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Hyers-Ulam stability of generalized ε-phase isometries

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Abstract. In this paper, let M and N be two real Hilbert spaces. We first prove that the standard generalized ε-phase isometries have a wider range than the standard ε-phase isometries, and then we prove the stability of standard generalized ε-phase isometries. Finally,we further study the Hyers-Ulam stability of standard generalized ε-phase isometries. That is, we prove that there is a linear surjective isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that

$$\|f(u) - \sigma(u)U(u)\| \leq 2\sqrt{2}\varepsilon, \forall u \in M$$

if $f : M \to N$ is a standard and almost surjective generalized ε-phase isometry.

1. Introduction

Let M and N be two real Hilbert spaces. Let X and Y be two real Banach spaces. Let $\langle \cdot, \cdot \rangle$ be the inner product on Hilbert spaces.

Definition 1.1. *A mapping $f : M \to N(X \to Y)$ is said to be standard if $f(0) = 0$.*

Definition 1.2. *A mapping $f : M \to N$ is called an almost surjective mapping if there exists $\delta > 0$ such that for all $v \in N$, $\exists u \in M$ satisfying*

$$\|f(u) - v\| < \delta.$$

Definition 1.3. *Let I be a nonempty set. Let $\mathcal{U}$ be family of subsets of I. $\mathcal{U}$ is called a non-principal (free) ultrafilter if the followings hold:*

1. *Non-triviality: $\emptyset \notin \mathcal{U}$ and the intersection of all sets in $\mathcal{U}$ is empty (i.e., $\bigcap_{W\in\mathcal{U}} W = \emptyset$);*
2. *Closure under finite intersections: If $W_1, W_2 \in \mathcal{U}$, then $W_1 \cap W_2 \in \mathcal{U}$;*
3. *Upward closure: For $W_1 \in \mathcal{U}$, if $W_1 \subseteq W_2$, then $W_2 \in \mathcal{U}$;*
4. *Ultra property: For any subset $W \subseteq I$, either $W \in \mathcal{U}$ or its complement $I \backslash W \in \mathcal{U}$.*

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Definition 1.4. *Let J be a Hausdorff space. A function $f : I \to J$ is said to $\mathcal{U}$-converge to a point $j \in J$ if, for every neighborhood W of j, the preimage $f^{-1}(W)$ belongs to $\mathcal{U}$. This convergence is denoted by*

$$\lim_{u,\mathcal{U}} f(u) = j.$$

A key result states that if J is compact,then every function $f : I \to J$ must $\mathcal{U}$-converge to some point in J.

Definition 1.5. *A mapping $f : M \to N$ is called a phase isometry if*

$$\{\|f(u) + f(v)\|, \|f(u) - f(v)\|\} = \{\|u + v\|, \|u - v\|\}, \forall u, v \in M.$$

Maksa and Páles [13] proved that $f : M \to N$ is a phase isometry if and only if $|(f(u), f(v))| = |(u, v)|$ for all $u, v \in M$. The Wigner unitarity-antiunitarity theorem in [12] indicates that $|(f(u), f(v))| = |(u, v)|$ if and only if f satisfies $f = \sigma U$, where $U : M \to N$ is a linear isometry and $\sigma : M \to \{-1, 1\}$ is a phase mapping. In [17], Sun et.al studied phase-isometries between the positive cones of c_0. In addition, phase-isometries between the positive cones of the Banach space of continuous real-valued functions have been characterized in [10].

Definition 1.6. *A mapping $f : X \to Y$ is said to be an isometry if*

$$\|f(u) - f(v)\| = \|u - v\|, \forall u, v \in X.$$

The Mazur-Ulam Theorem[14] asserts that if $f : X \to Y$ is a standard surjective isometry, then f must be linear. Figiel Theorem [9] deals with the non-surjective isometries. It constructs a norm-1 linear operator $F : \overline{\text{span}} f(X) \to X$ such that $F \circ f = Id$ (the identity on X). In [20], Wang and Yao studied isometries and phase-isometries of non-Archimedean normed spaces.

Definition 1.7. *Let $\varepsilon \geq 0$. A mapping $f : X \to Y$ is said to be an ε-isometry provided*

$$|\|f(u) - f(v)\| - \|u - v\|| \leq \varepsilon, \forall u, v \in X.$$

In the research of ε-isometries, the stability problem is frequently discussed. Qian proposed the following problem in [16].

