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New inequalities of q-numerical radius of operators

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Abstract. In this paper, we establish new q-numerical radius inequalities for bounded linear operators on a complex Hilbert space, where $q \in [0, 1]$. By introducing various notions and norms depending on q, we generalise some results related to the numerical radius.

1. Introduction

Let H be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$, and its associated norm $\| \cdot \|$. The C^*-algebra of all bounded linear operators acting on H is denoted $\mathcal{B}(H)$, with I represents the identity element of $\mathcal{B}(H)$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we denote by T^*, $\|T\|$ and $\sigma(T)$ the adjoint, the operator norm and the spectrum of T, respectively.

The numerical range $W(T)$ and the numerical radius $w(T)$ of the operator T are respectively defined by:

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\},$$

and

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

The famous Toeplitz-Hausdorff theorem asserts that $W(T)$ is always a convex subset of $\mathbb{C}$. It is important to note that $W(T)$ is a non-empty bounded subset of $\mathbb{C}$, which is not necessarily closed (except if $\dim H < +\infty$). It is well-known that the numerical radius $w(\cdot)$ is a norm on $\mathcal{B}(H)$ which is equivalent to the norm operator $\| \cdot \|$. Indeed, it satisfies the following inequality:

$$\frac{1}{2}\|T\| \le w(T) \le \|T\| \ \ (T \in \mathcal{B}(H)).$$

Over the years, these notions has been generalized in different directions. Among them is the q-numerical range and q-numerical radius (with $0 \le q \le 1$) which are respectively given by:

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in H, \|x\| = \|y\| = 1, \langle x, y\rangle = q\},$$

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and

$$w_q(T) = \sup\{|\lambda| : \lambda \in W_q(T)\}.$$

In 1979, Stolov [15] proved that the 0-numerical range $W_0(T)$ of a matrix T is exactly the closed disk centered at origin with radius $\min\{\|T - \lambda I\| : \lambda \in \mathbb{C}\}$. In particular, this set is convex. Five years later, in the finite dimensional case, Tsing [17] showed that the q-numerical range is always a convex subset of $\mathbb{C}$. In 1994, Li et al. [12] extended these results to the infinite-dimensional case. Furthermore, in the same paper, they showed that $q\sigma(T)$ is contained in the closure of $W_q(T)$. They also established that for each $q \in [0, 1]$, $w_q(\cdot)$ is a seminorm on $\mathcal{B}(H)$, and it is a norm if and only if $q \neq 0$. More precisely, they provided the following inequality:

$$qw(T) \leq w_q(T) \leq \beta_q w(T), \tag{1}$$

where $T \in \mathcal{B}(H)$, $0 < q \leq 1$ and $\beta_q = \min\{2, q + 2\sqrt{1 - q^2}\}$. From (1), we can obtain the following inequality (see [6]):

$$\frac{q}{2}\|T\| \leq w_q(T) \leq \|T\|. \tag{2}$$

In particular, for any normal operator T, we have:

$$q\|T\| \leq w_q(T) \leq \|T\|.$$

For more information regarding these concepts, and their applications, we invite the reader to consult the following references [9, 12, 13, 17] and the references therein.

It is well-known that two operators T and S acting on H (see for instance [10]) satisfy the following inequalities:

$$w(TS) \leq \|TS\| \leq 2\|T\|w(S) \leq 4w(T)w(S). \tag{3}$$

If $TS = ST$ then

$$w(TS) \leq 2w(T)w(S). \tag{4}$$

If T and S doubly commute (i.e., $TS = ST$ and $TS^* = S^*T$) then

$$w(TS) \leq \|T\|w(S). \tag{5}$$

$$w(TS + ST) \leq 4w(T)w(S). \tag{6}$$

$$w(TS + ST) \leq 2\sqrt{2}\|T\|w(S). \tag{7}$$

$$w(TS \pm ST^*) \leq 2\|T\|w(S), \tag{8}$$

During the years, these inequalities have been improved by several mathematicians. For example, in 2015, Abu-Omar and Kittaneh [1] used the quantity $R(T) = \inf_{\lambda\in\mathbb{C}} w(T - \lambda I)$ to refine the above inequalities. Later, in 2023, Taki and Kaadoud [16] defined a new quantity $K(T) = \inf_{\mu\in\mathbb{R}} w(T - i\mu z_T I)$, where z_T is the centre of the smallest disc containing $W(T)$, in order to improve the inequalitites obtained by Abu-Omar and Kittaneh in [1]. They also showed that the quantity $K(T)$ satisfies the following properties:

$$R(T) \leq K(T) \leq w(T),\ K(T^*) = K(T),\ K(\lambda T) = |\lambda|K(T) \text{ and } K(U^*TU) = K(T),$$

where $\lambda \in \mathbb{C}$ and U is an unitary operator. We refer the reader to [1, 16] and the references therein for further details.

In this paper, we provide some q-numerical radius inequalities which are similar to the inequalities (3)–(8). Using the new quantities $R_q(T) = \inf_{\lambda\in\mathbb{C}} w_q(T - \lambda I)$ and $K_q(T) = \inf_{\mu\in\mathbb{R}} w_q(T - i\mu z_T I)$, which are depend on $q \in [0, 1]$, we extend some results of [16] to the q-numerical radius. We also generalise some results of [4].

