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A new modification of the Bernstein operator based on shifted evaluations

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Abstract. This paper introduces a new modification of the classical Bernstein operator by replacing the point-wise evaluations $\psi(w/n)$ with a linear symmetric combination of neighboring values $\psi((w-1)/n)$ and $\psi((w+1)/n)$, weighted by expressions depending on x and w. The modified operator preserves the classical Bernstein structure while incorporating local smoothness into the evaluation process. We establish uniform convergence using Korovkin's theorem and derive a Voronovskaja-type asymptotic formula. Numerical comparisons confirm that the proposed operator provides an improved approximation accuracy.

1. Introduction

Approximation theory is a fundamental branch of mathematical analysis that focuses on representing complex functions using simpler mathematical operators. These approximations play a crucial role in simplifying analysis and supporting applications across various scientific and engineering domains. Among the most classical tools in this area is the Bernstein operator, introduced in 1912 [6], defined by

$$\mathcal{B}_n(\psi; y) = \sum_{w=0}^{n} \psi\left(\frac{w}{n}\right) q_{w,n}(y), \tag{1}$$

where

$$q_{w,n}(y) = \binom{n}{w} y^w (1-y)^{n-w}, \quad y \in [0,1], \tag{2}$$

which is widely used to approximate continuous functions on the interval $[0,1]$.

In 1953, Korovkin [9] provided a fundamental convergence criterion for sequences of positive linear operators. His theorem states that if a sequence of such operators converges on the test functions $\{1, y, y^2\}$,

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then it converges uniformly on all functions in $C([a,b])$. This result significantly simplified the study of operator convergence and spurred further research on constructing more effective approximation operators.

Over the years, many modifications have been proposed to the classical Bernstein operator to enhance its approximation behavior. For example, in 2020, Usta [11] introduced a modified operator that incorporates the second central moment:

$$\mathcal{B}_n^*(\psi;y) = \frac{1}{n}\sum_{w=0}^{n}\binom{n}{w}(w-ny)^2 y^{w-1}(1-y)^{n-w-1}\psi\left(\frac{w}{n}\right). \tag{3}$$

In 2022, Zhang et al. [12] presented a Bernstein–Kantorovich modification that reproduces affine functions via a specially scaled integral:

$$\mathcal{K}_n(\psi;y) = \psi(0)q_{0,n}(y) + \psi(1)q_{n,n}(y) + (n+1)\sum_{w=1}^{n-1} q_{w,n}(y)\int_{\frac{w}{n+1}}^{\frac{w+1}{n+1}} \psi\left(a_{n,w}t\right)dt, \tag{4}$$

where

$$a_{n,w} = \frac{n+1}{n}\cdot\frac{2w}{2w+1}. \tag{5}$$

More recently, in 2025, Chen et al. [7] proposed a modification of the classical Bernstein–Durrmeyer operator by altering the term to $k=n$. Instead of evaluating the function at the end point $f(1)$, it uses a linear combination of values at neighboring points:

$$U_n(\psi,y) = \left(2\psi\left(1-\frac{1}{n}\right)-\psi\left(1-\frac{2}{n}\right)\right)q_{n,n}(y) + n\sum_{w=0}^{n-1} q_{w,n}(y)\int_0^1 q_{w,n-1}(t)\psi(t)\,dt. \tag{6}$$

Recent developments in approximation theory have also focused on error estimation and affine function preservation in various operator families, as seen in the works on Kantorovich and Szász-Mirakjan type operators [3–5, 8]. However, most existing refinements of the Bernstein operator, including those mentioned above, primarily focus on integral averages or endpoint interpolation. While effective in specific contexts, these approaches often lack a mechanism to balance local errors symmetrically or may introduce computational complexity.

Motivated by these developments and to address these limitations, this paper introduces a new modification of the classical Bernstein operator. By utilizing a symmetric linear combination of neighboring nodes, the proposed method acts as a central difference stabilizer, aiming to reduce the error constant and preserve local smoothness more effectively than standard evaluations.The operator is defined by:

$$\mathcal{Z}_n(\psi;y) = \sum_{w=0}^{n}\mathcal{P}_{n,w}(\psi;y)\,q_{w,n}(y), \tag{7}$$

where

$$\mathcal{P}_{n,w}(\psi;y) = \frac{w+1-ny}{2}\,\psi\left(\frac{w-1}{n}\right) + \frac{ny-w+1}{2}\,\psi\left(\frac{w+1}{n}\right), \quad y\in[0,1], \tag{8}$$

and $\psi \in C([0,1])$, with norm

$$\|\psi\| = \sup_{y\in[0,1]}|\psi(y)|.$$

The paper is organized as follows: Section 2 presents the construction of the new operator and preliminary lemmas. Section 3 is devoted to the main convergence theorems and asymptotic analysis. Section 4 provides numerical examples comparing the proposed operator with existing methods, followed by conclusions in Section 5.

2. Primary Results

In this section, several key lemmas are established to characterize the fundamental properties of the operator $\mathcal{Z}_n(\psi; y)$ and its associated moment sequence $\mathcal{T}_{m,n}(y)$. These results are formulated in a concise form and will be used as a foundation for subsequent theoretical developments.

Lemma 2.1. *For the classical Bernstein weights $q_{w,n}(y)$, the following identities hold:*

$$(i) \quad \sum_{w=0}^{n} q_{w,n}(y) = 1,$$

$$(ii) \quad \sum_{w=0}^{n} q_{w,n}(y)\,\frac{w}{n} = y,$$

$$(iii) \quad \sum_{w=0}^{n} q_{w,n}(y)\,\frac{w^2}{n^2} = y^2 - \frac{y^2}{n} + \frac{y}{n}.$$

Proof. These identities are derived directly by expanding the binomial terms corresponding to the zeroth, first, and second moments of the Bernstein weights. Each expression follows immediately from the basic properties of the binomial distribution. □

Lemma 2.2. *The derivative of the Bernstein basis functions $q_{w,n}(y)$ satisfies the following identity:*

$$y(1-y)\,q'_{w,n}(y) = (w - ny)\,q_{w,n}(y).$$

Proof. By differentiating the Bernstein basis function $q_{w,n}(y)$ with respect to y, and then multiplying both sides by $y(1-y)$, the following is obtained:

$$\begin{aligned} y(1-y)q'_{w,n}(y) &= y(1-y)\binom{n}{w}\left[wy^{w-1}(1-y)^{n-w} - (n-w)y^{w}(1-y)^{n-w-1}\right] \\ &= \left[w(1-y) - (n-w)y\right] q_{w,n}(y) \\ &= (w - ny)\,q_{w,n}(y). \end{aligned}$$