Problem 1.8. *Whether there is a constant $\gamma > 0$ with the following property: For each standard ε-isometry $f : X \to Y$, there is a bounded linear operator $T : \overline{\text{span}} f(X) \to X$ such that*

$$\|Tf(u) - u\| \leq \gamma\varepsilon, \forall u \in X?$$

When Y is a Hilbert space, there is a positive answer to the stability problem, see [4]. More results can be found in [3] and [1].

In [2], Cheng et.al introduced the conception of the stability for standard ε-isometries:

Definition 1.9. *A standard ε-isometry $f : X \to Y$ is called (α, γ)-stable if there are $\alpha, \gamma > 0$ and a linear operator $T : L(f) \equiv \overline{\text{span}} f(X) \to X$ with $\|T\| \leq \alpha$ such that*

$$\|Tf(u) - u\| \leq \gamma\varepsilon, \forall u \in X.$$

The pair (X, Y) is called stable if every standard ε-isometry $f : X \to Y$ is (α, γ)-stable for some $\alpha, \gamma > 0$.

The results in [2] establish equivalences of some classical spaces by characterizing various stability conditions. In [8], Dai and Zheng studied the stability of a pair of Banach spaces(X, Y) when X is a $C(K)$ space, where K is a compact Hausdorff spaces.

In fact, there is a stronger stability problem regarding ε-isometries, which is the Hyers-Ulam stability. In [11], Hyers and Ulam had their eyes on surjective ε-isometries and presented a classical question:

Problem 1.10. *Whether for every standard surjective ε-isometry $f : X \to Y$, there is a surjective linear isometry $U : X \to Y$ and $\gamma > 0$ such that*

$$\|f(u) - U(u)\| \leq \gamma\varepsilon, \forall u \in X?$$

M. Omladič and P. Šemrl accurately estimated γ to 2 and proved the following result in [15]:

Theorem 1.11. *If $f : X \to Y$ is a standard surjective ε-isometry, then there is a surjective linear isometry $U : X \to Y$ such that*

$$\|f(u) - U(u)\| \leq 2\varepsilon, \forall u \in X.$$

In [19], Aleksej Turnšek introduced a generalized notion of phase isometries called ε-phase isometries as follows:

Definition 1.12. *A mapping $f : M \to N$ is called an ε-phase isometry if $f(0) = 0$ and*

$$\min\{|\|f(u) - f(v)\| - \|u - v\||, |\|f(u) - f(v)\| - \|u + v\||\} \leq \varepsilon, \forall u, v \in M. \tag{1}$$

Aleksej Turnšek proved the following result:

Theorem 1.13. *If $f : M \to N$ is a surjective ε-phase isometry, then there is a linear surjective isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that*

$$\|f(u) - \sigma(u)U(u)\| \leq 2\sqrt{2}\varepsilon, \forall u \in M.$$

In [18], for a standard almost surjective ε-phase isometry $f : M \to N$, Sun et.al showed that there is a linear surjective isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that

$$\|f(u) - \sigma(u)U(u)\| \leq 2\sqrt{2}\varepsilon, \forall u \in M.$$

In[7], Dai, Que, Sun and Zheng introduced a special class of ε-phase isometries.

Definition 1.14. *$f : X \to Y$ is called a standard ε-phase isometry for some $\varepsilon \geq 0$ if $f(0) = 0$, and*

$$||\|f(u) + f(v)\| \pm \|f(u) - f(v)\|| - |\|u + v\| \pm \|u - v\||| \leq \varepsilon, \forall u, v \in X.$$

In [6], Dai et.al studied the relationship between ε-phase isometries and phase isometries from X to Y.

We can consider the minimum value of the following four terms and define generalized ε-phase isometries as follows:

Definition 1.15. *A mapping $f : M \to N$ is called a generalized ε-phase isometry if*

$$\begin{aligned}\min\{&|\|f(u) - f(v)\| - \|u - v\||, |\|f(u) - f(v)\| - \|u + v\||,\\ &|\|f(u) + f(v)\| - \|u + v\||, |\|f(u) + f(v)\| - \|u - v\||\} \leq \varepsilon, \forall u, v \in M.\end{aligned} \tag{2}$$

Following Qian's problem, we can propose the stability problem of generalized ε-phase isometries between Hilbert spaces.

