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Rectangular functions and fractal measures, a general discussion of measures of Cartesian product sets

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Abstract. In the present work, we investigate the mixed generalized Hausdorff and packing measures. We emphasize the significance of these measures in analyzing the geometric and dimensional properties of product sets. Our study provides a thorough and detailed discussion on the measurement of product sets in general metric spaces $(\mathbb{X}, \rho)$. Additionally, we will explore the $0 - \infty$ case under a geometric condition on $\mathbb{X}$, which has been excluded in several previous studies.

1. Introduction

The notion of "weighted" in fractal geometry is crucial for understanding the intricate nature of geometric sets and their dimensions. Weighted measures manifest in the form of weighted covers and weighted packings, both of which provide essential insights into the approximation and quantification of complex structures. A weighted cover comprises a collection of closed balls, each associated with a non-negative weight, which collectively represents the set being studied. This methodology is particularly important for assessing key properties such as Hausdorff and packing dimensions, capturing the scaling behavior inherent in fractals. In contrast, weighted packings focus on the arrangement of disjoint subsets within a space, enabling a detailed analysis of their density and spatial distribution. The requirements for weights and sizes of these balls ensure that the packings exhibit desirable properties, allowing researchers to establish various inequalities and results pertinent to fractal analysis. By leveraging both weighted covers and packings, mathematicians can delve deeper into the geometric complexities of fractals, highlighting their relevance and importance in ongoing mathematical research. The reader can be referred to [4, 16, 23, 33, 34, 41] for more details of weighted Hausdorff and packing measures.

Weighted measures are crucial in fractal geometry as they allow for the assignment of varying importance to different regions of a set, reflecting the heterogeneity in geometric properties. Building on this

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idea, mixed multifractal analysis has recently attracted considerable attention. This approach examines the simultaneous scaling behavior of multiple finite measures, each potentially associated with different weights, thus offering a much deeper insight into the local geometric properties of fractal structures. In contrast, classical multifractal analysis is limited to the local scaling behavior of a single measure, which restricts its applicability to analyzing the multifractality of individual time series. Olsen [49] was the first to develop a comprehensive mixed multifractal theory, unifying the concepts of mixed Rényi dimensions and mixed coarse multifractal spectra.

Let $(\mathbb{X}, \rho)$ and $(\mathbb{X}', \rho')$ be two separable metric spaces. The Cartesian product space $\mathbb{X} \times \mathbb{X}'$ is defined with the metric $\rho \times \rho'$ given by

$$\rho \times \rho'\Big((x_1, x_1'), (x_2, x_2')\Big) = \max\Big\{\rho(x_1, x_2), \rho'(x_1', x_2')\Big\}.$$

Let $\mathcal{M}(\mathbb{X})$ denote the family of Borel probability measures on $\mathbb{X}$. We denote by $\mathcal{B}(\mathbb{X})$ the family of closed balls on $\mathbb{X}$ and by $\Phi(\mathbb{X})$ the class of premeasures, i.e., increasing functions $\zeta : \mathcal{B}(\mathbb{X}) \to [0, +\infty]$ such that $\zeta(\emptyset) = 0$. In this study, we explore general fractal measures by studying the fractal structure of product sets. A key tool in our study is the weighted Hausdorff and packing measures, denoted as $\mathcal{W}_{\mu}^{q,\zeta}$ and $\mathcal{Q}_{\mu}^{q,\zeta}$, which play a crucial role in analyzing the geometric and dimensional properties of product sets. A complete and comprehensive discussion is given especially when the measures $\boldsymbol{\mu}$ and ζ satisfy a doubling condition, the metric spaces satisfy some geometric condition or ζ is defined using Hausdorff function. Moreover, a special focus will be placed on the situation where the weighted Hausdorff measure coincides with the Hausdorff measure, i.e., $\mathcal{H}_{\mu}^{q,\zeta} = \mathcal{W}_{\mu}^{q,\zeta}$. In this framework, we will identify and explore several classic examples that have been extensively studied and analyzed in the literature. This will be referred to throughout this work to simplify the illustration and discussion:

- **case 1** : $\boldsymbol{q} = 0_{\mathbb{R}^k}$ and $\zeta(B(x, r)) = \operatorname{diam}(B(x, r))^s$, $s > 0$.
- **case 2** : $\boldsymbol{q} \in \mathbb{R}$ and $\zeta(B(x, r)) = h(2r)$, where h is a Hausdorff function (called also a dimension function).
- **case 3** : $\boldsymbol{q} \in \mathbb{R}$ and $\zeta(B(x, r)) = (2r)^t$, $t \in \mathbb{R}$.
- **case 4** : Hewitt-Stromberg measure are used instead of the Hausdorff and packing measures,

where $B(x, r) = \Big\{y \in \mathbb{X};\ \rho(x, y) \le r\Big\}$. Notice, if $\mathbb{X} = \mathbb{R}$, that the standard Hausdorff measure $\mathcal{H}^1$ coincides with $\mathcal{L}^1$, the 1-dimensional Lebesgue measure. Let Γ be Lebesgue-measurable subsets of $\mathbb{R}^2$ and denote by $\Gamma_x = \{(x, y) \in \Gamma;\ y \in \mathbb{R}\}$. Fubini's theorem implies that Γ_x is $\mathcal{L}^1$-measurable for almost all x, and the 2-dimensional Lebesgue measure of E can be expressed as

$$\begin{cases} \mathcal{L}^2(\Gamma) = \displaystyle\int_{\mathbb{R}} \mathcal{H}^1(\Gamma_x)\, d\mathcal{H}^1(x) \\ \\ \mathcal{L}^2(\Gamma) = \mathcal{H}^1(E)\mathcal{H}^1(F) \qquad\qquad \text{if } \Gamma = E \times F. \end{cases}$$

These prove, in particular that $\mathcal{H}^2(\Gamma) \neq \mathcal{H}^1(E)\mathcal{H}^1(F)$. The question of equality and the potential inequality of previously established equalities has been extensively studied across various forms of fractal measures, especially Hausdorff and packing measures. Researchers have explored this topic to examine the nature of inequality, which always holds, and the conditions under which equality occurs. This investigation spans multiple disciplines, including fractal geometry, measure theory, and mathematical analysis, aiming to uncover deeper insights into the properties and behaviors of complex structures. One initial approach, when considering $\Gamma = E \times F$, is to cover E and F with intervals of length less than δ, which would naturally lead to covering Γ with squares of side length less than δ. From there, one could attempt to estimate the corresponding Hausdorff measures. Let ζ_0 be the Cartesian product measure generated by the product of ζ and ζ' and defined on $\mathcal{B}(\mathbb{X} \times \mathbb{X}')$ by

$$\zeta_0(B \times B') = \zeta(B)\zeta'(B') \tag{1.1}$$

Under a general circumstance, we will establish in Section 3 the validity of the following inequality

$$c\,\mathcal{H}_{\mu}^{q,\zeta}(E)\,\mathcal{H}_{\nu}^{q,\zeta'}(E') \le \mathcal{H}_{\mu\times\nu}^{q,\zeta_0}(E\times E'), \tag{1.2}$$

where c is a positive constant and with the convention, that $0\times\infty = \infty\times 0 = 0$. We also prove that $c = 1$ when we assume that μ and ν satisfy the doubling condition. The inequality (1.2) will be established using the density approach. However, this method has a limitation: it cannot handle sets of infinite measure. To overcome this issue and extend our study to such sets, we impose the semi-finiteness of the mixed generalized measure. With this additional assumption, we can obtain a similar inequality without restrictions on finite measure sets. More precisely, if $\mathcal{H}_{\mu}^{q,\zeta}$ is semi-finite on E and $\mathcal{H}_{\nu}^{q,\zeta'}$ is semi-finite on E', then (1.2) holds (see Section 3). However, a significant issue arises: the interval lengths that yield an 'optimal' covering for E may produce a 'suboptimal' covering for F, making such estimates ineffective. In other words, the inverse inequality in (1.2) is not true as demonstrated in Example 3.6. Indeed, we will construct two sets A and B such that

$$\mathcal{H}_{\mu\times\nu}^{q,\zeta_0}(A\times B) \ge 1 > 0 = \mathcal{H}_{\mu}^{q,\zeta}(A)\mathcal{H}_{\nu}^{q,\zeta'}(B).$$

