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Extremal functions on the spaces $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$

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Abstract. In this paper, we consider the generalized Weinstein operator $\Delta^{d,\alpha,n}_W$, we introduce and study the Sobolev-Gevrey spaces associated with the generalized Weinstein operator and investigate their properties. Next, as application, we study the extremal functions on the spaces $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ using the theory of reproducing kernels.

1. Introduction

The classical Sobolev-Gevrey space $\mathscr{H}^{s}_{a,\sigma}(\mathbb{R}^d)$ is defined by the use of the classical Fourier transform as the set of all tempered distributions u such that their classical Fourier transform $\widehat{u}$ are functions satisfying :

$$\int_{\mathbb{R}^d}(1+\|\xi\|^2)^s\left|\widehat{u}(\xi)\right|^2 e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\xi<\infty. \tag{1}$$

Gevrey spaces and their Sobolev generalizations provide an essential granularity in regularity scales that captures phenomena invisible to both $\mathscr{C}^\infty$ and analytic frameworks. Their ability to quantify how infinite the loss of derivatives is, through the parameter σ, makes them indispensable in modern analysis of PDEs, harmonic analysis, and numerical mathematics.
In [3], the author use the spaces $\mathscr{H}^{s}_{a,\sigma}(\mathbb{R}^d)$ to get better explosion results of the maximal solution of incompressible Navier–Stokes equations. In [16], the authors establish the well-posedness in Gevrey function space with optimal class of regularity 2 for the three dimensional Prandtl system without any structural assumption. (See also [1] and [2]).
The study of these hybrid Sobolev-Gevrey spaces is motivated by their pivotal role in capturing intermediate regularity phenomena that are not precisely described by pure Sobolev or analytic frameworks. In PDE theory, they provide the natural functional setting for analyzing the well-posedness and long-time behavior of solutions to equations such as weakly hyperbolic systems or nonlinear dispersive flows, where one must simultaneously track polynomial energy decay and the propagation of exponential regularity in phase

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space [12, 21]. From the perspective of harmonic analysis, these spaces offer a refined tool for microlocal and time-frequency analysis, enabling precise descriptions of generalized wavefront sets and the boundedness of operators with non-smooth symbols [14]. Furthermore, in approximation theory, membership in a Sobolev-Gevrey class directly governs the transition from algebraic to exponential convergence rates for spectral methods, delivering sharp a priori estimates for numerical solutions [13, 27]. It is this interplay quantifying the precise blend of Sobolev and Gevrey characteristics that makes a detailed investigation of their structural properties fundamental to advancing problems across these interconnected fields.
We give generalisations of the Sobolev-Gevrey space $\mathscr{H}^{s,\alpha}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ by replacing in the relation (1), the classical Fourier transform $\widehat{u}$ by Weinstein transform $\mathscr{F}^{\alpha,d}_W$. (See [4], [5], [15] and [23]-[25]).
The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x^{2n}_{d+1} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}),$$

where $x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by :

$$\forall \xi \in \mathbb{C},\ j_\alpha(\xi) := \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{\xi}{2}\right)^{2n}. \tag{2}$$

(See [6] -[10]).
Using the Weinstein kernel $\Lambda_{\alpha,d,n}$, we define the Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ by :

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W(f)(\lambda) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x),$$

where $f \in \mathbb{L}^1(\mathbb{R}^{d+1}_+, \mu_{\alpha,d}(x))$ and $\mu_{\alpha,d}$ is the measure on $\mathbb{R}^{d+1}_+$ given by :

$$d\mu_{\alpha,d}(x) = x^{2\alpha+1}_{d+1}dx. \tag{3}$$

We denote by $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ the subspace of $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\},\ \frac{\partial^k f}{\partial x^k_{d+1}}(x', 0) = f(x', 0) = 0.$$

Let $s \in \mathbb{R}$, $a > 0$ and $\sigma > 1$, we define the Sobolev-Gevrey space that will be denoted $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ as the set of all $u \in \mathscr{S}'_{n,*}$ (the strong dual of the space $\mathscr{S}_{n,*}(\mathbb{R}^d)$) such that $\mathscr{F}^{\alpha,d,n}_W(u)$ is a function and

$$\mathscr{F}^{\alpha,d,n}_W(u) \in \mathbb{L}^2\left(\mathbb{R}^{d+1}_+,\ (1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}}\xi^{2\alpha+4n+1}_{d+1}d\xi\right).$$

We provide $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ with the inner product

$$\langle u, v\rangle_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^s \mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)(\xi)}e^{2a\|\xi\|^{\frac{1}{\sigma}}}\xi^{2\alpha+4n+1}_{d+1}d\xi,$$

and the norm $\|u\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = \left(\langle u, u\rangle_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\right)^{\frac{1}{2}}$ where $C_{\alpha,d}$ is the constant given by :

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}}\,2^\alpha\Gamma(\alpha+1)}. \tag{4}$$

The main objective of this paper is to investigate the properties of $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$. Using the theory of reproducing kernels, we study the extremal functions on the spaces $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.

The contents of the paper are as follows :
In the section 2, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.
The section 3 is devoted to define and study the generalized Weinstein Sobolev-Gevrey spaces $\mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.
Finally, in the last section, as application, using the theory of reproducing kernels, we give good estimates of extremal functions on the spaces $\mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.

2. Preliminaires

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by the relation :

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2} = \Delta_W^{\alpha,d} - \frac{4n(\alpha+n)}{x_{d+1}^2}, \tag{5}$$

where $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$ and $\Delta_W^{\alpha,d}$ the classical Laplace-Bessel operator (Weinstein operator) given by :

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial}{\partial x_{d+1}}. \tag{6}$$

(See [4], [5] and [23]-[25]).
Notations. In what follows, we need the following notations:
• $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
• $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
• $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
• $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.
• $\mathscr{M}_n$, the map defined by :

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x). \tag{7}$$

where $x = (x', x_{d+1})$ and $x' = (x_1, x_2, ..., x_d)$
• $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\begin{aligned} \|f\|_{\alpha,n,p} &= \left[\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1} f(x)|^p d\mu_{\alpha+2n,d}(x)\right]^{\frac{1}{p}} < +\infty, \text{ if } 1 \le p < +\infty, \\ \|f\|_{\alpha,n,\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^{d+1}} \left|\mathscr{M}_n^{-1} f(x)\right| < +\infty, \end{aligned}$$

where $\mu_{\alpha,d}$ is the measure defined on $\mathbb{R}_+^{d+1}$ by the relation (3).
• $\mathbb{L}_\alpha^p(\mathbb{R}_+^{d+1}) = \mathbb{L}_{\alpha,0}^p(\mathbb{R}_+^{d+1})$ and $\|f\|_{\alpha,p} = \|f\|_{\alpha,0,p}$, $1 \le p \le +\infty$.
• $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ respectively stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\},\ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

Definition 2.1. *The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :*

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x, y) = x_{d+1}^{2n} e^{-i\langle x', y'\rangle} j_{\alpha+2n}(x_{d+1} y_{d+1}), \tag{8}$$

where $x = (x', x_{d+1})$, $x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by the relation (2).

