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Asymptotic distribution of the largest entry of sample correlation matrices from 1-dependent normal population

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Abstract. In this paper, we investigate the limiting distribution of the largest entry of sample correlation matrices in the ultra-high-dimensional case. Let $\mathcal{M}_{n,p} = (X_{k,i})$ be an $n \times p$ random matrix whose n rows are n observations coming from the p-dimensional 1-dependent normal population. Suppose that both the population dimension p and the sample size n are large with $\log p = o(n^t)$ for any $t \in (0, 1/3]$. For $L_n = \max_{1\le i<j\le p} \hat{\rho}_{ij}$, where $\hat{\rho}_{ij}$ is the Pearson correlation coefficient between i-th column and j-th column of $\mathcal{M}_{n,p}$, the limiting distribution and the law of large numbers are obtained by the Chen–Stein Poisson approximation method and the moderate deviation principle. Additionally, an example is given to test the covariance structure.

1. Introduction

The present paper is inspired by the application in high-dimensional statistical inference problems, where the dimension p is always expected to be much larger than the sample size n. In such a setting, the classical limiting theorems for fixed p (see, e.g., [2]) are no longer applicable, and we need to call for new statistical theories and methods.

Consider a p-dimensional population represented by a random vector $\boldsymbol{X} = (X_1, \ldots, X_p)'$ with mean $\boldsymbol{\mu} \in \mathbb{R}^p$, covariance matrix $\boldsymbol{\Sigma}_{p\times p}$ and correlation coefficient matrix $\boldsymbol{R}_{p\times p}$. Let $\mathcal{M}_{n,p} = (X_{k,i})_{1\le k\le n, 1\le i\le p}$ be an $n \times p$ data matrix whose n rows are independent copies of $\boldsymbol{X}$. Define

$$\begin{aligned} L_n &= \max_{1\le i<j\le p} \hat{\rho}_{ij}, \quad L_n' = \max_{1\le i<j\le p} |\hat{\rho}_{ij}|, \\ L_{n1} &= \max_{1\le i<j\le p, i<j-1} \hat{\rho}_{ij}, \quad L_{n2} = \max_{1\le i<j\le p, i=j-1} \hat{\rho}_{ij}, \end{aligned} \tag{1}$$

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where

$$\hat{\rho}_{ij} = \frac{\sum_{k=1}^{n} (X_{k,i} - \bar{X}_i)(X_{k,j} - \bar{X}_j)}{\sqrt{\sum_{k=1}^{n} (X_{k,i} - \bar{X}_i)^2} \sqrt{\sum_{k=1}^{n} (X_{k,j} - \bar{X}_j)^2}}, \quad \bar{X}_i = \frac{\sum_{k=1}^{n} X_{k,i}}{n} \tag{2}$$

for $1 \le i, j \le p$. The first statistic L_n is the maximum magnitude of off-diagonal entries of the sample correlation matrix. To obtain the limiting distribution of L_n, it is necessary to consider the limiting properties of L_{n1} and L_{n2} in our work.

In the high-dimensional case where p is at most a polynomial function of n, at the origin of the related researches is Jiang [11] obtaining the limiting distribution of L_n' by Chen–Stein Poisson approximation method as follows.

Lemma 1.1. *(Theorem 1.2 in [11]) Assume $E|X_{1,1}|^r < \infty$ for some $r > 30$. If $\lim_{n\to\infty} n/p = \gamma \in (0, \infty)$, then*

$$\lim_{n\to\infty} P\left(nL_n'^2 - 4\log p + \log\log p \le y\right) = \exp\left(-\frac{e^{-y/2}}{\sqrt{8\pi}}\right) \tag{3}$$

for any $y \in \mathbb{R}$.

The limiting distribution in the above lemma is called the extreme–value distribution of type I. Zhou [31] obtained similar results to (3) under the more relaxed moment condition that

$$n^6 P(|X_{1,1}X_{1,2}| > n) \to 0 \quad \text{as } n \to \infty.$$

Li et al. [16, 17] showed necessary and sufficient conditions for the asymptotic distribution of L_n'. Furthermore, Li and Rosalsky [18] established strong limit theorems for L_n'. In particular, Liu at el. [20] considered a higher dimension case, where $p = O(n^\alpha)$ for any $\alpha > 0$. They proved similar results to (3) and showed thet the rate of convergence is $O((\log n)^{5/2}/\sqrt{n})$.

In the ultra-high-dimensional case, the asymptotic properties of L_n and L_n' have been extensively studied recently. Cai and Jiang [5] obtained similar results to (3) under the assumption $\log p = o(n^t)$ for any $t \in (0, 1/3]$. Furthermore, Shao and Zhou [23] established similar results to (3) under the assumption $\log p = o(n^\beta)$ for any $\beta \in (0, 1]$. Cai and Jiang [6] proved that the limiting behavior of the coherence differs in three regimes: $(\log p)/n \to 0$, $(\log p)/n \to \alpha \in (0, \infty)$ and $(\log p)/n \to \infty$ and exhibited interesting phase transition phenomena when p grows as a function of n. Cai et al. [4] studied the limiting distribution of the pairwise geodesic distances among n random points. Zhao and Zhang [28] assumed that n rows of $\mathcal{M}_{n,p}$ are strictly stationary α-mixing random vectors and obtained the limiting distribution of L_n'. Fan and Jiang [9] proved the following limit theorem of L_n under the assumption that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are a random sample coming from equi-correlation normal population and the entries of the population distribution have a common covariance ρ_n.

Lemma 1.2. *(Theorem 2.2 in [9]) Let $\rho_n \ge 0$ for each $n \ge 1$ and $\sup_{n\ge1} \rho_n < 1$. Assume that $p = p_n \to \infty$ and $\log p = o(n^{1/3})$ as $n \to \infty$. Then*

$$\begin{cases} 4\sqrt{\log p}\left(\sqrt{n-1}L_n - \mu_0\right) \xrightarrow{d} \xi, & \rho_n\sqrt{\log p} \to 0, \\ \frac{\sqrt{n-1}L_n - \mu_0}{\sqrt{2}\rho_n} \xrightarrow{d} \phi + \lambda_0\xi, & \rho_n\sqrt{\log p} \to \lambda_1 \in (0, \infty), \\ \frac{\sqrt{n-1}L_n - \mu_0}{\sqrt{2}\rho_n(1-\rho_n)} \xrightarrow{d} N(0,1), & \rho_n\sqrt{\log p} \to \infty, \end{cases}$$

where $\mu_0 = \sqrt{n-1}\rho_n + (1-\rho_n)(2\sqrt{\log p} - \frac{\log\log p}{4\sqrt{\log p}})\sqrt{1 + 2\rho_n - \rho_n^2}$, $\phi \sim N(0,1)$, $\lambda_0 = \frac{1}{4\sqrt{2}\lambda_1}$, and the distribution function of ξ is $F_\xi(x) = e^{-\frac{1}{4\sqrt{2\pi}}e^{-x/2}}$.

Very recently, Jiang and Pham [13] studied the limiting distribution of L_n when the covariance matrix of the population is the covariance matrix of the auto-regressive model $AR(1)$ or a Toeplitz matrix with a fixed sequence.

In addition, Bai et al. [3], Cai et al. [7], Lytova [21], Tieplova [24], and Xiao and Wu [25] studied the asymptotic behavior of the maximum off-diagonal entry of the sample covariance matrix. Jiang [12], Johnstone [15] and Péché [22] considered the limiting distribution of the largest eigenvalue of sample correlation matrix. Some limiting properties of random tensor were derived in Jiang and Xie [14], Lytova [21] and Tieplova [24].

In this paper, we assume $\boldsymbol{X}$ obeys the p-variate 1-dependent normal distribution, that is, two adjacent elements of $\boldsymbol{X}$ have a common correlation coefficient ρ_n. We will study the law of large numbers and the limiting distribution of the largest magnitude of the off-diagonal entries of the sample correlation matrices in the ultra-high-dimension case where the population dimension $p = p_n$ and the sample size n are sufficiently large with $\log p = o(n^t)$ for any $t \in (0, 1/3]$.

Similar to Fan and Jiang [9] and Jiang and Pham [13], we assume that $\boldsymbol{X}$ comes from a dependent population, and its covariance matrix has a special structure. Fan and Jiang [9] considered a strong dependent population, but this may have limitations in applications. In practical applications, it is often observed that the correlation between random variables decreases as their distance increases. Therefore, this paper considers a weak dependent case. Superficially, the 1-dependent structure is a special case of the Toeplitz matrix in Jiang and Pham [13]. The key difference is that the quantity of the correlation coefficient ρ_n in this paper varies with the sample size n.

The main probabilistic tools in this paper are (i) the moderation deviation of the partial sum of the independent but not necessarily identically distributed random variables (see [8], Proposition 4.5), and (ii) the Chen–Stein Poisson approximation method which is a special case of Arratia et al. [1], Theorem 1.

Throughout this paper, the symbol $\xrightarrow{P}$ means convergence in probability, $\xrightarrow{d}$ means convergence in distribution, and $\xi \overset{d}{=} \eta$ implies that ξ and η have the same distribution. Let $b_n = o(a_n)$ denote $\lim_{n\to\infty} b_n/a_n = 0$, let $b_n = O(a_n)$ denote $\limsup_{n\to\infty} |b_n/a_n| < \infty$. $\xi_n = o_p(a_n)$ and $\xi_n = O_p(a_n)$ are denoted by $\xi_n/a_n \to 0$ in probability as $n \to \infty$ and $\lim_{C\to\infty} \limsup_{n\to\infty} P(|\xi_n/a_n| > C) = 0$, respectively. And the symbol $\operatorname{sgn}(\rho_n)$ means

$$\operatorname{sgn}(\rho_n) = \begin{cases} 1\,, & \rho_n > 0, \\ 0\,, & \rho_n = 0, \\ -1, & \rho_n < 0. \end{cases}$$

In addition, C and C_1 are positive constants not depending n and p, and their values may be different from line to line.

The rest of the paper is organized as follows. We begin in Section **??** by studying the limiting laws of L_{n1}, L_{n2} and L_n in the ultra-high-dimensional case. Section 2 presents some technical tools and three propositions from which the main results can be derived obviously. And the proofs of the main results are given in Section 3. In Section 4 we give an example to test the covariance structure. Finally, four important lemmas are proved in Section 5.

Assumption 1.3. *Let the n rows of $\mathcal{M}_{n,p} = (X_{k,i})_{n\times p}$ be independent identically distributed (i.i.d.) random vectors with distribution $N_p(\boldsymbol{\mu}, \boldsymbol{R})$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ is arbitrary and $\boldsymbol{R}$ has the tridiagonal structure, that is,*

$$\boldsymbol{R} = \begin{pmatrix} 1 & \rho_n & 0 & \cdots & 0 \\ \rho_n & 1 & \rho_n & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \rho_n \\ 0 & \cdots & 0 & \rho_n & 1 \end{pmatrix}, \tag{4}$$

where $\sup_{n\geq 1}|\rho_n| < 1/2$.

Remark 1.4. *By Horn and Johnson [10], Theorem 6.1.10, we find that $\boldsymbol{R}$ is positive definite if $|\rho_n| < 1/2$.*

Assumption 1.5. *Let $\{p_n; n \geq 1\}$ be a sequence of positive integers such that $p_n \to \infty$ with $\log p = o(n^t)$ for any $t \in (0, 1/3]$ as $n \to \infty$.*

Let ξ be a random variable with distribution function

$$F_\xi(x) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right), \quad x \in \mathbb{R}. \tag{5}$$

Set

$$\mu_1 = 2\sqrt{\log p} - \frac{\log\,\log p}{4\sqrt{\log p}} \text{ and } \mu_2 = \sqrt{2\log p} - \frac{\log\,\log p}{2\sqrt{2\log p}} + \frac{\rho_n\sqrt{n-1}}{1-\rho_n^2} + \frac{\log 8}{2\sqrt{2\log p}}. \tag{6}$$

The following two theorems are our main results.

Theorem 1.6. *Under Assumptions 1.3-1.5, the following holds as $n \to \infty$:*

$$4\sqrt{\log p}\left(\sqrt{n-1}L_{n1} - \mu_1\right) \xrightarrow{d} \xi \text{ and } 2\sqrt{2\log p}\left(\frac{\sqrt{n-1}}{1-\rho_n^2}L_{n2} - \mu_2\right) \xrightarrow{d} \xi.$$

Theorem 1.7. *Under Assumptions 1.3-1.5, suppose $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda$ as $n \to \infty$. Then, the following holds as $n \to \infty$:*

$$\begin{cases} 4\sqrt{\log p}\left(\sqrt{n-1}L_n - \mu_1\right) \xrightarrow{d} \xi, & \lambda \in \left[-\infty, 2-\sqrt{2}\right], \\ 2\sqrt{2\log p}\left(\dfrac{\sqrt{n-1}}{1-\rho_n^2}L_n - \mu_2\right) \xrightarrow{d} \xi, & \lambda \in \left(2-\sqrt{2}, \infty\right]. \end{cases}$$

Remark 1.8. *Cai and Jiang [5] assumed that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are a random sample from $(\tau-1)$-dependent normal population and considered the statistic*

$$L_{n,\tau} = \max_{1\le i<j\le p, |i-j|\ge\tau} \left|\hat{\rho}_{ij}\right|,$$

where $\tau \geq 2$ is a integer. They found that $nL_{n,\tau}^2 - 4\log p + \log\log p$ converges weakly to an extreme-value distribution of type I. In this paper, we relax $L_{n,\tau}$ to L_n and get similar results to Cai and Jiang [5].