This construction is due to Falconer in [15] for case 1 and later adopted in [25] for case 4. Thus, the problem of equality in (1.2) is not obvious and requires modification or additional conditions. It is also natural to consider the general case where Γ is not necessarily $E\times F$. We can show in cases 1, 2, and 4 that inequality (1.2) extends to an integral version. For instance, in case 3, we obtain

$$\int \mathcal{H}_{\mu}^{q,s}(E^y)\,d\mathcal{H}_{\nu}^{q,t}(y) \le c\,\mathcal{H}_{\mu\times\nu}^{q,s+t}(E), \tag{1.3}$$

for $E\subseteq\mathbb{R}^{m+n}$ and $E^y = \{x\in\mathbb{R}^n,\ (x,y)\in E\}$ and for some positive constant c. This inequality was established in [48] and, for case 1, in [14] and [37]. But it is not reversible, as shown in Example 3.8. Turn back to the inequality (1.2) and explore possible modifications that would lead to an equality. First, note that the product measure $\mathcal{H}^t\ \mathcal{H}^s$ is not required to be a Hausdorff measure. Nonetheless, a straightforward reasoning shows that utilizing the covering by rectangles enables us to derive the inverse inequality since $\{B_i\times B_j\}_{i,j}$ is a cover of $E\times F$ whenever $\{B_i\}_i$ is a cover of E and $\{B_j\}_j$ is a cover of F. To formulate this, we will consider $\mathcal{H}_{\mu\times\nu}^{q,\zeta_0}$ where the Cartesian product measure ζ_0 is defined by (1.1) on the class of rectangles $\mathcal{B}(\mathbb{X})\times\mathcal{B}(\mathbb{X}')$ by (1.1). We will prove in Theorem 3.10 that under appropriate geometric conditions, we have

$$\mathcal{H}_{\mu\times\nu}^{q,\zeta_0}(E\times E') \le \mathcal{H}_{\mu}^{q,\zeta}(E)\mathcal{H}_{\nu}^{q,\zeta'}(E').$$

This inequality was proved for $q\in\mathbb{R}$ in [4]. When, we allow the Cartesian product measure ζ_0 to be defined on the class of rectangles $\mathcal{B}(\mathbb{X})\times\mathcal{B}(\mathbb{X}')$, inequality (1.2) fails to hold in general. However, a valid alternative can be established by introducing the weighted Hausdorff measure (see section 2). Specifically, we will prove in Theorem 3.11 that the following inequality holds

$$\mathcal{W}_{\mu}^{q,\zeta}(E)\mathcal{H}_{\nu}^{q,\zeta'}(E') \le \mathcal{H}_{\mu\times\nu}^{q,\zeta_0}(E\times E').$$

At this stage, and still pursuing our goal of turning inequality (1.2) into an equality, we show that if, at least one of the spaces $\mathbb{X}$ or $\mathbb{X}'$, the Hausdorff measure coincides with the weighted Hausdorff measure, then one obtains the equality. We further explore the application of this result in the case where the above condition is satisfied.

When analyzing the structure of fractal sets, packing measures on Cartesian products exhibit distinctive features compared to those observed with Hausdorff measures. Like the Hausdorff measure, the packing measure quantifies the "size" of a set, but it does so by focusing on the arrangement of disjoint balls within the set rather than covering it. This difference in approach provides unique insights into the geometric and dimensional properties of fractal sets. For two sets E and E' in metric spaces, an upper bound for the packing measure of the Cartesian product $E\times E'$ can be derived in terms of the individual packing measures of E and E'. Specifically, under certain conditions, the following inequality holds (see Section 3)

$$\mathcal{P}^{q,\zeta_0}_{\mu\times\nu}(E\times E') \le \mathcal{P}^{q,\zeta}_{\mu}(E)\,\mathcal{P}^{q,\zeta'}_{\nu}(E'), \tag{1.4}$$

provided it is true in the case when the term on the right side is not $0\times\infty$ or $\infty\times 0$. It is natural to consider the integral version of the inequality for packing measures, which takes a highly general form, as previously discussed for Hausdorff measures. The integral version of the inequality for packing measures faces significant difficulties due to the nature of these measure. Packing measures are defined as the supremum over sums evaluated on disjoint packings. This maximization approach means that an optimal packing in the product space does not decompose neatly into optimal packings in the individual spaces, making it difficult to derive a coherent integral representation. To overcome this issue, the author in [37] introduced the measure $\gamma^{s,t}$, which is constructed using a Carathéodory method with coverings by rectangles rather than general packings. This approach employs a gauge function adapted to the product setting, ensuring better control over the measure distribution across different coordinate directions. As a result, $\gamma^{s,t}$ provides a crucial link between packing measures and integral inequalities, enabling the derivation of a Fubini-type result that would not hold directly for classical packing measures. More precisely, we obtain

$$\int \mathcal{P}^{s}(E^x)\, d\mathcal{P}^t(x) \le \gamma^{s,t}(E),$$

for $E \subseteq \mathbb{R}^2$ and $E^x = \{y\in\mathbb{R},\ (x,y)\in E\}$. But it is not reversible, as shown in [37, Example 4.2.1]. Now to study inequality (1.4) without additional conditions or assumptions, we introduce a new fractal measure called the mixed weighted generalized packing measure, which is more general than the mixed generalized packing measure (see Section 2). Specifically, we will prove in section 3 the following inequality

$$\mathcal{P}^{q,\zeta_0}_{\mu\times\nu}(E\times E') \le \mathcal{Q}^{q,\zeta}_{\mu}(E)\,\mathcal{P}^{q,\zeta'}_{\nu}(E'), \tag{1.5}$$

provided that the product on the right-hand side is not of the form $0\times\infty$ or $\infty\times 0$.

2. Preliminaries results

Let $k\in\mathbb{N}$, the set of positive integers, be a fixed integer and let $\boldsymbol{\mu} = (\mu_1,\dots,\mu_k)$ be a vector of probability measures on $\mathbb{X}$ with a common support equal to K. Let

$$\boldsymbol{\mu}(B(x,r)) = \mu_1(B(x,r))\dots\mu_k(B(x,r))$$

and for $\boldsymbol{q} = (q_1,\dots,q_k)\in\mathbb{R}^k$, we write

$$\boldsymbol{\mu}(B(x,r))^{\boldsymbol{q}} = \mu_1(B(x,r))^{q_1}\dots\mu_k(B(x,r))^{q_k}.$$

We denote $\mathbb{R}^k_- = (-\infty,0]\times\cdots\times(-\infty,0]$, and $\mathbb{R}^k_+ = [0,+\infty)\times\cdots\times[0,+\infty)$. For $a>1$ and $1\le j\le k$, we write

$$M^j_a(\mu) = \limsup_{r\searrow 0}\left(\sup_{x\in\operatorname{supp}\mu}\frac{\mu_j\Big(B(x,ar)\Big)}{\mu_j\Big(B(x,r)\Big)}\right).$$

Now, we say that the measure $\boldsymbol{\mu}$ satisfies the doubling condition if there exists $a>1$ such that $M^j_a(\mu)<\infty$ for all $1\le j\le k$. It is easily seen that the exact value of the parameter a is unimportant since $M^j_a(\mu)<\infty$, for some $a>1$ if and only if $M^j_a(\mu)<\infty$, for all $a>1$. Also, we denote by $\mathcal{M}_D(\mathbb{X})$ the family of Borel probability measures on $\mathbb{X}$ which satisfy the doubling condition. We can cite as classical examples of measures that satisfy the doubling condition, the self-similar measures and the self-conformal ones [47].

2.1. Mixed generalized measures of Hausdorff type

For $\delta > 0$, a sequence of closed balls $\{B_i\}_i$ is called a centered $\delta-$cover of a set $E \subseteq \mathbb{X}$ if, for all $i \geq 1$, B_i is centered in E, $\operatorname{diam} B_i \leq 2\delta$ and $E \subseteq \bigcup_{i=1}^{\infty} B_i$. The mixed generalized Hausdorff measure with respect to $(\boldsymbol{\mu}, \zeta)$, is defined as follows. For $E \subseteq \mathbb{X}$, we write

$$\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) = \inf\Big\{ \sum_i \boldsymbol{\mu}(B(x_i, r_i))^{\boldsymbol{q}} \zeta(B(x_i, r_i))\Big\},$$

where the infinimum is taken over all centered δ-coverings of E and with the conventions $0^{\boldsymbol{q}} = \infty$ for $\boldsymbol{q} \leq 0$ and $\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E)$ is given infinite value if no centered δ-cover of E exists. The mixed generalized Hausdorff pre-measure is then given by

$$\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},0}(E) = \lim_{\delta\to 0} \mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) = \sup_{\delta>0} \mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E).$$

Since $\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},0}$ is σ-sub-additive but not increasing, one needs a standard modification to get an outer measure. Hence we modify the definition to

$$\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}(E) = \sup_{F\subseteq E} \mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},0}(F).$$

The function $\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}$ is a metric outer measure and thus, a measure on the Borel family of subsets on $\mathbb{X}$. We refer the reader to [53] for more information about the notion of metric outer measure. The function $\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}$, is of course, a generalization of the Hausdorff measures introduced in [4, 23, 27, 47, 50]. In the following, we define the mixed weighted generalized Hausdorff measure. A sequence $(c_i, B_i)_{i\geq 1}$ of pairs, where c_i is a non-negative real number and B_i is a closed ball of $\mathbb{X}$, is said to be a weighted cover of E if

$$\chi_E \leq \sum_{i=1}^{\infty} c_i \chi_{B_i},$$

that is, for all points x of E, we have

$$\sum_{i=1}^{\infty} \Big\{ c_i \ ; \ x \in B_i \Big\} \geq 1. \tag{2.1}$$

In addition, for $\delta > 0$, we say that $(c_i; B_i)_{i\geq 1}$ is a weighted and centered $\delta-$cover of E if

- it is a weighted cover of E;
- for all $i \geq 1$, B_i is centered in E and $\operatorname{diam} B_i \leq 2\delta$.