Problem 1.16. *Whether there is a constant $D > 0$ such that for each standard generalized ε-phase isometry $f : M \to N$, there is a bounded linear operator $T : \overline{\text{span}} f(M) \to M$ and a phase function $\sigma : M \to \{-1, 1\}$ satisfying*

$$\|Tf(u) - \sigma(u)u\| \leq D\varepsilon, \forall u \in M?$$

In addition, we can also propose the Hyers-Ulam stability problem of generalized ε-phase isometries between Hilbert spaces.

Problem 1.17. *Whether there is a constant $D > 0$ such that for each standard almost surjective generalized ε-phase isometry $f : M \to N$, there is a surjective linear isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ satisfying*

$$\|f(u) - \sigma(u)U(u)\| \le D\varepsilon, \forall u \in M?$$

In section 2, we construct a function to illustrate that a standard generalized ε-phase isometry is not necessary a standard ε-phase isometry.

In section 3, Proposition 3.1 establishes a recursive inequality for standard generalized ε-phase isometries; Proposition 3.2 leverages the recursive framework of Proposition 3.1 to approximate a standard generalized ε-phase isometry f by a phase isometry; Theorem 3.3 refines the decomposition further, showing that for a standard generalized ε-phase isometry $f : M \to N$, there is a linear isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that

$$\|U^* f(u) - \sigma(u)u\| \le 2\sqrt{2}\varepsilon.$$

This result gives a positive answer to the stability problem of standard generalized ε-phase isometries between Hilbert spaces.

In section 4, for a standard almost surjective generalized ε-phase isometry $f : M \to N$, we prove that there is a linear surjective isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that

$$\|f(u) - \sigma(u)U(u)\| \le 2\sqrt{2}\varepsilon.$$

This result gives a positive answer to the Hyers-Ulam stability problem of standard generalized ε-phase isometries between Hilbert spaces.

2. On ε-phase isometries and generalized ε-phase isometries

Clearly, every standard ε-phase isometry is a standard generalized ε-phase isometry. However, the following example will illustrate that a standard generalized ε-phase isometry is not necessary a standard ε-phase isometry.

Example 2.1. *Fix any $\varepsilon > 0$. We define $h_1, h_2 : R \to R$ by*

$$h_1(u) = \begin{cases} u + \varepsilon, & u > 0 \\ 0, & u = 0 \\ u - \varepsilon, & u < 0, \end{cases}$$

and

$$h_2(u) = \begin{cases} \varepsilon, & u > 0 \\ 0, & u = 0 \\ -\varepsilon, & u < 0. \end{cases}$$

Let $f : R \to \ell_\infty^2$ be defined by $f(u) = (h_1(u), h_2(u))\,, \forall u \in R$. Then f is a standard generalized ε-phase isometry but not a standard ε-phase isometry.

Proof. For all $u, v \in R$,

$$\|f(u) - f(v)\| = \max\{|h_1(u) - h_1(v)|\,, |h_2(u) - h_2(v)|\}\,,$$

and

$$\|f(u) + f(v)\| = \max\{|h_1(u) + h_1(v)|\,, |h_2(u) + h_2(v)|\}\,.$$

When $u = \varepsilon, v = -\varepsilon$,

$$\|f(u) - f(v)\| = 4\varepsilon, \|u - v\| = 2\varepsilon, \|u + v\| = 0,$$

$$\min\{|\|f(u) - f(v)\| - \|u - v\||, |\|f(u) - f(v)\| - \|u + v\||\} = 2\varepsilon > \varepsilon.$$

Therefore, f is not an ε-phase isometry. Next, we will discuss five cases separately to prove that f is a standard generalized ε-phase isometry.

Case1: If $u, v > 0$ or $u, v < 0$, then

$$\|f(u) - f(v)\| = \|u - v\|.$$

Case2:If $u > 0, v < 0$ or $u < 0, v > 0$, then

$$\|f(u) + f(v)\| = \|u + v\|.$$

Case3:If $u = 0, v = 0$, then

$$\|f(u) - f(v)\| = \|u - v\|.$$

Case4:If $u > 0, v = 0$ or $u = 0, v > 0$, then

$$|\|f(u) - f(v)\| - \|u - v\|| = \varepsilon.$$

Case5:If $u < 0, v = 0$ or $u = 0, v < 0$, then

$$|\|f(u) - f(v)\| - \|u - v\|| = \varepsilon.$$

Overall, all these cases satisfy the inequality (2). Therefore, the standard generalized ε-phase isometries are some extensions of standard ε-phase isometries. □