Remark 1.9. *This paper considers the case of weak dependence, specifically a 1-dependent normal population. Additionally, various forms of weak dependence among the elements of $\boldsymbol{X}$ remain to be explored. Moreover, the law of large numbers for the maximum statistic is an interesting problem, as demonstrated by Jiang [11], Zhao and Zhang [29, 30], and Zhang et al. [27].*

Theorems 1.6 and 1.7 imply immediately the following laws of large numbers.

Corollary 1.10. *Under Assumptions 1.3-1.5, the following holds as $n \to \infty$:*

$$\sqrt{\frac{n-1}{\log p}}L_{n1} \xrightarrow{P} 2 \quad \text{and} \quad \frac{1}{1-\rho_n^2}\sqrt{\frac{n-1}{\log p}}\left(L_{n2} - \rho_n\right) \xrightarrow{P} \sqrt{2}.$$

Corollary 1.11. *Under Assumptions 1.3-1.5, suppose $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda$ as $n \to \infty$. Then, the following holds as $n \to \infty$:*

$$\begin{cases} \sqrt{\dfrac{n-1}{\log p}}L_n \xrightarrow{P} 2, & \lambda \in \left[-\infty, 2-\sqrt{2}\right], \\ \dfrac{1}{1-\rho_n^2}\sqrt{\dfrac{n-1}{\log p}}\left(L_n - \rho_n\right) \xrightarrow{P} \sqrt{2}, & \lambda \in \left(2-\sqrt{2}, \infty\right]. \end{cases}$$

2. Auxiliary Results

The proofs of Theorems 1.6 and 1.7 are quite complicated. To prove the main results, we give some technical tools and three essential propositions in this section.

For any $\rho_n \in (-1/2, 1/2)$, there exists a $\theta_n \in [0, 1/2)$ such that $\sqrt{\theta_n (1-\theta_n)} = |\rho_n|$. So we will substitute $\sqrt{\theta_n (1-\theta_n)}$ for $|\rho_n|$ in the proofs of theorems. Assume that the random variables

$$\{\xi_k, \xi_{k,i}; k = 1, 2, \ldots, i = 0, 1, 2, \ldots\} \text{ are i.i.d. as } N(0,1). \tag{7}$$

Define

$$x_{k,i} = \sqrt{\theta_n}\xi_{k,i-1} + \operatorname{sgn}(\rho_n) \cdot \sqrt{1-\theta_n}\xi_{k,i}, \quad 1 \le k \le n-1,\ 1 \le i \le p. \tag{8}$$

Given $\theta_n \in [0, 1/2)$ for each $n \ge 1$, set

$$a_n = \theta_n,\ b_n = 1-\theta_n,\ c_n = \sqrt{\theta_n(1-\theta_n)},\ d_n = -\frac{\theta_n^{3/2}\sqrt{(1-\theta_n)}}{2},\ e_n = -\frac{(1-\theta_n)^{3/2}\sqrt{\theta_n}}{2}. \tag{9}$$

Define

$$\eta'_{kij} = \operatorname{sgn}(\rho_n) \cdot \left(d_n\xi_{k,i-1}^2 + \frac{c_n}{2}\xi_{k,i}^2 + e_n\xi_{k,i+1}^2\right) + a_n^2\xi_{k,i-1}\xi_{k,i} + b_n^2\xi_{k,i}\xi_{k,i+1} + \rho_n\xi_{k,i-1}\xi_{k,i+1}. \tag{10}$$

For $x \in \mathbb{R}$ and integer $p \ge 1$, set

$$s_p = \sqrt{4\log p - \log\log p + x}, \tag{11}$$

$$s'_p = \sqrt{2\log p - \log\log p + \log 8 + x}. \tag{12}$$

Given $\epsilon > 0$, set $\frac{\epsilon}{\sqrt{\log p}}$. Denote $m = n-1$.

2.1. Some Technical Tools

Lemma 2.1. *[Arratia et al. [1], Theorem 1] Let $\{\eta_\alpha; \alpha \in I\}$ be random variables on an index set I and $\{B_\alpha; \alpha \in I\}$ be a set of subsets of I, that is, for each $\alpha \in I, B_\alpha \subset I$. For any $t \in \mathbb{R}$, set $\lambda_p = \sum_{\alpha\in I} P(\eta_\alpha > t)$. Then, we have*

$$\left|P\left(\max_{\alpha\in I}\eta_\alpha \le t\right) - e^{-\lambda_p}\right| \le \left(1 \wedge \lambda_p^{-1}\right)(b_1 + b_2 + b_3),$$

where

$$b_1 = \sum_{\alpha\in I}\sum_{\beta\in B_\alpha} P(\eta_\alpha > t) P\left(\eta_\beta > t\right), \quad b_2 = \sum_{\alpha\in I}\sum_{\alpha\neq\beta\in B_\alpha} P\left(\eta_\alpha > t, \eta_\beta > t\right),$$

$$b_3 = \sum_{\alpha\in I} E|P(\eta_\alpha > t \mid \sigma(\eta_\beta; \beta \notin B_\alpha)) - P(\eta_\alpha > t)|,$$

and $\sigma(\eta_\beta; \beta \notin B_\alpha)$ is the σ-algebra generated by $\{\eta_\beta; \beta \notin B_\alpha\}$. In particular, if η_α is independent of $\{\eta_\beta; \beta \notin B_\alpha\}$ for each α, then b_3 vanishes.

Lemma 2.2. *[Chen et al. [8], Proposition 4.5] Let $\{\eta_i; 1 \le i \le n\}$ be independent random variables with $E\eta_i = 0$ and $Ee^{h_n|\eta_i|} < \infty$ for some $h_n > 0$ and $1 \le i \le n$. Assume that $\sum_{i=1}^n E\eta_i^2 = 1$. Then*

$$\frac{P\left(\sum_{i=1}^n \eta_i \ge x\right)}{1-\Phi(x)} = 1 + C_n\left(1+x^3\right)\gamma e^{4x^3\gamma}$$

for all $0 \le x \le h_n$ and $\gamma = \sum_{i=1}^n E(|\eta_i|^3 e^{x|\eta_i|})$, where $\sup_{n\ge1}|C_n| \le C$ and C is a constant.

According to Fan and Jiang [9], Lemma 3.8, we could get the following lemma, and its proof will not be described in detail in this paper.

Lemma 2.3. *Let M_n be a random variable for each $n \geq 1$ and $h = \{0, 1\}$. Let $g > 0$ be a constant satisfying*

$$\lim_{n\to\infty} P\left(M_n \leq \sqrt{g^2 \log p - \log\log p - h \cdot \log 8 + x}\right) = F(x)$$

for any $x \in \mathbb{R}$, where $F(x)$ is a continuous distribution function on $\mathbb{R}$. Then

$$M_n = g\sqrt{\log p} - \frac{\log\log p}{2g\sqrt{\log p}} - \frac{h \cdot \log 8}{2g\sqrt{\log p}} + \frac{1}{2g\sqrt{\log p}} U_n,$$

where U_n converges weakly to a probability measure with distribution function $F(x)$.

Lemma 2.4. *[Fan and Jiang [9], Lemma 3.2] Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be i.i.d. random vectors and $\boldsymbol{X}_1 \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}$ is a positive definite matrix. Let $\hat{\rho}_{ij}$ be as in (2). Suppose $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{n-1}$ are i.i.d. and $\boldsymbol{Y}_1 \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$. Write $(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{n-1})' = (\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p)_{(n-1)\times p}$. Then*

$$\left(\hat{\rho}_{ij}\right)_{p\times p} \overset{d}{=} \left(\frac{\boldsymbol{V}_i'\boldsymbol{V}_j}{\|\boldsymbol{V}_i\| \cdot \|\boldsymbol{V}_j\|}\right)_{p\times p}.$$

Lemma 2.5. *[Linnik [19]] Suppose that $\{\zeta, \zeta_1, \zeta_2, \ldots\}$ is a sequence of i.i.d. random variables with $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$. Define $S_n = \sum_{i=1}^n \zeta_i$. If $Ee^{t_0|\zeta|^\alpha} < \infty$ for some $0 < \alpha \leq 1$ and $t_0 > 0$, then*

$$\lim_{n\to\infty} \frac{1}{x_n^2} \log P\left(\frac{S_n}{\sqrt{n}} \geq x_n\right) = -\frac{1}{2}$$

for any $x_n \to \infty$, $x_n = o\left(n^{\frac{\alpha}{2(2-\alpha)}}\right)$.

By the similar argument as in Lemma 3.6 from Fan and Jiang [9], we obtain the following lemma.

Lemma 2.6. *Let U, V and W be i.i.d. $N(0, 1)$-distributed random variables. Let $\{a_i; i = 1, \ldots, 7\}$ be real numbers. Set $\eta = a_1U^2 + a_2V^2 + a_3W^2 + a_4UV + a_5VW + a_6UW + a_7$. Then*

$$E\left(|\eta|^3 e^{x|\eta|}\right) \leq C\sum_{i=1}^7 |a_i|^3 \cdot e^{x|a_7|}$$

as $0 < x \leq \frac{1}{8\sum_{i=1}^6 a_i}$, where C is a constant not depending on a_i.

Lemma 2.7. *Let $\theta_n \in [0, 1/2)$ for all $n \geq 1$. Under Assumption 1.5, suppose $i < j - 1$. Then*

$$\left(\frac{2}{\sqrt{m}}\sum_{k=1}^m x_{k,i}x_{k,j}\right) \cdot \left[1 - \frac{1}{4m}\sum_{k=1}^m \left(x_{k,i}^2 + x_{k,j}^2\right)\right] = \frac{1}{\sqrt{m}}\sum_{k=1}^m x_{k,i}x_{k,j} + \Delta_{mij},$$

where

$$P\left(\left|\Delta_{mij}\right| > \epsilon_p\right) = o\left(\frac{1}{p^4}\right)$$

for $\epsilon > 0$ as $n \to \infty$.

Proof. If $i < j-1$, write

$$\frac{1}{4m}\sum_{k=1}^{m}\left(x_{k,i}^2 + x_{k,j}^2\right) = \frac{1}{2} + \frac{S_{mij}}{4\sqrt{m}} + \frac{T_{mij}}{2\sqrt{m}}, \tag{13}$$

where

$$S_{mij} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[a_n\left(\xi_{k,i-1}^2 + \xi_{k,j-1}^2 - 2\right) + b_n\left(\xi_{k,i}^2 + \xi_{k,j}^2 - 2\right)\right],$$

$$T_{mij} = \frac{\rho_n}{\sqrt{m}}\sum_{k=1}^{m}\left(\xi_{k,i-1}\xi_{k,i} + \xi_{k,j-1}\xi_{k,j}\right).$$

Then, we have

$$\begin{aligned}&\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m}x_{k,i}x_{k,j}\right)\cdot\left[1 - \frac{1}{4m}\sum_{k=1}^{m}\left(x_{k,i}^2 + x_{k,j}^2\right)\right]\\ =&\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m}x_{k,i}x_{k,j}\right)\cdot\left(\frac{1}{2} - \frac{S_{mij}}{4\sqrt{m}} - \frac{T_{mij}}{2\sqrt{m}}\right) = M_{mij} + \Delta_{mij},\end{aligned}$$

where

$$M_{mij} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}x_{k,i}x_{k,j} \quad \text{and} \quad \Delta_{mij} = -\frac{M_{mij}S_{mij}}{2\sqrt{m}} - \frac{M_{mij}T_{mij}}{\sqrt{m}}.$$

Write

$$x_{k,i}x_{k,j} = a_n\xi_{k,i-1}\xi_{k,j-1} + b_n\xi_{k,1}\xi_{k,j} + \rho_n\left(\xi_{k,i}\xi_{k,j-1} + \xi_{k,i-1}\xi_{k,j}\right).$$

Then, we know

$$E\left(x_{k,i}x_{k,j}\right) = 0,\ \mathrm{Var}\left(x_{k,i}x_{k,j}\right) = 1 \text{ and } Ee^{|x_{k,i}x_{k,j}|/4} < \infty.$$

By Lemma 2.5, for $\varepsilon > 0$, we can obtain

$$P\left(\frac{\left|M_{mij}\right|}{\sqrt{\log p}} > 3+\varepsilon\right) \le Ce^{-\frac{(3+\varepsilon)^2}{2}\log p} = o\left(\frac{1}{p^4}\right).$$