2. q-Numerical radius inequalities involving $K_q(T)$

The purpose of this section is to generalize and improve the six inequalities (3)-(8). Within this framework, we extend several results of [16]. To achieve our goal, we need to define some quantities and present some of their properties.

Definition 2.1. *Let $T \in \mathcal{B}(H)$ and $q \in [0,1]$. The quantities $R_q(T)$ and $K_q(T)$ are respectively given by:*

$$R_q(T) = \inf\{w_q(T - \lambda I) : \lambda \in \mathbb{C}\}$$

and

$$K_q(T) = \inf\{w_q(T - i\alpha z_T I) : \alpha \in \mathbb{R}\},$$

where z_T is the centre of the smallest disc containing $W(T)$.

Note that the existence and uniqueness of z_T are guaranteed by [11]. It also follows from [11] that $R_q(T)$ is exactly the radius of the smallest disc containing $W_q(T)$. Clearly, for $q = 1$, we have $K_1(T) = K(T)$. From the definition of $K_q(T)$, we can immediately derive the following equalities:

$$K_q(T^*) = K_q(T),\ K_q(\lambda T) = |\lambda| K_q(T) \text{ and } K_q(U^*TU) = K_q(T),$$

where $\lambda \in \mathbb{C}$ and $U \in \mathcal{B}(H)$ is an unitary operators. Moreover, we can show the following inequality:

$$R_q(T) \leq K_q(T) \leq w_q(T).$$

We now present some properties concerning the quantity $K_q(T)$. It should be noted that the proofs of the following three propositions are similar to those of [16].

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

1. *There exists a real number $\alpha_{q,T}$ such that $K_q(T) = w_q(T - i\alpha_{q,T} z_T I)$.*
2. *If $z_T = 0$ then $R_q(T) = K_q(T) = w_q(T)$.*
3. *If $z_T \neq 0$ then $R_q(T) < K_q(T)$.*
4. *Also we have*

$$R_q(T) \leq \frac{R_q(T) + K_q(T)}{2} \leq \sqrt{\frac{R_q^2(T) + K_q^2(T)}{2}} \leq K_q(T) \leq w_q(T),$$

and

$$w_q(T) \leq \sqrt{R_q^2(T) + K_q^2(T)} \leq \sqrt{2} K_q(T) \leq \sqrt{2} w_q(T).$$

5. *If $R_q(T) = 0$ then T is a scalar operator.*
6. *$K_q(T) = 0$ if and only if $T = 0$.*

Proposition 2.3. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$|K_q(T) - K_q(S)| \leq w_q(T - S) + \max\{|\alpha_{q,T}|, |\alpha_{q,S}|\} |z_T - z_S|,$$

Proposition 2.4. *Let $q \in [0,1]$ and $\{T_n\}$ be a sequence of operators in $\mathcal{B}(\mathcal{H})$ wich converges to T in $\mathcal{B}(\mathcal{H})$. Then the real sequence $\{K_q(T_n)\}$ converges to $K_q(T)$.*

Proposition 2.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$.Under one of the following conditions:*

(1) $z_{T+S} = 0$.

(2) z_T*,* z_S*, and* z_{T+S} *are real numbers.*
(3) z_T*,* z_S*, and* z_{T+S} *belong to the same line which passes through the origin.*

the following inequality holds:

$$K_q(T + S) \leq K_q(T) + K_q(S).$$

Remark 2.6. *Note that the assertion* (3) *of the proposition 2.5 can be obtained from the fact that* $z_{e^{i\theta}T} = e^{i\theta}z_T$ *and* $K_q(e^{i\theta}T) = K_q(T)$ *for all* $\theta \in \mathbb{R}$*.*

Proposition 2.7. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$*. Then*

1. $qR(T) \leq R_q(T) \leq \beta_q R(T)$.
2. $qK(T) \leq K_q(T) \leq \beta_q K(T)$.

Here $\beta_q = \min\{2, q + 2\sqrt{1 - q^2}\}$*.*

Proof. 1. By applying the inequality (1) we get that:

$$qw(T - \alpha I) \leq w_q(T - \alpha I) \leq \beta_q w(T - \alpha I) \quad (\alpha \in \mathbb{C}).$$

Hence, by taking the infimum over all $\alpha \in \mathbb{C}$, we obtain the desired inequality.

2. Similarly to (1).

□

We are now in a position to present our first main result in this section. Before we move on, it is important to note that the inequality (5) does not hold if w is replaced by w_q. Indeed, taking $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $q = \frac{1}{5}$. By [14, Theorem 1.1], we have

$$w_q(U) = \frac{1}{2}\sqrt{2 - q^2} = \frac{7}{10}.$$

Since $w_q(I) = q$ it follows that

$$w_q(UI) = w_q(U) = \frac{7}{10} > \frac{1}{5} = \|U\| w_q(I).$$

Hereafter, let $\beta_q = \min\{2, q + 2\sqrt{1 - q^2}\}$. The following result presents a generalization of the inequalities (5) and (7).