Thus, the identity is established. □

Lemma 2.3. *For the function $\mathcal{P}_{n,w}(\psi; y)$, the following properties hold:*

$$(i) \quad \mathcal{P}_{n,w}(1; y) = 1,$$

$$(ii) \quad \mathcal{P}_{n,w}(t; y) = y,$$

$$(iii) \quad \mathcal{P}_{n,w}(t^2; y) = \frac{2wy}{n} - \frac{w^2}{n^2} + \frac{1}{n^2}.$$

Proof. Using the definition of function $\mathcal{P}_{n,w}(\psi; y)$, the following are verified:

$$(i) \quad \mathcal{P}_{n,w}(1; y) = \frac{w + 1 - ny}{2} + \frac{ny - w + 1}{2} = 1.$$

$$\begin{aligned} (ii) \quad \mathcal{P}_{n,w}(t; y) &= \frac{w + 1 - ny}{2}\left(\frac{w-1}{n}\right) + \frac{ny - w + 1}{2}\left(\frac{w+1}{n}\right) \\ &= \frac{w^2 - nwy + ny - 1 - w^2 + nwy + ny + 1}{2n} \\ &= y. \end{aligned}$$

$$(iii)\quad \mathcal{P}_{n,w}(t^2;y) = \frac{w+1-ny}{2}\left(\frac{w-1}{n}\right)^2 + \frac{ny-w+1}{2}\left(\frac{w+1}{n}\right)^2$$
$$= \frac{1}{2n^2}\Big(w^3 - w^2 - nw^2y - w + 2nwy - ny + 1$$
$$- w^3 - w^2 + nw^2y + w + 2nwy + ny + 1\Big)$$
$$= \frac{-2w^2 + 4nwy + 2}{2n^2}$$
$$= \frac{2wy}{n} - \frac{w^2}{n^2} + \frac{1}{n^2}.$$

□

Lemma 2.4. *For the sequence $\mathcal{Z}_n(\psi;y)$, the following identities hold:*

(i) $\mathcal{Z}_n(1;y) = 1,$

(ii) $\mathcal{Z}_n(t;y) = y,$

(iii) $\mathcal{Z}_n(t^2;y) = y^2 + \frac{y^2}{n} - \frac{y}{n} + \frac{1}{n^2}.$

Proof. The result follows directly by combining Lemmas 2.1 and 2.3:

$$(i)\quad \mathcal{Z}_n(1;y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(1;y)\, q_{w,n}(y) = \sum_{w=0}^{n} q_{w,n}(y) = 1.$$

$$(ii)\quad \mathcal{Z}_n(t;y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(t;y)\, q_{w,n}(y) = y\sum_{w=0}^{n} q_{w,n}(y) = y.$$

$$(iii)\quad \mathcal{Z}_n(t^2;y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(t^2;y)\, q_{w,n}(y)$$
$$= \sum_{w=0}^{n}\left(\frac{2wy}{n} - \frac{w^2}{n^2} + \frac{1}{n^2}\right) q_{w,n}(y)$$
$$= 2y\sum_{w=0}^{n}\frac{w}{n}\, q_{w,n}(y) - \sum_{w=0}^{n}\frac{w^2}{n^2} q_{w,n}(y) + \frac{1}{n^2}\sum_{w=0}^{n} q_{w,n}(y)$$
$$= 2y^2 - \left(y^2 - \frac{y^2}{n} + \frac{y}{n}\right) + \frac{1}{n^2}$$
$$= y^2 + \frac{y^2}{n} - \frac{y}{n} + \frac{1}{n^2}.$$

□

Lemma 2.5. *The function*

$$\mathcal{R}_{m,n,\alpha}(y) = \sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^m$$

has the following properties:

(*i*) $\mathcal{R}_{0,n,\alpha}(y) = 1$,

(*ii*) $\mathcal{R}_{1,n,\alpha}(y) = \frac{\alpha}{n}$,

(*iii*) $\mathcal{R}_{m+1,n,\alpha}(y) = \frac{1}{n}\left((y-y^2)\left(\mathcal{R}'_{m,n,\alpha}(y) + m\,\mathcal{R}_{m-1,n,\alpha}(y)\right) + \alpha\,\mathcal{R}_{m,n,\alpha}(y)\right)$,

(*iv*) $\mathcal{R}_{m,n,\alpha}(y) = O\left(n^{-[\frac{m+1}{2}]}\right)$.

Proof. The properties in parts (*i*) and (*ii*) can be derived directly from the definition. To establish part (*iii*), the following steps are carried out.

The derivative of $\mathcal{R}_{m,n,\alpha}(y)$ is computed as:

$$(iii)\quad \mathcal{R}'_{m,n,\alpha}(y) = \sum_{w=0}^{n}\left(q'_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^m - m\,q_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^{m-1}\right)$$
$$= \sum_{w=0}^{n} q'_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^m - m\,\mathcal{R}_{m-1,n,\alpha}(y).$$

Next, both sides are multiplied by $y(1-y)$, and Lemma 2.2 is applied. This yields:

$$(y-y^2)\,\mathcal{R}'_{m,n,\alpha}(y) = \sum_{w=0}^{n}(w-ny)\,q_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^m - (y-y^2)\,m\,\mathcal{R}_{m-1,n,\alpha}(y).$$

The identity $w - ny = n\left(\frac{w+\alpha}{n} - y\right) - \alpha$ is substituted into the summation. As a result:

$$(y-y^2)\,\mathcal{R}'_{m,n,\alpha}(y) = \sum_{w=0}^{n}\left[n\left(\frac{w+\alpha}{n} - y\right) - \alpha\right]q_{w,n}(y)\left(\frac{w+\alpha}{n} - y\right)^m$$
$$- (y-y^2)\,m\,\mathcal{R}_{m-1,n,\alpha}(y)$$
$$= n\,\mathcal{R}_{m+1,n,\alpha}(y) - \alpha\,\mathcal{R}_{m,n,\alpha}(y) - (y-y^2)\,m\,\mathcal{R}_{m-1,n,\alpha}(y).$$

Finally, the recurrence relation for the functional $\mathcal{R}_{m+1,n,\alpha}(y)$ is derived as follows:

$$\mathcal{R}_{m+1,n,\alpha}(y) = \frac{1}{n}\left((y-y^2)\left(\mathcal{R}'_{m,n,\alpha}(y) + m\,\mathcal{R}_{m-1,n,\alpha}(y)\right) + \alpha\,\mathcal{R}_{m,n,\alpha}(y)\right).$$

(*iv*) This part can be proven using mathematical induction. Assume that the statement holds for $m = 1$ and for some $m = \ell$. Then, applying the relationship established at point (*iii*) of the same lemma, we can deduce that the statement also holds for $m = \ell + 1$.