Furthermore, it is obvious that $E(S_{mij} + 2T_{mij}) = 0$, $\mathrm{Var}(S_{mij} + 2T_{mij}) = 4$ and $Ee^{|S_{mij}+2T_{mij}|/4} < \infty$. By Lemma 2.5, we see that

$$\begin{aligned}&P\left(\left|\triangle_{mij}\right| > \epsilon_p\right)\\ =&P\left(\left|\triangle_{mij}\right| > \epsilon_p, \frac{\left|M_{mij}\right|}{\sqrt{\log p}} > 3+\varepsilon\right) + P\left(\left|\triangle_{mij}\right| > \epsilon_p, \frac{\left|M_{mij}\right|}{\sqrt{\log p}} \le 3+\varepsilon\right)\\ \le&P\left(\frac{\left|M_{mij}\right|}{\sqrt{\log p}} > 3+\varepsilon\right) + P\left(\frac{\left|S_{mij} + 2T_{mij}\right|}{2\sqrt{m}} > \frac{\epsilon}{(3+\varepsilon)\log p}\right)\\ =&o\left(\frac{1}{p^4}\right) + 2\cdot P\left(\frac{S_{mij} + 2T_{mij}}{2} > \frac{\epsilon\sqrt{m}}{(3+\varepsilon)\log p}\right)\\ =&o\left(\frac{1}{p^4}\right) + 2\cdot\exp\left(-\frac{\epsilon^2 m}{2(3+\varepsilon)^2\left(\log p\right)^2}\right)\end{aligned} \tag{14}$$

for $\epsilon > 0$ and $\varepsilon > 0$. We find that the last item is identical to $o(1/p^4)$ by Assumption 1.5. These prove the lemma. □

Lemma 2.8. *Let $\theta_n \in [0, 1/2)$ for all $n \geq 1$. Under Assumption 1.5, suppose $i = j - 1$. Then*

$$\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}\right)\cdot\left[1-\frac{1}{4m}\sum_{k=1}^{m}\left(x_{k,i}^2+x_{k,j}^2\right)\right]=\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{kij}+\Lambda_{mij}+\rho_n\sqrt{m},$$

where η'_{kij} is defined in (10) and

$$P\left(\left|\Lambda_{mij}\right| > \epsilon_p\right) = o\left(\frac{1}{p^4}\right)$$

for all $\epsilon > 0$ as $n \to \infty$.

Proof. If $i = j - 1$, then we have

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j} = N_{mij} + \frac{\rho_n}{\sqrt{m}}\sum_{k=1}^{m}\xi_{k,i}^2,$$

where

$$N_{mij} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left(a_n\xi_{k,i-1}\xi_{k,i} + b_n\xi_{k,i}\xi_{k,i+1} + \rho_n\xi_{k,i-1}\xi_{k,i+1}\right).$$

Reviewing the proof the Lemma 2.7, we obtain that

$$\begin{aligned}
&\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}\right)\cdot\left[1-\frac{1}{4m}\sum_{k=1}^{m}\left(x_{k,i}^2+x_{k,j}^2\right)\right]\\
=&\left(N_{mij}+\frac{\rho_n}{\sqrt{m}}\sum_{k=1}^{m}\xi_{k,i}^2\right)\left(1-\frac{S_{mij}}{2\sqrt{m}}+\frac{T_{mij}}{\sqrt{m}}\right)\\
=&\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}-\frac{S_{mij}N_{mij}}{2\sqrt{m}}-\frac{T_{mij}N_{mij}}{\sqrt{m}}-\frac{\rho_n\left(S_{mij}+2T_{mij}\right)}{2m}\sum_{k=1}^{m}\xi_{k,i}^2\\
=&\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}-\frac{S_{mij}N_{mij}}{2\sqrt{m}}-\frac{T_{mij}N_{mij}}{\sqrt{m}}-\frac{\rho_n\left(S_{mij}+2T_{mij}\right)}{2m}\sum_{k=1}^{m}\left(\xi_{k,i}^2-1\right)-\frac{\rho_n\left(S_{mij}+2T_{mij}\right)}{2}\\
=&\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{kij}+\Lambda_{mij}+\rho_n\sqrt{m},
\end{aligned}$$

where

$$\Lambda_{mij} = -\frac{S_{mij}N_{mij}}{2\sqrt{m}}-\frac{T_{mij}N_{mij}}{\sqrt{m}}-\frac{\rho_n\left(S_{mij}+2T_{mij}\right)}{2m}\sum_{k=1}^{m}\left(\xi_{k,i}^2-1\right).$$

Given $\varepsilon > 0$, define

$$A_\varepsilon = \left\{\left|\frac{N_{mij}}{\sqrt{3\log p}}\right| \leq 3+\varepsilon, \left|\frac{\sum_{k=1}^{m}\left(\xi_{k,i}^2-1\right)}{\sqrt{2m\log p}}\right| \leq 3+\varepsilon\right\}.$$

Set $\zeta_k = \xi_{k,i-1}\xi_{k,i} + \xi_{k,i}\xi_{k,i+1} + \xi_{k,i-1}\xi_{k,i+1}$ and $\zeta'_k = \xi_{k,i}^2 - 1$. Obviously, $E\zeta_k = 0$, $\mathrm{Var}(\zeta_k) = 3$, $Ee^{|\zeta_k|/4} < \infty$, $E\zeta'_k = 0$, $\mathrm{Var}(\zeta'_k) = 2$ and $Ee^{|\zeta'_k|/4} < \infty$. By Lemma 2.5, we have

$$
\begin{aligned}
P\left(A_\varepsilon^c\right) \le & P\left(\left|\frac{N_{mij}}{\sqrt{3\log p}}\right| > 3+\varepsilon\right) + P\left(\left|\frac{\sum_{k=1}^m \left(\xi_{k,i}^2 - 1\right)}{\sqrt{2m\log p}}\right| > 3+\varepsilon\right)\\
\le & 2\cdot P\left(\frac{\sum_{k=1}^m \zeta_k}{\sqrt{3m\log p}} > 3+\varepsilon\right) + 2\cdot P\left(\frac{\sum_{k=1}^m \zeta'_k}{\sqrt{2m\log p}} > 3+\varepsilon\right)\\
= & 4\cdot e^{-\frac{(3+\varepsilon)^2}{2}\log p} = o\left(\frac{1}{p^4}\right)
\end{aligned}
$$

for $\varepsilon > 0$ as $n \to \infty$. For $\epsilon > 0$ and $\varepsilon > 0$, we see from (14) and Lemma 2.5 that

$$
\begin{aligned}
P\left(\left|\Lambda_{mij}\right| > \epsilon_p\right) = & P\left(\left|\Lambda_{mij}\right| > \epsilon_p, A_\varepsilon\right) + P\left(\left|\Lambda_{mij}\right| > \epsilon_p, A_\varepsilon^c\right)\\
\le & P\left(2\left|S_{mij} + 2T_{mij}\right| > \frac{\epsilon\sqrt{m}}{(3+\varepsilon)\log p}\right) + P\left(A_\varepsilon^c\right)\\
\le & 2\cdot P\left(\frac{S_{mij} + 2T_{mij}}{2} > \frac{\epsilon\sqrt{m}}{4(3+\varepsilon)\log p}\right) + 4\cdot e^{-\frac{(3+\varepsilon)^2}{2}\log p}\\
= & 2\cdot e^{-\frac{\epsilon^2 m}{32(3+\varepsilon)^2(\log p)^2}} + o\left(\frac{1}{p^4}\right) = o\left(\frac{1}{p^4}\right)
\end{aligned}
$$

as $n \to \infty$ by Assumption 1.5. Then the proof is completed. □

2.2. *Three Propositions*

The first proposition presents the asymptotic behavior of L_{n2}.

Proposition 2.9. *Let $\theta_n \in [0, 1/2)$ for each $n \ge 1$. Under Assumptions 1.3 and 1.5, set $\sigma_{n1}^2 = \theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1$ and $M'_n = \max_{1\le i<j\le p, i=j-1} \frac{1}{\sigma_{n1}}\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta'_{kij} + \Lambda_{mij}\right)$. Then*

$$
\lim_{n\to\infty} P\left(M'_n \le s'_p\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)
$$

for any $x \in \mathbb{R}$.

Proof. Set $I' = \{(i,j); 1 \le i < j \le p, i = j-1\}$. For $\alpha = (i,j) \in I'$, define

$$
\begin{aligned}
Z_\alpha &= \frac{1}{\sigma_{n1}}\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta'_{kij} + \Lambda_{mij}\right),\\
B'_\alpha &= \{(k,l) \in I';\ k \in \{i-2, i-1, i, i+1, i+2\}, \text{but } (k,l) \ne \alpha\},
\end{aligned}
$$

where Λ_{mij} is defined in Lemma 2.8. Note that Z_α is independent of $\{Z_\beta; \beta \notin B'_\alpha\}$. By Lemma 2.1, we have

$$
\left|P\left(\max_{\alpha\in I'} Z_\alpha \le s'_p\right) - e^{-\lambda_{p1}}\right| \le v_1 + v_2,
$$

where

$$
\lambda_{p1} = \sum_{\alpha\in I'} P\left(Z_\alpha > s'_p\right) = (p-1)\cdot P\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^m \eta'_{k12} + \frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p\right) := \mathcal{H}_1,
$$

$$v_1 = \sum_{\alpha \in I'} \sum_{\beta \in B'_\alpha} P\left(Z_\alpha > s'_p\right) P\left(Z_\beta > s'_p\right) \le (p-1) \cdot 5 \cdot P\left(\frac{1}{\sigma_{n1}\sqrt{m}} \sum_{k=1}^{m} \eta'_{k12} + \frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p\right)^2 := \mathcal{H}_2$$

and

$$\begin{aligned} v_2 =& \sum_{\alpha \in I'} \sum_{\alpha \neq \beta \in B'_\alpha} P\left(Z_\alpha > s'_p, Z_\beta > s'_p\right) \\ \le& 5(p-1) \cdot P\left(\frac{1}{\sigma_{n1}\sqrt{m}} \sum_{k=1}^{m} \eta'_{k12} + \frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p, \frac{1}{\sigma_{n1}\sqrt{m}} \sum_{k=1}^{m} \eta'_{k34} + \frac{\Lambda_{m34}}{\sigma_{n1}} > s'_p\right) := \mathcal{H}_3. \end{aligned}$$

Lemma 2.10. *Under the conditions of Proposition 2.9, we have* $\lim_{n\to\infty} \mathcal{H}_1 = \frac{1}{4\sqrt{2\pi}} e^{-x/2}$ *for any* $x \in \mathbb{R}$ *and* $\lim_{n\to\infty} \mathcal{H}_3 = 0$. *Obviously, we have* $\lim_{n\to\infty} \mathcal{H}_2 = 0$ *by* $\lim_{n\to\infty} \mathcal{H}_1 = \frac{1}{4\sqrt{2\pi}} e^{-x/2}$ *for any* $x \in \mathbb{R}$.

Lemma 2.10 says that $e^{-\lambda_{p1}} \to \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$, $v_1 \to 0$ and $v_2 \to 0$ as $n \to \infty$. These prove Proposition 2.9. $\square$

Propositions 2.11 and 2.14 will show the limiting behavior of L_n when $\frac{\rho_n \sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left[-\infty, 2-\sqrt{2}\right]$ and $\frac{\rho_n \sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left(2-\sqrt{2}, \infty\right]$, respectively.

Proposition 2.11. *Let* $\theta_n \in [0, 1/2)$ *for each* $n \ge 1$. *Under Assumptions 1.3 and 1.5, set*

$$M_n = \max_{1 \le i < j \le p} \left(\frac{2}{\sqrt{m}} \sum_{k=1}^{m} x_{k,i} x_{k,j}\right) \cdot \left[1 - \frac{1}{4m} \sum_{k=1}^{m} \left(x_{k,i}^2 + x_{k,j}^2\right)\right].$$

If $\frac{\rho_n \sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left[-\infty, 2-\sqrt{2}\right]$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} P\left(M_n \le s_p\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$$

for any $x \in \mathbb{R}$.