Throughout this article, the term of standard generalized ε-phase isometry means a mapping $f : M \to N$ satisfying the inequality (2) and $f(0) = 0$. Let $v = 0$ in (2). We deduce that:

$$|\|f(u)\| - \|u\|| \leq \varepsilon, \forall u \in M. \tag{3}$$

3. Stability of generalized ε-phase isometries

Proposition 3.1. *Consider a standard generalized ε-phase isometry $f : M \to N$, a nonnegative integer n and an element $u \in M$. There are two universal constants P and Q (independent of n and u) such that one of the following inequalities must hold:*

$$\left\|2^{-n} f(2^n u) - 2^{-n-1} f\left(2^{n+1} u\right)\right\| \leq P \cdot 2^{-\frac{n}{2}} \|u\|^{\frac{1}{2}} + Q \cdot 2^{-n} \tag{4}$$

or

$$\left\|2^{-n} f(2^n u) + 2^{-n-1} f\left(2^{n+1} u\right)\right\| \leq P \cdot 2^{-\frac{n}{2}} \|u\|^{\frac{1}{2}} + Q \cdot 2^{-n}. \tag{5}$$

Proof. Assume first that $\|u\| \geq \varepsilon$. From the condition (3), we deduce that:

$$\|u\| - \varepsilon \leq \|f(u)\| \leq \|u\| + \varepsilon.$$

Hence

$$(\|u\| - \varepsilon)^2 \leq \|f(u)\|^2 \leq (\|u\| + \varepsilon)^2. \tag{6}$$

Let $v = 2u$ in (2). This implies that one of the following four cases holds:

$$|\|f(u) + f(2u)\| - \|u\|| \leq \varepsilon$$

or

$$|\|f(u) + f(2u)\| - 3\|u\|| \leq \varepsilon$$

or

$$|\|f(u) - f(2u)\| - \|u\|| \leq \varepsilon$$

or

$$|\|f(u) - f(2u)\| - 3\|u\|| \leq \varepsilon.$$

Case 1: If $|\|f(u) + f(2u)\| - \|u\|| \leq \varepsilon$, then we use the parallelogram identity for $f(u) + f(2u)$ and $f(u)$:

$$\|2f(u) + f(2u)\|^2 = 2\|f(u) + f(2u)\|^2 + 2\|f(u)\|^2 - \|f(2u)\|^2.$$

Using (6), we deduce that

$$\begin{aligned} \|2f(u) + f(2u)\|^2 &\leq 2(\|u\| + \varepsilon)^2 + 2(\|u\| + \varepsilon)^2 - (2\|u\| - \varepsilon)^2 \\ &= 12\varepsilon\|u\| + 3\varepsilon^2. \end{aligned} \tag{7}$$

Thus,

$$\left\|f(u) + \frac{1}{2}f(2u)\right\| \leq 2\varepsilon^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + \varepsilon.$$

Case 2: If $|\|f(u) + f(2u)\| - 3\|u\|| \leq \varepsilon$, then we use the parallelogram identity twice. First, for $f(u) - f(2u)$ and $f(u)$:

$$\|2f(u) - f(2u)\|^2 = 2\|f(u) - f(2u)\|^2 + 2\|f(u)\|^2 - \|f(2u)\|^2.$$

Second, for $f(u) - f(2u)$ and $f(u) + f(2u)$:

$$2\|f(u) - f(2u)\|^2 = 4\|f(u)\|^2 + 4\|f(2u)\|^2 - 2\|f(u) + f(2u)\|^2.$$

Combining these results, we have

$$\|2f(u) - f(2u)\|^2 = 6\|f(u)\|^2 + 3\|f(2u)\|^2 - 2\|f(u) + f(2u)\|^2.$$

Combining the last equality with (6) gives

$$\|2f(u) - f(2u)\|^2 \leq 6(\|u\| + \varepsilon)^2 + 3(2\|u\| + \varepsilon)^2 - 2(3\|u\| - \varepsilon)^2 = 36\varepsilon\|u\| + 7\varepsilon^2.$$

Therefore,

$$\left\|f(u) - \frac{1}{2}f(2u)\right\| \leq 3\varepsilon^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + 2\varepsilon.$$