Proposition 2.8. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$*. Then*

1. *If T and S doubly commute then*

$$w_q(TS) \leq \frac{\beta_q}{q}\|T\| w_q(S).$$

2.

$$w_q(TS + ST) \leq \frac{2\sqrt{2}\beta_q}{q}\|T\| w_q(S).$$

Proof. 1. If T and S doubly commute, then using the inequalities (1) and (5), we get

$$w_q(TS) \leq \beta_q w(TS) \leq \beta_q \|T\| w(S) \leq \frac{\beta_q}{q}\|T\| w_q(S).$$

2. Applying the inequalities (1) and (7), we get $w_q(TS + ST) \leq \beta_q w(TS + ST) \leq 2\sqrt{2}\beta_q \|T\| w(S) \leq \frac{2\sqrt{2}\beta_q}{q}\|T\| w_q(S)$.

□

In order to present our main results in this section, we need the following definition.

Definition 2.9. *Let* $T \in \mathcal{B}(\mathcal{H})$*, set*

$$\Gamma(T) = \{A \in \mathcal{B}(\mathcal{H}) : \mathcal{A}\mathcal{T} - \mathcal{T}\mathcal{A}^* = 0\}.$$

It is clear that $\Gamma(T) = \Gamma(T^*)$. Furthermore, $\Gamma(T)$ is a closed real linear subspace of $\mathcal{B}(H)$ in the weak topology of operators, as the following result shows.

Proposition 2.10. *Let* $T, S \in \mathcal{B}(\mathcal{H})$*. If* $\{S_n\}$ *is a sequence of elements of* $\Gamma(T)$ *such that* $\lim_n \langle S_n x, x\rangle = \langle Sx, x\rangle$ *for all* $x \in H$*. Then, we have* $S \in \Gamma(T)$*. In particular,* $\Gamma(T)$ *is closed in the uniform operator topology.*

Proof. Similar to the proof of [16, Proposition 2.10]. □

We now present the second main result of this section. In the following result, we give a generalization of Inequality (8).

Theorem 2.11. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$*. Then*

$$\begin{aligned} w_q(TS + ST^*) &\leq \frac{2\beta_q}{q} \operatorname{dist}(T, i\Gamma(S)) w_q(S) \\ &\leq \frac{2\beta_q}{q} \inf_{\alpha \in \mathbb{R}} \|T - i\alpha C\| w_q(S) \quad (\textit{for all } C \in \Gamma(S)) \\ &\leq \frac{2\beta_q}{q} \|T\| w_q(S), \end{aligned}$$

where $\operatorname{dist}(T, i\Gamma(T)) = \inf\{\|T - iC\| : C \in \Gamma(T)\}$.

Proof. We show only the first inequality, as the other inequalities are trivial.
Fix $C \in \Gamma(S)$. We have

$$w_q(TS + ST^*) = w_q((T - iC)S + S(T - iC)^*).$$

Using the inequalities (1) and (8), we obtain

$$\begin{aligned} w_q((T - iC)S + S(T - iC)^*) &\leq \beta_q w((T - iC)S + S(T - iC)^*) \\ &\leq 2\beta_q \|T - iC\| w(S) \\ &\leq \frac{2\beta_q}{q} \|T - iC\| w_q(S). \end{aligned}$$

Hence

$$w_q(TS + ST^*) \leq \frac{2\beta_q}{q} \|T - iC\| w_q(S).$$

Thus, by taking the infimum over all $C \in \Gamma(S)$, we deduce that

$$w_q(TS + ST^*) \leq \frac{2\beta_q}{q} \operatorname{dist}(T, i\Gamma(S)) w_q(S).$$

□

Remark 2.12. *If we replace T by iT in Theorem 2.11, we obtain*

$$\begin{aligned} w_q(TS - ST^*) &\leq \frac{2\beta_q}{q} \operatorname{dist}(T, \Gamma(S)) w_q(S) \\ &\leq \frac{2\beta_q}{q} \|T\| w_q(S). \end{aligned}$$

In the following result, we provide a generalization and improvement of the inequalities (4) and (6).

Theorem 2.13. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$. *Then*

1.
$$\begin{aligned} w_q(TS+ST) &\leq \frac{2\beta_q}{q^2}\left(K_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{2\beta_q}{q^2}\left(w_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{4\beta_q}{q^2}w_q(T)w_q(S). \end{aligned}$$

2. *If* $TS = ST$ *then*
$$\begin{aligned} w_q(TS) &\leq \frac{\beta_q}{q^2}\left(K_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{\beta_q}{q^2}\left(w_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{2\beta_q}{q^2}w_q(T)w_q(S). \end{aligned}$$

Proof.
1. By [16, Theorem 2], we have
$$w(TS+ST) \leq 2\left(K(T)+R(T)\right)w(S).$$
Hence, by Inequality (1), we get
$$w_q(TS+ST) \leq 2\beta_q\left(K(T)+R(T)\right)w(S).$$
On the other hand, by applying Proposition 2.7, we get that
$$w_q(TS+ST) \leq \frac{2\beta_q}{q^2}\left(K_q(T)+R_q(T)\right)w_q(S).$$
Therefore
$$\begin{aligned} w_q(TS+ST) &\leq \frac{2\beta_q}{q^2}\left(K_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{2\beta_q}{q^2}\left(w_q(T)+R_q(T)\right)w_q(S) \\ &\leq \frac{4\beta_q}{q^2}w_q(T)w_q(S). \end{aligned}$$
This gives the desired result.
2. Follows directly from the first assertion.
□

In the next result, we generalize and refine the inequalities (3) and (6).