Proof. Set

$$I = \{(i,j); 1 \le i < j \le p\} \text{ and } I'' = \{(i,j); 1 \le i < j \le p, i < j-1\}.$$

For $\alpha = (i,j) \in I$, define

$$\begin{aligned} X_\alpha &= \left(\frac{2}{\sqrt{m}} \sum_{k=1}^{m} x_{k,i} x_{k,j}\right) \cdot \left[1 - \frac{1}{4m} \sum_{k=1}^{m} \left(x_{k,i}^2 + x_{k,j}^2\right)\right] = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \eta_{kij}, \\ B_\alpha &= \{(k,l) \in I;\ \{k,l\} \cap \{i-1, i, i+1, j-1, j, j+1\} \neq \emptyset, \text{but}\,(k,l) \neq \alpha\}, \\ B''_\alpha &= \{(k,l) \in I'';\ \{k,l\} \cap \{i-1, i, i+1, j-1, j, j+1\} \neq \emptyset, \text{but}\,(k,l) \neq \alpha\}. \end{aligned}$$

The crucial point is that X_α is independent of $\{X_\beta; \beta \notin B_\alpha\}$. Reviewing the definitions of I' and B'_α, by Lemma 2.1, we have

$$\left|P\left(\max_{\alpha \in I} X_\alpha \le s_p\right) - e^{-\lambda_{p2}}\right| \le u_1 + u_2,$$

where

$$\begin{aligned}\lambda_{p2} &= \sum_{\alpha\in I} P\left(X_\alpha > s_p\right) = \sum_{\alpha\in I'} P\left(X_\alpha > s_p\right) + \sum_{\alpha\in I''} P\left(X_\alpha > s_p\right)\\ &= (p-1)P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12} > s_p\right) + \frac{(p-1)(p-2)}{2}P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k13} > s_p\right) := \mathcal{F}_1 + \mathcal{G}_1,\end{aligned}$$

$$\begin{aligned}u_1 &= \sum_{\alpha\in I}\sum_{\beta\in B_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right)\\ &= \sum_{\alpha\in I'}\sum_{\beta\in B'_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right) + \sum_{\alpha\in I''}\sum_{\beta\in B''_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right)\\ &\le (p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12} > s_p\right)^2 + \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k13} > s_p\right)^2 := \mathcal{F}_2 + \mathcal{G}_2\end{aligned}$$

and

$$\begin{aligned}u_2 &= \sum_{\alpha\in I}\sum_{\alpha\neq\beta\in B_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right)\\ &= \sum_{\alpha\in I'}\sum_{\beta\in B'_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right) + \sum_{\alpha\in I''}\sum_{\beta\in B''_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right)\\ &\le (p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k34} > s_p\right)\\ &\quad + \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k13} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k15} > s_p\right) := \mathcal{F}_3 + \mathcal{G}_3.\end{aligned}$$

Lemma 2.12. *Under the conditions of Proposition 2.11, we have* $\lim_{n\to\infty}\mathcal{G}_1 = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$ *for any* $x\in\mathbb{R}$ *and* $\lim_{n\to\infty}\mathcal{G}_3 = 0$. *Trivially, we have* $\lim_{n\to\infty}\mathcal{G}_2 = 0$ *by* $\lim_{n\to\infty}\mathcal{G}_1 = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$ *for any* $x\in\mathbb{R}$.

Lemma 2.13. *Under the conditions of Proposition 2.11, we have* $\lim_{n\to\infty}\mathcal{F}_1 = 0$ *and* $\lim_{n\to\infty}\mathcal{F}_3 = 0$. *Obviously, we have* $\lim_{n\to\infty}\mathcal{F}_2 = 0$ *by* $\lim_{n\to\infty}\mathcal{F}_1 = 0$.

By Lemmas 2.12 and 2.13, we know that $e^{-\lambda_{p2}} \to \exp\left(-\frac{1}{4\sqrt{2\pi}}e^{-x/2}\right)$, $u_1\to 0$ and $u_2\to 0$ as $n\to\infty$. Then the proof of Proposition 2.11 is completed. □

Proposition 2.14. *Let* $\theta_n\in[0,1/2)$ *for each* $n\ge 1$. *Under Assumptions 1.3 and 1.5, set* $\sigma_{n1}^2 = \theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1$ *and* $M_n = \max_{1\le i<j\le p}\left(\frac{2}{\sqrt{m}}\sum_{k=1}^m x_{k,i}x_{k,j}\right)\cdot\left[1 - \frac{1}{4m}\sum_{k=1}^m\left(x_{k,i}^2 + x_{k,j}^2\right)\right]$. *If* $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left(2-\sqrt{2},\infty\right]$ *as* $n\to\infty$, *then*

$$\lim_{n\to\infty} P\left(M_n \le c_n\sqrt{m} + \sigma_{n1}s'_p\right) = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$$

for any $x\in\mathbb{R}$.

Proof. Set $\tau = c_n\sqrt{m} + \sigma_{n1}s'_p$. Recalling the proof of Proposition 2.11, by Lemma 2.1, we have

$$\left|P\left(\max_{\alpha\in I} X_\alpha \le \tau\right) - e^{-\lambda_{p3}}\right| \le m_1 + m_2,$$

where

$$\lambda_{p3} = (p-1)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12} > \tau\right) + \frac{(p-1)(p-2)}{2}\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > \tau\right) := \mathcal{H}_1 + \mathcal{J}_1,$$

$$m_1 \leq 5(p-1)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12} > \tau\right)^2 + \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > \tau\right)^2 := \mathcal{H}_2 + \mathcal{J}_2$$

and

$$\begin{aligned} m_2 \leq &(p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12} > \tau, \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k34} > \tau\right) \\ &+ \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > \tau, \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k15} > \tau\right) := \mathcal{H}_3 + \mathcal{J}_3. \end{aligned}$$

Lemma 2.15. *Under the conditions of Proposition 2.14, we have* $\lim_{n\to\infty}\mathcal{J}_1 = 0$ *and* $\lim_{n\to\infty}\mathcal{J}_3 = 0$. *Trivially, we have* $\lim_{n\to\infty}\mathcal{J}_2 = 0$ *by* $\lim_{n\to\infty}\mathcal{J}_1 = 0$.

Then, we see from Lemmas 2.10 and 2.15 that $e^{-\lambda_{p3}} \to \exp\left(-\frac{1}{4\sqrt{2\pi}}e^{-x/2}\right)$, $m_1 \to 0$ and $m_2 \to 0$ as $n \to \infty$. These imply Proposition 2.14. □

3. Proofs of Main Results

Now we are in a position to prove Theorems 1.6 and 1.7.

3.1. Proof of Theorem 1.6

Proof. Reviewing (7) and (8), we find that the $n-1$ rows of the matrix $(x_{k,i})_{(n-1)\times p}$ are $n-1$ observations and $x_{1,i} \sim N(0,1)$ for each $1 \leq i \leq p$. In addition, for $1 \leq i, j \leq p$, we have

$$\mathrm{Cov}\left(x_{1,i}, x_{1,j}\right) = \begin{cases} 0, & \left|i-j\right| > 1, \\ \mathrm{sgn}(\rho_n)\sqrt{\theta_n(1-\theta_n)}, & \left|i-j\right| = 1, \end{cases}$$

where $\mathrm{sgn}(\rho_n)\sqrt{\theta_n(1-\theta_n)} = \rho_n$. That is, each row of the matrix $(x_{k,i})_{(n-1)\times p}$ obeys $N_p(\mathbf{0}, \boldsymbol{R})$. Write $(x_{k,i})_{(n-1)\times p} = (\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p)$ such that $\boldsymbol{V}_i = (x_{1,i}, \ldots, x_{n-1,i})'$ for each $1 \leq i \leq p$. By Lemma 2.4, we have

$$\sqrt{m}\max_{1\leq i<j\leq p}\hat{\rho}_{ij} \overset{d}{=} \max_{1\leq i<j\leq p}\frac{\frac{1}{\sqrt{m}}\sum_{k=1}^{m}x_{k,i}x_{k,j}}{h_ih_j},$$

where

$$h_i = \sqrt{\frac{1}{m}\sum_{k=1}^{m}x_{k,i}^2} = \sqrt{1+\frac{\zeta_{ni}}{\sqrt{m}}} \quad \text{and} \quad \zeta_{n,i} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left(x_{k,i}^2 - 1\right).$$

By the Taylor expansion, we obtain

$$\frac{1}{h_ih_j} = \left[1 - \frac{\zeta_{n,i}}{2\sqrt{m}} + \phi\left(\frac{\zeta_{n,i}}{\sqrt{m}}\right)\right]\cdot\left[1 - \frac{\zeta_{n,j}}{2\sqrt{m}} + \phi\left(\frac{\zeta_{n,j}}{\sqrt{m}}\right)\right] = 1 - \frac{\zeta_{n,i}}{2\sqrt{m}} - \frac{\zeta_{n,j}}{2\sqrt{m}} + \varepsilon_{ij}, \tag{15}$$

where

$$\varepsilon_{ij} = \frac{\zeta_{n,i}\zeta_{n,j}}{4m} + \left(1 - \frac{\zeta_{n,i}}{2\sqrt{m}}\right)\cdot\varphi\left(\frac{\zeta_{n,j}}{\sqrt{m}}\right) + \left(1 - \frac{\zeta_{n,j}}{2\sqrt{m}}\right)\cdot\varphi\left(\frac{\zeta_{n,i}}{\sqrt{m}}\right) + \phi\left(\frac{\zeta_{n,i}}{\sqrt{m}}\right)\varphi\left(\frac{\zeta_{n,j}}{\sqrt{m}}\right).$$

If $|\zeta_{n,i}/\sqrt{m}| < \delta$ and $|\zeta_{n,j}/\sqrt{m}| < \delta$, then $\max_{k=i,j}|1 - \zeta_{n,k}/(2\sqrt{m})| < 2$ due to $\delta \in (0,1)$. Therefore,

$$\left|\varepsilon_{ij}\right| \le \frac{\left|\zeta_{n,i}\right|\cdot\left|\zeta_{n,j}\right|}{4m} + \frac{2\zeta_{n,i}^2}{m} + \frac{2\zeta_{n,j}^2}{m} + \frac{\zeta_{n,i}^2}{m}\cdot\frac{\zeta_{n,j}^2}{m} \le \frac{4\left(\zeta_{n,i}^2 + \zeta_{n,j}^2\right)}{m}.$$

It follows that

$$\max_{1\le i<j\le p}\left|\varepsilon_{ij}\right| \le \frac{8}{m}\cdot\max_{1\le i\le p}\zeta_{n,i}^2 \tag{16}$$

by $\max_{1\le i\le p}|\zeta_{n,i}/\sqrt{m}| < \delta$. Let $\zeta_k = (x_{k,1}^2 - 1)/\sqrt{2}$ for $1 \le k \le m$. Obviously, $E\zeta_k = 0$, $\operatorname{Var}(\zeta_k) = 1$ and $Ee^{|\zeta_k|/2} < \infty$. Set

$$\Upsilon_n = \left\{\max_{1\le i\le p}\left|\zeta_{n,i}\right| < 3\sqrt{\log p}\right\}.$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} P(\Upsilon_n) = 1. \tag{17}$$

Then, we see from (15) that

$$\sqrt{m}\hat{\rho}_{ij} = \left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}\right)\cdot\left[1 - \frac{1}{4m}\left(\sum_{k=1}^{m} x_{k,i}^2 + \sum_{k=1}^{m} x_{k,j}^2\right)\right] + \varepsilon_{ij}', \tag{18}$$

where

$$\varepsilon_{ij}' = \left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}\right)\cdot\varepsilon_{ij}.$$

Zhang et al. [26] proved that

$$\max_{1\le i<j\le p}\frac{1}{\sqrt{m}}\left|\sum_{k=1}^{m} x_{k,i}x_{k,j}\right| = O_p\left(\sqrt{m}\right)$$

as $n \to \infty$. Therefore, by (16), we have

$$\max_{1\le i<j\le p}\left|\varepsilon_{ij}'\right| \le \max_{1\le i<j\le p}\frac{1}{\sqrt{m}}\left|\sum_{k=1}^{m} x_{k,i}x_{k,j}\right|\cdot\max_{1\le i<j\le p}\left|\varepsilon_{ij}\right| \le \frac{8}{m}\cdot\max_{1\le i<j\le p}\frac{1}{\sqrt{m}}\left|\sum_{k=1}^{m} x_{k,i}x_{k,j}\right|\cdot\max_{1\le i\le p}\zeta_{n,i}^2$$

by $\max_{1\le i<j\le p}|\zeta_{n,i}/\sqrt{m}| < \delta$. Then,

$$I_{\Upsilon_n}\cdot\max_{1\le i<j\le p}\left|\varepsilon_{ij}'\right| = \frac{8}{m}\cdot O_p\left(\sqrt{m}\right)\cdot\left(3\sqrt{\log p}\right)^2 = O_p\left(\frac{\log p}{\sqrt{m}}\right).$$

Therefore, writing $I_{\Upsilon_n} = 1 - I_{\Upsilon_n^c}$, we see from (17) and (18) that

$$\begin{aligned}&\sqrt{m}\max_{1\le i<j\le p, i<j-1}\hat{\rho}_{ij}\\ &= \max_{1\le i<j\le p, i<j-1}\left\{\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m} x_{k,i}x_{k,j}\right)\cdot\left[1 - \frac{1}{4m}\left(\sum_{k=1}^{m} x_{k,i}^2 + \sum_{k=1}^{m} x_{k,j}^2\right)\right]\right\} + o_p\left(\frac{1}{\sqrt{\log p}}\right).\end{aligned} \tag{19}$$

By the same argument, we obtain that

$$\begin{aligned}&\sqrt{m}\max_{1\le i<j\le p, i=j-1}\hat{\rho}_{ij}\\ &=\max_{1\le i<j\le p, i=j-1}\left\{\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m}x_{k,i}x_{k,j}\right)\cdot\left[1-\frac{1}{4m}\left(\sum_{k=1}^{m}x_{k,i}^2+\sum_{k=1}^{m}x_{k,j}^2\right)\right]\right\}+o_p\left(\frac{1}{\sqrt{\log p}}\right).\end{aligned}$$

Reviewing (1) and Lemma 2.7, we have

$$\sqrt{m}L_{n1}=\sqrt{m}\max_{1\le i<j\le p, i<j-1}\hat{\rho}_{ij}=\max_{1\le i<j\le p, i<j-1}\frac{1}{\sqrt{m}}\sum_{k=1}^{m}x_{ki}x_{kj}+o_p\left(\frac{1}{\sqrt{\log p}}\right).$$

Then, by Proposition 6.4 from Cai and Jiang [5] and Lemma 2.3,

$$\sqrt{m}L_{n1}=2\sqrt{\log p}-\frac{\log\log p}{4\sqrt{\log p}}+\frac{1}{4\sqrt{\log p}}U_{n1}+o_p\left(\frac{1}{\sqrt{\log p}}\right),$$

where $U_{n1}\xrightarrow{d}\xi$ with distribution function $F_\xi(x)$ as in (5). It follows that

$$4\sqrt{\log p}\left(\sqrt{m}L_{n1}-\mu_1\right)=U_{n1}\xrightarrow{d}\xi,$$

where $\mu_1=2\sqrt{\log p}-\frac{\log\log p}{4\sqrt{\log p}}$ and ξ is defined as above.