Case 3 ($|\|f(u) - f(2u)\| - \|u\|| \leq \varepsilon$) and case 4 ($|\|f(u) - f(2u)\| - 3\|u\|| \leq \varepsilon$) have already been discussed in[19] as follows: If $|\|f(u) - f(2u)\| - \|u\|| \leq \varepsilon$, then

$$\left\|f(u) - \frac{1}{2}f(2u)\right\| \leq 2\varepsilon^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + \varepsilon.$$

If $|\|f(u) - f(2u)\| - 3\|u\|| \leq \varepsilon$, then

$$\left\|f(u) + \frac{1}{2}f(2u)\right\| \leq 3\varepsilon^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + 2\varepsilon.$$

For $\|u\| < \varepsilon$, we have

$$\begin{aligned}\left\|f(u) \pm \frac{1}{2}f(2u)\right\| &\leq \|f(u)\| + \frac{1}{2}\|f(2u)\| \\ &\leq (\|u\| + \varepsilon) + \frac{1}{2}(2\|u\| + \varepsilon) \\ &< 2\varepsilon^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + \frac{3}{2}\varepsilon. \end{aligned} \tag{8}$$

By choosing $P = 3\varepsilon^{\frac{1}{2}}$ and $Q = 2\varepsilon$, we have:

$$\min\left\{\left\|f(u) - \frac{1}{2}f(2u)\right\|, \left\|f(u) + \frac{1}{2}f(2u)\right\|\right\} \leq P\|u\|^{\frac{1}{2}} + Q, \forall u \in M.$$

Substituting $2^n u$ for u and dividing by 2^n, we complete the proof. □

Proposition 3.2. *Let $f : M \to N$ be a standard generalized ε-phase isometry, then there are universal constants P and Q and a phase isometry $F = \psi U$, where U is a linear isometry and ψ is a phase function such that*

$$\|f(u) - \psi(u)U(u)\| \leq P\|u\|^{\frac{1}{2}} + Q, \ \forall u \in M.$$

Proof. For $u \in M$, we define a sequence $(c_n(u))$ recursively. Let $c_0(u) = f(u)$. For $n \geq 0$, if the inequality

$$\|c_n(u) - 2^{-n-1}f\left(2^{n+1}u\right)\| \leq P \cdot 2^{-\frac{n}{2}}\|u\|^{\frac{1}{2}} + Q \cdot 2^{-n} \tag{9}$$

holds, we define

$$c_{n+1}(u) = \frac{f\left(2^{n+1}u\right)}{2^{n+1}}.$$

If the inequality (9) fails, Proposition 3.1 guarantees that the alternative inequality

$$\|c_n(u) + 2^{-n-1}f(2^{n+1}u)\| \leq P \cdot 2^{-\frac{n}{2}}\|u\|^{\frac{1}{2}} + Q \cdot 2^{-n}$$

must hold. In this case, we define

$$c_{n+1}(u) = -\frac{f\left(2^{n+1}u\right)}{2^{n+1}}.$$

This construction ensures that the sequence $(c_n(u))$ satisfies the recursive bound:

$$\|c_n(u) - c_{n+1}(u)\| \leq P \cdot 2^{-\frac{n}{2}}\|u\|^{\frac{1}{2}} + Q \cdot 2^{-n} \quad , \forall n \geq 0. \tag{10}$$

Take $s, n \in \mathbb{N}$ with $s > n$. Using the triangle inequality and (10), we get that:

$$\|c_s(u) - c_n(u)\| \leq P\|u\|^{\frac{1}{2}} \sum_{k=n}^{s-1} 2^{-\frac{k}{2}} + Q\sum_{k=n}^{s-1} 2^{-k}. \tag{11}$$

As $n \to \infty$, the series converge. Hence $(c_n(u))$ is a Cauchy sequence. Denote its limit by $G(u)$. Setting $n = 0$ and taking $s \to \infty$ in (11), which yields that

$$\|f(u) - G(u)\| \leq (2 + \sqrt{2})P\|u\|^{\frac{1}{2}} + 2Q.$$

Absorbing the constants into P and Q, we obtain the simplified bound:

$$\|f(u) - G(u)\| \leq P\|u\|^{\frac{1}{2}} + Q, \forall u \in M. \tag{12}$$

Let $n \in \mathbb{N}_+$. Because f is a standard generalized ε-phase isometry, for $u, v \in M$

$$\min\{|\|f(2^n u) - f(2^n v)\| - \|2^n u - 2^n v\||, |\|f(2^n u) - f(2^n v)\| - \|2^n u + 2^n v\||,$$

$$|\|f(2^n u) + f(2^n v)\| - \|2^n u + 2^n v\||, |\|f(2^n u) + f(2^n v)\| - \|2^n u - 2^n v\||\} \leq \varepsilon.$$

