left(w(T_{k-1}q)e^{\sigma(B_{T_k}-B_{T_{k-1}})+\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k} + (c\mathfrak{T}_k + w(T_{k-1})(1-q))e^{r\mathfrak{T}_k} < Y_k \middle| w(T_{k-1})\right) \\ =\ & P\left(e^{\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k+\sigma(B_{T_k}-B_{T_{k-1}})} < \frac{Y_k - \left(w(T_{k-1})(1-q) + c\mathfrak{T}_k\right)e^{r\mathfrak{T}_k}}{qw(T_{k-1})} \middle| w(T_{k-1})\right). \end{aligned} \tag{14}$$

Equivalently, we have

$$\begin{aligned} & P\left(w(T_k) < 0 | w(T_{k-1})\right) \\ =\ & P\left(e^{\sigma(B_{T_k}-B_{T_{k-1}})} < \frac{e^{-\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k}\left(Y_k - \left((1-q)w(T_{k-1}) + c\mathfrak{T}_k\right)e^{r\mathfrak{T}_k}\right)}{qw(T_{k-1})} \middle| w(T_{k-1})\right). \end{aligned} \tag{15}$$

The later expression is possible if $Y_k > \left(c\mathfrak{T}_k + (1-q)w(T_{k-1})\right)e^{\mathfrak{T}_k}$.
This completes the proof. □

The latter result shows that by choosing suitable premium rate c, ratio q and time interval $\mathfrak{T}_k$, ruin can be minimized.

Further, assume $Y_k > (\mathfrak{T}_k c + (1-q)w(T_{k-1}))\, e^{r\mathfrak{T}_k}$, for all $k = 1, 2, \ldots$, then using (11), one can write

$$\begin{aligned} & P\left(w(T_k) < 0 | w(T_{k-1}) > 0\right) \\ = \; & P\left(q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})} - Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k} < -e^{\left(r + \frac{\sigma^2}{2} - a\right)\mathfrak{T}_k} \left((1-q)w(T_{k-1}) + c\mathfrak{T}_k\right) \middle| w(T_{k-1}) > 0 \right) \qquad (16) \\ = \; & \int_{-\infty}^{-\left((1-q)w(T_{k-1}) + c\mathfrak{T}_k\right) e^{\left(r + \frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}} f_{q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})} - Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}}(z)\, dz. \end{aligned}$$

Suppose the claims Y_k are log-normally distributed (see for details Benckert (1962) and Zuanetti, Diniz and Leite (2006)) i.e., $Z_k = \ln(Y_k), k = 1, 2, \cdots$, where each Z_k is normally distributed with variance $\tilde{\sigma}^2$ and mean $\tilde{\mu}$, then for ruin probability (12) of the proposed model, we have the following theorem.

Theorem 3.2. *If $Y_k > (\mathfrak{T}_k c + (1-q)w(T_{k-1}))\, e^{r\mathfrak{T}_k}$, $w(T_{k-1}) > 0$, $k = 1, 2, \ldots$, then at time T_k we have the following explicit forms of the ruin probability function*

$$\begin{aligned} \psi\left(a, c, \sigma; w(T_{k-1}), T_k\right) &= P\left(w(T_k) < 0 | w(T_{k-1})\right) \\ &= \frac{1}{2\pi\sigma\tilde{\sigma}\sqrt{\mathfrak{T}_k}} \int_{-\infty}^{-\left((1-q)w(T_{k-1}) + c\mathfrak{T}_k\right) e^{\left(r + \frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}} \int_{z + \left(\mathfrak{T}_k c + (1-q)w(T_{k-1})\right) e^{\left(r - a + \frac{\sigma^2}{2}\right)\mathfrak{T}_k}}^{\infty} \frac{1}{x(x-z)} \\ &\times \exp\left[\frac{-1}{2\sigma^2\mathfrak{T}_k} \ln^2 \frac{x}{q w(T_{k-1})} - \frac{1}{2\tilde{\sigma}^2} \left(\ln(x - z) + (a - \frac{\sigma^2}{2})\mathfrak{T}_k - \tilde{\mu} \right)^2 \right] dx dz \\ &= \frac{1}{2\pi\sigma\tilde{\sigma}\sqrt{\mathfrak{T}_k}} \int_{-\infty}^{0} \int_{z}^{\infty} \frac{1}{x\left(x - z + (\mathfrak{T}_k c + (1-q)w(T_{k-1}))\, e^{\left(r - a + \frac{\sigma^2}{2}\right)\mathfrak{T}_k}\right)} \\ &\times e^{\frac{-1}{2\sigma^2\mathfrak{T}_k} \ln^2 \frac{x}{q w(T_{k-1})} - \frac{1}{2\tilde{\sigma}^2} \left(\ln\left(x - z + \left(\mathfrak{T}_k c + (1-q)w(T_{k-1})\right) e^{\left(r - a + \frac{\sigma^2}{2}\right)\mathfrak{T}_k} \right) + (a - \frac{\sigma^2}{2})\mathfrak{T}_k - \tilde{\mu} \right)^2} dx dz. \qquad (17) \end{aligned}$$