Theorem 2.14. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$. *Then*

1.

$$\begin{aligned} w_q(TS) &\leq \|TS\| \\ &\leq \frac{1}{q}\|T\|\left(K_q(S)+R_q(S)\right) \\ &\leq \frac{1}{q}\|T\|\left(w_q(S)+R_q(S)\right). \end{aligned}$$

2.

$$\begin{aligned} w_q(TS) &\leq \|TS\| \\ &\leq \frac{1}{q^2}\left(K_q(T)+R_q(T)\right)\left(K_q(S)+R_q(S)\right) \\ &\leq \frac{1}{q^2}\left(w_q(T)+R_q(T)\right)\left(K_q(S)+R_q(S)\right) \\ &\leq \frac{1}{q^2}\left(w_q(T)+R_q(T)\right)\left(w_q(S)+R_q(S)\right). \end{aligned}$$

Proof. 1. It follows from [16, Theorem 3] that

$$\|TS\| \leq \|T\|\left(K(S)+R(S)\right).$$

Then, by applying Proposition 2.7, we get

$$\begin{aligned} w_q(TS) &\leq \|TS\| \\ &\leq \|T\|\left(K(S)+R(S)\right) \\ &\leq \frac{1}{q}\|T\|\left(K_q(S)+R_q(S)\right) \\ &\leq \frac{1}{q}\|T\|\left(w_q(S)+R_q(S)\right). \end{aligned}$$

2. The required result is obtained by reapplying [16, Theorem 3] and Proposition 2.7. □

The following result is a direct consequence of Theorem 2.14.

Corollary 2.15. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $0 < q \leq 1$. *Then*

1.

$$w_q(TS) \leq \frac{2}{q}\|T\|K_q(S) \leq \frac{2}{q}\|T\|w_q(S).$$

2.

$$w_q(TS) \leq \frac{4}{q}K_q(T)K_q(S) \leq \frac{4}{q}w_q(T)w_q(S).$$

3. q-Numerical radius inequalities involving N_q

In this section, we extend several of Dragomir's results [4]. The following definition is needed.

Definition 3.1. *Let* $T \in \mathcal{B}(\mathcal{H})$, $q \in [0,1]$ *and* $\alpha \in \mathbb{C}$. *We define*

$$d_q(T,\alpha) = \sup\{\|Tx - \alpha y\| : (x,y) \in C_q\},$$

where $C_q = \{(x,y) \in H \times H : \|x\| = \|y\| = 1$ *and* $\langle x, y\rangle = q\}$.

The proofs of the following results are similar to the approach presented in [4].

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. If $\alpha \in \mathbb{C}^*$ and $r > 0$ satisfy*

$$d_q(T,\alpha) \leq r,$$

then

$$(0 \leq)\|T\| - w_q(T) \leq \frac{1}{2}\frac{r^2}{|\alpha|}.$$

Corollary 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Let $\phi, \psi \in \mathbb{C}$ with $\psi \notin \{\phi, -\phi\}$. Suppose*

$$Re\Big(\langle \psi y - Tx, Tx - \phi y\rangle\Big) \geq 0, \ \forall (x,y) \in C_q, \tag{9}$$

then

$$(0 \leq)\|T\| - w_q(T) \leq \frac{1}{4}\frac{|\psi - \phi|}{|\psi + \phi|}.$$

Corollary 3.4. *Suppose that T, α, r are as in Theorem 3.2. Assume also that,*

$$\Big||\alpha| - w_q(T)\Big| \geq t$$

for some $t \geq 0$, then

$$\|T\|^2 - w_q^2(T) \leq r^2 - t^2.$$

Theorem 3.5. *Let T be a nonzero operator on H and $q \in [0,1]$. Let $\alpha \in \mathbb{C}^*, r > 0$ such that $|\alpha| > r$. Suppose*

$$d_q(T,\alpha) \leq r,$$

then

$$\sqrt{1 - \frac{r^2}{|\alpha|^2}} \leq \frac{w_q(T)}{\|T\|}.$$

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Let $\phi, \psi \in \mathbb{C}$ satisfy (9) and $\Re(\psi\overline{\phi}) > 0$. Then*

$$\frac{2\sqrt{\Re(\psi\overline{\phi})}}{|\psi + \phi|} \leq \frac{w_q(T)}{\|T\|}, \tag{10}$$

and

$$\|T\|^2 - w_q^2(T) \leq \left|\frac{\psi - \phi}{\psi + \phi}\right|^2 \|T\|^2.$$

Proof. Take $\alpha = \frac{\psi+\phi}{2}$ and $r = \frac{1}{2}|\psi - \phi|$, then

$$|\alpha|^2 - r^2 = \left|\frac{\psi + \phi}{2}\right|^2 - \left|\frac{\psi - \phi}{2}\right|^2 = \Re(\psi\overline{\phi}) > 0.$$

Now, by Theorem 3.5, we get the desired result. □

Remark 3.7. *From Inequality* (2)*, we have*

$$w_q(T) \geq \frac{q}{2}\|T\|.$$

Assume that

$$|\psi - \phi| \leq \sqrt{1 - \frac{q}{2}}\,|\psi + \phi|, \ \ \Re(\psi\overline{\phi}) > 0,$$

then (10) is a refinement of the inequality $w_q(T) \geq \frac{q}{2}\|T\|$.