Remark 2.2. *i) We have*

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n}\Lambda_{\alpha+2n,d}(x,y), \tag{9}$$

where

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d}(x,y) = \Lambda_{\alpha,d,0}(x,y) = e^{-i\langle x',y'\rangle} j_\alpha(x_{d+1}y_{d+1}).$$

ii) We have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ \Lambda_{\alpha,d}(x,\lambda)\Lambda_{\alpha,d}(y,\lambda) = T_x^{\alpha,d}\left[\Lambda_{\alpha,d}(.,\lambda)\right](y). \tag{10}$$

The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ has a unique extention to $\mathbb{C}^{d+1}\times\mathbb{C}^{d+1}$ and satisifies the following properties.

Proposition 2.3. *i) We have*

$$\forall x, y \in \mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d,n}(x,y)| \le x_{d+1}^{2n}. \tag{11}$$

ii) We have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ L_{\alpha,n}\Lambda_{\alpha,d,n}(.,y)(x) = -y_{d+1}^2\Lambda_{\alpha,d,n}(x,y). \tag{12}$$

iii) The function $x \mapsto \Lambda_{\alpha,d,n}(x,y)$ satisifies the differentiel equation :

$$\triangle_W^{\alpha,d,n}(\Lambda_{\alpha,d,n}(.,y))(x) = -\left\|y\right\|^2\Lambda_{\alpha,d,n}(x,y). \tag{13}$$

Proof. i) The result follows from the relations (8) and the fact that

$$\forall \beta > -\frac{1}{2},\ \forall t \in \mathbb{R},\ \left|j_\beta(t)\right| \le 1.$$

ii) We start by showing

$$L_{\alpha,n} \circ \mathscr{M}_n = \mathscr{M}_n \circ L_{\alpha+2n}. \tag{14}$$

Let $f \in \mathscr{C}^2(\mathbb{R})$, for all $t > 0$, we have

$$\begin{aligned} L_{\alpha,n}(\mathscr{M}_n f)(t) &= \left(t^{2n}f\right)'' + \frac{2\alpha+1}{t}\left(t^{2n}f\right)' - 4n(\alpha+n)t^{2n-2}f(t) \\ &= t^{2n}f''(t) + (2\alpha+4n+1)t^{2n-1}f'(t) \\ &= t^{2n}\left[f''(t) + \frac{2\alpha+4n+1}{t}f'(t)\right] \\ &= \mathscr{M}_n(L_{\alpha+2n}f)(t). \end{aligned}$$

Now, by invoking (8) and (14), we can write

$$\begin{aligned} L_{\alpha,n}\Lambda_{\alpha,d,n}(.,y)(x) &= e^{-i\langle x',y'\rangle}L_{\alpha,n}\circ\mathscr{M}_n\left(j_{\alpha+2n}(x_{d+1}y_{d+1})\right) \\ &= e^{-i\langle x',y'\rangle}\mathscr{M}_n\circ L_{\alpha+2n}\left(j_{\alpha+2n}(x_{d+1}y_{d+1})\right) \\ &= e^{-i\langle x',y'\rangle}\mathscr{M}_n\left[-y_{d+1}^2 j_{\alpha+2n}(x_{d+1}y_{d+1})\right] \\ &= -y_{d+1}^2 x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}) \\ &= -y_{d+1}^2\Lambda_{\alpha,d,n}(.,y)(x,y). \end{aligned}$$

Which gives the desired result.
iii) From the relation (12), for all $x,\ y \in \mathbb{R}^{d+1}$, we get

$$\begin{aligned}\triangle_W^{\alpha,d,n}\left(\Lambda_{\alpha,d,n}(.,y)\right)(x) &= \left(\Delta_d + L_{\alpha,n}\right)\left(\Lambda_{\alpha,d,n}(.,y)\right)(x)\\ &= x_{d+1}^{2n}\Delta_d\Lambda_{\alpha+2n,d}(x,y) + L_{\alpha,n}x_{d+1}^{2n}\Lambda_{\alpha+2n,d}(x,y)\\ &= -x_{d+1}^{2n}\left(y_1^2+...+y_d^2\right)\Lambda_{\alpha+2n,d}(x,y) - y_{d+1}^2x_{d+1}^{2n}\Lambda_{\alpha+2n,d}(x,y)\\ &= -\left(y_1^2+...+y_d^2+y_{d+1}^2\right)x_{d+1}^{2n}\Lambda_{\alpha+2n,d}(x,y)\\ &= -\left\|y\right\|^2\Lambda_{\alpha,d,n}(x,y).\end{aligned}$$

Thus the proof is finished. □

Definition 2.4. *The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ is given for $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ by :*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x), \tag{15}$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by the relation (3).

Example 2.5. *Let $t > 0$ and $n \in \mathbb{N}$, we consider the two functions $\xi_t^{\alpha,d,n}$ and $\theta_t^{\alpha,d,n}$ given by :*

$$\forall x \in \mathbb{R}^{d+1},\ \xi_t^{\alpha,d,n}(x) = \frac{2x_{d+1}^{2n}e^{-\frac{\|x\|^2}{4t}}}{\pi^{\frac{d}{2}}\Gamma(\alpha+2n+1)(4t)^{\alpha+2n+\frac{d}{2}+1}}, \tag{16}$$

and

$$\forall x \in \mathbb{R}^{d+1},\ \theta_t^{\alpha,d,n}(x) = C_{\alpha+2n,d}\left(4t\left(\alpha+2n+\frac{d}{2}+1\right)-\|x\|^2\right)\frac{x_{d+1}^{2n}}{(2t)^{\alpha+2n+\frac{d}{2}+3}}e^{-\frac{\|x\|^2}{4t}},$$

where $C_{\alpha,d}$ is the constant given the relation (4).
Then we have :

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\xi_t^{\alpha,d,n}\right)(\lambda) = e^{-t\|\lambda\|^2}, \tag{17}$$

and

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\theta_t^{\alpha,d,n}\right)(\lambda) = \|\lambda\|^2 e^{-t\|\lambda\|^2}.$$

Remark 2.6. *i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ $\left(\text{resp. } \mathscr{S}_*(\mathbb{R}^{d+1})\right)$ onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ $\left(\text{resp. } \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\right)$.*
ii) The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ can be written in the form :

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}, \tag{18}$$

where $\mathscr{F}_W^{\alpha,d} = \mathscr{F}_W^{\alpha,d,0}$ is the classical Weinstein transform.

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results. We obtain the results from the relation (18).

Theorem 2.7. *i) Let $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. If $\mathscr{F}^{\alpha,d,n}_W(f) \in \mathbb{L}^1_{\alpha+2n}(\mathbb{R}^{d+1}_+)$, then we have*

$$f(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(y)\,\Lambda_{\alpha,d,n}(-x,y)d\mu_{\alpha+2n,d}(y),\ a.e\ x \in \mathbb{R}^{d+1}_+, \tag{19}$$

where $C_{\alpha,d}$ is the constant given by the relation (4).
ii) The Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ is a topological isomorphism from $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{S}_*(\mathbb{R}^{d+1})$ and from $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{H}_*(\mathbb{C}^{d+1})$.