Proof. Using the bound $Y_k > (\mathfrak{T}_k c + (1-q)w(T_{k-1}))\, e^{r\mathfrak{T}_k}$ and convolution theory on densities

$$f_{q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})}}(x) = \frac{1}{x\sigma\sqrt{2\mathfrak{T}_k \pi}} e^{\frac{-1}{2\sigma^2\mathfrak{T}_k} \ln^2 \frac{x}{q w(T_{k-1})}} \qquad (18)$$

and

$$\begin{aligned} f_{Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}}(y) &= f_{e^{Z_k} e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}}(y) \\ &= \frac{1}{\tilde{\sigma} y \sqrt{2\pi}} e^{\frac{-1}{2\tilde{\sigma}^2}\left(\ln y + (a - \frac{\sigma^2}{2})\mathfrak{T}_k - \tilde{\mu} \right)^2} \qquad (19) \end{aligned}$$

of the random variables $q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})}$ and $Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}$, we obtain

$$\begin{aligned} & f_{q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})} - Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}}(z) \\ = \; & \int_{z + \left(\mathfrak{T}_k c + (1-q)w(T_{k-1})\right) e^{\left(r - a + \frac{\sigma^2}{2}\right)\mathfrak{T}_k}}^{\infty} f_{q w(T_{k-1}) e^{\sigma(B_{T_k} - B_{T_{k-1}})}}(x) \cdot f_{Y_k e^{\left(\frac{\sigma^2}{2} - a\right)\mathfrak{T}_k}}(x - z)\, dx \qquad (20) \\ = \; & \frac{1}{2\pi\sigma\tilde{\sigma}\sqrt{\mathfrak{T}_k}} \int_{z + \left(\mathfrak{T}_k c + (1-q)w(T_{k-1})\right) e^{\left(r - a + \frac{\sigma^2}{2}\right)\mathfrak{T}_k}}^{\infty} \frac{1}{x(x-z)} \exp\left[\frac{-1}{2\sigma^2\mathfrak{T}_k} \ln^2 \frac{x}{q w(T_{k-1})} - \frac{1}{2\tilde{\sigma}^2} \left(\ln(x - z) + (a - \frac{\sigma^2}{2})\mathfrak{T}_k - \tilde{\mu} \right)^2 \right] dx. \end{aligned}$$

Similarly

$$f_{Y_k e^{\left(\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}-qw(T_{k-1})e^{\sigma(B_{T_k}-B_{T_{k-1}})}}(z) = \int_z^\infty f_{Y_k e^{\left(\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}}(x)\cdot f_{qw(T_{k-1})e^{\sigma(B_{T_k}-B_{T_{k-1}})}}(x-z)dx$$
$$= \frac{1}{2\pi\sigma\widetilde{\sigma}\sqrt{\mathfrak{T}_k}}\int_z^\infty \frac{1}{x(x-z)} e^{\frac{-1}{2\sigma^2\mathfrak{T}_k}\ln^2\frac{x-z}{qw(T_{k-1})}-\frac{1}{2\widetilde{\sigma}^2}\left(\ln y+\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k-\widetilde{\mu}\right)^2}dx. \tag{21}$$

Using expressions (20) and (21) in the form (16), we obtain the result.