Theorem 3.8. *Let T be a nonzero operator on H, $q \in [0,1]$, $\alpha \in \mathbb{C}^*$ and $r > 0$ with $|\alpha| > r$. Suppose*

$$d_q(T,\alpha) \leq r,$$

then

$$(0 \leq)\|T\|^2 - w_q^2(T) \leq \frac{2r^2}{|\alpha| + \sqrt{|\alpha|^2 - r^2}} w_q(T).$$

Proof. The proof is analogous to that in [4]. □

Corollary 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Let $\phi, \psi \in \mathbb{C}$ satisfy (9) and $Re(\psi\overline{\phi}) > 0$. Then*

$$\|T\|^2 - w_q(T)^2 \leq \left[|\psi + \phi| - 2Re(\psi\overline{\phi})\right] w_q(T).$$

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$, $q \in [0,1]$, $r \geq 1$ and $\alpha \in \mathbb{C}^*$ such that*

$$\|T\| \leq |\alpha|.$$

Then

$$\|T\|^{2r} + |\alpha|^{2r} \leq 2\|T\|^{r-1}|\alpha|^r w_q(T) + r^2|\alpha|^{2r-2} d_q^2(T,\alpha).$$

Proof. Let $r \geq 1$. In [8], the authors proved that

$$\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^{r-1}\|b\|^{r-1}\mathfrak{R}(\langle a,b\rangle) \leq r^2\|a\|^{2r-2}\|a-b\|^2, \tag{11}$$

for all $a, b \in H$ with $\|a\| \geq \|b\|$. Now, let $x \in H$ be of norm one. Then, there exists y such that $(x,y) \in C_q$. Taking $a = \alpha x$ and $b = Ty$ in (11), since $\|Ty\| \leq \|\alpha x\|$, we have

$$\begin{aligned}
&\|Tx\|^{2r} + |\alpha|^{2r} \leq 2\|Tx\|^{r-1}|\alpha|^r|\langle Tx,y\rangle| + r^2|\alpha|^{2r-2}\|Ty - \alpha x\|^2 \\
&\|Tx\|^{2r} + |\alpha|^{2r} \leq 2\|T\|^{r-1}|\alpha|^r w_q(T) + r^2|\alpha|^{2r-2} d_q^2(T,\alpha).
\end{aligned}$$

Taking the supremum over all unit vectors x, we obtain the desired result. □

Lemma 3.11. *[5] Let $a, b \in H$, $t \in \mathbb{R}$ and $\alpha \in [0,1]$. Then*

$$\|a\|^2\|b\|^2 - [(1-\alpha)\mathfrak{I}(\langle a,b\rangle) + \alpha\mathfrak{R}(\langle a,b\rangle)]^2 \leq \left[\alpha\|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right]\|b\|^2.$$

Theorem 3.12. *Let $T \in \mathcal{B}(\mathcal{H})$, $q, \alpha \in [0,1]$ and $t \in \mathbb{R}$. Then*

$$\|T\|^2 \leq \left[(1-\alpha)^2 + \alpha^2\right] w_q^2(T) + \alpha d_q^2(T,t) + (1-\alpha)d_q^2(T,it).$$

Proof. By Lemma 3.11, we have, for all $a, b \in H$, $t \in \mathbb{R}$, and $\alpha \in [0,1]$,

$$\|a\|^2\|b\|^2 - [(1-\alpha)\mathfrak{I}(\langle a,b\rangle) + \alpha\mathfrak{R}(\langle a,b\rangle)]^2 \leq \left[\alpha\|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right]\|b\|^2.$$

Hence

$$\begin{aligned}
\|a\|^2\|b\|^2 &\leq [(1-\alpha)\mathfrak{I}(\langle a,b\rangle + \alpha\mathfrak{R}(\langle a,b\rangle)]^2 + \left[\alpha\|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right]\|b\|^2 \\
&\leq \left[(1-\alpha)^2 + \alpha^2\right]|\langle a,b\rangle|^2 + \left[\alpha\|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right]\|b\|^2.
\end{aligned} \tag{12}$$

Now let $(x,y) \in C_q$. Choose $a = Tx$ and $b = y$ in (12), we get

$$\begin{aligned}
\|Tx\|^2 &\leq \left[(1-\alpha)^2 + \alpha^2\right]|\langle Tx,y\rangle|^2 + \left[\alpha\|ty - Tx\|^2 + (1-\alpha)\|ity - Tx\|^2\right] \\
&\leq \left[(1-\alpha)^2 + \alpha^2\right] w_q^2(T) + \left[\alpha d_q^2(T,t) + (1-\alpha)d_q^2(T,it)\right].
\end{aligned}$$

Then

$$\|T\|^2 \leq \left[(1-\alpha)^2 + \alpha^2\right] w_q^2(T) + \alpha d_q^2(T,t) + (1-\alpha)d_q^2(T,it).$$

□

Corollary 3.13. *Let $T \in \mathcal{B}(\mathcal{H})$, $q, \alpha \in [0,1]$ and $t \in \mathbb{R}$. Then*

1. $\|T\|^2 - qw_q^2(T) \leq \inf_{t\in\mathbb{R}} d_q^2(T,t)$.
2. $\|T\|^2 - qw_q^2(T) \leq \inf_{t\in\mathbb{R}} d_q^2(T,it)$.