Theorem 2.8. *(see [6] -[10])*
i) For all $f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have the following Parseval formula :

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(\lambda)}d\mu_{\alpha+2n,d}(\lambda). \tag{20}$$

ii) (Plancherel formula).
For all $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have :

$$\int_{\mathbb{R}^{d+1}_+} \left|f(x)\right|^2 d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda). \tag{21}$$

iii) (Plancherel Theorem) :
The transform $\mathscr{F}^{\alpha,d,n}_W$ extends uniquely to an isometric isomorphism from $\mathbb{L}^2(\mathbb{R}^{d+1}_+, d\mu_{\alpha,d}(x))$ onto $\mathbb{L}^2(\mathbb{R}^{d+1}_+, C^2_{\alpha+2n,d}d\mu_{\alpha+2n,d}(x))$.

Definition 2.9. *The translation operator $T^{\alpha,d,n}_x$, $x \in \mathbb{R}^{d+1}_+$, associated with the operator $\Delta^{\alpha,d,n}_W$ is defined on $\mathscr{E}_{n,*}(\mathbb{R}^{d+1}_+)$ by :*

$$\forall y \in \mathbb{R}^{d+1}_+,\ T^{\alpha,d,n}_x f(y) = x^{2n}_{d+1} y^{2n}_{d+1} T^{\alpha+2n,d}_x \mathscr{M}^{-1}_n f(y), \tag{22}$$

where

$$T^{\alpha,d}_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(x'+y',\ \sqrt{x^2_{d+1}+y^2_{d+1}+2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta, \tag{23}$$

and $x'+y' = (x_1+y_1, ..., x_d+y_d)$.

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 2.10. *i) For all $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$, the function $x \mapsto T^{\alpha,d,n}_x f(y)$ belongs to $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$.*
ii) Let $f \in \mathbb{L}^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $1 \le p \le +\infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $T^{\alpha,d,n}_x f$ belongs to $\mathbb{L}^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\|T^{\alpha,d,n}_x f\|_{\alpha,n,p} \le x^{2n}_{d+1}\|f\|_{\alpha,n,p}. \tag{24}$$

iii) The function $t \mapsto \Lambda_{\alpha,d,n}(t,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}^{d+1}_+$ the following product formula:

$$\forall x, y \in \mathbb{R}^{d+1}_+,\ \Lambda_{\alpha,d,n}(x,\lambda)\,\Lambda_{\alpha,d,n}(y,\lambda) = T^{\alpha,d,n}_x\left[\Lambda_{\alpha,d,n}(.,\lambda)\right](y). \tag{25}$$

iv) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}^{d+1}_+$, we have

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W\left(T^{\alpha,d,n}_x f\right)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\,\mathscr{F}^{\alpha,d,n}_W(f)(\lambda). \tag{26}$$

Proof. i) Let $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$, we can write

$$\forall x \in \mathbb{R}_+^{d+1},\ f(x) = x_{d+1}^{2n} g(x),\ g \in \mathscr{E}_*(\mathbb{R}^{d+1}).$$

Then using the relation (22), we have

$$\forall x \in \mathbb{R}_+^{d+1},\ T_x^{\alpha,d,n} f(y) = x_{d+1}^{2n} y_{d+1}^{2n} T_x^{\alpha+2n,d} g(y).$$

Since the function $x \mapsto y_{d+1}^{2n} T_y^{\alpha+2n,d} g(x)$ is in $\mathscr{E}_*(\mathbb{R}^{d+1})$ then the function $x \mapsto T_x^{\alpha,d,n} f(y)$ belongs to $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$.
ii) Let $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1}),\ 1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$.
If $f \in \mathbb{L}_{\alpha,n}^\infty(\mathbb{R}_+^{d+1})$, then from the relation (22), for all $y \in \mathbb{R}_+^{d+1}$, we get

$$\begin{aligned}\left|\mathscr{M}_{n,y}^{-1} T_x^{\alpha,d,n} f(y)\right| &= x_{d+1}^{2n} \left|T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f(y)\right| \le x_{d+1}^{2n} \|T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f\|_{\alpha,\infty} \\ &\le x_{d+1}^{2n} \|\mathscr{M}_n^{-1} f\|_{\alpha,\infty} = x_{d+1}^{2n} \|f\|_{\alpha,n,\infty}.\end{aligned}$$

Then $T_x^{\alpha,d,n} f$ belongs to $\mathbb{L}_{\alpha,n}^\infty(\mathbb{R}_+^{d+1})$ and we have

$$\|T_x^{\alpha,d,n} f\|_{\alpha,n,\infty} \le x_{d+1}^{2n} \|f\|_{\alpha,n,\infty}.$$

Now let $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1}),\ 1 \le p < +\infty$. We have

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_{n,y}^{-1} T_x^{\alpha,d,n} f(y)|^p d\mu_{\alpha+2n,d}(y) &= x_{d+1}^{2pn} \int_{\mathbb{R}_+^{d+1}} |T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f(y)|^p d\mu_{\alpha+2n,d}(y) \\ &= x_{d+1}^{2pn} \left\|T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f\right\|_{\alpha,p}^p \\ &\le x_{d+1}^{2pn} \left\|\mathscr{M}_n^{-1} f\right\|_{\alpha,p}^p \\ &\le x_{d+1}^{2pn} \|f\|_{\alpha,n,p}^p.\end{aligned}$$

Then $T_x^{\alpha,d,n} f$ belongs to $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and we get

$$\|T_x^{\alpha,d,n} f\|_{\alpha,n,p} \le x_{d+1}^{2n} \|f\|_{\alpha,n,p}.$$

iii) From the relations (8), (9) and (10), for all $x, y \in \mathbb{R}_+^{d+1}$, we obtain

$$\begin{aligned}\Lambda_{\alpha,d,n}(x,\lambda)\Lambda_{\alpha,d,n}(y,\lambda) &= x_{d+1}^{2n} \Lambda_{\alpha+2n,d}(x,\lambda)\, y_{d+1}^{2n} \Lambda_{\alpha+2n,d}(x,\lambda) \\ &= x_{d+1}^{2n} y_{d+1}^{2n} T_x^{\alpha+2n,d} \left[\Lambda_{\alpha+2n,d}(.,\lambda)\right](y) \\ &= x_{d+1}^{2n} y_{d+1}^{2n} T_x^{\alpha+2n,d} \left[\mathscr{M}_n^{-1} \Lambda_{\alpha,d,n}(.,\lambda)\right](y) \\ &= T_x^{\alpha,d,n} \left[\Lambda_{\alpha,d,n}(.,\lambda)\right](y).\end{aligned}$$

Which gives the desired result.
iv) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}_+^{d+1}$, using the relations (18) and (22), for all $\lambda \in \mathbb{R}_+^{d+1}$, we have

$$\begin{aligned}\mathscr{F}_W^{\alpha,d,n}\left(T_x^{\alpha,d,n} f\right)(\lambda) &= \mathscr{F}_W^{\alpha+2n,d}\left[\mathscr{M}_n^{-1} T_x^{\alpha,d,n} f\right](\lambda) \\ &= x_{d+1}^{2n} \mathscr{F}_W^{\alpha+2n,d}\left(T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f\right)(\lambda) \\ &= x_{d+1}^{2n} \Lambda_{\alpha+2n,d}(-x,\lambda)\, \mathscr{F}_W^{\alpha+2n,d}\left(\mathscr{M}_n^{-1} f\right)(\lambda) \\ &= \Lambda_{\alpha,d,n}(-x,\lambda)\, \mathscr{F}_W^{\alpha,d,n}(f)(\lambda).\end{aligned}$$

Which completes the proof. □

The convolution operators based on the construct $T_x^{\alpha,d,n}$ are known from the works by V.A. Kakichev and L.N. Lyakhov (see [17]-[20]).