Now, we write

$$\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) = \inf\Big\{ \sum_i c_i \boldsymbol{\mu}(B(x_i, r_i)^{\boldsymbol{q}} \zeta(B(x_i, r_i))\Big\},$$

where the infinimum is taken over all weighted and centered δ-covers of E and with the convention that $\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E)$ is given infinite value if no weighted and centered δ-cover of E exists. Now, applying the standard construction we obtain the mixed weighted generalized Hausdorff $\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}$ defined by

$$\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},0}(E) = \sup_{\delta>0} \mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) = \lim_{\delta\to 0} \mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E),$$
$$\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}(E) = \sup_{F\subseteq E} \mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},0}(F).$$

Remark 2.1. *If the weights c_i are restricted to the value unity, then* (2.1) *requires each point of E to be covered once. That is to be covered in the normal sense. It follows that*

$$\mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) \leq \mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E) \quad \text{and then} \quad \mathcal{W}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}(E) \leq \mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu}}(E).$$

Theorem 2.2. *Let $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$, $\boldsymbol{q} \in \mathbb{R}^k$ and $\zeta \in \Phi(\mathbb{X})$. Then,*

1. *the functions $\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$ and $\mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$ are metric outer measures on $\mathbb{X}$ and thus measures on the Borel family of subsets of $\mathbb{X}$;*
2. *for $E \subseteq \mathbb{X}$, we can find a Borel set B such that,*

$$\mathcal{H}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(B) = \mathcal{H}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(E) \quad \text{and} \quad \mathcal{W}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(B) = \mathcal{W}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(E).$$

Proof. The proof is straightforward and mimics that in [4, Theorem 2.1] and [4, Theorem 2.3]. □

In general, we have $\mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta} \leq \mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$. Therefore it is worth to compute sufficient condition to get the equivalence between $\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$ and $\mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$, that is,

Proposition 2.3. *Let $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$, $\boldsymbol{q} \in \mathbb{R}^k$ and $\zeta \in \Phi_D(\mathbb{X})$. Then, there exists a constant γ such that, for any $E \subseteq \mathbb{X}$ we have*

$$\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E) \leq \gamma\, \mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E). \tag{2.2}$$

provided that $\boldsymbol{q} \in \mathbb{R}_-^k$ or $\boldsymbol{q} \in \mathbb{R}_+^{*,k}$ and $\boldsymbol{\mu} \in (\mathcal{M}_D(\mathbb{X}))^k$.

Proof. Let $\delta > 0$, $F \subseteq E \subseteq \mathbb{X}$ and $\{c_i, B_i\}_{i\geq1}$ be a weighted and centered δ-covering of F. Since $\Big\{\boldsymbol{\mu}(B(x_i,r_i))^{\boldsymbol{q}}\zeta(B(x_i,r_i))\Big\}_{i\geq1}$ is a sequence of non-negative numbers, then using [13, Corollary 4.4], there exists a subfamily $\{B(x_{i_j}, r_{i_j}\}_{j\geq1}$ of balls such that $F \subseteq \bigcup_{j\geq1} B(x_{i_j}, 3r_{i_j})$ and

$$\sum_{j\geq1} \boldsymbol{\mu}\Big(B(x_{i_j}, r_{i_j})\Big)^{\boldsymbol{q}} \zeta\Big(B(x_{i_j}, r_{i_j})\Big) \leq 8 \sum_{i\geq1} c_i \boldsymbol{\mu}\Big(B(x_i, r_i)\Big)^{\boldsymbol{q}} \zeta\Big(B(x_i, r_i)\Big).$$

Therefore, there exists a constant C such that

$$\begin{aligned} \mathcal{H}_{\boldsymbol{\mu},\delta}^{\boldsymbol{q},\zeta}(F) &\leq \sum_{j\geq1} \boldsymbol{\mu}\Big(B(x_{i_j}, 3r_{i_j})\Big)^{\boldsymbol{q}} \zeta\Big(B(x_{i_j}, 3r_{i_j})\Big) \\ &\leq 8\,C \sum_{i\geq1} c_i \boldsymbol{\mu}\Big(B(x_i, r_i)\Big)^{\boldsymbol{q}} \zeta\Big(B(x_i, r_i)\Big) \end{aligned}$$

Now, taking the infimum over all weighted and centered δ-coverings $\{c_i, B_i\}_{i\geq1}$ of F we obtain

$$\mathcal{H}_{\boldsymbol{\mu},\delta}^{\boldsymbol{q},\zeta}(F) \leq 8\,C\, \mathcal{W}_{\boldsymbol{\mu},\delta}^{\boldsymbol{q},\zeta}(F).$$

Letting $\delta \to 0$, one gets

$$\mathcal{H}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(F) \leq 8\,C\, \mathcal{W}_{\boldsymbol{\mu},0}^{\boldsymbol{q},\zeta}(F) \leq 8\,C\, \mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E).$$

Since F is arbitrarily chosen, we obtain $\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E) \leq 8\,C\, \mathcal{W}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E)$. □

2.2. *Mixed generalized measures of packing type*

Let $\delta > 0$, a sequence $(x_i, r_i)_i$, $x_i \in E$ and $r_i > 0$, is a δ-packing of E if, and only if, for all $i, j = 1, 2, \ldots$ we have

$$i \neq j \implies \rho(x_i, x_j) > r_i + r_j,$$

and $r_i \leq \delta$. Now, we write

$$\mathcal{P}_{\boldsymbol{\mu},\delta}^{\boldsymbol{q},\zeta}(E) = \sup\Big\{\sum_i \mu\Big(B(x_i, r_i)\Big)^{\boldsymbol{q}} \zeta(B(x_i, r_i))\Big\},$$

where the supremum is taken over all centered δ-packing of E and with the conventions $0^q = \infty$ for $q \le 0$ and $\mathcal{P}^{q,\zeta}_{\mu,\delta}(E)$ is given infinite value if no centered δ-packing of E exists. The mixed generalized packing pre-measure is then given by

$$\mathcal{P}^{q,\zeta}_{\mu,0}(E) = \lim_{\delta\to 0} \mathcal{P}^{q,\zeta}_{\mu,\delta}(E) = \inf_{\delta>0} \mathcal{P}^{q,\zeta}_{\mu,\delta}(E).$$

The function $\mathcal{P}^{q,\zeta}_{\mu,0}$ is increasing but not σ-additive. By applying now the standard construction [46, 51, 57], we obtain the mixed generalized packing measure defined by

$$\mathcal{P}^{q,\zeta}_{\mu}(E) = \inf\Big\{\sum_{i=1}^{\infty} \mathcal{P}^{q,\zeta}_{\mu,0}(E_i);\quad E \subseteq \bigcup_{i=1}^{\infty} E_i\Big\}.$$

The function $\mathcal{P}^{q,\zeta}_{\mu}$ is, of course, a generalization of the packing measures introduced in [10, 23, 27, 36, 39, 47]. The mixed generalized packing measure is "dual" to mixed generalized Hausdorff measure. We will introduce the mixed weighted generalized packing measure, $\mathcal{Q}^{q,h}_{\mu}$, which may be "dual" to mixed weighted generalized Hausdorff measure. We say that (c_i, x_i, r_i) with $c_i > 0, x_i \in E$ and $r_i > 0$ is a weighted δ-packing of E if, and only if, for all $x \in E$ we have,

$$\sum \Big\{c_i,\ \rho(x_i, x) \le r_i\Big\} \le 1$$

and $r_i \le \delta$ for $i = 1, 2 \ldots$. We write,

$$\mathcal{Q}^{q,\zeta}_{\mu,\delta}(E) = \sup\Big\{\sum_i c_i\ \mu\Big(B(x_i, r_i)\Big)^q \zeta(B(x_i, r_i))\Big\},$$

where the supremum is taken over all centered δ-pseudo-packing of E and with the conventions $0^q = \infty$ for $q \le 0$ and $\mathcal{Q}^{q,\zeta}_{\mu,\delta}(E)$ is given infinite value if no centered δ-weighted-packing of E exists. Now, applying the standard construction we obtain the mixed weighted generalized packing $\mathcal{Q}^{q,\zeta}_{\mu}$ defined by

$$\mathcal{Q}^{q,\zeta}_{\mu,0}(E) = \lim_{\delta\to 0} \mathcal{Q}^{q,\zeta}_{\mu,\delta}(E) = \inf_{\delta>0} \mathcal{Q}^{q,\zeta}_{\mu,\delta}(E),$$

$$\mathcal{Q}^{q,\zeta}_{\mu}(E) = \inf\Big\{\sum_{i=1}^{\infty} \mathcal{Q}^{q,\zeta}_{\mu,0}(E_i);\quad E \subseteq \bigcup_{i=1}^{\infty} E_i\Big\}.$$

In general, we have $\mathcal{P}^{q,\zeta}_{\mu} \le \mathcal{Q}^{q,\zeta}_{\mu}$ since any δ-packing of E is also a δ-weighted packing of E. Therefore it is worth to compute sufficient condition to get the equivalence between $\mathcal{P}^{q,\zeta}_{\mu}$ and $\mathcal{Q}^{q,\zeta}_{\mu}$, that is,

Proposition 2.4. *Let $\mu \in (\mathcal{M}(\mathbb{X}))^k$, $q \in \mathbb{R}^k$ and $\zeta \in \Phi_D(\mathbb{X})$. Then, there exists a constant γ such that, for any $E \subseteq \mathbb{X}$ we have*

$$\mathcal{Q}^{q,\zeta}_{\mu}(E) \le \gamma \mathcal{P}^{q,\zeta}_{\mu}(E), \tag{2.3}$$

provided that $q \in \mathbb{R}^k_-$ or $q \in \mathbb{R}^{*,k}_+$ and $\mu \in (\mathcal{M}_D(\mathbb{X}))^k$.