If $\alpha = \frac{1}{2}$ we have

$$\|T\|^2 \leq \frac{1}{2}w_q^2(T) + \frac{1}{2}\inf_{t\in\mathbb{R}}\left(d_q^2(T,t) + d_q^2(T,it)\right).$$

Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. We define

$$d_q(T,S) = \sup\{\|Tx - Sy\| : (x,y) \in C_q\}.$$

It is clear that

$$d_q(T,\alpha) = d_q(T,\alpha I),$$

where $d_q(T,\alpha)$ is as defined in Definition 3.1. We set

$$N_q(T,S) = \sup\{\| \sqrt{\|Tx\|^2 + \|Sy\|^2} : (x,y) \in C_q\}.$$

Theorem 3.14. *Let $q \in [0,1]$, $N_q(T,S)$ is a norm on $\mathcal{B}(\mathcal{H})^{\in}$.*

Proof. Let $N_q(T,S) = 0$ and let $x \in \mathcal{H}$ be a unit vector. Then there exists $y \in \mathcal{H}$ such that $(x,y) \in C_q$. Hence

$$\|Tx\|^2 + \|Sy\|^2 \leq N_q^2(T,S) = 0.$$

Thus $\|Tx\| = 0$ for all $x \in \mathcal{H}$ with $\|x\| = 1$, so $T = 0$. By the same argument, we get $S = 0$. Then $(T,S) = (0,0)$.
It is clear that $N_q(\alpha(T,S)) = |\alpha| N_q(T,S), \forall \alpha \in \mathbb{C}$.
Let $T, S, U, V \in \mathcal{B}(\mathcal{H})$ and $(x,y) \in C_q$, we have

$$\begin{aligned}
\|(T+U)x\|^2 + \|(S+V)y\|^2 &= \|Tx\|^2 + \|Ux\|^2 + 2\Re(\langle Tx, Ux\rangle) \\
&+ \|Sy\|^2 + \|Vy\|^2 + 2\Re(\langle Sy, Vy\rangle) \\
&\leq \|Tx\|^2 + \|Ux\|^2 + 2\|Tx\|\|Ux\| \\
&+ \|Sy\|^2 + \|Vy\|^2 + 2\|Sy\|\|Vy\| \\
&\leq \|Tx\|^2 + \|Sy\|^2 + \|Ux\|^2 + \|Vy\|^2 \\
&+ 2\sqrt{(\|Tx\|^2 + \|Sy\|^2)(\|Ux\|^2 + \|Vy\|^2)} \\
&\leq N_q^2(T,S) + +2N_q(T,S)N_q(U,V) + N_q^2(U,V) \\
&\leq (N_q(T,S) + N_q(U,V))^2.
\end{aligned}$$

Taking the supremum over all $(x,y) \in C_q$, we get

$$N_q^2(T+U, S+V) \leq (N_q(T,S) + N_q(U,V))^2.$$

Then

$$N_q((T,S) + (U,V)) \leq N_q(T,S) + N_q(U,V).$$

□

Remark 3.15. *If $q = 1$, we have*

$$\|Tx\|^2 + \|Sy\|^2 = \|Tx\|^2 + \|Sx\|^2 = \langle T^*Tx, x\rangle + \langle S^*Sx, x\rangle = \langle (T^*T + S^*S)x, x\rangle.$$

Then

$$N_q^2(T,S) = \|T^*T + S^*S\|.$$

We give the relationship between the three notions w_q, d_q and the norm N_q.

Theorem 3.16. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Suppose $d_q(T,S) \leq r$ for some $r \geq 0$, then*

$$\frac{1}{2}N_q^2(T,S) \leq w_q(S^*T) + \frac{1}{2}r^2.$$

Proof. Let $x \in H$ of norm one, then there exists y such that $(x,y) \in C_q$. Since $d_q(T,S) \leq r$ then

$$\|Tx\|^2 + \|Sy\|^2 \leq 2\Re(\langle Tx, Sy\rangle) + r^2.$$

Hence

$$\|Tx\|^2 + \|Sy\|^2 \leq 2\Re(\langle S^*Tx, y\rangle) + r^2 \leq 2w_q(S^*T) + r^2.$$

Taking the supremum over all $(x,y) \in C_q$, we get

$$\frac{1}{2}N_q^2(T,S) \leq w_q(S^*T) + \frac{1}{2}r^2.$$

□

If $q = 1$, we obtain

$$\frac{1}{2}\|T^*T + S^*S\| \leq w(S^*T) + \frac{1}{2}r^2.$$

Corollary 3.17. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$\frac{1}{2}N_q^2(T,S) \leq w_q(S^*T) + \frac{1}{2}d_q^2(T,S).$$

Proposition 3.18. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Suppose $d_q(T,S) \leq r$ for some $r \geq 0$, then*

$$\frac{1}{4}\left(\|T\|^2 + \|S\|^2\right) \leq w_q(S^*T) + \frac{1}{2}r^2.$$

Proof. Let $x \in H$ of norm one, then there exists y such that $(x,y) \in C_q$. Then

$$\|Tx\|^2 \leq \|Tx\|^2 + \|Sy\|^2 \leq N_q^2(T,S).$$

Hence

$$\|Tx\|^2 \leq N_q^2(T,S).$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|T\|^2 \leq N_q^2(T,S).$$