Definition 2.11. *Let $f, g \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. The generalized Weinstein convolution product of f and g is given by :*

$$f *_{\alpha,n} g\,(x) = \int_{\mathbb{R}^{d+1}_+} T_x^{\alpha,d,n} f\,(-y)\, g\,(y)\, d\mu_{\alpha,d}(y),\ x \in \mathbb{R}^{d+1}_+. \tag{27}$$

Remark 2.12. *Let $f, g \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. using the relations (22) and (27), we obtain*

$$\begin{aligned} f *_{\alpha,n} g\,(x) &= x_{d+1}^{2n} \int_{\mathbb{R}^{d+1}_+} T_x^{\alpha+2n,d} \left(\mathscr{M}_n^{-1} f\right)(-y) \left(\mathscr{M}_n^{-1} g\right)(y)\, d\mu_{\alpha+2n,d}(y) \\ &= x_{d+1}^{2n} \left(\mathscr{M}_n^{-1} f\right) *_{\alpha+2n} \left(\mathscr{M}_n^{-1} g\right)(x), \end{aligned}$$

where for all $\varphi, \psi \in \mathbb{L}^1_\alpha(\mathbb{R}^{d+1}_+)$, we have

$$\varphi *_\alpha \psi\,(x) := \varphi *_{\alpha,0} \psi\,(x) = \int_{\mathbb{R}^{d+1}_+} T_x^{\alpha,d} \varphi\,(-y)\, \psi\,(y)\, d\mu_{\alpha,d}(y),\ x \in \mathbb{R}^{d+1}_+.$$

Proposition 2.13. *(See [6]-[10]) For all $f, g \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $f *_{\alpha,n} g \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have*

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g). \tag{28}$$

Example 2.14. *Let $s, t > 0$ and $n \in \mathbb{N}$, we consider the function $\xi_t^{\alpha,d,n}$ given by the relation (16). Using the relations (17) and (28), for all $\lambda \in \mathbb{R}^{d+1}_+$, we obtain*

$$\begin{aligned} \mathscr{F}_W^{\alpha,d,n}\left(\xi_t^{\alpha,d,n} *_{\alpha,n} \xi_s^{\alpha,d,n}\right)(\lambda) &= \mathscr{F}_W^{\alpha,d,n}\left(\xi_t^{\alpha,d,n}\right)(\lambda)\mathscr{F}_W^{\alpha,d,n}\left(\xi_s^{\alpha,d,n}\right)(\lambda) \\ &= e^{-t\|\lambda\|^2} \times e^{-s\|\lambda\|^2} \\ &= \mathscr{F}_W^{\alpha,d,n}\left(\xi_{s+t}^{\alpha,d,n}\right)(\lambda). \end{aligned}$$

Then, we get

$$\forall x \in \mathbb{R}^{d+1},\ \xi_t^{\alpha,d,n} *_{\alpha,n} \xi_s^{\alpha,d,n}\,(x) = \xi_{s+t}^{\alpha,d,n}\,(x) = \frac{x_{d+1}^{2n} e^{-\frac{\|x\|^2}{4(s+t)}}}{\pi^{\frac{d}{2}} 2^{2\alpha+4n+1}\Gamma(\alpha+2n+1)\,(s+t)^{\alpha+2n+\frac{d}{2}+1}}.$$

Notations. We denoted by :
- $\mathscr{S}'_*$, the strong dual of the space $\mathscr{S}_*(\mathbb{R}^{d+1})$.
- $\mathscr{S}'_{n,*}$, the strong dual of the space $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$.

Definition 2.15. *The generalized Fourier-Weinstein transform of a distribution $u \in \mathscr{S}'_{n,*}$ is defined by :*

$$\forall \phi \in \mathscr{S}_*(\mathbb{R}^{d+1}),\ \langle \mathscr{F}_W^{\alpha,d,n}(u),\ \phi\rangle = \langle u,\ \left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}(\phi)\rangle. \tag{29}$$

The following proposition is as an immediate consequence of Theorem 2.7.

Proposition 2.16. *The transform $\mathscr{F}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathscr{S}'_{n,*}$ onto $\mathscr{S}'_*$.*

Remark 2.17. *Let $m \in \mathbb{N}$ and $u \in \mathscr{S}'_{n,*}$, we have*

$$\left(\mathscr{F}_W^{\alpha,d,n}\right)\left[(\Delta_W^{\alpha,d,n})^m u\right] = (-1)^m \|x\|^{2m} \left(\mathscr{F}_W^{\alpha,d,n}\right)(u), \tag{30}$$

where

$$\forall \phi \in \mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right),\ \langle \Delta_W^{\alpha,d,n} u,\ \phi\rangle = \langle u,\ \Delta_W^{\alpha,d,n}\phi\rangle. \tag{31}$$

3. Sobolev-Gevrey spaces associated with the generalized Weinstein type operator

The goal of this section is to introduce and study the Sobolev-Gevrey spaces associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.

Definition 3.1. *For $s \in \mathbb{R}$, we define the generalized Sobolev-Weinstein space of order s, that will be denoted $\mathscr{H}^s_{\alpha,d,n}\left(\mathbb{R}^{d+1}_+\right)$, as the set of all $u \in \mathscr{S}'_{n,*}$ such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and*

$$\int_{\mathbb{R}^{d+1}_+} (1+\|\lambda\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) < \infty. \tag{32}$$

We provide $\mathscr{H}^s_{\alpha,d,n}\left(\mathbb{R}^{d+1}_+\right)$ with the inner product

$$\langle u,v\rangle_{\mathscr{H}^s_{\alpha,d,n}} = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} (1+\|\xi\|^2)^s \mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)}d\mu_{\alpha+2n,d}(\xi), \tag{33}$$

and the norm

$$\|u\|_{\mathscr{H}^s_{\alpha,d,n}} = \left[C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} (1+\|\xi\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\right]^{\frac{1}{2}}. \tag{34}$$

Definition 3.2. *For all $s \in \mathbb{R}$, $a > 0$ and $\sigma > 1$, we define the generalized Weinstein Sobolev-Gevrey space that will be denoted $\mathscr{H}^{s,\alpha,n}_{a,\sigma}\left(\mathbb{R}^{d+1}_+\right)$ as the set of all $u \in \mathscr{S}'_{n,*}$ such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and*

$$\mathscr{F}_W^{\alpha,d,n}(u) \in \mathbb{L}^2\left(\mathbb{R}^{d+1}_+,\ (1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}} d\mu_{\alpha+2n,d}(\xi)\right).$$

We provide $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ with the inner product

$$\langle u,v\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} (1+\|\xi\|^2)^s \mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)}e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi),$$

and the norm

$$\|u\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = \left[C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} (1+\|\xi\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 e^{2a\|\xi\|^{\frac{1}{\sigma}}} d\mu_{\alpha+2n,d}(\xi)\right]^{\frac{1}{2}}.$$

The following properties of the spaces $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ can easily be established.