□

Lemma 2.6. *For $m \geq 0$, the m-th order central moment of the sequence $\mathcal{Z}_n(\psi; y)$ is defined by*

$$\mathcal{T}_{m,n}(y) = \mathcal{Z}_n((t-y)^m; y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}((t-y)^m; y)\,q_{w,n}(y),$$

and possesses the following properties:

(*i*) $\mathcal{T}_{0,n}(y) = 1$,

(*ii*) $\mathcal{T}_{1,n}(y) = 0$,

(*iii*) $\mathcal{T}_{2,n}(y) = \frac{y(y-1)}{n} + \frac{1}{n^2}$,

(*iv*) $\mathcal{T}_{m,n}(y) = \frac{n}{2}\left(\mathcal{R}_{m+1,n,-1}(y) - \mathcal{R}_{m+1,n,1}(y)\right) + \mathcal{R}_{m,n,-1}(y) + \mathcal{R}_{m,n,1}(y)$.

Proof. The verification of the required central moments of the operator $\mathcal{T}_n$ is carried out as follows.
(i) For the zeroth central moment, by the definition of $\mathcal{T}_n$ and the application of Lemma 2.4, it is established that

$$\mathcal{T}_{0,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(1; y)\, q_{w,n}(y) = \sum_{w=0}^{n} q_{w,n}(y) = 1.$$

(ii) The first central moment is calculated by writing

$$\mathcal{T}_{1,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(t - y; y)\, q_{w,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(t; y)\, q_{w,n}(y) - y \sum_{w=0}^{n} \mathcal{P}_{n,w}(1; y)\, q_{w,n}(y).$$

Applying Lemma 2.4 yields

$$\mathcal{T}_{1,n}(y) = y - y = 0.$$

(iii) To compute the second central moment, consider

$$\mathcal{T}_{2,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}((t - y)^2; y)\, q_{w,n}(y).$$

Expanding the square, one obtains

$$\mathcal{T}_{2,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}(t^2; y)\, q_{w,n}(y) - 2y \sum_{w=0}^{n} \mathcal{P}_{n,w}(t; y)\, q_{w,n}(y) + y^2 \sum_{w=0}^{n} \mathcal{P}_{n,w}(1; y)\, q_{w,n}(y).$$

By using Lemma 2.4, the following equality is deduced:

$$\mathcal{T}_{2,n}(y) = \frac{y(y-1)}{n} + \frac{1}{n^2}.$$

(iv) For the general case, the m-th central moment is given by

$$\mathcal{T}_{m,n}(y) = \sum_{w=0}^{n} \mathcal{P}_{n,w}((t - y)^m; y)\, q_{w,n}(y).$$

Using the explicit form of $\mathcal{P}_{n,w}$, it follows that

$$\begin{aligned}
\mathcal{T}_{m,n}(y) &= \sum_{w=0}^{n} \left(\frac{w + 1 - ny}{2} \left(\frac{w-1}{n} - y \right)^m + \frac{ny - w + 1}{2} \left(\frac{w+1}{n} - y \right)^m \right) q_{w,n}(y) \\
&= \frac{n}{2} \sum_{w=0}^{n} \left(\frac{w-1}{n} - y \right)^{m+1} q_{w,n}(y) + \sum_{w=0}^{n} \left(\frac{w-1}{n} - y \right)^m q_{w,n}(y) \\
&\quad - \frac{n}{2} \sum_{w=0}^{n} \left(\frac{w+1}{n} - y \right)^{m+1} q_{w,n}(y) + \sum_{w=0}^{n} \left(\frac{w+1}{n} - y \right)^m q_{w,n}(y).
\end{aligned}$$

The application of Lemma 2.5 then yields

$$\mathcal{T}_{m,n}(y) = \frac{n}{2} \left(\mathcal{R}_{m+1,n,-1}(y) - \mathcal{R}_{m+1,n,1}(y) \right) + \mathcal{R}_{m,n,-1}(y) + \mathcal{R}_{m,n,1}(y).$$

□

3. Main Results

In this section, the fundamental convergence and asymptotic properties of the sequence $\mathcal{Z}_n(\psi; y)$ are established. Specifically, a convergence theorem is proved using Korovkin's criterion, and a Voronovskaja-type asymptotic formula is derived. In addition, an estimate of the rate of convergence is provided in terms of the modulus of continuity. These results form the theoretical foundation for the subsequent analytical and numerical investigations.

Theorem 3.1 (Convergence Theorem). *The sequence $\mathcal{Z}_n(\psi; y)$ is said to converge to the function ψ as $n \to \infty$, for every $\psi \in C([0, 1])$.*

Proof. To apply Korovkin's theorem [9], it suffices to verify the convergence of the operator $\mathcal{Z}_n$ on the test functions 1, t, and t^2. By Lemma 2.4, we have:

$$\lim_{n\to\infty} \mathcal{Z}_n(1; y) = 1, \quad \lim_{n\to\infty} \mathcal{Z}_n(t; y) = y, \quad \lim_{n\to\infty} \mathcal{Z}_n(t^2; y) = y^2,$$

uniformly on $[0, 1]$.

Therefore, by Korovkin's theorem, it follows that the sequence $\mathcal{Z}_n(\psi; y)$ converges uniformly to $\psi(y)$ for all $\psi \in C([0, 1])$. □

Definition 3.2. *Let $\epsilon > 0$. The modulus of continuity [2] of a function ψ is defined as:*

$$\omega_\epsilon(\psi) = \sup_{\substack{t,y\in[0,1] \\ |t-y|\leq\epsilon}} |\psi(t) - \psi(y)|. \tag{9}$$

Theorem 3.3. *Let $\psi \in C([0, 1])$. Then, the following inequality holds:*

$$\left|\mathcal{Z}_n(\psi; y) - \psi(y)\right| \leq 2\,\omega_\epsilon(\psi), \tag{10}$$

where $\epsilon = \dfrac{n}{2\sqrt{1 + ny(1 - y)}}$.

