Filomat 40:5 (2026), 1699–1711
https://doi.org/10.2298/FIL2605699T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conformable Laplace transform on time scales

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Abstract. In this paper, we define the conformable Laplace transform on time scales. Conditions for the absolute convergence, and some fundamental properties such as linearity, transform of conformable integral, shifting theorem, transform of conformable derivative are given. The concept of conformable convolution is discussed, and the convolution theorem is proved. At last a conformable dynamic equation is solved.

1. Introduction

Fractional calculus is an extension of classical calculus which deals with derivative and integration of non-integer order. The core concept of fractional calculus is summarized by two approaches, the first is Riemann-Liouville, which uses integrals, and the Gamma function, and the second is Grünwald-Letnikov, which is based on iterated derivatives and fractionalization through binomial coefficients. These traditional fractional derivatives are complex and lack some of the properties of classical derivatives, such as the product and chain rule. However, they do have good semigroup properties in some cases. In [19] Khalil R. et al. introduced a simpler and more intuitive derivative based on the basic limit definition of the derivative as a conformable fractional derivative. Consequently, in [2] Anderson et al. generalized the concept of the conformable differential operator, which shows that it is more appropriate to regard the conformable derivative as a concept in its own right, distinct from the fractional derivative theory.

While conducting research on conformable calculus, Abdeljawad T., in [1] introduced the conformable Laplace transform of order α, $0 < \alpha \leq 1$ for $f : [t_0, \infty) \to \mathbb{R}$ as

$$L_\alpha^{t_0}\{f(t)\}(s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)d\alpha(t, t_0). \tag{1}$$

Further, Silva F. S. et al. [20] established a sufficient condition to guarantee the reliability of solving the constant coefficient fractional differential equation by the conformable Laplace transform.

2020 *Mathematics Subject Classification*. Primary 26E70; Secondary 44A35; 26A33.
Keywords. time scales, Laplace transform, conformable dynamic equations
Received: 07 March 2025; Accepted: 21 December 2025
Communicated by Dragan S. Djordjević
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A time scale $\mathcal{T}$ is any non-empty closed subset of the real numbers, equipped with the subspace topology inherited from the standard topology of real numbers. The theory of time scales was introduced in 1988 through Hilger's Ph.D. thesis [18]. This theory unifies the definitions and results from discrete and continuous dynamical systems while extending these concepts to broader classes of dynamical systems. In [4, 5, 20, 28] a conformable fractional calculus on arbitrary time scales is developed as an extension of the classical conformable fractional calculus. Time scale studies mainly focus on dynamic equations. Therefore, it is important to develop integral transform methods to solve these problems. Thus far, the generalization of various integral transforms has been performed on time scales [3, 7, 11, 14, 21–26].

The motivation of this study is the generalization of the conformable Laplace transform for solving conformable dynamic equations on time scales. The paper is organized as follows. In Section 2, we recollect some basic concepts and notations of conformable fractional calculus on time scales. In Section 3, we generalize the conformable Laplace transform. In Section 4, some fundamental properties are given. In Section 5, we show applicability of the conformable Laplace transform for solving linear conformable fractional dynamic equation with constant coefficients.

2. Preliminaries

All results in this section are taken from [4–9, 12, 13, 15, 16, 27, 28]. Here, we assume that the time scale $\mathcal{T}$ under consideration is unbounded above and $t_0 \in \mathcal{T}$ is fixed. For $t \in \mathcal{T}$, the forward jump operator $\sigma : \mathcal{T} \to \mathcal{T}$ is given as $\sigma(t) := \inf\{s \in \mathcal{T} : s > t\}$ while the backward jump operator is $\rho : \mathcal{T} \to \mathcal{T}$ is $\rho(t) := \sup\{s \in \mathcal{T} : s < t\}$. Furthermore, the forward graininess function $\mu : \mathcal{T} \to [0, \infty)$ is given as $\mu(t) := \sigma(t) - t$.

Definition 2.1. *If a function $g : \mathcal{T} \to \mathbb{R}$ is continuous at the right dense points in $\mathcal{T}$, and has a finite limit at the left dense points in $\mathcal{T}$, then g is said to be rd-continuous. We denote the set of all rd-continuous functions by $\mathcal{C}_{\mathrm{rd}}(\mathcal{T}, \mathbb{R})$.*

Definition 2.2. *Assume $g : \mathcal{T} \to \mathbb{R}$, $t \in \mathcal{T}^\kappa$, and $\alpha \in (0, 1]$. For $t > 0$, if $\mathcal{T}_\alpha(g)(t)$ is the number (provided it exists) with the property that, given any $\epsilon > 0$, there exists a $\delta-$neighborhood $\mathcal{U} \subset \mathcal{T}$ of t,with $\delta > 0$, such that*

$$|[g(\sigma(t)) - g(s)]t^{1-\alpha} - \mathcal{T}_\alpha(g)(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in \mathcal{U}$. We call $\mathcal{T}_\alpha(g)(t)$ the conformable fractional derivative of g of order α at t. The conformable fractional derivative at 0 is given as $\mathcal{T}_\alpha(g)(0) = \lim_{t\to 0^+} \mathcal{T}_\alpha(g)(t)$.

Throughout the paper, we use the notation $f^\alpha(t) = \mathcal{T}_\alpha(g)(t)$.