Divide by 2^n and note that $\frac{f(2^n u)}{2^n}$ is equal to $\pm c_n(u)$ and that $\frac{f(2^n v)}{2^n}$ is equal to $\pm c_n(v)$. Thus, we get that

$$\begin{aligned}\min\{&|\|c_n(u) - c_n(v)\| - \|u - v\||, |\|c_n(u) - c_n(v)\| - \|u + v\||,\\ &|\|c_n(u) + c_n(v)\| - \|u + v\||, |\|c_n(u) + c_n(v)\| - \|u - v\||\} \leq \frac{\varepsilon}{2^n}, \forall n \in \mathbb{N}_+.\end{aligned}$$

Taking $n \to \infty$, the limit $G(u)$ satisfies one of the following four cases:

$$\|G(u) - G(v)\| = \|u - v\|$$

or

$$\|G(u) - G(v)\| = \|u + v\|$$

or

$$\|G(u) + G(v)\| = \|u + v\|$$

or

$$\|G(u) + G(v)\| = \|u - v\|.$$

This ensures that G satisfies the Wigner equation $|\langle G(u), G(v)\rangle| = |\langle u, v\rangle|$. By the Wigner theorem, G must satisfy $G = \psi U$, where $U : M \to N$ is a linear isometry and $\psi : M \to \{-1, 1\}$ is a phase function. □

Theorem 3.3. *If $f : M \to N$ is a standard generalized ε-phase isometry, then there is a linear isometry $U : M \to N$ such that*

$$\left\|U^* f(u) - \sigma(u)u\right\| \leq 2\sqrt{2}\varepsilon, \forall u \in M.$$

Proof. Let $u, v \in M$ and $n \in \mathbb{N}_+$. Since f is a standard generalized ε-phase isometry, we have

$$\begin{aligned}\min\{&|\|f(u) - f(nv)\| - \|u - nv\||, |\|f(u) - f(nv)\| - \|u + nv\||,\\ &|\|f(u) + f(nv)\| - \|u + nv\||, |\|f(u) + f(nv)\| - \|u - nv\||\} \leq \varepsilon.\end{aligned}$$

Case1: If $|\|f(u) - f(nv)\| - \|u - nv\|| \leq \varepsilon$, then

$$\begin{aligned}\left|\|f(u) - f(nv)\|^2 - \|u - nv\|^2\right| &\leq \varepsilon(\|f(u) - f(nv)\| + \|u - nv\|)\\ &\leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|).\end{aligned} \tag{13}$$

Case2: If $|\|f(u) - f(nv)\| - \|u + nv\|| \leq \varepsilon$, then

$$\begin{aligned}\left|\|f(u) - f(nv)\|^2 - \|u + nv\|^2\right| &\leq \varepsilon(\|f(u) - f(nv)\| + \|u + nv\|) \\ &\leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|). \end{aligned} \tag{14}$$

Case3: If $|\|f(u) + f(nv)\| - \|u + nv\|| \leq \varepsilon$, then

$$\begin{aligned}\left|\|f(u) + f(nv)\|^2 - \|u + nv\|^2\right| &\leq \varepsilon(\|f(u) + f(nv)\| + \|u + nv\|) \\ &\leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|). \end{aligned} \tag{15}$$

Case4: If $|\|f(u) + f(nv)\| - \|u - nv\|| \leq \varepsilon$, then

$$\begin{aligned}\left|\|f(u) + f(nv)\|^2 - \|u - nv\|^2\right| &\leq \varepsilon(\|f(u) + f(nv)\| + \|u - nv\|) \\ &\leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|). \end{aligned} \tag{16}$$