Proof. By using the standard property of the modulus of continuity [2], one has

$$|\psi(t) - \psi(y)| \leq \omega_\epsilon(\psi)\left(\frac{|t - y|}{\epsilon} + 1\right).$$

Since the operator $\mathcal{Z}_n(\cdot; y)$ is linear, it follows that

$$\begin{aligned}\left|\mathcal{Z}_n(\psi(t); y) - \psi(y)\right| &\leq \omega_\epsilon(\psi)\left(\frac{\mathcal{Z}_n(|t - y|; y)}{\epsilon} + 1\right) \\ &= \omega_\epsilon(\psi)\left(\frac{1}{\epsilon}\sum_{w=0}^{n} q_{w,n}(y)\,\mathcal{P}_{n,w}(|t - y|; y) + 1\right).\end{aligned}$$

By substituting the explicit form of $\mathcal{P}_{n,w}$ and using Lemma 2.5, the inequality becomes:

$$\begin{aligned}\left|\mathcal{Z}_n(\psi(t);y)-\psi(y)\right| &\le \omega_\epsilon(\psi)\Bigg(\frac{1}{\epsilon}\sum_{w=0}^{n} q_{w,n}(y)\Bigg[\frac{w+1-ny}{2}\left|\frac{w-1}{n}-y\right| \\ &\quad + \frac{ny-w+1}{2}\left|\frac{w+1}{n}-y\right|\Bigg]+1\Bigg) \\ &= \omega_\epsilon(\psi)\Bigg(\frac{1}{\epsilon}\Bigg[\frac{n}{2}\sum_{w=0}^{n}\left(\frac{w-1}{n}-y\right)^2 q_{w,n}(y)-\frac{n}{2}\sum_{w=0}^{n}\left(\frac{w+1}{n}-y\right)^2 q_{w,n}(y) \\ &\quad + \sum_{w=0}^{n}\left|\frac{w-1}{n}-y\right| q_{w,n}(y)+\sum_{w=0}^{n}\left|\frac{w+1}{n}-y\right| q_{w,n}(y)\Bigg]+1\Bigg) \\ &= \omega_\epsilon(\psi)\Bigg(\frac{1}{\epsilon}\Big[\frac{n}{2}\left(\mathcal{R}_{2,n,-1}(y)-\mathcal{R}_{2,n,1}(y)\right) \\ &\quad + \sum_{w=0}^{n}\left|\frac{w-1}{n}-y\right| q_{w,n}(y)+\sum_{w=0}^{n}\left|\frac{w+1}{n}-y\right| q_{w,n}(y)\Big]+1\Bigg).\end{aligned}$$

Using Lemma 2.5 (iii), it can be seen that $\mathcal{R}_{2,n,-1}(y)-\mathcal{R}_{2,n,1}(y)=0$. Applying the Cauchy–Schwarz inequality [10], the following estimate is derived:

$$\begin{aligned}\left|\mathcal{Z}_n(\psi(t);y)-\psi(y)\right| &\le \omega_\epsilon(\psi)\Bigg(\frac{1}{\epsilon}\Bigg[\left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w-1}{n}-y\right)^2\right)^{1/2} \\ &\quad + \left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+1}{n}-y\right)^2\right)^{1/2}\Bigg]+1\Bigg) \\ &= \omega_\epsilon(\psi)\left(\frac{1}{\epsilon}\left[(\mathcal{R}_{2,n,-1}(y))^{1/2}+(\mathcal{R}_{2,n,1}(y))^{1/2}\right]+1\right) \\ &= \omega_\epsilon(\psi)\left(\frac{1}{\epsilon}\cdot\frac{2\sqrt{1+ny(1-y)}}{n}+1\right).\end{aligned}$$

Finally, by substituting $\epsilon=\dfrac{n}{2\sqrt{1+ny(1-y)}}$, we obtain:

$$\left|\mathcal{Z}_n(\psi(t);y)-\psi(y)\right| \le 2\,\omega_\epsilon(\psi).$$

□

Theorem 3.4. *For $\psi \in C([0,1])$ with ψ'' existing and continuous, we have*

$$\lim_{n\to\infty} n\Big(\mathcal{Z}_n\big(\psi(t);y\big)-\psi(y)\Big)=\frac{y(y-1)}{2}\psi''(y). \tag{11}$$

Proof. The Taylor expansion of $\psi(t)$ about y is given by [1]

$$\psi(t)=\psi(y)+(t-y)\psi'(y)+\frac{(t-y)^2\psi''(y)}{2}+(t-y)^2\zeta(t,y),$$

where $\zeta(t,y)\to 0$ as $t\to y$.

Applying the operator $\mathcal{Z}_n(\cdot;y)$ on both sides and using Lemma 2.6, it follows that

$$\mathcal{Z}_n\Big(\psi(t);y\Big) = \psi(y)\,\mathcal{Z}_n(1;y) + \psi'(y)\,\mathcal{T}_{1,n}(y) + \frac{1}{2}\psi''(y)\,\mathcal{T}_{2,n}(y) + \mathcal{Z}_n\Big((t-y)^2\zeta(t,y);y\Big).$$

Multiplying both sides by n and taking the limit as $n \to \infty$, one obtains

$$\lim_{n\to\infty} n\Big(\mathcal{Z}_n\Big(\psi(t);y\Big) - \psi(y)\Big) = \psi'(y)\lim_{n\to\infty}\Big(n\mathcal{T}_{1,n}(y)\Big) + \frac{1}{2}\psi''(y)\lim_{n\to\infty}\Big(n\mathcal{T}_{2,n}(y)\Big) + \lim_{n\to\infty}\Big(n\,\mathcal{Z}_n\Big((t-y)^2\zeta(t,y);y\Big)\Big).$$

It is known that

$$\lim_{n\to\infty} n\,\mathcal{T}_{1,n}(y) = 0 \quad \text{and} \quad \lim_{n\to\infty}\Big(n\mathcal{T}_{2,n}(y)\Big) = y(y-1),$$

which yields

$$\lim_{n\to\infty} n\Big(\mathcal{Z}_n\Big(\psi(t);y\Big) - \psi(y)\Big) = \frac{y(y-1)}{2}\psi''(y) + \lim_{n\to\infty}\Big(n\,\mathcal{Z}_n\Big((t-y)^2\zeta(t,y);y\Big)\Big).$$

It remains to show that

$$\lim_{n\to\infty}\left(n\mathcal{Z}_n\left((t-y)^2\zeta(t,y);y\right)\right) = 0.$$