Definition 2.3. *Assume $g : \mathcal{T} \to \mathbb{R}$ is a regulated function. Then the α-fractional integral of g, for $\alpha \in (0, 1]$, is given as,*

$$\mathcal{I}_\alpha(g)(t) = \int g(t)\Delta^\alpha t = \int g(t)t^{\alpha-1}\Delta t.$$

Definition 2.4. *A function $g : \mathcal{T} \to \mathbb{R}$ is said to be 'α-regressive' provided*

$$1 + \mu(t)g(t)t^{\alpha-1} \neq 0, \quad \textit{for all } \ t \in \mathcal{T}^\kappa,$$

holds. Similarly, $g : \mathcal{T} \to \mathbb{R}$ is said to be 'α-positively regressive' provided

$$1 + \mu(t)g(t)t^{\alpha-1} > 0, \quad \textit{for all } \ t \in \mathcal{T}^\kappa.$$

The set of all α-regressive and rd-continuous functions $g : \mathcal{T} \to \mathbb{R}$ is denoted by $\mathcal{R}^\alpha$.
$\mathcal{R}^{\alpha+}$ is subset of $\mathcal{R}^\alpha$ containing all α-positively regressive functions.

Definition 2.5. *In $\mathcal{R}^\alpha$ 'α-circle plus' addition $\oplus_\alpha$ is defined as below*

$$(g_1 \oplus_\alpha g_2)(t) = g_1(t) + g_2(t) + \mu(t)g_1(t)g_2(t)t^{\alpha-1}, \ \ \textit{for all } t \in \mathcal{T}^\kappa.$$

The set $\mathcal{R}^\alpha$ forms an abelian group under $\oplus_\alpha$.
For $g \in \mathcal{R}^\alpha$, the inverse of g is given as

$$\ominus_\alpha g = \frac{-g(t)}{1 + \mu(t)g(t)t^{\alpha-1}}, \ \ \textit{for all } t \in \mathcal{T}^\kappa.$$

Definition 2.6. *For $\mathcal{R}^\alpha$, 'α-circle minus' substraction $\ominus_\alpha$ is defined as*

$$(g_1 \ominus_\alpha g_2)(t) = \frac{g_1(t) - g_2(t)}{1 + \mu(t)g_2(t)t^{\alpha-1}}, \textit{for all } t \in \mathcal{T}^\kappa.$$

The following theorem, gives important algebraic properties of α-regressive functions.

Theorem 2.7. *Let $g_1, g_2 \in \mathcal{R}^\alpha$. Then*

(1) $(g_1 \ominus_\alpha g_1)(t) = 0$;

(2) $\ominus_\alpha(\ominus_\alpha g_1)(t) = g_1(t)$;

(3) $\ominus_\alpha(g_1 \ominus_\alpha g_2)(t) = (g_2 \ominus_\alpha g_1)(t)$;

(4) $\ominus_\alpha(g_1 \oplus_\alpha g_2)(t) = [(\ominus_\alpha g_1) \oplus_\alpha (\ominus_\alpha g_2)](t)$;

(5) $g_1(t) \oplus_\alpha \frac{g_2(t)}{1+\mu(t)g_1(t)t^{\alpha-1}} = g_1(t) + g_2(t)$.

Definition 2.8 (Generalized exponential function). *Let $g \in \mathcal{R}^\alpha$, then the generalized exponential function is defined by*

$$\mathbb{E}_g(t,s) = exp\Big(\int_s^t \xi_{\mu(\tau)}\Big(g(\tau)\tau^{\alpha-1}\Big)\Delta\tau\Big), \ \ \textit{for all } t,s \in \mathcal{T}. \tag{2}$$

Using the definition for the cylindrical transformation, [6, Definition 2.21] Equation (2) can be rewritten as

$$\mathbb{E}_g(t,s) = exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log(1 + \mu(\tau)g(\tau)\tau^{\alpha-1})\Delta\tau\Big), \ \ \text{for all } t,s \in \mathcal{T}. \tag{3}$$

Next theorem collects some important properties of the generalized exponential function.

Theorem 2.9. *If $g_1, g_2 \in \mathcal{R}^\alpha$, then for all $s,t \in \mathcal{T}$,*

(1) $\mathbb{E}_0(t,s) = 1$ *and* $\mathbb{E}_{g_1}(t,t) = 1$.

(2) $\mathbb{E}_{g_1}(t,s)\mathbb{E}_{g_1}(s,r) = \mathbb{E}_{g_1}(t,r)$.

(3) $\mathbb{E}_{g_1}(t,s) = \frac{1}{\mathbb{E}_{g_1}(s,t)} = \mathbb{E}_{\ominus_\alpha g_1}(s,t)$.

(4) $\mathbb{E}_{g_1}(t,s)\mathbb{E}_{g_2}(t,s) = \mathbb{E}_{g_1 \oplus_\alpha g_2}(t,s)$.

(5) $\frac{\mathbb{E}_{g_1}(t,s)}{\mathbb{E}_{g_2}(t,s)} = \mathbb{E}_{g_1 \ominus_\alpha g_2}(t,s)$.

(6) $\mathbb{E}_{g_1}(\sigma(t),s) = (1 + \mu(t)g_1(t)t^{\alpha-1})\mathbb{E}_{g_1}(t,s)$.

(7) $\mathbb{E}_{\ominus_\alpha g_1}(\sigma(t),s) = \frac{\mathbb{E}_{\ominus_\alpha g_1}(t,s)}{1+\mu(t)g_1(t)t^{\alpha-1}}$.