Proposition 3.3. *i) For all $s \in \mathbb{R}$, we have*

$$\mathscr{S}_*(\mathbb{R}^d) \subset \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+).$$

ii) For all $\sigma > 1$, we have

$$\mathscr{H}^{0,\alpha,n}_{0,\sigma}\left(\mathbb{R}^{d+1}_+\right) = \mathscr{H}^0_{\alpha,d,n}\left(\mathbb{R}^{d+1}_+\right) = \mathbb{L}^2_\alpha(\mathbb{R}^{d+1}_+).$$

iii) For all $s, t \in \mathbb{R}$, $t > s$, the space $\mathscr{H}^{t,\alpha,n}_{a,\sigma}\left(\mathbb{R}^{d+1}_+\right)$ is continuously contained in $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.
iv) If $0 < a < b$ then the space $\mathscr{H}^{s,\alpha}_{b,\sigma}\left(\mathbb{R}^{d+1}_+\right)$ is continuously contained in $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.
v) For $s \in \mathbb{R}$, $a > 0$ and $\sigma > 1$, we have

$$\mathscr{H}^s_{\alpha,d,n}\left(\mathbb{R}^{d+1}_+\right) \subset \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+).$$

Proposition 3.4. *The space $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ provided with the norm $\|.\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}}$ is a Banach space.*

Proof. Let $(u_m)_{m\in\mathbb{N}}$ be a Cauchy sequence of $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$. From the definition of the norm $\|.\|_{(1),\mathscr{H}^{s,\alpha}_{a,\sigma}}$, it is clear that $\left(\mathscr{F}^{\alpha,d,n}_W(u_m)\right)_{m\in\mathbb{N}}$ is a Cauchy sequence of $\mathbb{L}^2_{\sigma,s}\left(\mathbb{R}^{d+1}_+\right) := \mathbb{L}^2_\alpha(\mathbb{R}^{d+1}_+,\ (1+\|x\|^2)^s e^{2a\|x\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(x))$. Since $\mathbb{L}^2_{\sigma,s}\left(\mathbb{R}^d\right)$ is complete, there exists a function $u\in\mathbb{L}^2_{\sigma,s}\left(\mathbb{R}^d\right)$ such that

$$\lim_{m\to+\infty}\|\mathscr{F}^{\alpha,d,n}_W(u_m)-u\|_{\mathbb{L}^2_{\sigma,s}(\mathbb{R}^d)}=0.$$

Then $u\in\mathscr{S}'$ and $h=\left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1}(u)\in\mathscr{S}$.
So, $\mathscr{F}^{\alpha,d,n}_W(v)=u\in\mathbb{L}^2_{\sigma,s}\left(\mathbb{R}^{d+1}_+\right)$, which proves that $v\in\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and we have

$$\|u_m-v\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}=\|\mathscr{F}^{\alpha,d,n}_W(u_m)-u\|_{\mathbb{L}^2_{\sigma,s}(\mathbb{R}^d)}\underset{m\to+\infty}{\to}0.$$

Hence, $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ is complete. □

Proposition 3.5. *Let $s_1,s,s_2\in\mathbb{R}$, satisfying $s_1<s<s_2$. Then, for all $\varepsilon>0$, there exists a nonnegative constant C_ε such that for all $u\in\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we have*

$$\|u\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\le C_\varepsilon\|u\|_{\mathscr{H}^{s_1,\alpha,n}_{a,\sigma}}+\varepsilon\|u\|_{\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}}. \tag{35}$$

Proof. Let $s_1,s_2\in\mathbb{R}$ and $s=(1-t)s_1+ts_2$, $t\in]0,\ 1[$. Let $u\in\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$. We put $t=\dfrac{1}{p}$ and $1-t=\dfrac{1}{q}$, applying the Hölder's inequality, we get

$$\begin{aligned}\|u\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}}&\le\|u\|^{1-t}_{\mathscr{H}^{s_1,\alpha,n}_{a,\sigma}}\times\|u\|^t_{\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}}\\&\le\left(\varepsilon^{\frac{-t}{1-t}}\|u\|_{\mathscr{H}^{s_1,\alpha,n}_{a,\sigma}}\right)^{1-t}\times\left(\varepsilon\|u\|_{\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}}\right)^t\\&\le\varepsilon^{\frac{s-s_1}{s-s_2}}\|u\|_{\mathscr{H}^{s_1,\alpha,n}_{a,\sigma}}+\varepsilon\|u\|_{\mathscr{H}^{s_2,\alpha,n}_{a,\sigma}}.\end{aligned}$$

Thus the proof is finished. □

Proposition 3.6. *Let $s,t\in\mathbb{R}$. Then, the operator*

$$\mathscr{A}_t:\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)\to\mathscr{H}^{s-t,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+),$$

defined for all $x\in\mathbb{R}^{d+1}_+$ by :

$$\mathscr{A}_tu(x)=C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|)^t\Lambda_{\alpha,d,n}(-x,\xi)\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\,d\mu_{\alpha+2n,d}(\xi),$$

is an isomorphism.

Proof. Let $s,t\in\mathbb{R}$ and $u\in\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$. The function :
$\xi\mapsto(1+\|\xi\|)^t(1+\|\xi\|^2)^{\frac{s-t}{2}}\mathscr{F}^{\alpha,d,n}_W(u)(\xi)$ belongs to $\mathbb{L}^2_{\alpha+2n}(\mathbb{R}^{d+1}_+)$ and have

$$\forall\xi\in\mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W(\mathscr{A}_tu)(\xi)=(1+\|\xi\|)^t\mathscr{F}^{\alpha,d,n}_W(u)(\xi).$$

Thus

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^{s-t}\left|\mathscr{F}^{\alpha,d,n}_W(\mathscr{A}_tu)(\xi)\right|^2e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi)\\&\le\sup_{x\in\mathbb{R}^{d+1}_+}\left[\frac{(1+\|x\|)^{2t}}{(1+\|x\|^2)^t}\right]\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^s\left|\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\right|^2e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi)\\&\le2^{|t|}\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^s\left|\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\right|^2e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi).\end{aligned}$$

Then, $\mathscr{A}_t u \in \mathscr{H}^{s-t,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and we have

$$\|\mathscr{A}_t u\|_{\mathscr{H}^{s-t,\alpha,n}_{a,\sigma}} \leq 2^{\frac{|t|}{2}} \|u\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}}.$$

Now, let $v \in \mathscr{H}^{s-t,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and put

$$u = \left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1}\left((1+\|\xi\|)^{-t}\mathscr{F}^{\alpha,d,n}_W(v)\right).$$

From the definition of the operator $\mathscr{A}_t$, we have $\mathscr{A}_t u = v$ and we get

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^s\left|\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\right|^2 e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi)\\ &=\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^s(1+\|\xi\|)^{-2t}\left|\mathscr{F}^{\alpha,d,n}_W(v)(\xi)\right|^2 e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi)\\ &\leq 2^{|t|}\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^{s-t}\left|\mathscr{F}^{\alpha,d,n}_W(v)(\xi)\right|^2 e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi).\end{aligned}$$

Hence, $u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and we obtain

$$\|u\|_{\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \leq 2^{\frac{|t|}{2}} \|\mathscr{A}_t u\|_{\mathscr{H}^{s-t,\alpha,n}_{a,\sigma}}.$$

Which completes the proof. □

The following theorem gives a relation between the dual of $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and $\mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.