To prove Proposition 2.4, we introduce a new fractal measure called the mixed generalized pseudo-packing measure, denoted by $\mathcal{R}^{q,\zeta}_{\mu}$. This measure is defined using δ-pseudo-packing of balls. A sequence $(x_i, r_i)_i$, where $x_i \in E$ and $r_i > 0$, is considered a δ-pseudo-packing of E if, for all $i, j = 1, 2, \ldots$ the following conditions are satisfied:

$$i \ne j \implies \rho(x_i, x_j) > \max(r_i, r_j),$$

and $r_i \le \delta$. Now, we write

$$\mathcal{R}^{q,\zeta}_{\mu,\delta}(E) = \sup\Big\{\sum_i \mu\Big(B(x_i, r_i)\Big)^q \zeta(B(x_i, r_i))\Big\},$$

where the supremum is taken over all centered δ-pseudo-packing of E and with the conventions $0^q = \infty$ for $q \leq 0$ and $\mathcal{R}^{q,\zeta}_{\mu,\delta}(E)$ is given infinite value if no centered δ-pseudo-packing of E exists. The mixed generalized packing pre-measure is then given by

$$\mathcal{R}^{q,\zeta}_{\mu,0}(E) = \lim_{\delta\to 0} \mathcal{R}^{q,\zeta}_{\mu,\delta}(E) = \inf_{\delta>0} \mathcal{R}^{q,\zeta}_{\mu,\delta}(E).$$

The function $\mathcal{R}^{q,\zeta}_{\mu,0}$ is increasing but not σ-additive. By applying now the standard construction [46, 51, 57], we obtain the mixed generalized pseudo-packing measure defined by

$$\mathcal{R}^{q,\zeta}_{\mu}(E) = \inf\Big\{ \sum_{i=1}^{\infty} \mathcal{R}^{q,\zeta}_{\mu,0}(E_i); \quad E \subseteq \bigcup_{i=1}^{\infty} E_i \Big\}.$$

The function $\mathcal{R}^{q,\zeta}_{\mu}$ is, of course, a generalization of the pseudo-packing measures introduced in [12, 23, 33].

2.3. Proof of Proposition 2.4

Let $\delta > 0$ and $E \subseteq \mathbb{X}$. We start by proving that $\mathcal{Q}^{q,\zeta}_{\mu}(E) \leq \mathcal{R}^{q,\zeta}_{\mu}(E)$. We may assume that $\mathcal{R}^{q,\zeta}_{\mu}(E) < \infty$. Suppose that we have shown

$$\mathcal{Q}^{q,\zeta}_{\mu,\delta}(E) \leq \mathcal{R}^{q,h}_{\mu,\delta}(E). \tag{2.4}$$

Then, for $\epsilon > 0$, choose a sequence of sets E_i such that

$$E \subseteq \bigcup_i E_i \qquad \text{and} \qquad \sum_i \mathcal{R}^{q,\zeta}_{\mu,0}(E_i) \leq \mathcal{R}^{q,\zeta}_{\mu}(E) + \epsilon.$$

It follows, using (2.4), that

$$\mathcal{Q}^{q,\zeta}_{\mu}(E) \leq \sum_i \mathcal{Q}^{q,\zeta}_{\mu,0}(E_i) \leq \sum_i \mathcal{R}^{q,\zeta}_{\mu,0}(E_i) \leq \mathcal{R}^{q,\zeta}_{\mu}(E) + \epsilon$$

and we get the desire result by letting ϵ to 0. Let us prove (2.4). Let $l < \mathcal{Q}^{q,\zeta}_{\mu,\delta}(E)$. Choose $\{c_i, x_i, r_i\}_i$ a weighted δ-packing of E. By choosing N large enough we may approximate c_i by rational α_i/N such that $\alpha_i/N \leq c_i$ and $\sum_{i=1}^{\infty} \alpha_i/N\boldsymbol{\mu}(B(x_i,r_i))^q\zeta(B(x_i,r_i) > l$. In addition, by relabelling and choosing n sufficiently large we may assume that

$$\sum_{i=1}^{n} \alpha_i/N\boldsymbol{\mu}(B(x_i,r_i))^q\zeta(B(x_i,r_i) > l. \tag{2.5}$$

Now, we define the function $m_0 : \{1,\dots,n\} \to \mathbb{N}_0$ by $m_0(i) = \alpha_i$, where $\mathbb{N}_0$ is the set of the natural numbers including 0. We consider the set of indices

$$J_1 = \Big\{ i \in \{1,\dots,n\},\ m_0(i) \geq 1 \Big\}.$$

It follows, using [23, Lemma 4], that we can choose $I_1 \subseteq J_1$ so that $\{(x_i,r_i), i \in I_1\}$ is maximal pseudo-packing from the family of pairs $\{(x_i,r_i), m_0(i) \geq 1\}$ that covers $\{x_i, m_0(i) \geq 1\}$. Inductively, for $j \geq 1$, we choose $I_j \subseteq J_j$ and define

$$m_j(i) = \begin{cases} m_{j-1}(i) - 1 & \text{if } i \in I_j \\ m_{j-1}(i) & \text{otherwise} \end{cases}$$

where

$$J_j = \Big\{ i \in \{1,\dots,n\},\ m_{j-1}(i) \geq 1 \Big\}.$$

Now, we define the function, for $j \geq 0$, $\zeta_j : \mathbb{X} \to \mathbb{N}_0$ by

$$\zeta_j(x) = \sum \{m_j(i); \quad \rho(x_i, x) \leq r_j\}.$$

It is clear, since I_j covers $\{x_i, m_{j-1}(i) \geq 1\}$, that, for $i \in J_j$, there exists $k \in I_j$ such that $\rho(x_i, x_k) \leq r_k$. It follows that, for each $i \in \{1, \dots, n\}$, we have

$$\zeta_j(x_i) \leq \zeta_{j-1}(x_i) - 1,$$

provides that $m_{j-1}(i) \geq 1$. Suppose that $J_N \neq \emptyset$ and let $k \in J_N \subseteq J_{N-1} \dots \subseteq J_1$. Thus

$$\zeta_0(x_k) \geq N + \zeta_N(x_k) \geq N + m_N(k) \geq N + 1. \tag{2.6}$$

By definition of the weighted packing, we have

$$\zeta_0(x_k) = \sum \left\{\alpha_i, \ \rho(x_i, x_k) \leq r_i\right\} \leq N \sum \left\{\alpha_i, \ \rho(x_i, x) \leq r_i\right\} \leq N$$

contradicting (2.6). Then $J_N = \emptyset$ and

$$\sum_{j=1}^{N} (m_{j-1}(i) - m_j(i)) = \alpha_i.$$

As a consequence, since for each $j \geq 1$, $\sum_{i \in I_j} \boldsymbol{\mu}(B(x_i, r_i))^{\boldsymbol{q}} \zeta(B(x_i, r_i)) \leq \mathcal{R}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E)$, we have

$$\begin{aligned} l < \sum_{i=1}^{n} \alpha_i / N \boldsymbol{\mu}(B(x_i, r_i))^{\boldsymbol{q}} \zeta(B(x_i, r_i)) &= \sum_{j=1}^{N} \sum_{i=1}^{n} \frac{m_{j-1}(i) - m_j(i)}{N} \mu(B(x_i, r_i))^{q} \zeta(B(x_i, r_i)) \\ &\leq \sum_{j=1}^{N} \frac{1}{N} \mathcal{R}^{\boldsymbol{q},\zeta}_{\mu,\delta}(E) = \mathcal{R}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E), \end{aligned}$$

as desired to get (2.4). Now, if (x_i, r_i) is a δ-pseudo-packing of E then $(x_i, r_i/2)$ is a δ-packing of E. Hence, there exists γ such that

$$\mathcal{R}^{\boldsymbol{q},\zeta}_{\mu}(E) \leq \gamma \mathcal{P}^{\boldsymbol{q},\zeta}_{\mu}(E),$$

since $\zeta \in \Phi_D(\mathbb{X})$ and $\boldsymbol{\mu} \in \mathcal{P}_D(\mathbb{X})$. Finely, we get the desired result.