(8) $\mathbb{E}^{\Delta}_{g_1}(t,s) = g_1(t)\mathbb{E}_{g_1}(t,s)t^{\alpha-1}$.

(9) $\mathbb{E}^{(\alpha)}_{g_1}(t,s) = g_1(t)\mathbb{E}_{g_1}(t,s)$.

Definition 2.10. *For $h > 0$, we define the Hilger complex numbers, and the Hilger real axis as* $\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}$ *and,* $\mathbb{R}_h = \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and, } z > -\frac{1}{h} \right\}$ *respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$ and $\mathbb{R}_0 := \mathbb{R}$.*

Definition 2.11. *Let $h > 0$, and $z \in \mathbb{C}_h$. The Hilger real part of z is defined as*

$$Re_h(z) := \frac{|hz+1| - 1}{h}.$$

Definition 2.12. *For given time scale $\mathcal{T}$, the minimal graininess function $\mu_* : \mathcal{T} \to \mathbb{R}^+ \cup \{0\}$ is defined as*

$$\mu_*(t_0) := \inf_{\tau \in [t_0,\infty)_{\mathcal{T}}} \mu(\tau) \quad \text{for } t_0 \in \mathcal{T}.$$

Further, for $h \geq 0$ and $\theta \in \mathbb{R}$, $\mathbb{C}_h(\theta) := \{z \in \mathbb{C}_h : Re_h(z) > \theta\}$.

3. Main results

In this section, we present the concept of conformable Laplace transform on time scales. At first, we define the following.

Definition 3.1 (Conformable exponentially bounded function). *A function $g : \mathcal{T} \to \mathbb{R}$ is said to be conformable exponentially bounded, if there exists constants $\mathtt{M}, \mathtt{c} > 0$ such that*

$$|g(t)| \leq \mathtt{M}\mathbb{E}_{\mathtt{c}}(t, t_0).$$

Definition 3.2 (Conformable Laplace transform). *Let $g : \mathcal{T} \to \mathbb{C}$ is rd-continuous. Then, the conformable Laplace transform of order α with $\alpha \in (0,1]$ of g is defined by*

$$\mathfrak{L}_\alpha\{g(t)\}(z) = \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0) g(t) \Delta^\alpha t, \quad \text{for } z \in \mathcal{D}\{g\}, \tag{4}$$

where $\mathcal{D}\{g\}$ consists of all $t \in \mathcal{T}$ for which the improper integral exists, and for which $1 + \mu(t)zt^{\alpha-1} \neq 0$ for all $t \in \mathcal{T}$.

3.1. Asymptotic decay of the generalized exponential function

In this subsection, we see some useful results to prove the asymptotic decay property of the generalized exponential function.

Consider a function $p(\tau) : \mathbb{C} \to \mathbb{C}$ and a real constant λ. Let us set

$$\mathcal{M}_\lambda(t, t_0) = \int_{t_0}^{t} \frac{\tau^{\alpha-1}}{1 + \mu(\tau)\lambda} \Delta\tau \quad \text{for } t_0, t \in \mathcal{T} \text{ and } \lambda \in \mathcal{R}^\alpha(\mathcal{T}, \mathbb{R}). \tag{5}$$

Lemma 3.3. *For $\sup \mathcal{T} = \infty$. Let $t_0 \in \mathcal{T}$, and $\lambda \in \mathcal{R}^\alpha([t_0,\infty)_{\mathcal{T}}, \mathbb{R})$. Then, we have*

(i) $\mathcal{M}^{\Delta_1}_\lambda(t, t_0) > 0$ *for all $t \in (t_0, \infty)_{\mathcal{T}}$, where Δ_1 is delta derivative with respect to t.*

(ii) $\lim_{t\to\infty} \mathcal{M}_\lambda(t, t_0) = \infty$.

Proof. The proof follows similarly, to the proof of the [10, Lemma 3.1]. □

Lemma 3.4. *Let $t_0 \in \mathfrak{T}$ and $g \in \mathcal{R}^{\alpha+}([t_0,\infty)_{\mathcal{T}},\mathbb{R})$. Then,*

$$0 < \mathbb{E}_g(t,t_0) \leq exp\Big\{\int_{t_0}^{t} g(\tau)\tau^{\alpha-1}\Delta\tau\Big\}, \ \ \forall\, t \in [t_0,\infty)_{\mathcal{T}}.$$

Proof. We have,

$$\mathbb{E}_g(t,t_0) = exp\Big\{\int_{t_0}^{t} \xi_{\mu(\tau)}(g(\tau)\tau^{\alpha-1})\Delta\tau\Big\}, \ \ \forall\, t_0, t \in \mathfrak{T} \ \text{ with } t \geq t_0. \tag{6}$$

As, $1+\mu(\tau)g(\tau)\tau^{\alpha-1} > 0$, we have $Log(1+\mu(\tau)g(\tau)\tau^{\alpha-1}) \in \mathbb{R}$, and therefore, $\xi_{\mu(\tau)}(g(\tau)\tau^{\alpha-1}) \in \mathbb{R}$.
Now, if $\mu(\tau) = 0$, then we have

$$\xi_{\mu(\tau)}(g(\tau)\tau^{\alpha-1}) = g(\tau)\tau^{\alpha-1}.$$

Further, if $\mu(\tau) > 0$, then we have

$$\begin{aligned}
\xi_{\mu(\tau)}(g(\tau)\tau^{\alpha-1}) &= \frac{1}{\mu(\tau)}Log(1+\mu(\tau)g(\tau)\tau^{\alpha-1}) \\
&= \frac{1}{\mu(\tau)}\log(1+\mu(\tau)g(\tau)\tau^{\alpha-1}) \\
&= \frac{\mu(\tau)g(\tau)\tau^{\alpha-1} - [\mu(\tau)g(\tau)\tau^{\alpha-1} - \log(1+\mu(\tau)g(\tau)\tau^{\alpha-1})]}{\mu(\tau)} \\
&\leq g(\tau)\tau^{\alpha-1}.
\end{aligned}$$