Similarly, we have

$$\|S\|^2 \leq N_q^2(T,S).$$

Then it follows that

$$\frac{1}{4}\left(\|T\|^2 + \|S\|^2\right) \leq \frac{1}{2}N_q^2(T,S).$$

We finish by using Theorem 3.16. □

Corollary 3.19. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Suppose $d_q(T,S) \leq r$ for some $r \geq 0$, then*

$$\frac{1}{2}\|T\|\|S\| \leq w_q(S^*T) + \frac{1}{2}r^2.$$

Proof. It is a simple consequence of the inequality $\frac{1}{2}\|T\|\|S\| \le \frac{1}{4}\left(\|T\|^2 + \|S\|^2\right)$ and Proposition 3.18. □

Theorem 3.20. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$\left[\frac{1}{2}d_q(T,-S)\right]^2 \le \frac{1}{2}\left[\frac{N_q^2(T,S)}{2} + w_q(S^*T)\right].$$

Proof. We have

$$\begin{aligned} d_q^2(T,-S) &= \sup\{\|Tx\|^2 + \|Sy\|^2 + 2\Re(\langle S^*Tx, y\rangle) : (x,y) \in C_q\} \\ &\le N_q^2(T,S) + 2w_q(S^*T). \end{aligned}$$

Then we get the desired result. □

If $q = 1$, we obtain

$$\|\frac{T+S}{2}\|^2 \le \frac{1}{2}\left[\frac{\|T^*T + S^*S\|}{2} + w_q(S^*T)\right].$$

Theorem 3.21. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $q \in [0,1]$ and $p \ge 2$. Then*

$$\left[\frac{N_q^2(T,S)}{2}\right]^{\frac{p}{2}} \le \frac{1}{4}\left[d_q^p(T,S) + d_q^p(T,-S)\right].$$

Proof. It was established in [3] that

$$2(\|a\|^p + \|b\|^p) \le \|a+b\|^p + \|a-b\|^p, \forall a, b \in H.$$

Let $(x,y) \in C_q$. By setting $a = Tx$ and $b = Sy$ in the last inequality, we get

$$\begin{aligned} &2(\|Tx\|^p + \|Sy\|^p) \le \|Tx + Sy\|^p + \|Tx - Sy\|^p \\ &2(\|Tx\|^p + \|Sy\|^p) \le d_q^p(T,S) + d_q^p(T,-S). \end{aligned}$$

Since $p \ge 2$, the function $x \mapsto x^{\frac{p}{2}}$ is convex on $\mathbb{R}^+$. Thus

$$\left[\frac{\|Tx\|^2 + \|Sy\|^2}{2}\right]^{\frac{p}{2}} \le \frac{\|Tx\|^p + \|Sy\|^p}{2}.$$

It follows that

$$\left[\frac{\|Tx\|^2 + \|Sy\|^2}{2}\right]^{\frac{p}{2}} \le \frac{1}{4}\left[d_q^p(T,S) + d_q^p(T,-S)\right].$$

We finish by taking the supremum over all $(x,y) \in C_q$. □

If $q = 1$, we have

$$\left[\frac{\|T^*T + S^*S\|}{2}\right]^{\frac{p}{2}} \le \frac{1}{4}\left[\|T-S\|^p + \|T+S\|^p\right].$$

We need the following lemma.

Lemma 3.22. *[7] Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator whose spectrum is contained in a interval J, and let $x \in \mathcal{H}$ be a unit vector. If f is a convex function on J, then*

$$f(\langle Tx, x\rangle) \le \langle f(T)x, x\rangle.$$

Let

$$c(T) = \inf\{|\alpha| : \alpha \in W(T)\}.$$

Theorem 3.23. *Let $T \in \mathcal{B}(\mathcal{H})$, $q \in [0,1]$ and f be a non-negative increasing operator convex function on $[0,\infty)$. Then*

$$f(c(T)w_q(T)) \leq \frac{1}{4}N_q^2\left((f(|T|^2))^{\frac{1}{2}},(f(|T^*|^2))^{\frac{1}{2}}\right)+\frac{1}{2}f(w_q(T^2)).$$

Proof. Let $(x,y) \in C_q$. We have

$$c(T) \leq |\langle y, T^*y\rangle|.$$

By the Buzano inequality [2],

$$|\langle Tx,y\rangle\langle y,T^*y\rangle| \leq \frac{1}{2}(\|Tx\|\|T^*y\|+|\langle Tx,T^*y\rangle|).$$

Hence

$$c(T)|\langle Tx,y\rangle| \leq \frac{\|Tx\|\|T^*y\|+|\langle T^2x,y\rangle|}{2}.$$