Recalling (1), Proposition 2.9, Lemmas 2.3 and 2.8, one can get that

$$\begin{aligned}\sqrt{m}L_{n2}&=\sqrt{m}\max_{1\le i<j\le p, i=j-1}\hat{\rho}_{ij}\\ &=\max_{1\le i<j\le p, i=j-1}\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{kij}+\Lambda_{mij}\right)+\rho_n\sqrt{m}+o_p\left(\frac{1}{\sqrt{\log p}}\right)\\ &=\left(1-\rho_n^2\right)\left(\sqrt{2\log p}-\frac{\log\log p}{2\sqrt{2\log p}}+\frac{\log 8}{2\sqrt{2\log p}}+\frac{1}{2\sqrt{2\log p}}U_{n2}\right)+\rho_n\sqrt{m}+o_p\left(\frac{1}{\sqrt{\log p}}\right),\end{aligned}$$

where $U_{n2}\xrightarrow{d}\xi$ with distribution function $F_\xi(x)$ as in (5). It follows that

$$2\sqrt{2\log p}\left(\frac{\sqrt{m}}{1-\rho_n^2}L_{n2}-\mu_2\right)=U_{n2}\xrightarrow{d}\xi,$$

where $\mu_2=\frac{\rho_n\sqrt{m}}{1-\rho_n^2}+\sqrt{2\log p}-\frac{\log\log p}{2\sqrt{2\log p}}+\frac{\log 8}{2\sqrt{2\log p}}$ and ξ is defined as above. □

3.2. Proof of Theorem 1.7

Proof. In this section, we continue to use the notations in the proof of Theorem 1.6. Then, by Lemma 2.3, Propositions 2.11 and 2.14, we have the results as follows.

Case (i): If $\lambda\in[-\infty,2-\sqrt{2}]$, then

$$M_n=2\sqrt{\log p}-\frac{\log\log p}{4\sqrt{\log p}}+\frac{1}{4\sqrt{\log p}}U_{n3},$$

where $U_{n3}\xrightarrow{d}\xi$ with distribution function $F_\xi(x)$ as in (5).

Case (ii): If $\lambda \in (2-\sqrt{2}, \infty]$, then

$$\frac{M_n}{1-\rho_n^2} = \frac{\rho_n\sqrt{m}}{1-\rho_n^2} + \sqrt{2\log p} - \frac{\log\log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}} + \frac{1}{2\sqrt{2\log p}}U_{n4},$$

where $U_{n4} \xrightarrow{d} \xi$ with distribution function $F_\xi(x)$ as in (5).

Recalling (19) and the definitions of μ_1, μ_2, M_n and L_n, we have

$$\sqrt{m}L_n = \max_{1\le i<j\le p}\left\{\left(\frac{2}{\sqrt{m}}\sum_{k=1}^{m} x_{ki}x_{kj}\right)\cdot\left[1-\frac{1}{4m}\left(\sum_{k=1}^{m} x_{ki}^2 + \sum_{k=1}^{m} x_{kj}^2\right)\right]\right\} + o_p\left(\frac{1}{\sqrt{\log p}}\right) = M_n + o_p\left(\frac{1}{\sqrt{\log p}}\right).$$

Then, one can get the following conclusions by the above relations.

Case (i): If $\lambda \in [-\infty, 2-\sqrt{2}]$, then

$$4\sqrt{\log p}\left(\sqrt{m}L_n - \mu_1\right) \xrightarrow{d} \xi,$$

where μ_1 and the distribution function of ξ are given in (6) and (5), respectively.

Case (ii): If $\lambda \in (2-\sqrt{2}, \infty]$, then

$$4\sqrt{\log p}\left(\frac{\sqrt{m}L_n}{1-\rho_n^2} - \mu_2\right) \xrightarrow{d} \xi,$$

where μ_2 and the distribution function of ξ are given in (6) and (5), respectively. □

4. Testing the covariance structure

We apply Theorem 1.7 to the problem of testing the covariance structure of a high-dimensional random vector. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a random sample from the population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with the corresponding correlation matrix $\boldsymbol{R}'_{p\times p}$. As the application of Section **??**, we wish to test whether $\boldsymbol{X}$ comes from the 1-dependent normal population, or equivalently in terms of the correlation matrix, we wish to test whether $\boldsymbol{R}'$ has the tridiagonal structure like $\boldsymbol{R}$ defined in (4). The specific hypotheses are

$$H_0 : \boldsymbol{R}' = \boldsymbol{R} \quad \text{v.s.} \quad H_1 : \boldsymbol{R}' \neq \boldsymbol{R}.$$

We find that L_n can be used as the test statistic. Under H_0, the conclusion of Theorem 1.7 still hold. For a given $\alpha \in (0, 1)$, set $q_\alpha = -\log(32\pi) - 2\log\log(1-\alpha/2)^{-1}$. It is obvious that q_α is the $(1-\alpha/2)$-quantile of the distribution $F_\xi(x)$ as in (5). Then, we can get rejection regions $\mathcal{K}_1$ and $\mathcal{K}_2$ when $\lambda \in [-\infty, 2-\sqrt{2}]$ and $\lambda \in (2-\sqrt{2}, \infty]$, respectively.

$$\mathcal{K}_1 = \left\{\sqrt{n-1}L_n - 2\sqrt{\log p} + \frac{\log\log p}{4\sqrt{\log p}} \ge \frac{q_\alpha}{4\sqrt{\log p}}\right\},$$

$$\mathcal{K}_2 = \left\{\frac{\sqrt{n-1}L_n}{\sqrt{1+\rho_n^2}} - \frac{\rho_n\sqrt{n-1}}{\sqrt{1+\rho_n^2}} - \sqrt{2\log p} + \frac{\log\log p}{2\sqrt{2\log p}} - \frac{\log 8}{2\sqrt{2\log p}} \ge \frac{q_\alpha}{2\sqrt{2\log p}}\right\}.$$

Specially, we also can test for independence in the normal case. That is, we wish to test whether $\boldsymbol{R} = \boldsymbol{I}_p$, where $\boldsymbol{I}_p$ is the $p \times p$ identity matrix. By the same argument as above, we find that the corresponding rejection region is $\mathcal{K}_1$.

5. Proofs of Lemmas 2.10, 2.12, 2.13 and 2.15

5.1. Proof of Lemma 2.10

Proof. *Step* 1: proof of $\lim_{n\to\infty} \mathcal{H}_1 = \frac{1}{4\sqrt{2\pi}} e^{-x/2}$ for any $x \in \mathbb{R}$. Review (10). Write

$$\eta'_{k12} = \operatorname{sgn}(\rho_n) \cdot \left(d_n \xi_{k,0}^2 + \frac{c_n}{2} \xi_{k,1}^2 + e_n \xi_{k,2}^2 \right) + a_n^2 \xi_{k,0} \xi_{k,1} + b_n^2 \xi_{k,1} \xi_{k,2} + \rho_n \xi_{k,0} \xi_{k,2}.$$

It is obvious that

$$E\left(\frac{1}{\sqrt{m}\sigma_{n1}} \sum_{k=1}^{m} \eta'_{k12} \right) = 0 \quad \text{and} \quad \operatorname{Var}\left(\frac{1}{\sqrt{m}\sigma_{n1}} \sum_{k=1}^{m} \eta'_{k12} \right) = 1. \tag{20}$$

Define

$$a = \frac{\operatorname{sgn}(\rho_n) \cdot d_n}{\sqrt{m}\sigma_{n1}}, \quad b = -\frac{\operatorname{sgn}(\rho_n) \cdot c_n}{2\sqrt{m}\sigma_{n1}}, \quad c = -\frac{\operatorname{sgn}(\rho_n) \cdot e_n}{\sqrt{m}\sigma_{n1}},$$
$$d = \frac{a_n^2}{\sqrt{m}\sigma_{n1}}, \quad e = \frac{b_n^2}{\sqrt{m}\sigma_{n1}}, \quad f = \frac{\rho_n}{\sqrt{m}\sigma_{n1}}.$$

Set $\eta_k = a\xi_{k,0}^2 + b\xi_{k,1}^2 + c\xi_{k,2}^2 + d\xi_{k,0}\xi_{k,1} + e\xi_{k,1}\xi_{k,2} + f\xi_{k,0}\xi_{k,2}$. Then it follows from (20) that

$$E(\eta_k) = 0 \quad \text{and} \quad \sum_{k=1}^{m} \operatorname{Var}(\eta_k) = 1 \tag{21}$$

for each k. Furthermore, we have

$$\max\{|a|, |b|, |c|, |d|, |e|, |f|\} \le \frac{1}{\sqrt{m}}. \tag{22}$$

Then, use the Hölder inequality, the fact that $|\xi_{k,1}\xi_{k,2}| \le \xi_{k,1}^2 + \xi_{k,2}^2$ and independence to see

$$\begin{aligned} Ee^{h|\eta_k|} \le & E\exp\left[h\left(|a|\xi_{k,0}^2 + |b|\xi_{k,1}^2 + |c|\xi_{k,2}^2 + |d\xi_{k,0}\xi_{k,1}| + |e\xi_{k,1}\xi_{k,2}| + |f\xi_{k,0}\xi_{k,2}| \right) \right] \\ \le & E\exp\left[\frac{3h}{\sqrt{m}} \left(\xi_{k,0}^2 + \xi_{k,1}^2 + \xi_{k,2}^2 \right) \right] \\ \le & E\exp\left(\frac{3h}{\sqrt{m}} \xi_{k,0}^2 \right) \cdot E\exp\left(\frac{3h}{\sqrt{m}} \xi_{k,1}^2 \right) \cdot E\exp\left(\frac{3h}{\sqrt{m}} \xi_{k,2}^2 \right) < \infty \end{aligned} \tag{23}$$

for all h, k, m satisfying $0 < h < h_m := \frac{\sqrt{m}}{12}$ and $1 \le k \le m$. By Lemma 2.6, we have

$$\gamma := \sum_{k=1}^{m} E\left(|\eta_k| e^{s'_p|\eta_k|} \right) \le \sum_{k=1}^{m} C \cdot \left(|a|^3 + |b|^3 + |c|^3 + |d|^3 + |e|^3 + |f|^3 \right) \le \sum_{k=1}^{m} \frac{C}{m^{3/2}} \le \frac{C}{\sqrt{m}}.$$