Here we have used the property $log(1+a) \leq a$ for all $a \in (-1,1)$. Thus, $\xi_{\mu(\tau)}(g(\tau)\tau^{\alpha-1}) \leq g(\tau)\tau^{\alpha-1}$ for all $\tau \in [t_0,t)_{\mathcal{T}}$, which proves the claim by using this in 6. □

We define a function $\Phi_h(p(\tau),\lambda) : \mathbb{C}_h \times \mathbb{R}_h \to \mathbb{R}$ by

$$\Phi_h(p(\tau),\lambda) := \begin{cases} \frac{1}{h}\log\left|\frac{1+h\lambda}{1+hp(\tau)}\right| & \text{if } h > 0 \\ \lambda - Re(p(\tau)) & \text{if } h = 0. \end{cases} \tag{7}$$

Lemma 3.5. *Let $h \geq 1$, and $\lambda \in \mathbb{R}_h$. Then,*

(1) For all $p(\tau) \in \mathbb{C}$,

$$\Phi_h(p(\tau),\lambda) = \Phi_h(Re_h(p(\tau),\lambda)).$$

(2) Suppose $1+h\lambda > 0$ and $x_1, x_2 \in \mathbb{R}$, $1+hx_1 > 0$, $1+hx_2 > 0$ if $x_1 < x_2$, then

$$\Phi_h(x_1,\lambda) > \Phi_h(x_2,\lambda).$$

Proof. The proof follows similarly, to the proof of [10, Lemma 3.3]. □

Theorem 3.6 (Decay of the generalized exponential function). *Let $t_0 \in \mathfrak{T}$ and $\lambda \in \mathcal{R}^{\alpha+}([t_0,\infty)_{\mathcal{T}},\mathbb{R})$. Then for any $p(\tau) \in \mathbb{C}_{\mu_*(t_0)}(\lambda)$, we have the following properties.*

(1) $|\mathbb{E}_{\lambda\ominus_\alpha p(\tau)}(t,t_0)| \leq \mathbb{E}_{\lambda\ominus_\alpha Re_{\mu_*(t_0)}(p(\tau))}(t,t_0)$ for all $t \in [t_0,\infty)_{\mathcal{T}}$.

(2) $\lim\limits_{t\to\infty} \mathbb{E}_{\lambda\ominus_\alpha Re_{\mu_*(t_0)}(p(\tau))}(t,t_0) = 0$.

(3) $\lim\limits_{t\to\infty} \mathbb{E}_{\lambda\ominus_\alpha p(\tau)}(t,t_0) = 0$.

Proof. Consider,

$$\phi_h(z) := \begin{cases} \frac{1}{h}\log|1+hz| & \text{if } h>0, \\ Re(z) & \text{if } h=0. \end{cases} \tag{8}$$

for $z \in \mathbb{C}_h$, then from Equation (3) for any $f \in \mathcal{R}^{\alpha}(\mathcal{T},\mathbb{C})$ we get

$$|\mathbb{E}_f(t,t_0)| = exp\Big\{\int_{t_0}^{t}\phi_{\mu(\tau)}(f(\tau)\tau^{\alpha-1})\Delta\tau\Big\} \quad \text{for all} \quad t \in [t_0,\infty)_{\mathcal{T}}.$$

Therefore,

$$|\mathbb{E}_{\lambda\ominus_\alpha p(\tau)}(t,t_0)| = exp\Big\{\int_{t_0}^{t}\phi_{\mu(\tau)}[(\lambda\ominus_\alpha p(\tau))\tau^{\alpha-1}]\Delta\tau\Big\} \quad \text{for all} \quad t \in [t_0,\infty)_{\mathcal{T}}.$$

We know that, for all $\tau \in [t_0,t)_{\mathcal{T}}$,

$$\lambda\ominus_\alpha p(\tau) = \frac{\lambda - p(\tau)}{1+\mu(\tau)p(\tau)\tau^{\alpha-1}}.$$

Then from Equations (7) and (8), we get

$$\begin{aligned}\phi_{\mu(\tau)}[(\lambda\ominus_\alpha p(\tau))\tau^{\alpha-1}] &= \begin{cases} \frac{1}{\mu(\tau)}\log|1+\mu(\tau)(\lambda\ominus_\alpha p(\tau))\tau^{\alpha-1}| & \text{if } \mu(\tau)>0, \\ Re((\lambda-p(\tau))\tau^{\alpha-1}) & \text{if } \mu(\tau)=0. \end{cases} \\ &= \begin{cases} \frac{1}{\mu(\tau)}\log\left|\frac{1+\mu(\tau)\lambda\tau^{\alpha-1}}{1+\mu(\tau)p(\tau)\tau^{\alpha-1}}\right| & \text{if } \mu(\tau)>0 \\ Re((\lambda-p(\tau))\tau^{\alpha-1}) & \text{if } \mu(\tau)=0. \end{cases} \\ &= \Phi_{\mu(\tau)}(p(\tau)\tau^{\alpha-1},\lambda\tau^{\alpha-1}).\end{aligned}$$