Since f is increasing, we infer that

$$\begin{aligned}
f(c(T)|\langle Tx,y\rangle|) &\leq f\left(\frac{\|Tx\|\|T^*y\|+|\langle T^2x,y\rangle|}{2}\right)\\
&\leq f\left(\frac{\|Tx\|\|T^*y\|+w_q(T^2)}{2}\right)\\
&= f\left(\frac{\sqrt{\langle Tx,Tx\rangle\langle T^*y,T^*y\rangle}+w_q(T^2)}{2}\right)\\
&= f\left(\frac{\sqrt{\langle |T|^2x,x\rangle\langle |T^*|^2y,y\rangle}+w_q(T^2)}{2}\right)\\
&\leq f\left(\frac{\frac{1}{2}(\langle |T|^2x,x\rangle+\langle |T^*|^2y,y\rangle)+w_q(T^2)}{2}\right)\\
&\quad \text{(by the arithmetic-geometric mean inequality)}\\
&= f\left(\frac{\frac{1}{2}(\langle (|T|^2+w_q(T^2))x,x\rangle+\langle (|T^*|^2+w_q(T^2))y,y\rangle)}{2}\right).
\end{aligned}$$

By the Hermite-Hadamard inequality, we have

$$f(\frac{a+b}{2}) \leq \int_0^1 f(ta+(1-t)b)dt \leq \frac{f(a)+f(b)}{2}.$$

Hence

$$\begin{aligned}
&f\left(\frac{\frac{1}{2}(\langle (|T|^2+w_q(T^2))x,x\rangle+\langle (|T^*|^2+w_q(T^2))y,y\rangle)}{2}\right)\\
&\leq \int_0^1 f\left(\frac{t}{2}\langle (|T|^2+w_q(T^2))x,x\rangle+\frac{1-t}{2}\langle (|T^*|^2+w_q(T^2))y,y\rangle\right)dt\\
&\leq \int_0^1 tf\left(\frac{\langle (|T|^2+w_q(T^2))x,x\rangle}{2}\right)+(1-t)f\left(\frac{\langle (|T^*|^2+w_q(T^2))y,y\rangle}{2}\right)dt,
\end{aligned}$$

because f is convex. Using again the convexity of f, we get

$$\begin{aligned}
f\left(\frac{\langle (|T|^2+w_q(T^2))x,x\rangle}{2}\right) &= f\left(\frac{1}{2}\langle (|T|^2+w_q(T^2))x,x\rangle\right)\\
&\leq \frac{1}{2}f\left(\langle |T|^2x,x\rangle\right)+\frac{1}{2}f\left(\langle w_q(T^2)x,x\rangle\right)\\
&\leq \frac{1}{2}f\left(\langle |T|^2x,x\rangle\right)+\frac{1}{2}f\left(w_q(T^2)\right).
\end{aligned}$$

By Lemma 3.22, we have

$$f\left(\langle |T|^2 x, x\rangle\right) \leq \langle f\left(|T|^2\right) x, x\rangle.$$

Thus

$$f\left(\frac{\langle (|T|^2 + w_q(T^2))x, x\rangle}{2}\right) \leq \frac{1}{2}\langle f\left(|T|^2\right) x, x\rangle + \frac{1}{2} f\left(w_q(T^2)\right).$$

By the same argument, we find

$$f\left(\frac{\langle (|T^*|^2 + w_q(T^2))y, y\rangle}{2}\right) \leq \frac{1}{2}\langle f\left(|T^*|^2\right) y, y\rangle + \frac{1}{2} f\left(w_q(T^2)\right).$$

Then

$$\begin{aligned} f(c(T)|\langle Tx, y\rangle|) &\leq \int_0^1 \frac{t}{2}\langle f\left(|T|^2\right) x, x\rangle + \frac{1-t}{2}\langle f\left(|T^*|^2\right) y, y\rangle + \frac{1}{2} f\left(w_q(T^2)\right) dt \\ &\leq \frac{1}{4}\langle f\left(|T|^2\right) x, x\rangle + \frac{1}{4}\langle f\left(|T^*|^2\right) y, y\rangle + \frac{1}{2} f\left(w_q(T^2)\right) \\ &\leq \frac{1}{4}\langle (f\left(|T|^2\right))^{\frac{1}{2}} x, (f\left(|T|^2\right))^{\frac{1}{2}} x\rangle + \frac{1}{4}\langle (f\left(|T^*|^2\right))^{\frac{1}{2}} y, (f\left(|T^*|^2\right))^{\frac{1}{2}} y\rangle \\ &+ \frac{1}{2} f\left(w_q(T^2)\right). \end{aligned}$$

Hence

$$f(c(T)|\langle Tx, y\rangle|) \leq \frac{1}{4}(\|(f\left(|T|^2\right))^{\frac{1}{2}} x\|^2 + \|(f\left(|T^*|^2\right))^{\frac{1}{2}} y\|^2) + \frac{1}{2} f\left(w_q(T^2)\right).$$

Then taking the supremum over all $(x, y) \in C_q$, we have

$$f(c(T) w_q(T)) \leq \frac{1}{4} N_q^2 \left((f(|T|^2))^{\frac{1}{2}}, (f(|T^*|^2))^{\frac{1}{2}}\right) + \frac{1}{2} f(w_q(T^2)).$$

□

Corollary 3.24. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0, 1]$. Then*

$$c(T) w_q(T) \leq \frac{1}{4} N_q^2 (|T|, |T^*|) + \frac{1}{2} w_q(T^2).$$

Proof. Applying Theorem 3.23 with $f(t) = t$ for all $t \in \mathbb{R}^+$, we obtain the desired inequality. □

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