Then, we see from (21), (23) and Lemma 2.2 that

$$P\left(\frac{1}{\sqrt{m}\sigma_{n1}} \sum_{k=1}^{m} \eta'_{k12} > s'_p \right) = P\left(\sum_{k=1}^{m} \eta_k > s'_p \right) = \left[1 - \Phi(s'_p) \right] \cdot \left[1 + O(1)\left(1 + s'^3_p \right) \gamma e^{4s'^3_p \gamma} \right] \tag{24}$$

due to $s'_p < h_m := \frac{\sqrt{m}}{12}$. Notice that $s'^3_p \gamma = O\left(s'^3_p m^{-1/2} e^{s'_p/\sqrt{m}} \right) \to 0$ as $n \to \infty$ by Assumption 1.5. Hence, we obtain

$$
\begin{aligned}
P\left(\frac{1}{\sqrt{m}\sigma_{n1}}\sum_{k=1}^{m}\eta'_{k12}+\frac{\Lambda_{m12}}{\sigma_{n1}}>s'_p\right)=&P\left(\frac{1}{\sqrt{m}\sigma_{n1}}\sum_{k=1}^{m}\eta'_{k12}>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}}\right)\\
=&P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}}\right)\\
=&P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}},\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right|>\epsilon_p\right)\\
&+P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}},\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right|\le\epsilon_p\right)
\end{aligned}\tag{25}
$$

for $\epsilon > 0$. By Lemma 2.8 and the fact $\sigma_{n1} \in (3/4, 1]$, we have

$$
p\cdot P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}},\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right|>\epsilon_p\right)\le p\cdot P\left(\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right|>\epsilon_p\right)\le p\cdot P\left(|\Lambda_{m12}|>\frac{4}{3}\epsilon_p\right)\to 0 \tag{26}
$$

for $\epsilon > 0$ as $n \to \infty$. If $\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right| \le \epsilon_p$, we assume $\frac{\Lambda_{m12}}{\sigma_{n1}} = \frac{\varepsilon}{\sqrt{\log p}}$ for some $\varepsilon \in [-\epsilon, \epsilon]$. Review (21), (23) and Lemma 2.2. Then, by choosing $\epsilon > 0$ small enough, we have

$$
\begin{aligned}
&p\cdot P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\Lambda_{m12}}{\sigma_{n1}},\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right|\le\epsilon_p\right)\\
=&p\cdot P\left(\sum_{k=1}^{m}\eta_k>s'_p-\frac{\varepsilon}{\sqrt{\log p}}\right)\\
=&p\cdot\left[1-\Phi\left(s'_p-\frac{\varepsilon}{\sqrt{\log p}}\right)\right]\cdot\left[1+O\left(\frac{1}{\sqrt{m}}\right)\right]\\
=&\frac{p}{\sqrt{2\pi}\left(s'_p-\frac{\varepsilon}{\sqrt{\log p}}\right)}e^{-\frac{\left(s'_p-\frac{\varepsilon}{\sqrt{\log p}}\right)^2}{2}}\left[1+o(1)\right]\to\frac{1}{4\sqrt{2\pi}}e^{-x/2}
\end{aligned}\tag{27}
$$

as $n \to \infty$. Combining the above equalities, (25), (26) and (27), we get $\lim_{n\to\infty}\mathcal{H}_1 = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$ for any $x \in \mathbb{R}$.

Step 2: proof of $\lim_{n\to\infty}\mathcal{H}_3 = 0$. Let P_1 stand for the conditional probability given $\{\xi_{k,2}; 1 \le k \le m\}$. Then,

$$
\begin{aligned}
&P\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}+\frac{\Lambda_{m12}}{\sigma_{n1}}>s'_p,\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k34}+\frac{\Lambda_{m34}}{\sigma_{n1}}>s'_p\right)\\
=&E\left[P_1\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}+\frac{\Lambda_{m12}}{\sigma_{n1}}>s'_p\right)^2\right].
\end{aligned}\tag{28}
$$

We see from independence that

$$
\frac{1}{\sqrt{m}\sigma_{n1}}\sum_{k=1}^{m}\eta'_{k12}\sim N\left(\mu_{n2},\sigma_{n2}^2\right), \tag{29}
$$

where

$$\mu_{n2} = -\frac{\rho_n(1-\theta_n)}{2\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\left(\xi_{k,2}^2-1\right),$$
$$\sigma_{n2}^2 = \frac{\theta_n^4+\theta_n^3-\theta_n^2+\theta_n}{2\sigma_{n1}^2}+\frac{1-\theta_n}{m\sigma_{n1}^2}\sum_{k=1}^{m}\xi_{k,2}^2.$$

Note that

$$\frac{\theta_n^4+\theta_n^3-\theta_n^2+\theta_n}{2(1-\theta_n)} < \frac{7}{16}$$

due to $\theta_n \in [0, 1/2)$.

Given $\delta \in (0,1)$ and $\epsilon > 0$, set

$$\begin{aligned} B_\delta &= \left\{1-\delta+\frac{\theta_n^4+\theta_n^3-\theta_n^2+\theta_n}{2(1-\theta_n)} < \frac{\sigma_{n1}^2\sigma_{n2}^2}{1-\theta_n} < 1+\delta+\frac{\theta_n^4+\theta_n^3-\theta_n^2+\theta_n}{2(1-\theta_n)} \text{ and } \left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right| \le \epsilon_p\right\} \\ &= \left\{1-\delta < \frac{1}{m}\sum_{k=1}^{m}\xi_{k,2}^2 < 1+\delta \text{ and } \left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right| \le \epsilon_p\right\}. \end{aligned}$$

Then, by the large deviations for the sum of i.i.d. random variables and Lemma 2.8, we obtain

$$P\left(B_\delta^c\right) \le P\left(\frac{1}{m}\sum_{k=1}^{m}\xi_{k,2}^2 \in [1-\delta, 1+\delta]^c\right) + P\left(\left|\frac{\Lambda_{m12}}{\sigma_{n1}}\right| > \epsilon_p\right) \le e^{-nC_\delta} + o\left(\frac{1}{p^4}\right)$$

for $\delta \in (0,1)$ and $\epsilon > 0$, where $C_\delta > 0$ for each $\delta \in (0,1)$. By the inequality $P(N(0,1) \ge y) \le \frac{1}{\sqrt{2\pi}y}e^{-y^2/2} \le \frac{1}{2}e^{-y^2/2}$ for all $y \ge 1$, we have from (29) that, on B_δ,

$$\begin{aligned} E\left[P_1\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}+\frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p\right)^2\right] &\le E\left[P_1\left(\frac{1}{\sqrt{m}\sigma_{n1}}\sum_{k=1}^{m}\eta'_{k12}+\epsilon_p > s'_p\right)^2\right] \\ &= E\left[P_1\left(N\left(\mu_{n2},\sigma_{n2}^2\right) > s'_p-\epsilon_p\right)^2\right] \\ &\le E\exp\left(-\frac{\left(s'_p-\epsilon_p-\mu_{n2}\right)^2}{2\sigma_{n2}^2}\right) \\ &= E\exp\left[-\alpha_1\left(\frac{\sum_{k=1}^{m}\left(\xi_{k2}^2-1\right)}{\sqrt{2m}}-\beta_1\right)^2\right], \end{aligned}$$

where

$$\alpha_1 = \frac{(1-\theta_n)^3\theta_n}{2\sigma_{n1}^2\sigma_{n2}^2} \quad \text{and} \quad \beta_1 = \frac{\sqrt{2}\sigma_{n1}\left(s'_p-\epsilon_p\right)}{(\theta_n-1)\rho_n}.$$

Review the fact that $Ee^{-\alpha'(\xi'-\beta')^2} = \frac{1}{\sqrt{2\alpha'+1}}\exp\left(-\frac{\alpha'\beta'^2}{2\alpha'+1}\right)$ for any $\alpha' > 0$, $\beta' \in \mathbb{R}$ and $\xi' \sim N(0,1)$ from Fan and Jiang [9], Lemma 3.11. Then, we obtain

$$E\exp\left[-\alpha_1\left(\frac{\sum_{k=1}^{m}\left(\xi_{k2}^2-1\right)}{\sqrt{2m}}-\beta_1\right)^2\right] \le \exp\left(-\frac{\left(s'_p-\epsilon_p\right)^2}{\frac{(\theta_n-1)^2\rho_n^2}{\sigma_{n1}^2}+\sigma_{n2}^2}\right) \tag{30}$$

for $\epsilon > 0$. By some calculations, we have

$$\begin{aligned}
&\frac{(\theta_n - 1)^2 \rho_n^2}{\sigma_{n1}^2} + \sigma_{n2}^2 \\
\leq &\frac{-\theta_n^4 + 7\theta_n^3 - 7\theta_n^2 + \theta_n + 2}{2\left(\theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1\right)} + \frac{(1-\theta_n)\,\delta}{\theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1} \\
\leq &\frac{3}{2}(1+\delta)
\end{aligned}$$

on B_δ. Combining the above inequalities and (28), we conclude that

$$\begin{aligned}
&P\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12} + \frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p, \frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k34} + \frac{\Lambda_{m34}}{\sigma_{n1}} > s'_p\right) \\
\leq &E\left[P_1\left(\frac{1}{\sigma_{n1}\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12} + \frac{\Lambda_{m12}}{\sigma_{n1}} > s'_p\right)^2 I_{B_\delta^c}\right] + \exp\left(-\frac{2\left(s'_p - \epsilon_p\right)^2}{3(1+\delta)}\right) \\
\leq &P\left(B_\delta^c\right) + \exp\left(-\frac{2\left(s'_p - \epsilon_p\right)^2}{3(1+\delta)}\right) \\
\leq &o\left(\frac{1}{p^4}\right) + \exp\left(-\frac{2\left(s'_p - \epsilon_p\right)^2}{3(1+\delta)}\right).
\end{aligned}$$

We find that the last expectation is identical to $o(1/p)$ by choosing $\delta > 0$ and $\epsilon > 0$ small enough.

Then, combining *Step* 1 with *Step* 2, the proof of Lemma 2.10 is completed. □

5.2. *Proof of Lemma 2.12*

Proof. *Step* 1: proof of $\lim_{n\to\infty} \mathcal{G}_1 = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$ for any $x \in \mathbb{R}$. From Lemma 2.7, we have

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}x_{k,1}x_{k,3} + \Delta_{m13},$$

where

$$P\left(|\Delta_{m13}| > \epsilon_p\right) = o\left(\frac{1}{p^4}\right) \tag{31}$$

for all $\epsilon > 0$ as $n \to \infty$. Write

$$x_{k,1}x_{k,3} = a_n\xi_{k,0}\xi_{k,2} + b_n\xi_{k,1}\xi_{k,3} + \rho_n\left(\xi_{k,0}\xi_{k,3} + \xi_{k,1}\xi_{k,2}\right).$$

Then,

$$E\left(\sum_{k=1}^{m}x_{k,1}x_{k,3}\right) = 0 \quad \text{and} \quad \mathrm{Var}\left(\sum_{k=1}^{m}x_{k,1}x_{k,3}\right) = ma_n^2 + mb_n^2 + 2m\rho_n^2 = m. \tag{32}$$

Next, we take $a' = a_n/\sqrt{m}$, $b' = b_n/\sqrt{m}$ and $c' = \rho_n/\sqrt{m}$. Set $\eta'_k = a'\xi_{k,0}\xi_{k,2} + b'\xi_{k,1}\xi_{k,3} + c'\left(\xi_{k,0}\xi_{k,3} + \xi_{k,1}\xi_{k,2}\right)$. Then it follows from (32) that

$$E\left(\eta'_k\right) = 0 \quad \text{and} \quad \sum_{k=1}^{m}\mathrm{Var}\left(\eta'_k\right) = 1 \tag{33}$$

for each k. By the same argument as in the proof of Lemma 2.10, we have

$$Ee^{h|\eta_k'|} < \infty \quad \text{and} \quad \sum_{k=1}^{m} E\left(|\eta_k'|^3 e^{s_p|\eta_k'|}\right) \le \frac{C}{\sqrt{m}} \tag{34}$$

for all h, k, m satisfying $0 < h < h_m' := \frac{2}{3}\sqrt{m}$ and $1 \le k \le m$. Then,

$$\begin{aligned}\frac{p^2}{2}\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p\right) =&\frac{p^2}{2}\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,1}x_{k,3} > s_p - \Delta_{m13}\right)\\ =&\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \Delta_{m13}\right)\\ =&\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \Delta_{m13}, |\Delta_{m13}| > \epsilon_p\right)\\ &+\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \Delta_{m13}, |\Delta_{m13}| \le \epsilon_p\right)\end{aligned} \tag{35}$$

for $\epsilon > 0$. By Lemma 2.7, we have

$$\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \Delta_{m13}, |\Delta_{m13}| > \epsilon_p\right) \le \frac{p^2}{2}\cdot P\left(|\Delta_{m13}| > \epsilon_p\right) \to 0 \tag{36}$$

as $n \to \infty$. Review (33), (34), Lemma 2.2 and the proof of Lemma 2.10. If $|\Delta_{m13}| \le \epsilon_p$, then we assume $\Delta_{m13} = \frac{\varepsilon}{\sqrt{\log p}}$ for some $\varepsilon \in [-\epsilon, \epsilon]$. Then, by choosing $\epsilon > 0$ small enough, we have

$$\begin{aligned}&\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \Delta_{m13}, |\Delta_{m13}| \le \epsilon_p\right)\\ =&\frac{p^2}{2}\cdot P\left(\sum_{k=1}^{m}\eta_k' > s_p - \frac{\varepsilon}{\sqrt{\log p}}\right)\\ =&\frac{p^2}{2}\cdot\left[1-\Phi\left(s_p - \frac{\varepsilon}{\sqrt{\log p}}\right)\right]\cdot\left[1+O\left(\frac{1}{\sqrt{m}}\right)\right]\\ =&\frac{p^2}{2\sqrt{2\pi}\left(s_p - \frac{\varepsilon}{\sqrt{\log p}}\right)}e^{-\frac{\left(s_p - \frac{\varepsilon}{\sqrt{\log p}}\right)^2}{2}}\left[1+o(1)\right] \to \frac{1}{4\sqrt{2\pi}}e^{-x/2}\end{aligned} \tag{37}$$

as $n \to \infty$. Combining the above equalities, (35), (36) and (37), we complete the proof.