Therefore,

$$|\mathbb{E}_{\lambda\ominus_\alpha p(\tau)}(t,t_0)| = exp\Big\{\int_{t_0}^{t}\Phi_{\mu(\tau)}(p(\tau)\tau^{\alpha-1},\lambda\tau^{\alpha-1})\Delta\tau\Big\} \text{ for all } t \in [t_0,\infty)_{\mathcal{T}}. \tag{9}$$

From Equation (9) and Lemma 3.5, we get for all $\lambda \in \mathcal{R}_c^{\alpha+}([t_0,\infty)_{\mathcal{T}},\mathbb{R})$ and $p(\tau) \in \mathbb{C}_{\mu_*(t_0)}(\lambda)$,

$$\begin{aligned}|\mathbb{E}_{\lambda\ominus_\alpha p(\tau)}(t,t_0)| &= exp\Big\{\int_{t_0}^{t}\Phi_{\mu(\tau)}(Re_{\mu(\tau)}(p(\tau))\tau^{\alpha-1},\lambda\tau^{\alpha-1})\Delta\tau\Big\} \\ &\le exp\Big\{\int_{t_0}^{t}\Phi_{\mu(\tau)}(Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1},\lambda\tau^{\alpha-1})\Delta\tau\Big\} \\ &= \mathbb{E}_{\lambda\ominus_\alpha Re_{\mu_*(t_0)}(p(\tau))}(t,t_0)\end{aligned}$$

for all $t \in [t_0,\infty)_{\mathcal{T}}$. Thus, statement (1) is proved.

Now since,

$$\begin{aligned}\lambda\ominus_\alpha Re_{\mu_*(t_0)}(p(\tau)) &= \frac{\lambda - Re_{\mu_*(t_0)}(p(\tau))}{1+\mu(\tau)Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1}} \\ &= \frac{\lambda - Re_{\mu_*(t_0)}(p(\tau))}{1+\mu(\tau)\lambda\tau^{\alpha-1}+\mu(\tau)[Re_{\mu_*(t_0)}(p(\tau)-\lambda)]\tau^{\alpha-1}},\end{aligned}$$

and

$$1+\mu(\tau)[\lambda \ominus_\alpha Re_{\mu_*(t_0)}(p(\tau))] = \frac{1+\mu(\tau)\lambda}{1+\mu(\tau)Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1}}$$

$$= \frac{1+\mu(\tau)\lambda}{1+\mu(\tau)\lambda\tau^{\alpha-1}+\mu(\tau)[Re_{\mu_*(t_0)}(p(\tau))-\lambda]\tau^{\alpha-1}},$$

we have for $\tau \in [t_0,\infty)_{\mathcal{T}}$

$$\lambda \ominus_\alpha Re_{\mu_*(t_0)}(p(\tau)) < 0 \text{ and } \lambda \ominus_\alpha Re_{\mu_*(t_0)}(p(\tau)) \in \mathcal{R}^{\alpha+}[(t_0,\infty)_{\mathcal{T}},\mathbb{R}).$$

Therefore, an application of Lemma 3.4 gives

$$\begin{aligned}\mathbb{E}_{\lambda\ominus_\alpha Re_{\mu_*(t_0)}(p(\tau))}(t,t_0) &\le exp\Big\{\int_{t_0}^{t}(\lambda \ominus_\alpha Re_{\mu_*(t_0)}(p(\tau)))\tau^{\alpha-1}\Delta\tau\Big\}\\ &= exp\Big\{[\lambda - Re_{\mu_*(t_0)}(p(\tau))]\int_{t_0}^{t}\frac{\tau^{\alpha-1}}{1+\mu(\tau)Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1}}\Delta\tau\Big\}\\ &= exp\Big\{-[Re_{\mu_*(t_0)}(p(\tau))-\lambda]\,\mathcal{M}_{Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1}}(t,t_0)\Big\}\end{aligned} \tag{10}$$

for all $t \in [t_0,\infty)_{\mathcal{T}}$.
Furthermore,

$$1+\mu(\tau)Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1} = 1+\mu(\tau)\lambda\tau^{\alpha-1}+\mu(\tau)[Re_{\mu_*(t_0)}(p(\tau))-\lambda]\tau^{\alpha-1},$$

thus, $p(\tau) \in \mathbb{C}_{\mu_*(t_0)}(\lambda)$ and $Re_{\mu_*(t_0)}(p(\tau))\tau^{\alpha-1} \in \mathcal{R}^{\alpha+}([t_0,\infty)_{\mathcal{T}},\mathbb{R})$. Therefore, from part (*ii*) of Lemma 3.3, and from (10), we get $\lim_{t\to\infty}\mathbb{E}_{\lambda\ominus_\alpha Re_{\mu_*(t_0)}}(p(\tau)) = 0$. This proves (2).
Finally, part (3) follows from (1) and (2), and thus, the proof is completed. □

Lemma 3.7. *Let $t_0 \in \mathcal{T}$ and $g \in \mathcal{C}_{rd}([t_0,\infty)_{\mathcal{T}},\mathbb{C})$ is conformable exponential of order $\mathtt{c}$. Then,*

$$\lim_{t\to\infty} g(t)\mathbb{E}_{\ominus_\alpha z}(t,t_0) = 0, \tag{11}$$

where $z \in \mathbb{C}_{\mu_*(t_0)}(\mathtt{c})$.