Theorem 3.7. *The dual of $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ can be identified with $\mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$. The relation of the identification is as follows :*

$$\langle u, v\rangle_{(0)} = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)(\xi)}e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\xi), \tag{36}$$

with $u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and $v \in \mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.

Proof. For all $u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and $v \in \mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we have

$$\left|\langle u, v\rangle_{(0)}\right| \leq \|u\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\|v\|_{(1),\mathscr{H}^{-s,\alpha,n}_{a,\sigma}}. \tag{37}$$

Then, $(u, v) \mapsto \langle u, v\rangle_{(0)}$ is a continuous bilinear form on
$\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+) \times \mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.
Let $v \in \mathscr{H}^{-s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we consider the function $\phi_v : u \mapsto \langle u, v\rangle_{(0)}$.
From the relation (37), we see that ϕ_v is a continuous linear form on $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ and we have

$$\left\|\phi_v\right\| \leq \|v\|_{(1),\mathscr{H}^{-s,\alpha,n}_{a,\sigma}}.$$

On the other hand for $u_0(\lambda) = \left[\mathscr{F}^{\alpha,d,n}_W\right]^{-1}\left((1+\|\lambda\|^2)^{-s}\mathscr{F}^{\alpha,d,n}_W(v)\right)(\lambda)$, we obtain

$$u_0 \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+) \text{ and } \langle u_0, v\rangle_{(0)} = \|v\|^2_{(1),\mathscr{H}^{-s,\alpha,n}_{a,\sigma}}.$$

Then $\left\|\phi_v\right\| = \|v\|_{(1),\mathscr{H}^{-s,\alpha,n}_{a,\sigma}}$.
Let now $v^* \in \left(\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)\right)'$. By the Riesz representation theorem and the relation (33), one can see that there exists $w \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, such that for all $u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we have

$$\begin{aligned}v^*(u) &= \langle u, w\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\\ &= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}(1+\|\lambda\|^2)^s\mathscr{F}^{\alpha,d,n}_W(w)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(u)(\lambda)}e^{2a\|\xi\|^{\frac{1}{\sigma}}}d\mu_{\alpha+2n,d}(\lambda).\end{aligned}$$

We put $v(\lambda) = \left[\mathscr{F}_W^{\alpha,d,n}\right]^{-1}\left((1+\|\lambda\|^2)^s \mathscr{F}_W^{\alpha,d,n}(w)(\lambda)\right)$.
Then, $v \in \mathscr{H}_{a,\sigma}^{-s,\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\forall u \in \mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1}),\ v^*(u) = \langle u,\ v\rangle_{(0)}.$$

Hence the map $v \mapsto \langle .,\ v\rangle_{(0)}$ is an isometry from $\mathscr{H}_{a,\sigma}^{-s,\alpha,n}(\mathbb{R}_+^{d+1})$ into $\left(\mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})\right)'$.
Thus the proof is finished. □

Proposition 3.8. *For $s \in \mathbb{R}$, the Hilbert space $\mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ admits the reproducing kernel:*

$$\forall x, y \in \mathbb{R}_+^{d+1},\ \mathscr{R}_{s,\sigma}^{\alpha,d,n}(x,y) = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^{-s} e^{-2a\|\xi\|^{\frac{1}{\sigma}}} \Lambda_{\alpha,d,n}(-x,\xi))\Lambda_{\alpha,d,n}(y,\xi) d\mu_{\alpha+2n,d}(\xi), \tag{38}$$

That is :
i) For every $y \in \mathbb{R}_+^{d+1}$, the distribution given by the function
$x \mapsto \mathscr{R}_{s,\sigma}^{\alpha,d,n}(x,y)$ belongs to $\mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.
ii) For every $f \in \mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ \langle f,\ \mathscr{R}_{s,\sigma}^{\alpha,d,n}(.,y)\rangle_{(1),\mathscr{H}_{\alpha,d,n}^s} = f(y).$$

Proof. i) It easy to see that

$$\forall a > 0, \forall s \in \mathbb{R},\ \int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^{-s} e^{-2a\|\xi\|^{\frac{1}{\sigma}}} d\mu_{\alpha+2n,d}(\xi) < +\infty. \tag{39}$$

Then using the relation (11), we deduce that the function
$(x,y) \mapsto \mathscr{R}_{s,\sigma}^{\alpha,d,n}(x,y)$ is well-defined.
Moreover for all $y \in \mathbb{R}_+^{d+1}$ and $s \in \mathbb{R}$, the function
$\xi \mapsto (1+\|\xi\|^2)^{-s} e^{-2a\|\xi\|^{\frac{1}{\sigma}}} \Lambda_{\alpha,d,n}(y,\xi)$ belongs to $\mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1}) \cap \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$. Then, from the relation (19), the function $x \mapsto \mathscr{R}_{s,\sigma}^{\alpha,d,n}(x,y)$ belongs to $\mathbb{L}_\alpha^2(\mathbb{R}_+^{d+1})$ and we have

$$\forall \xi \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left[\mathscr{R}_{s,\sigma}^{\alpha,d,n}(.,y)\right](\xi) = (1+\|\xi\|^2)^{-s} e^{-2a\|\xi\|^{\frac{1}{\sigma}}} \Lambda_{\alpha,d,n}(y,\xi). \tag{40}$$

Then

$$\left\|\mathscr{R}_{s,\sigma}^{\alpha,d,n}(.,y)\right\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \leq k_s y_{d+1}^{2n}, \tag{41}$$

where for all $s \in \mathbb{R}$, we have

$$k_s = k_s(\alpha, d, n, a, \sigma) = C_{\alpha+2n,d}\left(\int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^{-s} e^{-2a\|\xi\|^{\frac{1}{\sigma}}} d\mu_{\alpha+2n,d}(\xi)\right)^{\frac{1}{2}}. \tag{42}$$

Hence for all $y \in \mathbb{R}_+^{d+1}$ and $s \in \mathbb{R}$, $\mathscr{R}_{s,\sigma}^{\alpha,d,n}(.,y) \in H_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.
ii) Let $f \in H_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$. Using the relations (33), (40) and (19), we obtain

$$\begin{aligned}\langle f, \mathscr{R}_{s,\sigma}^{\alpha,d,n}(.,y)\rangle_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} &= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\xi)\, \Lambda_{\alpha,d,n}(-y,\xi) d\mu_{\alpha+2n,d}(\xi) \\ &= f(y).\end{aligned}$$

Which achieves the proof. □

4. Extremal functions on $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$

The theory of reproducing kernels started with two papers from 1921 (see [26]) and 1922 (see [11]) which dealt with typical reproducing kernels of Szegö and Bergman and since then the theory has been developed into a large and deep theory in complex analysis by many mathematicians.
In this section, using the theory of reproducing kernels, we study the extremal functions on the Hilbert space $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$.