3. Product inequalities of fractal measures

3.1. Densities Theorem

In the following we give a version of density theorem with respect to the mixed generalized Hausdorff and packing measures which will be useful to prove our results this section.

Definition 3.1. *Let $\theta \in \mathcal{M}(\mathbb{X})$, $x \in \operatorname{supp}(\theta)$, $\boldsymbol{q} \in \mathbb{R}^k$ and $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$. The upper, respectively lower, $(\boldsymbol{q}, \zeta)$-density of θ at x with respect to μ, is defined by*

$$\overline{d}_{\mu}^{\,\boldsymbol{q},\zeta}(x, \theta) = \limsup_{r \searrow 0} \frac{\theta\,(B(x, r))}{\boldsymbol{\mu}\,(B(x, r))^{\boldsymbol{q}}\,\zeta\,(B(x, r))},$$

respectively,

$$\underline{d}_{\mu}^{\,\boldsymbol{q},\zeta}(x, \theta) = \liminf_{r \searrow 0} \frac{\theta\,(B(x, r))}{\boldsymbol{\mu}\,(B(x, r))^{\boldsymbol{q}}\,\zeta\,(B(x, r))}.$$

Whenever $\overline{d}_{\mu}^{\,\boldsymbol{q},\zeta}(x, \theta) = \underline{d}_{\mu}^{\,\boldsymbol{q},\zeta}(x, \theta)$, we denote by $d_{\mu}^{\,\boldsymbol{q},\zeta}(x, \theta)$ their common value, which will be called the $(\boldsymbol{q}, \zeta)$-density of θ.

Theorem 3.2. *Let $q \in \mathbb{R}^k$, $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$, $\theta \in \mathcal{M}(\mathbb{X})$ and E be a Borel subset of* $\operatorname{supp}(\theta)$.

1. (a) *If $\boldsymbol{\mu} \in (\mathcal{M}_D(\mathbb{X}))^k$ and $\mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}(E) < \infty$, then*

$$\mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}(E) \inf_{x\in E} \overline{d}_{\boldsymbol{\mu}}^{q,\zeta}(x,\ \theta) \le \theta(E). \tag{3.1}$$

   (b) *If $\mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}(E) < \infty$, then*

$$\theta(E) \le \mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}(E) \sup_{x\in E} \overline{d}_{\boldsymbol{\mu}}^{q,\zeta}(x,\ \theta). \tag{3.2}$$

2. (a) *We have*

$$\mathcal{P}_{\boldsymbol{\mu}}^{q,\zeta}(E) \inf_{x\in E} \underline{d}_{\boldsymbol{\mu}}^{q,\zeta}(x,\theta) \le \theta(E). \tag{3.3}$$

   (b) *If θ has the strong-Vitali property, then*

$$\theta(E) \le \mathcal{P}_{\boldsymbol{\mu}}^{q,\zeta}(E) \inf_{x\in E} \underline{d}_{\boldsymbol{\mu}}^{q,\zeta}(x,\theta). \tag{3.4}$$

Proof. The proof of this Theorem is straightforward and mimics that Lemma 2.10 in [4] and in Theorem 4 in [23]. □

Example 3.3. *Theorem 3.2 will be used to analyze product inequalities for various fractal measures. In this example, we will demonstrate that inequality* (3.3) *remains invalid when considering the measure $\mathcal{R}_{\boldsymbol{\mu}}^{q,\zeta}$. Consequently, our density-based approach cannot be extended to encompass product inequalities involving pseudo-packing measures $\mathcal{R}_{\boldsymbol{\mu}}^{q,\zeta}$. To this end, let $\mathcal{X} = \prod_{n=1}^{\infty} G(N_n)$ be the Davies metric space [10, 23] constructed using the sequence $\{\gamma_n\}$. Now, we consider the set of Hausdorff function $\mathcal{F}$; that is, the set of continuous and increasing functions $h : [0,\infty) \to]0,\infty)$ satisfied $h(0) = 0$. It is proven in [10] that for all $h \in \hat{\mathcal{F}}$, where*

$$\hat{\mathcal{F}} = \Big\{h \in \mathcal{F},\ \ h(2^{-n+1}) = \gamma_n\Big\},$$

we have $\mathcal{R}^h(\mathcal{X}) = 1 \ge \mathcal{P}^h(\mathcal{X}) = 0$. Take n such that $(1/2)^n \le r < (1/2)^{n-1}$ and let μ be the uniform measure on $\mathcal{X}$; that is $\mu(B(u,r)) = 2\gamma_n$. Let E be the set of eventually peripheral points of $\mathcal{X}$, so $\mu(E) = 1$ [10]. If follows that,

$$\frac{\mu(B(u,r))}{h(r)} \ge \frac{2\gamma_n}{h(2^{-n+1})} = \frac{2\gamma_n}{\gamma_n} = 2. \tag{3.5}$$

Therefore $\inf_{u\in E} \underline{d}_{\mu}^{q,h}(x,\mu) \ge 2$. Observe that, we have the equality in (3.5) *for $r = 2^{-n+1}$. Then we obtain $\inf_{u\in E} \underline{d}_{\mu}^{q,h}(x,\mu) = 2$. Moreover, since $h \in \hat{\mathcal{F}}$ then $\mathcal{R}^h(E) = \mu(E) = 1$ [10] which implies that*

$$\mathcal{R}^h(E) \inf_{u\in E} \underline{d}_{\mu}^{q,h}(x,\nu) = 2.$$

On the other hand, we have $\mu(E) = 1$. So, we obtain $2 < 1$ which is impossible.

Next, we present the following lemma, which will play a crucial role in proving the main results discussed in the next results.

Lemma 3.4. *Let $\zeta \in \Phi_D(\mathbb{X})$, $\boldsymbol{q} \in \mathbb{R}^k$, $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$ and $E \subseteq \operatorname{supp} \boldsymbol{\mu}$ be a Borel set such that $\mathcal{P}_{\boldsymbol{\mu}}^{q,\zeta}(E) < +\infty$. Let $\theta_1 = \mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}{}_{\llcorner E}$ and $\theta_2 = \mathcal{P}_{\boldsymbol{\mu}}^{q,\zeta}{}_{\llcorner E}$.*

1. *If $\boldsymbol{\mu} \in (\mathcal{M}_D(\mathbb{X}))^k$, we have*

$$\overline{d}_{\boldsymbol{\mu}}^{q,\zeta}(x,\theta_1) = 1, \qquad \mathcal{H}_{\boldsymbol{\mu}}^{q,\zeta}\text{-a.a. on } E.$$

2. *If θ_2 has the strong-Vitali property, then*

$$\underline{d}_{\mu}^{q,\zeta}(x,\theta_2)=1,\qquad \mathcal{P}_{\mu}^{q,h}\text{-a.a. on } E.$$

3. *If there exists $\theta\in\mathcal{M}(\mathbb{X})$ such that $\sup_{x\in E}\overline{d}_{\mu}^{q,\zeta}(x,\ \theta)\le\gamma<\infty$, then*

$$\mathcal{H}_{\mu}^{q,\zeta}(E)\ge\theta(E)/\gamma.$$

4. *If there exists $\theta\in\mathcal{M}(\mathbb{X})$ such that $\inf_{x\in E}\underline{d}_{\mu}^{q,\zeta}(x,\ \theta)\ge\gamma>0$, then*

$$\mathcal{P}_{\mu}^{q,\zeta}(E)\le\theta(E)/\gamma.$$

Proof. The proof of this Lemma is straightforward and mimics that Corollary 2 in [23] and Corollary 2.11 in [4]. □

3.2. Product inequalities related to Hausdorff measures

In this section, we consider first the case when ζ_0 is defined on $\mathcal{B}(\mathbb{X}\times\mathbb{X}')$ by (1.1). In this situation, we may density theorem so that, we will prove Theorem 3.5.