Proof. We have,

$$\begin{aligned}|g(t)\mathbb{E}_{\ominus_\alpha z}(t,t_0)| &\le \mathtt{M}\mathbb{E}_{\mathtt{c}}(t,t_0)|\mathbb{E}_{\ominus_\alpha}(t,t_0)|\\ &= \mathtt{M}|\mathbb{E}_{\mathtt{c}\ominus_\alpha z}(t,t_0)|\end{aligned} \tag{12}$$

for all $t \in [t_0,\infty)_{\mathcal{T}}$ for some $\mathtt{M} > 0$. Then by Theorem 3.6 (3), and letting $t \to \infty$ on both sides of (12), we get (11). □

3.2. Convergence of the conformable Laplace transform.

In the following theorem, we give conditions for the absolute convergence of the conformable Laplace transform.

Theorem 3.8. *Let $g : \mathcal{T} \to \mathbb{C}$ is rd-continuous function of exponential order $\mathtt{c}$. Then, the conformable Laplace transform $\mathfrak{L}_\alpha\{g(t)\}(z)$ of $g(t)$ exists for all $z \in \mathbb{C}_{\mu_*(t_0)}(\mathtt{c})$ and converges absolutely.*

Proof. We have,

$$
\begin{aligned}
\Big|\int_{t_0}^{t}\mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g(t)\Delta^\alpha t\Big| &\le \int_{t_0}^{t}\Big|\mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g(t)\Big|\Delta^\alpha t\\
&\le \mathtt{M}\int_{t_0}^{t}\left|\frac{\mathbb{E}_{\ominus_\alpha z}(t,t_0)}{1+\mu(t)zt^{\alpha-1}}\mathbb{E}_{\mathtt{c}}(t,t_0)\right|\Delta^\alpha t\\
&= \mathtt{M}\int_{t_0}^{t}\left|\frac{\mathbb{E}_{\mathtt{c}\ominus_\alpha z}(t,t_0)}{1+\mu(t)zt^{\alpha-1}}\right|\Delta^\alpha t\\
&\le \mathtt{M}\int_{t_0}^{t}\frac{\mathbb{E}_{\mathtt{c}\ominus_\alpha Re_{\mu_*(t_0)}(z)}(t,t_0)}{1+\mu(t)Re_{\mu_*(t_0)}(z)t^{\alpha-1}}\Delta^\alpha t\\
&= \frac{\mathtt{M}}{[\mathtt{c}-Re_{\mu_*(t_0)}(z)]}\int_{t_0}^{t}\frac{[\mathtt{c}-Re_{\mu_*(t_0)}(z)]}{1+\mu(t)Re_{\mu_*(t_0)}(z)t^{\alpha-1}}\mathbb{E}_{\mathtt{c}\ominus_\alpha Re_{\mu_*(t_0)}(t,t_0)}\Delta^\alpha t\\
&= \frac{\mathtt{M}}{[\mathtt{c}-Re_{\mu_*(t_0)}(z)]}\int_{t_0}^{t}[\mathtt{c}\ominus_\alpha Re_{\mu_*(t_0)}(z)]\mathbb{E}_{\mathtt{c}\ominus_\alpha Re_{\mu_*(t_0)}(z)}(t,t_0)\Delta^\alpha t\\
&= \frac{\mathtt{M}}{[\mathtt{c}-Re_{\mu_*(t_0)}(z)]}\int_{t_0}^{t}\mathbb{E}^{(\alpha)}_{\mathtt{c}\ominus_\alpha Re_{\mu_*}(z)}(t,t_0)\Delta^\alpha t\\
&= \frac{\mathtt{M}}{[\mathtt{c}-Re_{\mu_*(t_0)}(z)]}\Big[\mathbb{E}^{(\alpha)}_{\mathtt{c}\ominus_\alpha Re_{\mu_*(t_0)}(z)}(t,t_0)-1\Big].
\end{aligned}
$$

Taking $t \to \infty$ and using Theorem 3.6, we get

$$
\begin{aligned}
|\mathfrak{L}_\alpha\{g(t)\}(z)| &= \left|\int_{t_0}^{t}\mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g(t)\Delta^\alpha t\right|\\
&\le \frac{\mathcal{M}}{Re_{\mu_*(t_0)}(z)-\mathtt{c}}
\end{aligned}
$$

for all $z \in \mathbb{C}_{\mu_*(t_0)}(\mathtt{c})$. □

3.3. *Conformable Laplace transform of some elementary functions*

The conformable Laplace transform of some elementary functions found using Definition 3.2 are given in the Table 1.

Table 1: The conformable Laplace transform of some elementary functions

$g(t)$	$\mathfrak{L}_\alpha\{g(t)\}(z)$
1	$\frac{1}{z}$
$\mathbb{E}_a(t,t_0)$	$\frac{1}{z-a}$
$\sin_a(t,t_0)$	$\frac{a}{z^2+a^2}$
$\cos_a(t,t_0)$	$\frac{z}{z^2+a^2}$
$\sinh_a(t,t_0)$	$\frac{a}{z^2-a^2}$
$\cosh_a(t,t_0)$	$\frac{z}{z^2-a^2}$

4. Fundamental properties

In this section, we discuss some elementary properties of the conformable Laplace transform.