Definition 4.1. *Let $r > 0$ and $\mathscr{L} : \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+) \to \mathcal{H}$ be bounded linear operator. For all $f, h \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we define the inner product in $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ by :*

$$\langle f, h\rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = r\langle f, h\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \langle \mathscr{L}f, \mathscr{L}h\rangle_{\mathcal{H}}. \tag{43}$$

The norm associated with this inner product is given by :

$$\|f\|^2_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = r\|f\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \|\mathscr{L}f\|^2_{\mathcal{H}}. \tag{44}$$

Lemma 4.2. *The norms $\|.\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}$ and $\|.\|_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}$ are equivalent.*

Proof. Let $u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$, we have

$$\sqrt{r}\|u\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \le \|u\|_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \le \sqrt{r + \|\mathscr{L}\|^2}\|u\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}.$$

This clearly yields the result. □

Proposition 4.3. *Let $r > 0$ and $s \in \mathbb{R}$. The space $\left(\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+), \langle ., .\rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\right)$ possesses a reproducing $\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}$ satisfying the identity :*

$$\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y) = (rI + \mathscr{L}^*\mathscr{L})^{-1}\mathscr{R}^{\alpha,d,n}_{s,\sigma}(., y), \tag{45}$$

where $I = Id$ and $\mathscr{L}^* : \mathcal{H} \to \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ is the adjoint of $\mathscr{L}$ given by :

$$\forall f \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+), \forall h \in \mathcal{H}, \ \langle \mathscr{L}f, h\rangle_{\mathcal{H}} = \langle f, \mathscr{L}^*h\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}.$$

Proof. From [22], the space $\left(\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+), \langle ., .\rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}\right)$ has a reproducing kernel denoted by $\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}$ and we have

$$\begin{aligned} f(y) &= \langle f, \mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y)\rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \\ &= r\langle f, \mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y)\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \langle \mathscr{L}f, \mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y)\rangle_{\mathcal{H}} \\ &= \langle f, (rI + \mathscr{L}^*\mathscr{L})\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y)\rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}. \end{aligned}$$

Then for all $y \in \mathbb{R}^{d+1}_+$, we have

$$(rI + \mathscr{L}^*\mathscr{L})\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(., y) = \mathscr{R}^{\alpha,d,n}_{s,\sigma}(., y). \tag{46}$$

Thus the proof is finished. □

The following proposition summarizes some properties of the kernel $\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}$.

Proposition 4.4. *The kernel $\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}$ satisfies the following properties*
i) We have

$$\|\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|_{(1),\mathscr{H}^{s}_{\alpha,d,n}} \leq \frac{k_s}{r} y^{2n}_{d+1}, \tag{47}$$

where k_s is the constant given by the relation (42).
ii) We have

$$\|\mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|_{\mathcal{H}} \leq \frac{k_s}{\sqrt{2r}} y^{2n}_{d+1}. \tag{48}$$

iii) For all $y \in \mathbb{R}^{d+1}_{+}$, we have

$$\|\mathscr{L}^{*}\mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \leq k_s y^{2n}_{d+1}. \tag{49}$$

Proof. Using the relation (41) and (46), for all $y \in \mathbb{R}^{d+1}_{+}$, we get

$$r^2\|\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + 2r\|\mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|^2_{\mathcal{H}} + \|\mathscr{L}^{*}\mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}$$

$$= \|\mathscr{R}^{\alpha,d,n}_{s,\sigma}(.,y)\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} \leq k_s^2 y^{4n}_{d+1}.$$

Then the assertions i), ii) and iii) are an immediate consequence of the above result. □

The main result of this section can be stated as follows.

Theorem 4.5. *Let $s \in \mathbb{R}$.*
For all $h \in \mathcal{H}$ and for all $r > 0$, the infimum

$$\inf_{f \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_{+})} \left[r\|f\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \|h - \mathscr{L}f\|^2_{\mathcal{H}} \right], \tag{50}$$

is attained by a unique function $f^{*}_{r,h}$ given by :

$$\forall y \in \mathbb{R}^{d+1}_{+},\ f^{*}_{r,h}(y) = \langle h, \mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\rangle_{\mathcal{H}}. \tag{51}$$

Moreover, the extremal function $f^{*}_{r,h}$ satisfies the following inequality :

$$\forall y \in \mathbb{R}^{d+1}_{+},\ \left|f^{*}_{r,h}(y)\right| \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\mathcal{H}} y^{2n}_{d+1}. \tag{52}$$

Proof. The existence and unicity of extremal function $f^{*}_{r,h}$ represented by the relation (50) is given by [22]. On the other hand from the relation (48), we get

$$\forall y \in \mathbb{R}^{d+1}_{+}, \left| f^{*}_{r,h}(y)\right| \leq \|\mathscr{L}\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(.,y)\|_{\mathcal{H}}\|h\|_{\mathcal{H}} \leq \frac{k_s}{\sqrt{2r}}\|h\|_{\mathcal{H}} y^{2n}_{d+1}.$$

Which gives the desired result. □

Corollary 4.6. *Let $s \in \mathbb{R}$ and $r > 0$. If $\mathscr{L}$ is isometry ($\mathscr{L}^{*}\mathscr{L} = Id$) then*
i) $\langle .,. \rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = (r+1)\langle .,. \rangle_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}$.
ii) For all $x, y \in \mathbb{R}^{d+1}_{+}$, we have

$$\mathscr{K}^{r,\mathscr{L}}_{s,a,\sigma}(x,y) = \frac{1}{r+1}\mathscr{R}^{\alpha,d,n}_{s,\sigma}(.,y).$$

iii) For all $h \in \mathcal{H}$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ f_{r,h}^*(y) = \frac{1}{r+1}\mathscr{L}^* h(y).$$

iv) For all $f \in \mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ f_{r,\mathscr{L}f}^*(y) = \frac{1}{r+1} f(y).$$

Corollary 4.7. *Let $s \in \mathbb{R}$ and $r > 0$. Let $f \in \mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and $h = \mathscr{L}f$.*
i) For all $y \in \mathbb{R}_+^{d+1}$, we have

$$f(y) = \lim_{r \to 0^+} f_{r,h}^*(y).$$

ii) We have

$$\forall y \in \mathbb{R}_+^{d+1}, \left| f(y) - f_{r,h}^*(y) \right| \le k_s \|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} y_{d+1}^{2n}.$$

iii) We have

$$\forall y \in \mathbb{R}_+^{d+1}, \left| f_{r,h}^*(y) \right| \le k_s \|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} y_{d+1}^{2n}.$$