This can be rewritten as

$$\mathcal{Z}_n(n(t-y)^2\zeta(t,y);y) = n\sum_{w=0}^{n} q_{w,n}(y)\mathcal{P}_{n,w}\left((t-y)^2\,\zeta(t,y);y\right).$$

By substituting the explicit form of $\mathcal{P}_{n,w}$, the expression becomes:

$$\begin{aligned}\mathcal{Z}_n(n(t-y)^2\zeta(t,y);y) &= n\sum_{w=0}^{n} q_{w,n}(y)\Bigg[\frac{w+1-ny}{2}\left(\frac{w-1}{n}-y\right)^2\zeta(\frac{w-1}{n},y) \\ &\quad + \frac{ny-w+1}{2}\left(\frac{w+1}{n}-y\right)^2\zeta(\frac{w+1}{n},y)\Bigg] \\ &= n\Bigg[\frac{n}{2}\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w-1}{n}-y\right)^3\zeta(\frac{w-1}{n},y) \\ &\quad + \sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w-1}{n}-y\right)^2\zeta(\frac{w-1}{n},y) \\ &\quad - \frac{n}{2}\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+1}{n}-y\right)^3\zeta(\frac{w+1}{n},y) \\ &\quad + \sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+1}{n}-y\right)^2\zeta(\frac{w+1}{n},y)\Bigg].\end{aligned}$$

Applying Lemma 2.5 and Using the Cauchy–Schwarz inequality [10], the expression can be estimated as:

$\mathcal{Z}_n(n(t-y)^2\zeta(t,y);y) \leq$

$$n\Bigg[\frac{n}{2}\left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w-1}{n}-y\right)^{6}\right)^{\frac{1}{2}}\times\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w-1}{n},y)^{2}\right)^{\frac{1}{2}}$$

$$+\left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w-1}{n}-y\right)^{4}\right)^{\frac{1}{2}}\times\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w-1}{n},y)^{2}\right)^{\frac{1}{2}}$$

$$-\frac{n}{2}\left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+1}{n}-y\right)^{6}\right)^{\frac{1}{2}}\times\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w+1}{n},y)^{2}\right)^{\frac{1}{2}}$$

$$+\left(\sum_{w=0}^{n} q_{w,n}(y)\left(\frac{w+1}{n}-y\right)^{4}\right)^{\frac{1}{2}}\times\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w+1}{n},y)^{2}\right)^{\frac{1}{2}}\Bigg]$$

$$=n\Bigg[\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w-1}{n},y)^{2}\right)^{\frac{1}{2}}\left(\frac{n}{2}(\mathcal{R}_{6,n,-1})^{\frac{1}{2}}+(\mathcal{R}_{4,n,-1})^{\frac{1}{2}}\right)$$

$$+\left(\sum_{w=0}^{n} q_{w,n}(y)\zeta(\frac{w+1}{n},y)^{2}\right)^{\frac{1}{2}}\left(\frac{-n}{2}(\mathcal{R}_{6,n,1})^{\frac{1}{2}}+(\mathcal{R}_{4,n,1})^{\frac{1}{2}}\right)\Bigg]$$

Using the classical Bernstein approximation properties, it can be shown that

$$\sum_{w=0}^{n} q_{w,n}(y)\,\zeta\left(\frac{w\pm 1}{n},y\right)^{2}\to\zeta^{2}(y,y)=0\quad\text{as } n\to\infty.$$

and from Lemma 2.5 it holds that $\lim_{n\to\infty} n\mathcal{R}_{m,n,\alpha}(y)=0$ for $m>2$. Hence,

$$\lim_{n\to\infty}\left(n\mathcal{Z}_{n}\left((t-y)^{2}\zeta(t,y);y\right)\right)=0.$$

□

Theorem 3.5. *Let $\psi\in C([0,1])$ and $n\in\mathbb{N}$. Then, the operator $\mathcal{Z}_n$ satisfies the following inequality:*

$$|\mathcal{Z}_{n}(\psi;\cdot)|\leq\|\psi\|\left(\frac{\sqrt{4+n}}{2}\right).$$

Proof. Using the definition of $\mathcal{Z}_n(\psi;y)$, we have:

$$\begin{aligned}\left|\mathcal{Z}_{n}(\psi;y)\right| &=\left|\sum_{w=0}^{n}\mathcal{P}_{n,w}(\psi;y)\,q_{w,n}(y)\right|\\ &\leq\sum_{w=0}^{n}|\mathcal{P}_{n,w}(\psi;y)|\,q_{w,n}(y)\\ &\leq\sum_{w=0}^{n}\left(\left|\frac{w+1-ny}{2}\right|\left|\psi\left(\frac{w-1}{n}\right)\right|+\left|\frac{ny-w+1}{2}\right|\left|\psi\left(\frac{w+1}{n}\right)\right|\right)q_{w,n}(y)\\ &\leq\|\psi\|\sum_{w=0}^{n}\left(\left|\frac{w+1-ny}{2}\right|+\left|\frac{ny-w+1}{2}\right|\right)q_{w,n}(y).\end{aligned}$$

By applying the Cauchy-Schwarz inequality [10] and using Lemma 2.1, we get:

$$\left|\mathcal{Z}_n(\psi;y)\right| \leq \|\psi\| \left(\left(\sum_{w=0}^{n} \left(\frac{w+1-ny}{2} \right)^2 q_{w,n}(y) \right)^{\frac{1}{2}} + \left(\sum_{w=0}^{n} \left(\frac{ny-w+1}{2} \right)^2 q_{w,n}(y) \right)^{\frac{1}{2}} \right)$$

$$= \|\psi\| \left(1 + ny(1-y)\right)^{\frac{1}{2}} \leq \|\psi\| \left(1 + \frac{n}{4}\right)^{\frac{1}{2}} = \|\psi\| \left(\frac{\sqrt{4+n}}{2} \right)$$

□

Theorem 3.6. *Let $\psi \in C([0,1])$ and assume that ψ'' exists and is continuous on $[0,1]$. Then,*

$$\left|\psi(y) - \mathcal{Z}_n(\psi;y)\right| \leq \|\psi''\| \left(\frac{1}{8n} + \frac{1}{2n^2} \right), \qquad \text{for all } y \in [0,1].$$