For the second form, we substitute the variable y by

$$y = z + (\mathfrak{T}_k c + (1-q)w(T_{k-1}))\, e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}$$

to get the answer. □

Note: It is noted that the explicit form can be used to calculate optimal values of a and σ (it will lead to select optimal market), value of ratio q, premium rate c, claims payment time interval $[T_{k-1}; T_k]$, initial surplus $w(T_{k-1})$ to minimize ruin.

Let's come to a practical example to illustrate some of the rationalities of our proposed model.

Practical Example: To verify our results, we consider some statistical data from short term motor insurance company in Zimbabwe (Mazviona et al. [18]). For confidentiality reasons they have not disclosed the name of the company in their article. The two years, i.e., from January 2009 to December 2010, claim sizes data in Zimbabwean dollar is the following:

12523, 18544, 18745, 26506, 16300, 18455, 12323, 26076, 17123, 15689, 14078, 16432, 48244, 12650, 16603, 18645, 36067, 13963, 23711, 27088, 12700, 19122, 12490, 11802, 23045, 19641, 15504, 19806, 27566, 34087, 19566, 12904, 10009, 19790, 13554, 12250, 16000, 18134, 24067, 19669, 16809, 12809, 16905, 18309, 16704.

If we let the starting time T_0 to be the first January 2009 with $w(T_0) = \$100,000$, T_1-the first February 2009, T_2-the first March 2009 and so on then the length of time interval $[T_{k-1}, T_k)$ is one month. Let the risky asset S_t, evolves as (1), is chosen to be West Texas Intermediate (WTI or NYMEX) crude oil [26] then we find $S_{T_0} = \$57.05$, $S_{T_1} = \$60.96$, $S_{T_2} = \$67.49$,.... This data shows that the average rate of return in the month of January is about $a = 0.0663$. If the interest rate on a bank account is 12% per year then the average rate of return on the risky asset is 0.01005, thus the average rate of return from the risky asset is more than the rate of return from the bank account for the month of January therefore, we conclude that more ratio of the surplus wealth is optimal to invest in the risky asset. Assume the ratio $q = \frac{3}{4}$ of $\$100,000$ is invested in S_t at time T_0, then using (1) we obtain $\triangle(T_0) = 1314.64$, that is, the number of assets during the month of January is about 1314.64. Moreover, using the data, we find $\lambda = 1.875$ and $c = 18955$. Next, using (1), we find wealth of the company at the end of January is $W(T_1-) = \$148400.24$. Assume the first two claims \$12523 and \$18544 are occurred during the month of January 2009 and the company wants to pay both of them then $Y_1 = \$31067$ and therefore, wealth of the company at time T_1 is \$117333.24. The same procedure at claim payment times T_2, T_3, ...

In the following result, we identify condition on some parameters of the model used to maximize expected value of wealth function $w(T_k), k = 1, 2, 3, \ldots$ with respect to the ratio q of surplus wealth $w(T_{k-1})$ invested in the risky asset (1) during the time interval $[T_{k-1}, T_k)$.

Theorem 3.3. *Assume $w(T_{k-1}) > 0, k = 1, 2, 3, \ldots$, expected value of wealth $w(t), t \in [T_{k-1}, T_k)$, of the proposed model, increases with respect to ratio q in time interval $[T_{k-1}, T_k), k = 1, 2, 3, \ldots$ if $a > r$ and vice versa.*

Proof. Inserting $\Delta(T_{k-1})S_{T_{k-1}} = qw(T_{k-1})$ and $Y_k = \sum_{i=N_{T_{k-1}}}^{N_{T_k-}} X_i$ in (11) and taking mathematical expectation, we write

$$E\left[w(T_k)|F_{T_{k-1}}\right] = e^{\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k} E\left[qw(T_{k-1})e^{\sigma(B_{T_k}-B_{T_{k-1}})} + ((1-q)w(T_{k-1}) + c\mathfrak{T}_k)e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k} - Y_k e^{\left(\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}\right]. \tag{22}$$

Using densities (18) and (19), we obtain

$$E\left[qw(T_{k-1})e^{\sigma(B_{T_k}-B_{T_{k-1}})}\right] = qw(T_{k-1})e^{\frac{\sigma^2\mathfrak{T}_k}{2}}, \tag{23}$$

and

$$E\left[Y_k e^{\left(\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}\right] = e^{\tilde{\mu}-\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k+\frac{\tilde{\sigma}}{2}}. \tag{24}$$