Theorem 4.1 (Linearity). *Let, $g_1, g_2 : \mathcal{T} \to \mathbb{C}$ are rd-continuous functions with conformable Laplace transforms $\mathfrak{L}_\alpha\{g_1(t)\}(z)$ for $z \in \mathcal{D}\{g_1\}$ and $\mathfrak{L}_\alpha\{g_2(t)\}(z)$ for $z \in \mathcal{D}\{g_2\}$ respectively. Then, for constants a_1 and a_2*

$$\mathfrak{L}_\alpha\{a_1 g_1(t) + a_2 g_2(t)\}(z) = a_1 \mathfrak{L}_\alpha\{g_1(t)\}(z) + a_2 \mathfrak{L}_\alpha\{g_2(t)\}(z)$$

for all $z \in \mathcal{D}\{g_1\} \cap \mathcal{D}\{g_2\}$.

Proof. The proof follows directly from Definition 3.2. □

Theorem 4.2 (Shifting property). *Let $\omega \in \mathcal{T},\ \omega > 0$, the unit step function $\mathcal{H}_\omega(t)$ is defined as*

$$\mathcal{H}_\omega(t) = \begin{cases} 0 & \text{if } t \in \mathcal{T} \cap (-\infty, \omega) \\ 1 & \text{if } t \in \mathcal{T} \cap [\omega, \infty), \end{cases}$$

and $g : \mathcal{T} \to \mathbb{C}$ is rd-continuous, then $\mathfrak{L}_\alpha\{\mathcal{H}_\omega(t) g(t)\}(z) = \mathbb{E}_{\ominus_\alpha z}(\omega, t_0) \mathfrak{L}_\alpha\{g(t)\}(z)$.

Proof.

$$\begin{aligned}
\mathfrak{L}_\alpha\{g(t)\}(z) &= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0) \mathcal{H}_\omega(t) g(t) \Delta^\alpha t \\
&= \int_{t_0}^{\infty} \frac{\mathbb{E}_{\ominus_\alpha z}(t, t_0)}{1 + \mu(t) z t^{\alpha-1}} \mathcal{H}_\omega(t) g(t) \Delta^\alpha t \\
&= \int_{\omega}^{\infty} \frac{\mathbb{E}_{\ominus_\alpha z}(t, t_0)}{1 + \mu(t) z t^{\alpha-1}} g(t) \Delta^\alpha t \\
&= \int_{\omega}^{\infty} \frac{\mathbb{E}_{\ominus_\alpha z}(t, \omega) \mathbb{E}_{\ominus_\alpha z}(\omega, t_0)}{1 + \mu(t) z t^{\alpha-1}} g(t) \Delta^\alpha t \\
&= \mathbb{E}_{\ominus_\alpha z}(\omega, t_0) \int_{\omega}^{\infty} \frac{\mathbb{E}_{\ominus_\alpha z}(t, \omega)}{1 + \mu(t) z t^{\alpha-1}} \Delta^\alpha t \\
&= \mathbb{E}_{\ominus_\alpha z}(\omega, t_0) \int_{\omega}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t), \omega) g(t) \Delta^\alpha t \\
&= \mathbb{E}_{\ominus_\alpha z}(\omega, t_0) \mathfrak{L}_\alpha\{g(t)\}(z).
\end{aligned}$$

□

Theorem 4.3 (Transform of integrals). *If $g : \mathcal{T} \to \mathbb{C}$ is regulated, then*

$$\mathfrak{L}_\alpha\Big\{ \int_{t_0}^{t} g(\gamma) \Delta^\alpha \gamma \Big\}(z) = \frac{1}{z} \mathfrak{L}_\alpha\{g(t)\}(z),$$

for all $z \in \mathcal{R}^\alpha(\mathcal{T}, \mathbb{R})$, provided $\lim_{t \to \infty} \mathbb{E}_{\ominus_\alpha z}(t, t_0) \int_{t_0}^{t} g(\gamma) \Delta^\alpha \gamma = 0$.

Proof. Consider, $G(t) = \int_{t_0}^{t} g(\gamma) \Delta^\alpha \gamma$ for $t, t_0 \in \mathcal{T}$. So, $G^{(\alpha)}(t) = g(t)$ and $G(t_0) = 0$.
Then,

$$\begin{aligned}
\mathfrak{L}_\alpha\{G(t)\}(z) &= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0) G(t) \Delta^\alpha t \\
&= \int_{t_0}^{t} \frac{\mathbb{E}_{\ominus_\alpha z}(t, t_0)}{1 + \mu(t) z t^{\alpha-1}} G(t) \Delta^\alpha t \\
&= -\frac{1}{z} \int_{t_0}^{\infty} \frac{-z}{1 + \mu(t) z t^{\alpha-1}} \mathbb{E}_{\ominus_\alpha}(t, t_0) G(t) \Delta^\alpha t
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{z}\int_{t_0}^{\infty} \ominus_\alpha z\, \mathbb{E}_{\ominus_\alpha z}(t,t_0)G(t)\Delta^\alpha t\\
&= -\frac{1}{z}\int_{t_0}^{\infty} \mathbb{E}^{\alpha}_{\ominus_\alpha z}(t,t_0)G(t)\Delta^\alpha t\\
&= -\frac{1}{z}\int_{t_0}^{\infty} \Big[\Big(\mathbb{E}_{\ominus_\alpha z}(t,t_0)G(t)\Big)^{(\alpha)} - \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)G^{(\alpha)}(t)\Big]\Delta^\alpha t\\
&= -\frac{1}{z}\int_{t_0}^{\infty} \Big[-G(t_0) - \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)\, G^{\alpha}(t)\Delta^\alpha t\Big]\\
&= \frac{1}{z}\int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g(t)\Delta^\alpha t\\
&= \frac{1}{z}\mathfrak{L}_\alpha\{g(t)\}(z).
\end{aligned}
$$