Step 2: proof of $\lim_{n\to\infty}\mathcal{G}_2 = 0$. Let P_2 stand for the conditional probability given $\{\xi_{k,0}, \xi_{k,1}; 1 \le k \le n\}$. By independence,

$$P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k15} > s_p\right) = E\left[P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p\right)^2\right]. \tag{38}$$

Given $\{\xi_{k,0}, \xi_{k,1}; 1 \le k \le n\}$, we have from independence that

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m} x_{k,1}x_{k,3} \sim N\left(0, \sigma_{n4}^2\right), \tag{39}$$

where

$$\sigma_{n4}^2 = \frac{1}{m}\left[\sum_{k=1}^{m}\left(a_n\xi_{k,0} + \operatorname{sgn}(\rho_n)\cdot c_n\xi_{k,1}\right)^2 + \sum_{k=1}^{m}\left(b_n\xi_{k,1} + \operatorname{sgn}(\rho_n)\cdot c_n\xi_{k,0}\right)^2\right]$$
$$= \frac{1}{m}\sum_{k=1}^{m}\left(\sqrt{a_n^2 + c_n^2}\,\xi_{k,0} + \operatorname{sgn}(\rho_n)\cdot\sqrt{b_n^2 + c_n^2}\,\xi_{k,1}\right)^2.$$

Given $\delta \in (0,1)$ and $\epsilon > 0$, set

$$A_\delta = \left\{1 - \delta < \frac{\sigma_{n4}^2}{a_n^2 + b_n^2 + 2c_n^2} < 1 + \delta \text{ and } |\Delta_{m13}| \le \epsilon_p\right\}.$$

Observe $\sqrt{a_n^2 + c_n^2}\,\xi_{k,0} + \operatorname{sgn}(\rho_n)\cdot\sqrt{b_n^2 + c_n^2}\,\xi_{k,1} \overset{d}{=} \sqrt{a_n^2 + b_n^2 + 2c_n^2}\,\xi_k$ due to (7). Thus, $\frac{\sigma_{n4}^2}{a_n^2+b_n^2+2c_n^2} \overset{d}{=} \frac{1}{m}\sum_{k=1}^{m}\xi_k^2$. Then, by the large deviations for the sum of i.i.d. random variables and Lemma 2.7, we obtain

$$P\left(A_\delta^c\right) \le P\left(\frac{1}{m}\sum_{k=1}^{m}\xi_k^2 \in [1-\delta, 1+\delta]^c\right) + P\left(|\Delta_{m13}| > \epsilon_p\right) \le e^{-nC_\delta} + o\left(\frac{1}{p^4}\right)$$

for all $\delta \in (0,1)$ and $\epsilon > 0$, where $C_\delta > 0$ for each $\delta \in (0,1)$.

By the inequality $P(N(0,1) \ge y) \le \frac{1}{\sqrt{2\pi}y}e^{-y^2/2} \le \frac{1}{2}e^{-y^2/2}$ for all $y \ge 1$, we have from (39) that, on A_δ,

$$P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p\right) \le P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}x_{k,1}x_{k,3} + \epsilon_p > s_p\right) = P_2\left(N\left(0,\sigma_{n4}^2\right) > s_p - \epsilon_p\right) \le \exp\left(-\frac{\left(s_p - \epsilon_p\right)^2}{2(1+\delta)}\right).$$

We use the fact that $\sigma_{n4}^2 < (1+\delta)\left(a_n^2 + b_n^2 + 2c_n^2\right) = 1 + \delta$ on A_δ in the above inequality. Recalling (38), we conclude

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k15} > s_p\right)\\
\le& E\left[P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > s_p\right)^2 I_{A_\delta^c}\right] + \exp\left(-\frac{\left(s_p - \epsilon_p\right)^2}{1+\delta}\right)\\
\le& e^{-nC_\delta} + o\left(\frac{1}{p^4}\right) + \exp\left(-\frac{s_p^2 + \epsilon_p^2 - 2s_p\cdot\epsilon_p}{1+\delta}\right).
\end{aligned}$$

We see that the last item is identical to $o(1/p^3)$ by choosing $\delta > 0$ and $\epsilon > 0$ small enough. These imply $\lim_{n\to\infty}\mathcal{G}_2 = 0$.

Combining *Step* 1 with *Step* 2, the proof of Lemma 2.12 is completed. □

5.3. *Proof of Lemma 2.13*

Proof. *Step* 1: proof of $\lim_{n\to\infty}\mathcal{F}_1 = 0$. In order to prove Lemma 2.13, we consider two different cases, that is, $\rho_n \ge 0$ and $\rho_n < 0$.

Case (i): $\rho_n \ge 0$. By Lemma 2.8, write

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left(\eta_{k12} - c_n\right) = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12}' + \Lambda_{m12},$$

where

$$P\left(|\Lambda_{m12}| > \epsilon_p\right) = o\left(\frac{1}{p^4}\right)$$

for all $\epsilon > 0$ as $n \to \infty$. Then,

$$E\left(\sum_{k=1}^{m} \eta'_{k12}\right) = 0 \quad \text{and} \quad \mathrm{Var}\left(\sum_{k=1}^{m} \eta'_{k12}\right) = m\left(\theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1\right). \tag{40}$$

Set $\sigma_{n1}^2 = \theta_n^4 - 2\theta_n^3 + 3\theta_n^2 - 2\theta_n + 1$. Then, $9/16 < \sigma_{n1}^2 \le 1$. Reviewing the proof of Lemma 2.10, we have $\eta_k = a\xi_{k,0}^2 + b\xi_{k,1}^2 + c\xi_{k,2}^2 + d\xi_{k,0}\xi_{k,1} + e\xi_{k,1}\xi_{k,2} + f\xi_{k,0}\xi_{k,2}$. Now, take $x_0 = (s_p - c_n\sqrt{m})/\sigma_{n1}$. Trivially, $x_0 \le s_p$. Observe that $x_0^3\gamma = O\left(s_p^3 m^{-1/2} e^{s_p/\sqrt{m}}\right) \to 0$ as $n \to \infty$ by Assumption 1.5. Then, we have

$$\begin{aligned} P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12} > s_p\right) =& P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12} - c_n) > s_p - c_n\sqrt{m}\right) \\ =& P\left(\sum_{k=1}^{m}\eta_k + \frac{\Lambda_{m12}}{\sigma_{n1}} > x_0\right) \\ =& P\left(\sum_{k=1}^{m}\eta_k + \frac{\Lambda_{m12}}{\sigma_{n1}} > x_0, \frac{|\Lambda_{m12}|}{\sigma_{n1}} > \epsilon_p\right) + P\left(\sum_{k=1}^{m}\eta_k + \frac{\Lambda_{m12}}{\sigma_{n1}} > x_0, \frac{|\Lambda_{m12}|}{\sigma_{n1}} \le \epsilon_p\right) \end{aligned} \tag{41}$$

for $\epsilon > 0$. By Lemma 2.8, one can get

$$p \cdot P\left(\sum_{k=1}^{m}\eta_k + \frac{\Lambda_{m12}}{\sigma_{n1}} > x_0, \frac{|\Lambda_{m12}|}{\sigma_{n1}} > \epsilon_p\right) \le p \cdot P\left(\frac{|\Lambda_{m12}|}{\sigma_{n1}} > \epsilon_p\right) \le p \cdot P\left(|\Lambda_{m12}| > \frac{3}{4}\epsilon_p\right) \to 0 \tag{42}$$

for $\epsilon > 0$ as $n \to \infty$. From (21), (23) and Lemma 2.2, we obtain that

$$\begin{aligned} & p \cdot P\left(\sum_{k=1}^{m}\eta_k + \frac{\Lambda_{m12}}{\sigma_{n1}} > x_0, \frac{|\Lambda_{m12}|}{\sigma_{n1}} \le \epsilon_p\right) \\ \le& p \cdot P\left(\sum_{k=1}^{m}\eta_k + \epsilon_p > x_0\right) \\ =& p \cdot \left[1 - \Phi\left(x_0 - \epsilon_p\right)\right] \cdot \left[1 + O\left(\frac{1}{\sqrt{m}}\right)\right] \\ =& \frac{p \cdot \sigma_{n1}}{\sqrt{2\pi}\left(s_p - c_n\sqrt{m} - \epsilon_p\sigma_{n1}\right)} e^{-\left(s_p - c_n\sqrt{m} - \epsilon_p\sigma_{n1}\right)^2/2\sigma_{n1}^2}\left[1 + o(1)\right] \\ \le& \frac{p}{\sqrt{2\pi}\left(s_p - c_n\sqrt{m} - \epsilon_p\right)} \cdot \exp\left(-\frac{\left(s_p - c_n\sqrt{m}\right)^2}{2} - \frac{9}{32}\epsilon_p^2 + \left(s_p - c_n\sqrt{m}\right)\epsilon_p\right)\left[1 + o(1)\right] \end{aligned} \tag{43}$$

for $\epsilon > 0$ as $n \to \infty$. If $\lambda \in [-\infty, 2 - \sqrt{2}]$, then

$$-\frac{s_p^2}{2} - \frac{c_n^2 m}{2} + c_n\sqrt{m}s_p + \log p - \frac{1}{2}\log\log p \to -\infty$$

as n is sufficiently large. Combining this with (43) and choosing $\epsilon > 0$ small enough, we obtain the desired conclusion.

Case (ii): $\rho_n < 0$. Write

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}+c_n)=\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}+\Lambda_{m12}.$$

Recalling (40), (41) and *Case* (i), we have

$$\begin{aligned}P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12}>s_p\right)\le&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}+c_n)>s_p\right)\\ =&P\left(\sum_{k=1}^{m}\eta_k+\frac{\Lambda_{m12}}{\sigma_{n1}}>\frac{s_p}{\sigma_{n1}}\right)\\ =&P\left(\sum_{k=1}^{m}\eta_k+\frac{\Lambda_{m12}}{\sigma_{n1}}>\frac{s_p}{\sigma_{n1}},\frac{|\Lambda_{m12}|}{\sigma_{n1}}>\epsilon_p\right)+P\left(\sum_{k=1}^{m}\eta_k+\frac{\Lambda_{m12}}{\sigma_{n1}}>\frac{s_p}{\sigma_{n1}},\frac{|\Lambda_{m12}|}{\sigma_{n1}}\le\epsilon_p\right)\end{aligned}$$

for $\epsilon>0$. Then, by (42) and Lemma 2.8, we know

$$p\cdot P\left(\sum_{k=1}^{m}\eta_k+\frac{\Lambda_{m12}}{\sigma_{n1}}>\frac{s_p}{\sigma_{n1}},\frac{|\Lambda_{m12}|}{\sigma_{n1}}>\epsilon_p\right)\to 0$$

for $\epsilon>0$ as $n\to\infty$. Recalling (43) and the definition of σ^2_{n1}, we obtain that

$$\begin{aligned}&p\cdot P\left(\sum_{k=1}^{m}\eta_k+\frac{\Lambda_{m12}}{\sigma_{n1}}>\frac{s_p}{\sigma_{n1}},\frac{|\Lambda_{m12}|}{\sigma_{n1}}\le\epsilon_p\right)\\ \le&p\cdot P\left(\sum_{k=1}^{m}\eta_k>s_p-\epsilon_p\right)\\ =&p\cdot\left[1-\Phi\left(s_p-\epsilon_p\right)\right]\cdot(1+o(1))\\ =&\frac{p}{\sqrt{2\pi}\left(s_p-\epsilon_p\right)}e^{-\frac{(s_p-\epsilon_p)^2}{2}}\to 0\end{aligned}$$

for $\epsilon>0$ as $n\to\infty$. Then, combining the above two cases, we get $\lim_{n\to\infty}\mathcal{F}_1=0$.

Step 2: proof $\lim_{n\to\infty}\mathcal{F}_3=0$. Let P_1 stand for the conditional probability given $\{\xi_{k,2};1\le k\le n\}$.