Proof. By expanding ψ at the point y using Taylor's formula and applying the linear operator $\mathcal{Z}_n$ to both sides, and utilizing the moment results from Lemma 2.6, we obtain:

$$\mathcal{Z}_n(\psi;y) = \psi(y)\,\mathcal{T}_{0,n}(y) + \psi'(y)\,\mathcal{T}_{1,n}(y) + \frac{1}{2}\psi''(\xi)\,\mathcal{T}_{2,n}(y)$$
$$= \psi(y) + \frac{\psi''(\xi)}{2} \left(\frac{y(y-1)}{n} + \frac{1}{n^2} \right),$$

where ξ lies between t and y. Taking the absolute value and the supremum norm over $[0,1]$, we get:

$$\left|\psi(y) - \mathcal{Z}_n(\psi;y)\right| \leq \frac{\|\psi''\|}{2} \left| \frac{y(y-1)}{n} + \frac{1}{n^2} \right|.$$

Since $\max_{y\in[0,1]} |y(y-1)| = \frac{1}{4}$, it follows that:

$$\left|\psi(y) - \mathcal{Z}_n(\psi;y)\right| \leq \|\psi''\| \left(\frac{1}{2} \cdot \frac{1}{4n} + \frac{1}{2n^2} \right) = \|\psi''\| \left(\frac{1}{8n} + \frac{1}{2n^2} \right).$$

□

Definition 3.7. *Let $\psi \in C([0,1])$, and let $C^2([0,1])$ denote the space of functions in $C([0,1])$ that are twice continuously differentiable. The K-functional is defined for any $\delta > 0$ as:*

$$K(\psi,\delta) = \inf_{\varphi \in C^2([0,1])} \left\{ \|\psi - \varphi\| + \delta \|\varphi''\| \right\}. \tag{12}$$

Theorem 3.8. *Let $\psi \in C([0,1])$. Then, there exists a constant $\mathcal{M} > 0$ such that for all $n \in \mathbb{N}$:*

$$|\psi(y) - \mathcal{Z}_n(\psi;y)| \leq \mathcal{M} \cdot K\left(\psi, \frac{1}{8n} + \frac{1}{2n^2} \right), \quad \text{for all } y \in [0,1].$$

Proof. Let $\varphi \in C^2([0,1])$ be arbitrary. By the linearity of $\mathcal{Z}_n$, we can write:

$$|\psi(y) - \mathcal{Z}_n(\psi;y)| \leq |\psi(y) - \varphi(y)| + |\mathcal{Z}_n(\psi - \varphi;y)| + |\mathcal{Z}_n(\varphi;y) - \varphi(y)|.$$

Using Theorem 3.5 and Theorem 3.6, we have the following estimates:

$$|\mathcal{Z}_n(\psi - \varphi;y)| \leq \left(\frac{\sqrt{4+n}}{2} \right) \|\psi - \varphi\| \text{ and } |\mathcal{Z}_n(\varphi;y) - \varphi(y)| \leq \left(\frac{1}{8n} + \frac{1}{2n^2} \right) \|\varphi''\|.$$

Combining these inequalities yields:

$$|\psi(y) - \mathcal{Z}_n(\psi;y)| \leq \left(1 + \frac{\sqrt{4+n}}{2} \right) \|\psi - \varphi\| + \left(\frac{1}{8n} + \frac{1}{2n^2} \right) \|\varphi''\|.$$

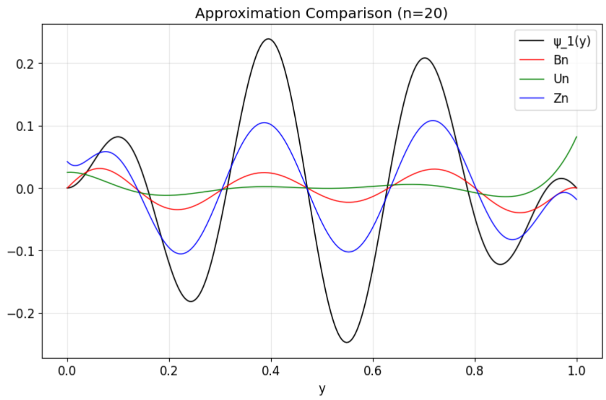
This confirms the existence of a constant $\mathcal{M}$ (dependent on n) which validates the upper bound in terms of the K-functional. Taking the infimum over all $\varphi \in C^2([0,1])$ completes the proof. □

4. Numerical Experiments

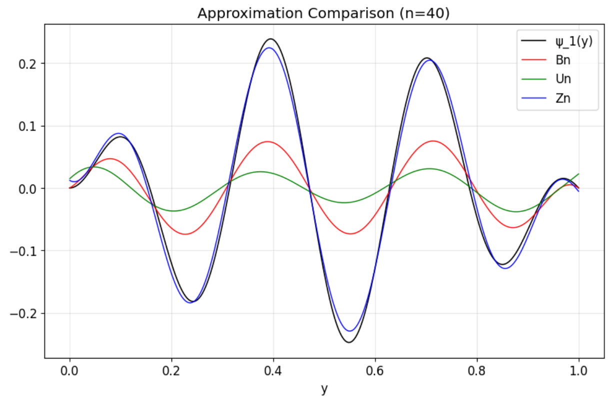
In this section, numerical examples are presented to evaluate the approximation behavior of the proposed operator $\mathcal{Z}_n$ in comparison with the classical operator $\mathcal{B}_n$ and the modified operators $\mathcal{K}_n$ and $\mathcal{U}_n$. Three test functions exhibiting distinct geometric characteristics are considered. For each function, the approximations are plotted alongside the exact function for selected values of n, and the corresponding absolute error distributions are visualised. Furthermore, to facilitate a quantitative assessment, the maximum and average absolute errors are tabulated.

Example 4.1. $\psi_1(y) = y(1-y)\sin(20y)$, $y \in [0,1]$.

The approximation plots and error distributions for Example 4.1 are shown in Figures 1 and 2.