With these values (22) takes the form

$$E\left[w(T_k)|F_{T_{k-1}}\right] = qw(T_{k-1})\left(e^{a\mathfrak{T}_k}-e^{r\mathfrak{T}_k}\right) + (w(T_{k-1})+c\mathfrak{T}_k)e^{r\mathfrak{T}_k} - e^{\tilde{\mu}+\frac{\sigma^2}{2}}. \tag{25}$$

Coefficient of q is positive if $a > r$. Thus wealth function $w(t), t \in [T_{k-1}, T_k), k = 1, 2, 3, \dots$ approaches its maximum at T_k with respect to q if $a > r$ and vice versa. □

Note: For simplicity, we put $q = 1$ i.e., when the total wealth $w(T_{k-1})$ is invested in risky asset at time T_{k-1}, then (17) takes the following form

$$\begin{aligned}\psi\left(a, c, \sigma; w(T_{k-1}), T_k\right) &= \frac{1}{2\pi\sigma\widetilde{\sigma}\sqrt{T_k - T_{k-1}}}\int_{-\infty}^{-c\mathfrak{T}_k e^{\left(r+\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}}\int_{z+c\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}}^{\infty}\frac{1}{x(x-z)}\\ &\times \exp\left[\frac{-1}{2\sigma^2\mathfrak{T}_k}\ln^2\frac{x}{w(T_{k-1})} - \frac{1}{2\widetilde{\sigma}^2}\left(\ln(x-z) + \left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k - \tilde{\mu}\right)^2\right]dxdz. \end{aligned}\tag{26}$$

With the above expressions, we come to the following second order partial differential equation:

Theorem 3.4. *The ruin probability given in (26) satisfies the second order partial differential equation*

$$\begin{aligned}\frac{\partial^2\psi}{\partial w^2} + \frac{1}{w}\frac{\partial\psi}{\partial w} + \frac{1}{\mathfrak{T}_k\sigma^2 w^2}\psi &= \frac{1}{\sigma^5\tilde{\sigma}\mathfrak{T}_k^{\frac{5}{2}}w^2}\int_{-\infty}^{-\mathfrak{T}_k c\, e^{\mathfrak{T}_k\left(r-a+\frac{\sigma^2}{2}\right)}}\int_{z+\mathfrak{T}_k c\, e^{\mathfrak{T}_k\left(r+\frac{\sigma^2}{2}-a\right)}}^{\infty}\frac{\ln^2\frac{x}{w}}{x(x-z)}\\ &\times \phi\left(\frac{\ln\frac{x}{w}}{\sigma\sqrt{T_k - T_{k-1}}}\right)\phi\left(\frac{\ln(x-z) + \left(a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k - \tilde{\mu}}{\tilde{\sigma}}\right)dxdz,\end{aligned}\tag{27}$$

where $\phi(\cdot)$ denotes density function of standard normal random variable.

Proof. The first partial derivative of ruin probability function (26) is

$$\begin{aligned}\frac{\partial\psi}{\partial w} &= \frac{1}{2\pi\mathfrak{T}_k^{3/2}\sigma^3\tilde{\sigma}w}\int_{-\infty}^{-c\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}}\int_{z+\mathfrak{T}_k c\, e^{\mathfrak{T}_k\left(r+\frac{\sigma^2}{2}-a\right)}}^{\infty}\frac{\ln\frac{x}{w}}{x(x-z)}\\ &\times \exp\left[\frac{-1}{2\sigma^2\mathfrak{T}_k}\ln^2\frac{x}{w} - \frac{1}{2\widetilde{\sigma}^2}\left(\ln(x-z)+\left(a-\frac{\sigma^2}{2}\right)\mathfrak{T}_k - \tilde{\mu}\right)^2\right]dxdz.\end{aligned}$$

Differentiating again the later expression with respect to w and using the values of $\frac{\partial\psi}{\partial w}$ and ψ, we complete the proof. □

Next, we are ready to state the following smoothness result:

Lemma 3.5. *The ruin probability function ψ, given in (26), is $C^2(0,\infty)$ with respect to the initial wealth $w(T_{k-1})$ and the second derivative satisfies the bounds*

$$\frac{-1}{\sigma^2 w^2(T_{k-1})\mathfrak{T}_k} \le \frac{\partial^2\psi}{\partial w^2} \le \frac{-1}{w(T_{k-1})}\frac{\partial\psi}{\partial w} + \frac{1}{\sigma^5\tilde{\sigma} w^2(T_{k-1})\mathfrak{T}_k^{\frac{5}{2}}} \int_{-\infty}^{-\mathfrak{T}_k c\, e^{\mathfrak{T}_k\left(r-a+\frac{\sigma^2}{2}\right)}} \int_{z+\mathfrak{T}_k c\, e^{\mathfrak{T}_k\left(r+\frac{\sigma^2}{2}-a\right)}}^{\infty} \frac{\ln^2 \frac{x}{w(T_{k-1})}}{x(x-z)} dx dz.$$

Proof. For the lower bound of $\frac{\partial^2\psi}{\partial w^2}$, we use the right-hand side of the equation (17), Theorem 3.3 and the bound $\psi \le 1$. For the upper bound, we use the bounds $\psi \ge 0$ and $\phi(\cdot) \le 1$. □

The latter result shows that the ruin probability function (26) is neither convex nor concave with respect to the initial wealth. The following result gives behavior of the ruin probability with respect to the premium rate c.

Theorem 3.6. *The ruin probability ψ is differentiable and decreasing with respect to the premium rate c.*

Proof. Using the Fundamental Theorem of Calculus, first derivative of the ruin probability function ψ with respect to the premium parameter c is

$$\begin{aligned} \frac{\partial\psi}{\partial c} &= \frac{-1}{\sigma\tilde{\sigma}\sqrt{T_k - T_{k-1}}}\Bigg[\int_0^\infty \frac{\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}}{x\left(x + c\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}\right)} \phi\left(\frac{\ln\frac{x}{w(T_{k-1})}}{\sigma\sqrt{T_k - T_{k-1}}}\right)\phi\left(\frac{\ln\left(x + c\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}\right) + \left(a - \frac{\sigma^2}{2}\right)\mathfrak{T}_k - \tilde{\mu}}{\tilde{\sigma}}\right) dx \\ &+ \frac{1}{c}\int_{-\infty}^{-c\mathfrak{T}_k e^{\left(r-a+\frac{\sigma^2}{2}\right)\mathfrak{T}_k}} \frac{1}{z + c\mathfrak{T}_k e^{\left(r+\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}} \phi\left(\frac{1}{\sigma\sqrt{T_k - T_{k-1}}} \ln\frac{z + c\mathfrak{T}_k e^{\left(r+\frac{\sigma^2}{2}-a\right)\mathfrak{T}_k}}{w(T_{k-1})}\right)\phi\left(\frac{\ln c + \ln\mathfrak{T}_k + r\mathfrak{T}_k - \tilde{\mu}}{\tilde{\sigma}}\right) dz\Bigg]. \end{aligned}$$

Proof follows from the later derivative. □

Note: We observe, from Theorem 3.2, that ψ strongly depends on the parameters, i.e., premium rate c, time interval $T_k - T_{k-1}$, initial wealth $w(T_{k-1})$, $k = 1, 2, \ldots$, drift term a and stock volatility σ. These parameters can be used to identify the optimal claim size, premium rate, suitable time interval and the financial markets for investments to minimize their ruin.

Comparison: Existing literature shows no one has studied discrete time investments partial differential equations and variational inequalities satisfied by ruin probability function.

Conclusion: Under delay claims payment, we considered a discrete time risk model on the wealth of insurance companies with investments in risky and non-risky assets where the claims sizes are taken to be log-normally distributed and investigated explicit expression of ruin probability. It has been found that ruin can be minimized through investments. Moreover, under the proposed model, ruin probability function is smooth and satisfying second order partial differential equation in initial wealth. It is also differentiable and decreasing with respect to the premium rate.

Data Availability Statement: No data were used to support this study.

Competing interests: The authors declare no Competing of interest.

Acknowledgement: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2603).

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