From (13)-(16), we deduce that

$$\begin{aligned}&\min\{|\|f(u) - f(nv)\|^2 - \|u - nv\|^2|, |\|f(u) - f(nv)\|^2 - \|u + nv\|^2|, \\ &\quad |\|f(u) + f(nv)\|^2 - \|u + nv\|^2|, |\|f(u) + f(nv)\|^2 - \|u - nv\|^2|\} \\ &\quad \leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|).\end{aligned}$$

$$\begin{aligned}&\min\{|\|f(u)\|^2 - \|u\|^2 + \|f(nv)\|^2 - \|nv\|^2 - 2\langle f(u), f(nv)\rangle + 2\langle u, nv\rangle|, \\ &\quad |\|f(u)\|^2 - \|u\|^2 + \|f(nv)\|^2 - \|nv\|^2 - 2\langle f(u), f(nv)\rangle - 2\langle u, nv\rangle|, \\ &\quad |\|f(u)\|^2 - \|u\|^2 + \|f(nv)\|^2 - \|nv\|^2 + 2\langle f(u), f(nv)\rangle - 2\langle u, nv\rangle|, \\ &\quad |\|f(u)\|^2 - \|u\|^2 + \|f(nv)\|^2 - \|nv\|^2 + 2\langle f(u), f(nv)\rangle + 2\langle u, nv\rangle|\} \\ &\quad \leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|).\end{aligned}$$

Using the triangle inequality and the bound

$$|\|f(nv)\| - \|nv\|| \leq \varepsilon,$$

we further simplify:

$$\begin{aligned}&\min\{|\|f(u)\|^2 - \|u\|^2 - 2\langle f(u), f(nv)\rangle + 2\langle u, nv\rangle|, |\|f(u)\|^2 - \|u\|^2 - 2\langle f(u), f(nv)\rangle \\ &\quad - 2\langle u, nv\rangle|, |\|f(u)\|^2 - \|u\|^2 + 2\langle f(u), f(nv)\rangle - 2\langle u, nv\rangle|, \\ &\quad |\|f(u)\|^2 - \|u\|^2 + 2\langle f(u), f(nv)\rangle + 2\langle u, nv\rangle|\} \\ &\quad \leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|) + |\|f(nv)\|^2 - \|nv\|^2| \\ &\quad \leq \varepsilon(\|f(u)\| + \|f(nv)\| + \|u\| + n\|v\|) + \varepsilon(\|f(nv)\| + \|nv\|) \\ &\quad \leq \varepsilon(\|f(u)\| + \|u\| + 2\varepsilon + 4n\|v\|).\end{aligned} \tag{17}$$

By Proposition 3.2, there are universal constants P and Q, a linear isometry $U : M \to N$ and a phase function ψ such that

$$\|f(nv) - \psi(nv)U(nv)\| \leq P\|nv\|^{\frac{1}{2}} + Q.$$

Hence

$$\|\frac{f(nv)}{n} - \psi(nv)U(v)\| \leq Pn^{-\frac{1}{2}}\|v\| + \frac{Q}{n}.$$

Taking the ultrafilter limit, we deduce that

$$\lim_{n,\mathcal{U}} \psi(nv)\frac{f(nv)}{n} = U(v). \tag{18}$$

Because the set $\{1, -1\}$ is compact, the ultrafilter limit of $\psi(nv)$ exists and equals to 1 or -1. Without loss of generality, we assume that

$$\lim_{n,\mathcal{U}} \psi(nv) = 1.$$

Hence

$$\lim_{n,\mathcal{U}} \frac{f(nv)}{n} = U(v). \tag{19}$$

Next, dividing the inequality in (17) by n and taking the ultrafilter limit, we combine the result with (19) to yield that

$$\min\{|-2\langle f(u), U(v)\rangle + 2\langle u, v\rangle|, |-2\langle f(u), U(v)\rangle - 2\langle u, v\rangle|,$$
$$|2\langle f(u), U(v)\rangle + 2\langle u, v\rangle|, |2\langle f(u), U(v)\rangle - 2\langle u, v\rangle|\} \le 4\varepsilon\|v\|,$$

which simplifies to

$$\min\left\{\left|\langle U^* f(u) + u, v\rangle\right|, \left|\langle U^* f(u) - u, v\rangle\right|\right\} \le 2\varepsilon\|v\|, \tag{20}$$

where U^* is the adjoint of U. Using Lemma 2.3 in [19] for (20), we get that

$$\min\left\{\left\|U^* f(u) + u\right\|, \left\|U^* f(u) - u\right\|\right\} \le 2\sqrt{2}\varepsilon, \forall u \in M. \tag{21}$$