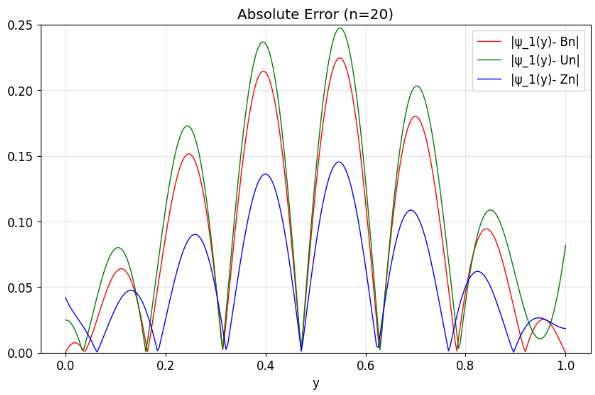


(a) Approximation at $n = 20$

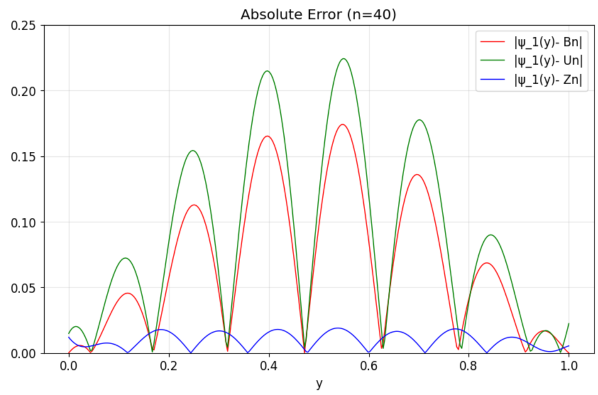


(b) Approximation at $n = 40$

Figure 1: Comparison between $\mathcal{B}_n$, $\mathcal{U}_n$ and $\mathcal{Z}_n$ for $\psi_1(y) = y(1-y)\sin(20y)$.



(a) Absolute Error at $n = 20$



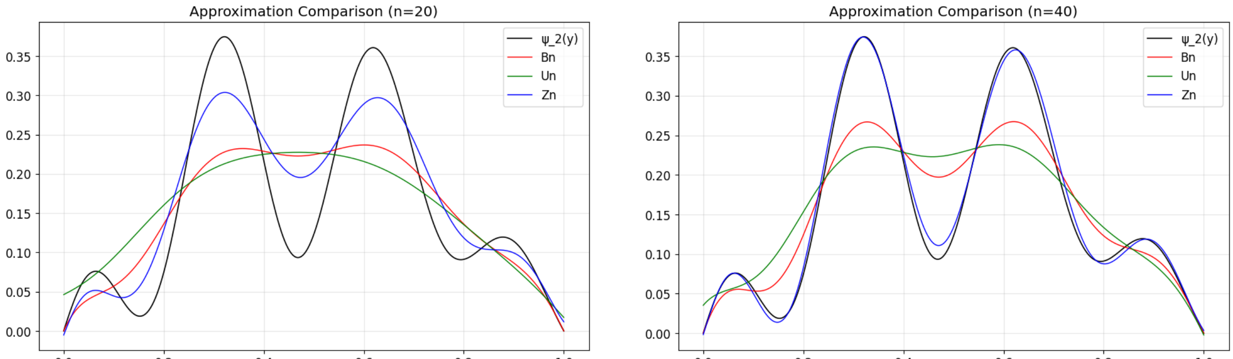
(b) Absolute Error at $n = 40$

Figure 2: Absolute errors for $\psi_1(y)$. The proposed operator $\mathcal{Z}_n$ shows reduced error magnitude.

Table 1: Maximum and Average Absolute Errors for $\psi_1(y)$ at various values of n.

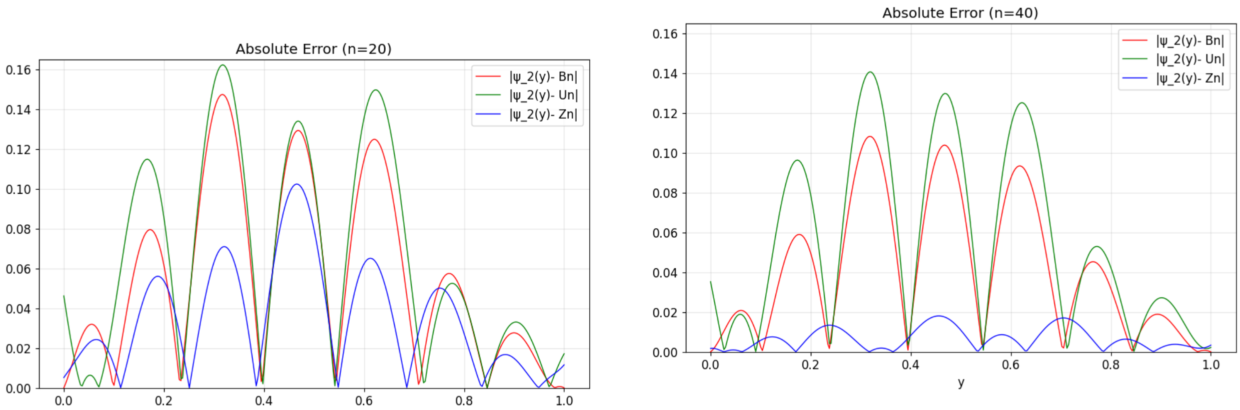
n	$\mathcal{B}_n$		$\mathcal{K}_n$		$\mathcal{U}_n$		$\mathcal{Z}_n$	
	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err
10	0.24600	0.10200	0.24600	0.10200	0.24800	0.10800	0.23700	0.09950
20	0.22500	0.08970	0.22500	0.09020	0.24700	0.10400	0.14500	0.05650
50	0.15500	0.05870	0.15500	0.05900	0.21000	0.08350	0.02160	0.00928
100	0.09700	0.03590	0.09720	0.03600	0.15300	0.05910	0.04600	0.01860
150	0.07020	0.02570	0.07030	0.02580	0.11800	0.04510	0.04460	0.01720
200	0.05490	0.02000	0.05500	0.02010	0.09620	0.03630	0.03970	0.01500

Example 4.2. $\psi_2(y) = y(1-y)\,(\cos(20y)\,e^{-y} + 1)$, $y \in [0, 1]$.



(a) Approximation at $n = 20$ (b) Approximation at $n = 40$

Figure 3: Comparison between $\mathcal{B}_n$, $\mathcal{U}_n$ and $\mathcal{Z}_n$ for $\psi_2(y)$.