Theorem 3.5. *Let $\boldsymbol{\mu}\in(\mathcal{M}_D(\mathbb{X}))^k$, $\boldsymbol{\nu}\in(\mathcal{M}_D(\mathbb{X}'))^k$, $\zeta\in\Phi_D(\mathbb{X})$, $\zeta'\in\Phi_D(\mathbb{X}')$, $\boldsymbol{q}\in\mathbb{R}^k$, E and E' be two Borel sets of* $\operatorname{supp}(\boldsymbol{\mu})$ *and* $\operatorname{supp}(\boldsymbol{\nu})$ *respectively. Assume that $\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E)<+\infty$ and $\mathcal{H}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}(E')<+\infty$. Then,*

$$\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E)\mathcal{H}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}(E')\le\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}(E\times E').$$

Proof. Let θ_1 be the restriction of $\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}$ to E and θ_2 be the restriction of $\mathcal{H}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}$ to E'. We set

$$\widetilde{E}=\left\{x\in E,\quad \overline{d}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(x,\theta_1)=1\right\}$$

and

$$\widetilde{E'}=\left\{x\in E',\quad \overline{d}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}(x,\theta_2)=1\right\}.$$

Then, using Lemma 3.4, we have $\theta_1(E)=\theta_1(\widetilde{E})$ and $\theta_2(E')=\theta_2(\widetilde{E'})$. Therefore, for $(x,y)\in\widetilde{E}\times\widetilde{E}$, we have

$$\begin{aligned}\overline{d}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}\Big((x,y),\theta_1\times\theta_2\Big) &= \limsup_{r\to0}\left[\frac{\theta_1\big(B(x,r)\big)}{\boldsymbol{\mu}\big(B(x,r)\big)^{\boldsymbol{q}}\zeta(B(x,r))}\ \frac{\theta_2\big(B(y,r)\big)}{\boldsymbol{\nu}\big(B(y,r)\big)^{\boldsymbol{q}}\zeta'(B(y,r))}\right]\\ &\le \overline{d}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}\Big(x,\theta_1\Big)\ \overline{d}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}\Big(y,\theta_2\Big)=1.\end{aligned}$$

Therefore, by Lemma 3.4,

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}\Big(\widetilde{E}\times\widetilde{E'}\Big)\ge\theta_1\times\theta_2\Big(\widetilde{E}\times\widetilde{E'}\Big)=\theta_1\Big(\widetilde{E}\Big)\theta_2\Big(\widetilde{E'}\Big)=\theta_1(E)\theta_2(E').$$

Hence,

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}(E\times E')\ge\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}\Big(\widetilde{E}\times\widetilde{E'}\Big)\ge\theta_1(E)\theta_2(E')=\mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E)\mathcal{H}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}(E').$$

□

It is natural to ask whether an inequality in the opposite direction holds. More precisely, does there exist a constant c such that

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\boldsymbol{q},\zeta_0}(E\times E')\le c\ \mathcal{H}_{\boldsymbol{\mu}}^{\boldsymbol{q},\zeta}(E)\mathcal{H}_{\boldsymbol{\nu}}^{\boldsymbol{q},\zeta'}(E')?$$

The following example shows that the answer is negative.

Example 3.6. *In this example, we take $\mathbb{X} = \mathbb{X}' = \mathbb{R}$, and*

$$\boldsymbol{\mu} = \boldsymbol{\nu} = (\mathcal{L}^1, \dots, \mathcal{L}^1), \qquad k \geq 2.$$

Let $\boldsymbol{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$, $\zeta(B(x,r)) = (\mathrm{diam}(B(x,r))^s$ and $\zeta'(B(x,r)) = (\mathrm{diam}(B(x,r))^t$ such that

$$\Big(\sum_{i=1}^{k} q_i\Big) + t > 0 \qquad \text{and} \qquad \Big(\sum_{i=1}^{k} q_i\Big) + s > 0. \tag{3.6}$$

We show that there exist Borel subsets A and B of $\mathbb{R}$ of Hausdorff measure 0 such that $\mathcal{H}^{\boldsymbol{q},\zeta_0}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A \times B) > 0$. Let $\{\alpha_j\}$ be a decreasing sequence of numbers with $\lim_{j\to\infty} \alpha_j = 0$ and let $\{m_j\}$ be increasing sequence of integers. We can choose $m_0 = 0$ and $\{m_j\}$ rapidly enough to ensure that, for all $j \geq 1$

$$\begin{cases} \displaystyle\sum_{k=0}^{j-1} (m_{2k+1} - m_{2k}) & \leq \alpha_j \, m_{2j} \\ \displaystyle\sum_{k=1}^{j} (m_{2k} - m_{2k-1}) & \leq \alpha_j \, m_{2j} \end{cases} \tag{3.7}$$

Consider the set $A \subset [0,\ 1]$ such that, if r is odd and $m_j + 1 \leq r \leq m_{j+1}$ then the rth decimal place is zero, i.e., A is the set of x such that

$$x = 0, x_1 \dots x_{m_1} \underbrace{0 \dots 0}_{(m_2 - m_1)\ times} x_{m_2+1} \dots x_{m_3} \underbrace{0 \dots 0}_{(m_4 - m_3)\ times} \dots$$

where $x_i \in \{0, 1, \dots, 9\}$. Similarly take the set $B \subset [0,\ 1]$ such that, if r is even and $m_{j+1} \leq r \leq m_j + 1$ then the rth decimal place is zero, i.e., B is the set of x such that

$$x = 0, \underbrace{0 \dots 0}_{m_1\ times} x_{m_1+1} \dots x_{m_2} \underbrace{0 \dots 0}_{(m_3 - m_2)\ times} x_{m_3+1} \dots x_{m_4} \dots$$

where $x_i \in \{0, 1, \dots, 9\}$. It is clear that we can cover A by 10^k intervals of length $10^{-m_{2j}}$ where

$$k = (m_1 - m_0) + (m_3 - m_2) + \dots + (m_{2j-1} - m_{2j-2}).$$

It follows from (3.7) and (3.6) that $\mathcal{H}^{q,\zeta}_{\mu,0}(A) = 0$. This is true for all set $\widetilde{A} \subseteq A$, then $\mathcal{H}^{q,\zeta}_{\mu}(A) = 0$, and similarly we have $\mathcal{H}^{q,\zeta'}_{\nu}(B) = 0$. Now let ϕ denote orthogonal projection from the plane onto the line $L : y = x$. Then $\phi(x, y)$ is the point of L at distance $2^{-1/2}(x + y)$ from the origin. Take $v \in [0,\ 1]$ we may find two numbers $x \in A$ and $y \in B$ such that $v = x + y$, indeed some of the decimal digits of v are provided by x, the rest by y. Thus $\phi(A \times B)$ is a sub-interval of L of length $2^{-1/2}$. Using [14, Corollary 2.4], we have

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu}(A \times B) \geq 1 > 0 = \mathcal{H}^{q,\zeta}_{\mu}(A)\mathcal{H}^{q,\zeta'}_{\nu}(B).$$

Remark 3.7. *The $0 - \infty$ case does not present any problem in our situation, as we have adopted the convention that $0 \times \infty = 0$. However, we can still raise the question if $\mathcal{H}^{\boldsymbol{q},\zeta_0}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(E \times E')$ is always equal to ∞? This question is important for understanding the behavior of product measures in different contexts. To answer this question, the author in [18] construct four sets $\mathcal{K}_1$ and $\mathcal{K}_2^j$, $j = 1, 2, 3$, where $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be the one-dimensional generalized Cantor set. For $\boldsymbol{q} = 0$, $\zeta(B(x,r)) = r^t$ and $\zeta'(B(x,r)) = r^s$ for all $B(x,r)$ in $\mathbb{R}^n$, they proved that under appropriate condition, $\mathcal{H}^t_0(\mathcal{K}_1) = \infty$ and $\mathcal{H}^s_0(\mathcal{K}_2^j) = 0$ while $\mathcal{H}^{t+s}_0(\mathcal{K}_1 \times \mathcal{K}_2^j)$ is infinite, positive finite, zero according as $j = 1, 2, 3$ respectively.*

Example 3.8. *This example proves that the reverse inequality of* (1.3) *does not holds. Let E be the segment in the plane with endpoints* $(0,0)$ *and* $(1,1)$*. Take* $\mu(B(x,r)) = \nu(B(x,r)) = \mathcal{L}^1(B(x,r))$ *and* $\zeta(B(x,r)) = (2r)^t, \zeta'(B(x,r)) = (2r)^s$ *for all ball* $B(x,r)$ *in* $\mathbb{R}$*. Let* $m(B(x,y),r) = \min\{\mu(B(x,r)), \nu(B(y,r))\}$ *and* $\theta = \mathcal{H}^1_{|E}$*. Take* $q,t,s \in \mathbb{R}$ *such that* $2 = 2q + t + s$ *and notice that*

$$\begin{aligned}\frac{\theta(B((x,y),r))}{\mu\times\nu(B((x,y),r))^q\zeta_0(B((x,y),r))} &\le \frac{\sqrt{2}\; m(B((x,y),r))}{\mu\times\nu(B((x,y),r))^q\zeta_0(B((x,y),r))}\\ &= \sqrt{2}\;(2r)^{2-2q-t-s} = \sqrt{2}\end{aligned}$$

Hence, $\sup_{(x,y)\in E} \overline{d}^{q,\zeta_0}_{\mu\times\nu}((x,y),\theta) \le \sqrt{2}$*. It follows, using Lemma 3.4, that*

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu}(E) \ge \theta(E)/\sqrt{2} = 1.$$

In the other hand, for all $x \in (0,1)$*,* E^x *is a singleton so for all* $s,t > 0$*, we have*

$$\int \mathcal{H}^{q,s}_{\mu}(E^y)\, d\mathcal{H}^{q,t}_{\nu}(y) = \int 0\, d\mathcal{H}^{q,t}_{\nu}(y) = 0.$$