Finally, we define a phase function $\sigma : M \to \{-1, 1\}$ as follows: Let $\sigma(u) = 1$ if

$$\left\|U^* f(u) - u\right\| \le 2\sqrt{2}\epsilon$$

or else $\sigma(u) = -1$. According to (21), we deduce that

$$\left\|U^* f(u) - \sigma(u)u\right\| \le 2\sqrt{2}\epsilon, \forall u \in M.$$

□

4. Hyers-Ulam stability of generalized ε-phase isometries

Let f be a convex function on X. For a point $u \in X$, the subdifferential $\partial f(u)$ is defined as the set of all continuous linear functionals $u^* \in X^*$ that satisfy the following condition:

$$f(v) \ge f(u) + u^*(v - u), \forall v \in X.$$

Note that if $f = \|\cdot\|$ (the norm of X), then $u^* \in \partial\|u\|$ holds if and only if $u^*(u) = \|u\|$ with $\|u^*\| = 1$.

Now we present a weaker condition than almost surjective to prove the Hyers-Ulam stability of standard generalized ε-phase isometries.

Theorem 4.1. *If $f : M \to N$ is a standard generalized ε-phase isometry and*

$$\liminf_{t\to\infty} \frac{d(tv, f(M))}{|t|} < \frac{1}{2}, \forall v \in N,$$

then there is a linear surjective isometry $U : M \to N$ and a phase function $\sigma : M \to \{-1, 1\}$ such that

$$\|f(u) - \sigma(u)U(u)\| \le 2\sqrt{2}\varepsilon, \forall u \in M. \tag{22}$$

Proof. By Theorem 3.3, we deduce that there is a linear isometry $U : M \to N$ such that

$$\left\|U^* f(u) - \sigma(u)u\right\| \leq 2\sqrt{2}\varepsilon, \forall u \in M. \tag{23}$$

Hence

$$|\sigma(u)\langle u, z\rangle - \langle f(u), U(z)\rangle| \leq 2\sqrt{2}\|z\|\varepsilon, \forall u, z \in M.$$

Since U is a linear isometry, U is injective. Next, we will prove that U is surjective. Suppose, for contradiction, that U is not surjective. Then there is a point $v \in N \backslash U(M)$. Because $U(M)$ is a closed subset of N, $d(v, U(M)) > 0$. Without loss of generality, assume that $d(v, U(M)) = 1$. By Corollary 6.8 in [5] and the Rieze Representation Theorem, there is a point $v_1 \in N, \|v_1\| = 1$, such that $\langle v, v_1\rangle = 1, \langle u, v_1\rangle = 0, \forall u \in U(M)$.

Since

$$\liminf_{n\to\infty} \frac{d(nv_1, f(M))}{n} < \frac{1}{2},$$

there is a sequence $(u_n)_{n\in\mathbf{N}_+} \subset M$ such that

$$\liminf_{n\to\infty} \frac{\|f(u_n) - nv_1\|}{n} < \frac{1}{2}.$$

Choose any $z_n \in \partial\|u_n\|$ and let $\phi_n = Uz_n$. Then,

$$\begin{aligned}
2\sqrt{2}\varepsilon &\geq |\sigma(u)\langle u_n, z_n\rangle - \langle f(u_n), \phi_n\rangle| \\
&= |\sigma(u)\langle u_n, z_n\rangle - \langle f(u_n) - nv_1, \phi_n\rangle| \\
&\geq \|u_n\| - \|f(u_n) - nv_1\| \\
&\geq (\|f(u_n)\| - \varepsilon) - \|f(u_n) - nv_1\| \\
&\geq \|nv_1\| - 2\|f(u_n) - nv_1\| - \varepsilon \\
&= n(1 - 2\frac{\|f(u_n) - nv_1\|}{n}) - \varepsilon \to \infty.
\end{aligned} \tag{24}$$

This contradiction implies that U is surjective. According to the inequality (23), we deduce that

$$\begin{aligned}
\|f(u) - \sigma(u)U(u)\| &= \|UU^* f(u) - \sigma(u)U(u)\| \\
&\leq \|U\|\|U^* f(u) - \sigma(u)u\| \\
&\leq 2\sqrt{2}\varepsilon, \forall u \in M.
\end{aligned} \tag{25}$$

□

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