Case (i): $\rho_n\ge 0$. By indenpendence,

$$P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12}>s_p,\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k34}>s_p\right)=E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}-c_n)>s_p-c_n\sqrt{m}\right)^2\right].\tag{44}$$

Given $\{\xi_{k,2};1\le k\le n\}$, we see that

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}\sim N\left(\mu_{n3},\sigma^2_{n3}\right),\tag{45}$$

where

$$\begin{aligned}\mu_{n3}&=\frac{(\theta_n-1)\,\rho_n}{2\sqrt{m}}\sum_{k=1}^{m}\left(\xi^2_{k2}-1\right),\\ \sigma^2_{n3}&=\frac{\theta_n^4+\theta_n^3-\theta_n^2+\theta_n}{2}+\frac{1-\theta_n}{m}\sum_{k=1}^{m}\xi^2_{k2}:=A_{\theta_n}+\frac{1-\theta_n}{m}\sum_{k=1}^{m}\xi^2_{k2}.\end{aligned}$$

Given $\delta \in (0,1)$ and $\epsilon > 0$, set

$$D_\delta = \left\{1-\delta+\frac{A_{\theta_n}}{(1-\theta_n)} < \frac{\sigma_{n3}^2}{1-\theta_n} < 1+\delta+\frac{A_{\theta_n}}{(1-\theta_n)} \text{ and } |\Lambda_{m12}| \le \epsilon_p\right\}$$
$$= \left\{1-\delta < \frac{1}{m}\sum_{k=1}^m \xi_{k2}^2 < 1+\delta \text{ and } |\Lambda_{m12}| \le \epsilon_p\right\}.$$

Then, by (36), Lemma 2.8 and the large deviations for the sum of i.i.d. random variables, we obtain

$$P\left(D_\delta^c\right) \le P\left(\frac{1}{m}\sum_{k=1}^m \xi_{k2}^2 \in [1-\delta, 1+\delta]^c\right) + P\left(|\Lambda_{m12}| \le \epsilon_p\right) \le e^{-nC_\delta} + o\left(\frac{1}{p^4}\right)$$

for all $\delta \in (0,1)$ and $\epsilon > 0$, where $C_\delta > 0$ for each $\delta \in (0,1)$.

By the same argument as in (30) and the inequality $P(N(0,1) \ge y) \le \frac{1}{\sqrt{2\pi}y}e^{-y^2/2} \le \frac{1}{2}e^{-y^2/2}$ for all $y \ge 1$, we have from (45) that, on D_δ,

$$\begin{aligned}
&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m (\eta_{k12}-c_n) > s_p - c_n\sqrt{m}\right)^2\right]\\
=&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12}' + \Lambda_{m12} > s_p - c_n\sqrt{m}\right)^2\right]\\
\le&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12}' + \epsilon_p > s_p - c_n\sqrt{m}\right)^2\right]\\
=&E\left[P_1\left(N\left(\mu_{n3},\sigma_{n3}^2\right) > s_p - c_n\sqrt{m}-\epsilon_p\right)^2\right]\\
\le&E\exp\left(-\frac{\left(s_p - c_n\sqrt{m}-\epsilon_p-\mu_{n3}\right)^2}{2\sigma_{n3}^2}\right)\\
\le&\exp\left(-\frac{\left(s_p - c_n\sqrt{m}-\epsilon_p\right)^2}{(\theta_n-1)^2\rho_n^2+\sigma_{n3}^2}\right).
\end{aligned} \tag{46}$$

Note that $(\theta_n-1)^2\rho_n^2+\sigma_{n3}^2 \le 3/2+\delta$ on D_δ. Reviewing (28), we then obtain

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k34} > s_p\right)\\
\le&E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n (\eta_{k12}-c_n) > s_p - c_n\sqrt{m}\right)^2 I_{D_\delta^c}\right] + \exp\left(-\frac{\left(s_p - c_n\sqrt{m}-\epsilon_p\right)^2}{(\theta_n-1)^2\rho_n^2+\sigma_{n3}^2}\right)\\
\le&e^{-nC_\delta} + o\left(\frac{1}{p^4}\right) + \exp\left(-\frac{\left(s_p - c_n\sqrt{m}-\epsilon_p\right)^2}{3/2+\delta}\right).
\end{aligned}$$

Choosing $\delta > 0$ and $\epsilon > 0$ small enough, we know the last item is identical to $o(1/p)$.

Case (ii): $\rho_n < 0$. By independence,

$$P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k12} > s_p, \frac{1}{\sqrt{m}}\sum_{k=1}^m \eta_{k34} > s_p\right) = E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m (\eta_{k12}+c_n) > s_p + c_n\sqrt{m}\right)^2\right]. \tag{47}$$

Reviewing the definitions of μ_{n3}, σ_{n3}^2 and D_δ, we find that $\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}+c_n)\sim N\left(\mu_{n3},\sigma_{n3}^2\right)$. By the same argument as in (46), we have that, on D_δ,

$$\begin{aligned}
&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}+c_n)>s_p+c_n\sqrt{m}\right)^2\right]\\
\leq&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta'_{k12}+\epsilon_p>s_p+c_n\sqrt{m}\right)^2\right]\\
=&E\left[P_1\left(N\left(\mu_{n3},\sigma_{n3}^2\right)>s_p+c_n\sqrt{m}-\epsilon_p\right)^2\right]\\
\leq&\exp\left(-\frac{\left(s_p-c_n\sqrt{m}-\epsilon_p\right)^2}{3/2+\delta}\right).
\end{aligned}$$

Recalling (47), we then have

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k12}>s_p,\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k34}>s_p\right)\\
=&E\left[P_1\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}(\eta_{k12}+c_n)>s_p+c_n\sqrt{m}\right)^2 I_{D_\delta^c}\right]+\exp\left(-\frac{\left(s_p+c_n\sqrt{m}-\epsilon_p\right)^2}{3/2+\delta}\right)\\
\leq&e^{-nC_\delta}+o\left(\frac{1}{p^4}\right)+\exp\left(-\frac{\left(s_p+c_n\sqrt{m}-\epsilon_p\right)^2}{3/2+\delta}\right).
\end{aligned}$$

We see that the last expectation is identical to $o\left(1/p\right)$ by choosing $\delta>0$ and $\epsilon>0$ small enough.

Combining *Step* 1 with *Step* 2, the proof of Lemma 2.13 is completed. □

5.4. Proof of Lemma 2.15

Proof. Step 1: proof of $\lim_{n\to\infty}\mathcal{J}_1=0$. Reviewing (35) and the proof of Lemma 2.12, we have

$$p^2\cdot P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13}>\tau\right)=p^2\cdot P\left(\sum_{k=1}^{m}\eta'_k+\Delta_{m13}>\tau,|\Delta_{m13}|>\epsilon_p\right)+p^2\cdot P\left(\sum_{k=1}^{m}\eta'_k+\Delta_{m13}>\tau,|\Delta_{m13}|\leq\epsilon_p\right)\tag{48}$$

for $\epsilon>0$. By Lemma 2.7, for $\epsilon>0$, we obtain

$$p^2\cdot P\left(\sum_{k=1}^{m}\eta'_k+\Delta_{m13}>\tau,|\Delta_{m13}|>\epsilon_p\right)\to 0\tag{49}$$

as $n\to\infty$. Then, it is necessary to estimate $P\left(\sum_{k=1}^{m}\eta'_k+\Delta_{m13}>\tau,|\Delta_{m13}|\leq\epsilon_p\right)$ from two different cases.

Case (i): $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}}\to\lambda\in\left(2-\sqrt{2},\infty\right)$. By (33), (34) and Lemma 2.2, we have

$$\begin{aligned}
&p^2\cdot P\left(\sum_{k=1}^{m}\eta'_k+\Delta_{m13}>\tau,|\Delta_{m13}|\leq\epsilon_p\right)\\
\leq&p^2\cdot P\left(\sum_{k=1}^{m}\eta'_k>c_n\sqrt{m}+\sigma_{n1}s'_p-\epsilon_p\right)\\
=&\frac{p^2}{\sqrt{2\pi}\left(c_n\sqrt{m}+\sigma_{n1}s'_p-\epsilon_p\right)}e^{-\frac{\left(c_n\sqrt{m}+\sigma_{n1}s'_p-\epsilon_p\right)^2}{2}}
\end{aligned}$$

for $\epsilon > 0$. To prove the lemma, it is sufficent to show

$$\exp\left(-\frac{\left(c_n\sqrt{m}+\sigma_{n1}s_p'-\epsilon_p\right)^2}{2}\right)=o\left(\frac{1}{p^2}\right)$$

for $\epsilon > 0$ as $n \to \infty$. If $\lambda \in \left(2-\sqrt{2},\infty\right)$, then $\rho_n = O\left(\frac{\sqrt{\log p}}{\sqrt{n}}\right) = o\left(\frac{1}{\sqrt{\log p}}\right)$. Furthermore, $\sigma_{n1} = \sqrt{1-\rho_n^2} = 1-\frac{\rho_n^2}{2}+o\left(\rho_n^2\right) = 1+o\left(\frac{1}{\log p}\right)$. Then, without loss of generality, we assume $c_n\sqrt{m} = \left(2-\sqrt{2}+\varepsilon\right)\sqrt{\log p}$ for some $\varepsilon > 0$. Note that $c_n\sqrt{m}+s_p' \geq \left(2-\sqrt{2}+\varepsilon\right)\sqrt{\log p}+s_p' > \frac{2-\sqrt{2}+\varepsilon}{\sqrt{2}}s_p'+s_p' = \left(\sqrt{2}+\varepsilon\right)s_p'$. Then, we obtain that

$$\begin{aligned}
&\exp\left(-\frac{\left(c_n\sqrt{m}+\sigma_{n1}s_p'-\epsilon_p\right)^2}{2}\right)\\
=&\exp\left\{-\frac{\left[c_n\sqrt{m}+\left(1+o\left(\frac{1}{\log p}\right)\right)s_p'\right]^2}{2}\right\}\cdot\exp\left\{-\frac{1}{2}\epsilon_p^2+c_n\sqrt{m}\epsilon_p+\left[1+o\left(\frac{1}{\log p}\right)\right]s_p'\epsilon_p\right\}\\
=&\exp\left(-\frac{\left(c_n\sqrt{m}+s_p'\right)^2}{2}+o\left(1\right)\right)\cdot e^{C\epsilon}\\
=&\exp\left(-\frac{\left(\sqrt{2}+\varepsilon\right)^2s_p'^2}{2}+o\left(1\right)\right)\cdot e^{C\epsilon}=o\left(\frac{1}{p^2}\right)
\end{aligned}\tag{50}$$

as $n \to \infty$, where C is an absolute constant.

Case (ii): $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \infty$. By the same argument as above, it is obvious that

$$p^2\cdot P\left(\sum_{k=1}^{m}\eta_k'+\Delta_{m13}>\tau,|\Delta_{m13}|\leq\epsilon_p\right)\leq\frac{p^2}{\sqrt{2\pi}\left(c_n\sqrt{m}+\sigma_{n1}s_p'-\epsilon_p\right)}e^{-\frac{\left(c_n\sqrt{m}+\sigma_{n1}s_p'-\epsilon_p\right)^2}{2}}\to 0\tag{51}$$

for $\epsilon > 0$ as $n \to \infty$.

In summary, for $\epsilon > 0$,

$$p^2\cdot P\left(\sum_{k=1}^{m}\eta_k'+\Delta_{m13}>\tau,|\Delta_{m13}|\leq\epsilon_p\right)\to 0\tag{52}$$

as n is sufficiently large. Then, the desired conclusion follows from (48), (49) and (52).

Step 2: proof of $\lim_{n\to\infty}\mathcal{J}_3 = 0$. Let P_1 stand for the conditional probability given $\{\xi_{k,0},\xi_{k,1}; 1 \leq k \leq n\}$. By independence,

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13}>c_n\sqrt{m}+\sigma_{n1}s_p',\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k15}>c_n\sqrt{m}+\sigma_{n1}s_p'\right)\\
=&E\left[P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13}>c_n\sqrt{m}+\sigma_{n1}s_p'\right)^2\right].
\end{aligned}$$

Reviewing the proof of Lemma 2.12 and the definition of A_δ, choosing $\delta > 0$ and $\epsilon > 0$ small enough, we have from (52) that

$$P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > c_n\sqrt{m} + \sigma_{n1}s_p'\right) \leq \exp\left(-\frac{\left(c_n\sqrt{m} + \sigma_{n1}s_p' - \epsilon_p\right)^2}{2(1+\delta)}\right) = o\left(\frac{1}{p^2}\right)$$

on A_δ as $n \to \infty$. Then, we see

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > c_n\sqrt{m} + \sigma_{n1}s_p', \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k15} > c_n\sqrt{m} + \sigma_{n1}s_p'\right)\\
\leq &E\left[P_2\left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\eta_{k13} > c_n\sqrt{m} + \sigma_{n1}s_p'\right)^2 I_{A_\delta^c}\right] + \exp\left(-\frac{\left(c_n\sqrt{m} + \sigma_{n1}s_p' - \epsilon_p\right)^2}{1+\delta}\right)\\
\leq &e^{-nC_\delta} + o\left(\frac{1}{p^4}\right) + \exp\left(-\frac{\left(c_n\sqrt{m} + \sigma_{n1}s_p' - \epsilon_p\right)^2}{1+\delta}\right).
\end{aligned}$$

Then the desired conclusion follows from the arbitrariness of δ and ϵ.

Combining *Step* 1 with *Step* 2, the proof of Lemma 2.15 is completed. □

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