Now, in order to obtain the reversed inequality of (1.2), we need to modify the construction of the generalized Hausdorff measure in the product space. Specifically, we define the Cartesian product measure ζ_0 on the class of rectangles $\mathcal{B}(\mathbb{X}) \times \mathcal{B}(\mathbb{X}')$ by

$$\zeta_0(B \times B') = \zeta(B)\;\zeta'(B').$$

In order to include the $0-\infty$ case, we shall require some restrictions in the metric space $\mathbb{X}$ as indicated in the following Remark. We assume that, for a given $\boldsymbol{q} \in \mathbb{R}^k$, $(\mathbb{X}, \zeta, \boldsymbol{\mu})$ satisfy the following assumption

$$\begin{aligned}&\mathbb{X} \text{ can be covered by a countable collection of balls } (B_i)_i \text{ with arbitrarily small}\\ &\text{diameters and that the expression } \mu(B_i)^q\zeta(B_i) \text{ remains finite for each } i.\end{aligned} \tag{3.8}$$

Remark 3.9. *Let* $\boldsymbol{q} \in \mathbb{R}^k$*,* $\delta > 0$*,* $E \subseteq \mathbb{X}$ *and* $E' \subseteq \mathbb{X}'$*. Assume that* $(\mathbb{X}, \zeta, \boldsymbol{\mu})$ *and* $(\mathbb{X}', \zeta', \boldsymbol{\nu})$ *both satisfy condition* (3.8)*. Then, if* $\mathcal{H}^{\boldsymbol{q},\zeta}_{\boldsymbol{\mu},\delta}(E)$ *is infinite and* $\mathcal{H}^{\boldsymbol{q},\zeta'}_{\boldsymbol{\nu},\delta}(E')$ *is zero then* $\mathcal{H}^{\boldsymbol{q},\zeta_0}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\delta}(E\times E')$ *must also be zero. Indeed, let us consider* $(B_i)_{i\ge1}$ *a centered* δ*-cover of E such that* $\{\boldsymbol{\mu}(B_i)^{\boldsymbol{q}}\zeta(B_i)\}$ *is finite for each* $i \ge 1$*. Then, for* $\epsilon > 0$*, we may choose for each i, a centered* δ*-cover* $(B'_{i_j})_{j\ge1}$ *of* E' *such that*

$$\big(\boldsymbol{\mu}(B_i)\big)^{\boldsymbol{q}}\,\zeta(B_i)\sum_{j=1}^{\infty}\big(\boldsymbol{\nu}(B'_{i_j})\big)^{\boldsymbol{q}}\,\zeta'(B'_{i_j}) < \frac{\epsilon}{2^i}. \tag{3.9}$$

Consequently, $(B_i \times B'_{i_j})_{i,j\ge1}$ *is a centered* δ*-cover of* $E\times E'$*. Using (3.9), we get :*

$$\begin{aligned}\mathcal{H}^{\boldsymbol{q},\zeta_0}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\delta}(E\times E') &\le \sum_{i,j=1}^{\infty}\boldsymbol{\mu}\times\boldsymbol{\nu}(B_i\times B'_{i_j})^{\boldsymbol{q}}\zeta_0(B_i\times B'_{i_j})\\ &= \sum_{i,j=1}^{\infty}\boldsymbol{\mu}(B_i)^{\boldsymbol{q}}\boldsymbol{\nu}(B'_{i_j})^{\boldsymbol{q}}\zeta(B_i)\zeta'(B'_{i_j})\\ &= \sum_{i=1}^{\infty}\boldsymbol{\mu}(B_i)^{\boldsymbol{q}}\zeta(B_i)\Big(\sum_{j=1}^{\infty}\boldsymbol{\nu}(B'_{i_j})^{\boldsymbol{q}}\zeta'(B'_{i_j})\Big)\\ &< \sum_{i=1}^{\infty}\frac{\epsilon}{2^i} = \epsilon\,.\end{aligned}$$

Theorem 3.10. *Let $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$, $\boldsymbol{\nu} \in (\mathcal{M}(\mathbb{X}'))^k$, $\zeta \in \Phi(\mathbb{X})$, $\zeta' \in \Phi(\mathbb{X}')$ and $\boldsymbol{q} \in \mathbb{R}^k$. Assume that both triples $(\mathbb{X}, \zeta, \boldsymbol{\mu})$ and $(\mathbb{X}', \zeta', \boldsymbol{\nu})$ satisfy* (3.8)*, then*

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu}(E \times E') \leq \mathcal{H}^{q,\zeta}_{\mu}(E)\mathcal{H}^{q,\zeta'}_{\nu}(E'). \tag{3.10}$$

Proof. We first prove that

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu,\delta}(E \times E') \leq \mathcal{H}^{q,\zeta}_{\mu,\delta}(E)\mathcal{H}^{q,\zeta'}_{\nu,\delta}(E'). \tag{3.11}$$

We assume that $\mathcal{H}^{q,\zeta}_{\mu,\delta}(E)\mathcal{H}^{q,\zeta'}_{\nu,\delta}(E')$ is finite. Let $\{B_i\}_{i\geq1}$ and $\{B'_i\}_{i\geq1}$ be centered δ-covers for E and E' respectively. The Cartesian product $E \times E'$ is then covered by the sets $\{B_i \times B'_j\}_{i,j\geq1}$ is a centered δ-cover for $E \times E'$. As a consequence, we have

$$\begin{aligned}\mathcal{H}^{q,\zeta_0}_{\mu\times\nu,\delta}(E \times E') &\leq \sum_{i,j=1}^{\infty} \boldsymbol{\mu} \times \boldsymbol{\nu}(B_i \times B'_j)^{q}\zeta_0(B_i \times B'_j)\\ &= \sum_{i,j=1}^{\infty} \boldsymbol{\mu}(B_i)^{q}\boldsymbol{\nu}(B'_j)^{q}\zeta(B_i)\zeta'(B'_j)\\ &= \Big(\sum_{i=1}^{\infty} \boldsymbol{\mu}(B_i)^{q}\zeta(B_i)\Big)\Big(\sum_{j=1}^{\infty} \boldsymbol{\nu}(B_i)^{q}\zeta'(B'_i)\Big).\end{aligned}$$

Then, (3.11) holds. Now, let $E_1 \subseteq E$ and $E'_1 \subseteq E'$. By letting $\delta \to 0$,

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu,0}(E_1 \times E'_1) \leq \mathcal{H}^{q,\zeta}_{\mu,0}(E_1)\mathcal{H}^{q,\zeta'}_{\nu,0}(E'_1) \leq \mathcal{H}^{q,\zeta}_{\mu}(E)\mathcal{H}^{q,\zeta'}_{\nu}(E')\,.$$

Therefore

$$\mathcal{H}^{q,\zeta_0}_{\mu\times\nu}(E \times E') \leq \mathcal{H}^{q,\zeta}_{\mu}(E)\mathcal{H}^{q,\zeta'}_{\nu}(E')\,.$$

□

Note that inequality fails to hold in general when applied to the class of rectangles. However, a valid alternative can be established by introducing the weighted Hausdorff measure (see section 2). Specifically, we have the following result

Theorem 3.11. *Let $\boldsymbol{q} \in \mathbb{R}^k$, $\zeta \in \Phi(\mathbb{X})$ and $\zeta' \in \Phi(\mathbb{X}')$. Assume that both $(\mathbb{X}, \zeta, \boldsymbol{\mu})$ and $(\mathbb{X}', \zeta', \boldsymbol{\nu})$ satisfy* (3.8)*, then*

$$\mathcal{W}^{q,\zeta}_{\mu}(E)\mathcal{H}^{q,\zeta'}_{\nu}(E') \leq \mathcal{H}^{q,\zeta_0}_{\mu\times\nu}(E \times E') \tag{3.12}$$

Proof. Without loss of generality, we suppose that $\mathcal{H}^{q,\zeta_0}_{\mu\times\nu,\delta}(E \times E')$ is finite and $\mathcal{H}^{q,\zeta'}_{\nu,\delta}(E')$ is positive. Let p any number such that $0 < p < \mathcal{H}^{q,\zeta'}_{\nu,\delta}(E')$ and $(B_i \times B'_i)_{i\geq1}$ be any centered δ-cover of $E \times E'$. It follows that $(B'_i)_{i\geq1}$ is a centered δ-cover for E'. For each i, let $u_i = \nu(B'_i)^q\zeta'(B'_i)/p$, then

$$p < \sum_{i=1}^{\infty} \nu(B'_i)^q\zeta'(B'_i) \quad \text{and} \quad \sum_{\substack{i=1\\ x\in B_i}}^{\infty} u_i > 1\,.$$

Since this holds for each $x \in E$, we obtain a weighted and centered δ-cover $(u_i, B_i)_{i\geq1}$ of E and we have :

$$\begin{aligned}\sum_{i=1}^{\infty} \boldsymbol{\mu} \times \nu(B_i \times B'_i)^q\zeta_0(B_i \times B'_i) &= \sum_{i=1}^{\infty} \boldsymbol{\mu}(B_i)^q\zeta(B_i)\nu(B'_i)^q\zeta'(B'_i)\\ &= p\sum_{i=1}^{\infty} u_i\boldsymbol{\mu}(B_i)^q\zeta(B_i) \geq p\mathcal{W}^{q,\zeta}_{\mu,\delta}(E)\,.\end{aligned}$$

Since $(B_i \times B_i')_{i\geq 1}$ and p are arbitrarily chosen, we obtain

$$\mathcal{W}^{q,\zeta}_{\mu,\delta}(E)\mathcal{H}^{q,\zeta'}_{\nu,\delta}(E') \leq \mathcal{H}^{q,\zeta_0}_{\mu\times\nu,\delta}(E\times E').$$

□

3.3. Product inequalities related to packing measures

We now turn to the study of packing measures on Cartesian products. The following theorem establishes an important inequality that provides an upper bound for the packing measure of a product set in terms of the individual packing measures of its factors. Its proof relies on the density approach, which was introduced earlier and will allow us to derive precise estimates on the measure of the product set.