(a) Absolute Error at $n = 20$ (b) Absolute Error at $n = 40$

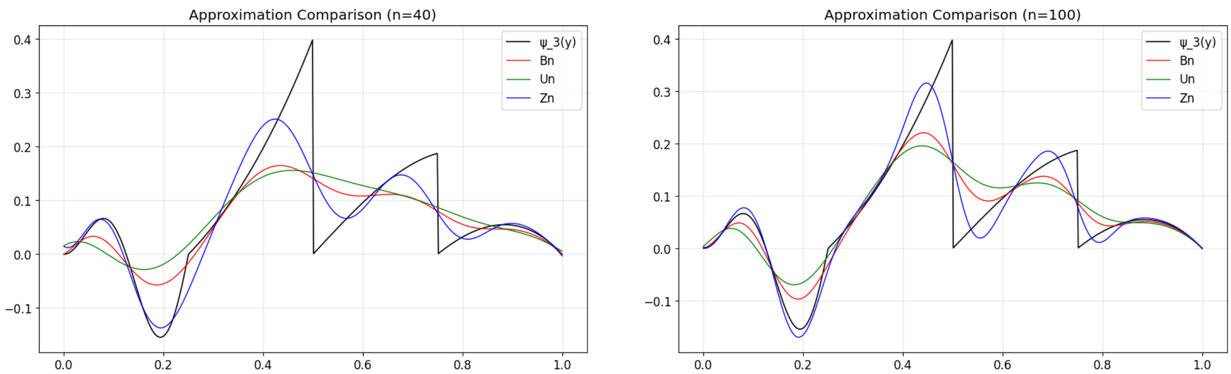
Figure 4: Absolute errors for $\psi_2(y)$.

Table 2: Maximum and Average Absolute Errors for $\psi_2(y)$ at various values of n.

n	$\mathcal{B}_n$		$\mathcal{K}_n$		$\mathcal{U}_n$		$\mathcal{Z}_n$	
	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err
10	0.17600	0.06450	0.17700	0.06490	0.17600	0.07500	0.16500	0.06120
20	0.14700	0.05600	0.14800	0.05640	0.16200	0.06470	0.10200	0.03500
50	0.09490	0.03650	0.09540	0.03670	0.13000	0.05200	0.01650	0.00624
100	0.05890	0.02230	0.05900	0.02240	0.09310	0.03670	0.03050	0.01160
150	0.04270	0.01600	0.04280	0.01600	0.07170	0.02790	0.02790	0.01070
200	0.03350	0.01240	0.03350	0.01250	0.05850	0.02250	0.02430	0.00929

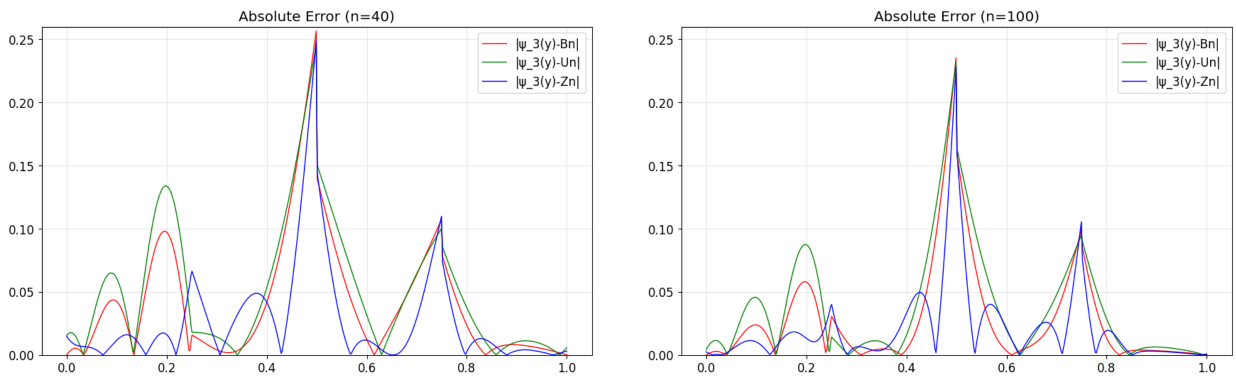
Example 4.3.

$$\psi_3(y) = y(1-y) \times \begin{cases} \sin(8\pi y), & 0 \le y < 0.25, \\ \sinh\big(5(y-0.25)\big), & 0.25 \le y < 0.5, \\ 4\big|y-0.5\big|, & 0.5 \le y < 0.75, \\ 4(y-0.75), & 0.75 \le y \le 1. \end{cases}$$



(a) Approximation at $n = 40$ (b) Approximation at $n = 100$

Figure 5: Comparison between $\mathcal{B}_n$ and $\mathcal{Z}_n$ for the piecewise function $\psi_3(y)$.



(a) Absolute Error at $n = 40$ (b) Absolute Error at $n = 100$

Figure 6: Absolute errors for $\psi_3(y)$.

n	$\mathcal{B}_n$		$\mathcal{K}_n$		$\mathcal{U}_n$		$\mathcal{Z}_n$	
	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err	Max Err	Avg Err
10	0.31600	0.06480	0.26900	0.06070	0.28800	0.06760	0.29100	0.06230
20	0.28200	0.05460	0.24700	0.05240	0.26500	0.06020	0.25800	0.04080
50	0.25100	0.03860	0.22900	0.03840	0.24200	0.04900	0.24400	0.02550
100	0.23500	0.02840	0.22000	0.02840	0.23200	0.03850	0.22800	0.02100
150	0.22800	0.02310	0.21500	0.02310	0.22700	0.03260	0.22000	0.01790
200	0.22300	0.01990	0.21200	0.01990	0.22400	0.02860	0.21500	0.01590

Table 3: Maximum and Average Absolute Errors for $\psi_3(y)$ at various values of n.

From the numerical results across all three examples, it is evident that the modified operator $\mathcal{Z}_n$ consistently provides a more accurate approximation than the classical Bernstein operator $\mathcal{B}_n$. This improvement is attributed to the symmetric linear combination of neighboring nodes, which effectively acts as a stabilizer reducing the local error.

5. Conclusion

In this study, a novel modification of the Bernstein operator based on symmetric shifted evaluations was introduced. In the construction of the proposed operator, denoted by $\mathcal{Z}_n$, standard pointwise evaluations were replaced by symmetric linear combinations involving neighboring nodes. It was confirmed via theoretical analysis that the positive linear structure is preserved, and uniform convergence to the target function is achieved, exhibiting a Voronovskaja-type asymptotic behavior analogous to the classical case.

Through numerical experiments, which included tests on both smooth and piecewise functions, it was demonstrated that the approximation error is significantly reduced by the symmetric structure compared to the classical Bernstein operator. Consequently, the practical utility of utilizing neighboring nodes to stabilize local approximation was highlighted, particularly for functions exhibiting high local variations.

Regarding future research, the extension of this symmetric modification to bivariate operators on tensor-product domains and simplexes is envisaged. Such generalizations are anticipated to be of significant relevance for applications in image processing and multidimensional data analysis, where local smoothness and edge preservation are considered critical.

Conflict of interest

The authors declared that they have no conflict of interest.

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