Proof. Let $f \in \mathscr{H}_{a,\sigma}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and $h = \mathscr{L}f$.
i) From the relations (46) and (51), we get

$$\forall y \in \mathbb{R}_+^{d+1},\ f_{r,h}^*(y) = \langle f, \mathscr{L}^* \mathscr{L} \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \rangle_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}}. \tag{53}$$

Then, for all $y \in \mathbb{R}_+^{d+1}$, we obtain

$$f_{r,h}^*(y) = \langle f, \mathscr{R}_{s,\sigma}^{\alpha,d,n}(., y) - r\mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \rangle_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}}.$$

Hence

$$\forall y \in \mathbb{R}_+^{d+1},\ f_{r,h}^*(y) = f(y) - r\langle f, \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \rangle_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}}, \tag{54}$$

and we have

$$\lim_{r \to 0^+} f_{r,h}^*(y) = \lim_{r \to 0^+} \left[f(y) - r\langle f, \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \rangle_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \right] = f(y).$$

ii) By invoking (47) and (54), for all $y \in \mathbb{R}_+^{d+1}$, we can write

$$\begin{aligned}
\left| f(y) - f_{r,h}^*(y) \right| &= r \left| \langle f, \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \rangle_{(1),\mathscr{H}_{\alpha,d,n}^{s}} \right| \\
&\le r\|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \\
&\le k_s \|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} y_{d+1}^{2n}.
\end{aligned}$$

iii) Using the relations (49) and (53), for all $y \in \mathbb{R}_+^{d+1}$, we obtain

$$\begin{aligned}
\left| f_{r,h}^*(y) \right| &\le \|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \|\mathscr{L}^* \mathscr{L} \mathscr{K}_{s,a,\sigma}^{r,\mathscr{L}}(., y) \|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} \\
&\le k_s \|f\|_{(1),\mathscr{H}_{a,\sigma}^{s,\alpha,n}} y_{d+1}^{2n}.
\end{aligned}$$

Which finishes the proof. □

Example 4.8. *For all $s \in \mathbb{R}$, the identity operator id $: \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+) \to \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ is bounded ad we have*

$$\forall u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+),\ \|id\,(u)\|_{\alpha,n,2} \leq \|u\|_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}}.$$

Its adjoint operator $id^* : \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \to \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ is given by :

$$\forall v \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+),\ id^*(v) = \left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1}\left[(1+\|\xi\|^2)^{-s}e^{-2a\|\xi\|^{\frac{1}{\sigma}}}\mathscr{F}^{\alpha,d,n}_W(v)\right].$$

On the other hand, the inner product associated with the operator id can be written

$$\langle u,v\rangle_{(2),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\left[1+r(1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}}\right]\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)}(\xi)d\mu_{\alpha+2n,d}(\xi).$$

In this case, the Hilbert space $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ admits the following reproducing kernel :

$$\mathscr{K}^{r,id}_{s,a,\sigma}(x,y) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\frac{\Lambda_{\alpha,d,n}(-x,\xi)\Lambda_{\alpha,d,n}(y,\xi)}{1+r(1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}}}d\mu_{\alpha+2n,d}(\xi).$$

For all $h \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and for all $r > 0$, the infimum

$$\inf_{f\in\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)}\left[r\|f\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \|h-f\|^2_{\alpha,n,2}\right],$$

exists and it is attained by a unique function $f^*_{r,h}$ given by :

$$\begin{aligned} f^*_{r,h}(y) &= \langle h, \mathscr{K}^{r,id}_{s,a,\sigma}(.,y)\rangle_{L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)} \\ &= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\frac{\Lambda_{\alpha,d,n}(y,\xi)\mathscr{F}^{\alpha,d,n}_W(h)(\xi)}{1+r(1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}}}d\mu_{\alpha+2n,d}(\xi) \\ &= \langle h, \mathscr{F}^{\alpha,d,n}_W\left(\mathscr{K}^{r,id}_{s,a,\sigma}(.,y)\right)\rangle_{L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

Moreover, the extremal function $f^*_{r,h}$ satisfies the following inequality :

$$\forall y \in \mathbb{R}^{d+1}_+, \left|f^*_{r,h}(y)\right| \leq \frac{k_s}{\sqrt{2r}}\|h\|_{\alpha,n,2}\,y^{2n}_{d+1},$$

where k_s is the constant given by the relation (42).

Example 4.9. *For $m \in \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we define the multiplier operator $\mathscr{L}_m$ by :*

$$\forall u \in \mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+),\ \mathscr{L}_m u := \left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1}\left[m\mathscr{F}^{\alpha,d,n}_W(u)\right].$$

For all $s \geq 0$, the operator $\mathscr{L}_m$ is bounded from $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ into $\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$.
The Hilbert space $\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)$ admits the following reproducing kernel :

$$\mathscr{K}^{r,\mathscr{L}_m}_{s,a,\sigma}(x,y) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\frac{\Lambda_{\alpha,d,n}(-x,\xi)\Lambda_{\alpha,d,n}(y,\xi)}{r(1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}} + |m(\xi)|^2}d\mu_{\alpha+2n,d}(\xi).$$

For all $h \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and for all $r > 0$, the infimum

$$\inf_{f\in\mathscr{H}^{s,\alpha,n}_{a,\sigma}(\mathbb{R}^{d+1}_+)}\left[r\|f\|^2_{(1),\mathscr{H}^{s,\alpha,n}_{a,\sigma}} + \|h-\mathscr{L}_m f\|^2_{\alpha,n,2}\right],$$

exists an it is attained by a unique function $f^*_{r,h}$ given by :

$$\begin{aligned} f^*_{r,h}(y) &= \langle h, \mathscr{L}_m \mathscr{K}^{r,\mathscr{L}_m}_{s,a,\sigma}(.,y)\rangle_{\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)} \\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \frac{\overline{m(\xi)}\Lambda_{\alpha,d,n}(y,\xi)\mathscr{F}^{\alpha,d,n}_W(h)(\xi)}{r(1+\|\xi\|^2)^s e^{2a\|\xi\|^{\frac{1}{\sigma}}} + |m(\xi)|^2} d\mu_{\alpha+2n,d}(\xi). \end{aligned}$$

Moreover, the extremal function $f^*_{r,h}$ satisfies the following inequality :

$$\forall y \in \mathbb{R}^{d+1}_+, \left|f^*_{r,h}(y)\right| \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\alpha,n,2} y^{2n}_{d+1},$$

where k_s is the constant given by the relation (42).

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Declarations

Ethical Approval
Not applicable.
Competing interest
The authors declare that they have no conflict of interest.
Authors' Contributions
Conceptualization, methodology, validation, formal analysis, investigation, resources, data curation, writing-original, draft preparation, writing-review and editing, visualization, supervision and project administration, the authors have read and approved the published version of manuscript.
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