Theorem 3.12. *Let $\boldsymbol{\mu} \in (\mathcal{M}_D(\mathbb{X}))^k$, $\boldsymbol{\nu} \in (\mathcal{M}_D(\mathbb{X}'))^k$, $\zeta \in \Phi_D(\mathbb{X})$, $\zeta' \in \Phi_D(\mathbb{X}')$ and $\boldsymbol{q} \in \mathbb{R}^k$. For $E \subseteq \mathbb{X}$ and $E' \subseteq \mathbb{X}'$, we have*

$$\mathcal{P}^{q,\zeta_0}_{\mu\times\nu}(E\times E') \leq \mathcal{P}^{q,\zeta}_{\mu}(E)\,\mathcal{P}^{q,\zeta'}_{\nu}(E'),$$

provided it is true in the case when the term on the right side is not $0\times\infty$ or $\infty\times 0$.

Proof. If $\mathcal{P}^{q,\zeta}_{\mu}(E)=\infty$ or $\mathcal{P}^{q,\zeta'}_{\nu}(E')=\infty$ there is nothing to prove. So, we assume they are both finite. Let θ_1 be the restriction of $\mathcal{P}^{q,\zeta}_{\mu}$ to E and θ_2 be the restriction of $\mathcal{P}^{q,\zeta'}_{\nu}$ to E'. We set

$$\widetilde{E} = \left\{x\in E,\quad \underline{d}^{q,\zeta}_{\mu}(x,\theta_1)=1\right\},$$

and

$$\widetilde{E'} = \left\{x\in E',\quad \underline{d}^{q,\zeta'}_{\nu}(x,\theta_2)=1\right\}.$$

Then, using Lemma 3.4, we have $\theta_1(E)=\theta_1(\widetilde{E})$ and $\theta_2(E')=\theta_2(\widetilde{E'})$. Therefore, for $(x,y)\in\widetilde{E}\times\widetilde{E}$, we have

$$\begin{aligned}\underline{d}^{q,\zeta_0}_{\mu\times\nu}\Big((x,y),\theta_1\times\theta_2\Big) &= \liminf_{r\to 0}\left[\frac{\theta_1\Big(B(x,r)\Big)}{\mu\Big(B(x,r)\Big)^q\zeta(B(x,r))}\,\frac{\theta_2\Big(B(y,r)\Big)}{\nu\Big(B(y,r)\Big)^q\zeta'(B(y,r))}\right]\\ &\geq \underline{d}^{q,\zeta}_{\mu}\Big(x,\theta_1\Big)\,\underline{d}^{q,\zeta'}_{\nu}\Big(y,\theta_2\Big)=1.\end{aligned}$$

Therefore, Lemma 3.4, we have

$$\begin{aligned}\mathcal{P}^{q,\zeta_0}_{\mu\times\nu}\Big(\widetilde{E}\times\widetilde{E'}\Big) &\leq \theta_1\times\theta_2\Big(\widetilde{E}\times\widetilde{E'}\Big)=\theta_1\Big(\widetilde{E}\Big)\,\theta_2\Big(\widetilde{E'}\Big)\\ &= \theta_1(E)\,\theta_2(E')=\mathcal{P}^{q,\zeta}_{\mu}(E)\,\mathcal{P}^{q,\zeta'}_{\nu}(E').\end{aligned}$$

□

Transitioning from Theorem 3.12 to Theorem 3.13, we replace one of the generalized packing measures with its weighted version. This modification allows the product inequality to hold without the need for any doubling conditions on the measures $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, resulting in a more general and broadly applicable formulation

Theorem 3.13. *Let $\boldsymbol{\mu} \in (\mathcal{M}(\mathbb{X}))^k$, $\boldsymbol{\nu} \in (\mathcal{M}(\mathbb{X}'))^k$, $\zeta \in \Phi(\mathbb{X})$, $\zeta' \in \Phi(\mathbb{X}')$ and $\boldsymbol{q} \in \mathbb{R}^k$. For $E \subseteq \mathbb{X}$ and $E' \subseteq \mathbb{X}'$, we have*

$$\mathcal{P}^{q,\zeta_0}_{\mu\times\nu}(E\times E') \leq \mathcal{Q}^{q,\zeta}_{\mu}(E)\,\mathcal{P}^{q,\zeta'}_{\nu}(E'),$$

provided that the product on the right-hand side is not of the form $0\times\infty$ or $\infty\times 0$.

Proof. We may assume that $\mathcal{Q}_{\mu}^{q,\zeta}(E) < \infty$ and $\mathcal{P}_{\nu}^{q,\zeta'}(E') < \infty$. For $\varepsilon > 0$, we choose sequences of sets $\{E_i\}_{i\geq 1}$ and $\{E'_j\}_{j\geq 1}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \sum_{i=1}^{\infty} \mathcal{Q}_{\mu,0}^{q,\zeta}(E_i) \leq \mathcal{Q}_{\mu}^{q,\zeta}(E) + \epsilon$$

$$E' \subseteq \bigcup_{j=1}^{\infty} E'_j \quad \text{and} \quad \sum_{j=1}^{\infty} \mathcal{P}_{\mu,0}^{q,\zeta}(E'_i) \leq \mathcal{P}_{\mu}^{q,\zeta}(E') + \epsilon.$$

Now, we will prove that

$$\mathcal{P}_{\mu\times\nu,\delta}^{q,\zeta\zeta'}(E \times E') \leq \mathcal{Q}_{\mu,\delta}^{q,\zeta}(E)\, \mathcal{P}_{\nu,\delta}^{q,\zeta'}(E'). \tag{3.13}$$

Let $\delta > 0$ and $l < \mathcal{P}_{\mu\times\nu,\delta}^{q,\zeta\zeta'}(E \times F)$. Choose $\{(x_i,\ y_i), r_i\}_i$ a δ-packing of $E \times E'$ such that

$$\sum_{i=1}^{\infty} \mu(B(x_i, r_i))^q \nu(B(y_i, r_i))^q \zeta(B(x_i, r_i))\zeta'(B(x_i, r_i)) > l. \tag{3.14}$$

Let $N, \eta \in \mathbb{R}$ and, for each $i = 1, 2, \ldots,$

$$a_i = N\mu(B(x_i, r_i))^q \zeta(B(x_i, r_i)) - \eta \quad \text{and} \quad b_i = N\nu(B(y_i, r_i))^q \zeta'(B(x_i, r_i)) - \eta.$$

We can choose N big enough and η small enough, so that

$$\sum_{i=1}^{\infty} \frac{a_i b_i}{N^2} > l.$$

In addition, by relabeling and choosing n sufficiently large we may assume that $\sum_{i=1}^{n} \frac{a_i b_i}{N^2} > l$, with $a_i > 0$ and $b_i > 0$. Let $x \in E$, then

$$\Big\{(y_i, r_i),\ \ \rho(x_i, x) \leq r_i\Big\}$$

is a δ-packing of F. It follows that

$$\begin{aligned}\sum_i \{b_i,\ \ \rho(x_i, x) \leq r_i\} &\leq \sum_i N\nu(B(y_i, r_i))^q \zeta'(B(x_i, r_i)) \\ &\leq N\mathcal{P}_{\nu,\delta}^{q,\zeta'}(E').\end{aligned}$$

Thus, $(b_i/N,\ x_i,\ r_i)$ is a weighted δ-packing of E. Hence (3.13) follows. Therefore, for all $i, j = 1, 2, \cdots$

$$\mathcal{P}_{\mu\times\nu,0}^{q,\zeta\zeta'}(E_i \times E'_j) \leq \mathcal{Q}_{\mu,0}^{q,\zeta}(E_i)\, \mathcal{P}_{\nu,0}^{q,\zeta'}(E'_j). \tag{3.15}$$

Thus summing over i and j, we have

$$\begin{aligned}\sum_{i,j} \mathcal{P}_{\mu\times\nu,0}^{q,\zeta\zeta'}(E \times E') &\leq \sum_{i,j} \mathcal{Q}_{\mu,0}^{q,\zeta}(E_i)\mathcal{P}_{\nu,0}^{q,\zeta'}(E'_i) \\ &\leq \Big(\mathcal{Q}_{\mu}^{q,\zeta}(E) + \epsilon\Big)\Big(\mathcal{P}_{\nu}^{q,\zeta}(E') + \epsilon\Big).\end{aligned}$$

The result follows on letting $\epsilon \to 0$. □

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