□

Theorem 4.4 (Transform of conformable derivative). *Assume that $g : \mathcal{T} \to \mathbb{C}$ is rd-continuous such that $g^{(\alpha)} : \mathcal{T} \to \mathbb{C}$ is rd-continuous, and $\lim\limits_{t\to\infty} \mathbb{E}_{\ominus_\alpha z}(t,t_0)g(t) = 0$, then*

(1) $\mathfrak{L}_\alpha\{g^{(\alpha)}\}(z) = z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0)$.

(2) $\mathfrak{L}_\alpha\{g^{(\alpha)^2}(t)\}(z) = z^2\mathfrak{L}_\alpha\{g(t)\}(z) - zg(t_0) - g^{(\alpha)}(t_0)$.

Proof.

$$
\begin{aligned}
(1)\ \mathfrak{L}_\alpha\{g^{(\alpha)}(t)\}(z) &= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g^{(\alpha)}(t)\Delta^\alpha t\\
&= \int_{t_0}^{\infty} \Big[\Big(\mathbb{E}_{\ominus_\alpha z}(t,t_0)g(t)\Big)^{(\alpha)} - g(t)\mathbb{E}^{(\alpha)}_{\ominus_\alpha z}(t,t_0)\Big]\Delta^\alpha t\\
&= \Big[\mathbb{E}_{\ominus_\alpha z}(t,t_0)g(t)\Big]_{t=t_0}^{t\to\infty} - \int_{t_0}^{\infty} g(t)\mathbb{E}^{(\alpha)}_{\ominus_\alpha z}(t,t_0)\Delta^\alpha t\\
&= -g(t_0) - \int_{t_0}^{\infty} g(t)\ominus_\alpha z\, \mathbb{E}_{\ominus_\alpha z}(t,t_0)\Delta^\alpha t\\
&= -g(t_0) - \int_{t_0}^{\infty} g(t)\Big[\frac{-z}{1+\mu(t)zt^{\alpha-1}}\mathbb{E}_{\ominus_\alpha z}(t,t_0)\Big]\Delta^\alpha t\\
&= -g(t_0) + z\int_{t_0}^{\infty} g(t)\frac{\mathbb{E}_{\ominus_\alpha z}(t,t_0)}{1+\mu(t)zt^{\alpha-1}}\Delta^\alpha t\\
&= -g(t_0) + z\int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g(t)\Delta^\alpha t\\
&= z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0).
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \mathfrak{L}_\alpha\{g^{(\alpha)^2}(t)\}(z) &= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g^{(\alpha)^2}(t)\Delta^\alpha t\\
&= \int_{t_0}^{\infty} \Big[\Big(\mathbb{E}_{\ominus_\alpha z}(t,t_0)g^{(\alpha)}(t)\Big)^{(\alpha)} - g^{(\alpha)}(t)\, \mathbb{E}^{(\alpha)}_{\ominus_\alpha z}(t,t_0)\Big]\Delta^\alpha t
\end{aligned}
$$

$$= \Big[\mathbb{E}_{\ominus_\alpha z}(t,t_0)g^{\alpha}(t)\Big]_{t=t_0}^{t\to\infty} - \int_{t_0}^{\infty} g^{(\alpha)}(t)\ \ominus_\alpha z\ \mathbb{E}^{(\alpha)}_{\ominus_\alpha z}(t,t_0)\Delta^\alpha t$$
$$= -g^{(\alpha)}(t_0) - \int_{t_0}^{\infty} g^{(\alpha)}(t)\Big(\frac{-z}{1+\mu(t)zt^{\alpha-1}}\Big)E^{(\alpha)}_{\ominus_\alpha z}(t,t_0)\Delta^\alpha t$$
$$= -g^{(\alpha)}(t_0) + z\int_{t_0}^{\infty} g^{(\alpha)}(t)\frac{\mathbb{E}^{(\alpha)}_{\ominus_\alpha z}(t,t_0)}{1+\mu(t)zt^{\alpha-1}}\Delta^\alpha t$$
$$= -g^{(\alpha)}(t_0) + z\int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)g^{\alpha}(t)\Delta^\alpha t$$
$$= -g^{(\alpha)}(t_0) + z\mathfrak{L}_\alpha\{g^{(\alpha)}(t)\}(z)$$

$$= -g^{(\alpha)}(t_0) + z\Big[z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0)\Big]$$
$$= z^2\mathfrak{L}_\alpha\{g(t)\}(z) - zg(t_0) - g^{(\alpha)}(t_0).$$

More generally, we have

$$\mathfrak{L}_\alpha\{g^{(\alpha)^n}\}(z) = z^n\mathfrak{L}_\alpha\{g(t)\}(z) - \sum_{j=0}^{n-1} z^{n-(j+1)}g^{(\alpha)^j}(t_0).$$

□

Theorem 4.5 (Initial and final values). *Let $g:\mathcal{T}\to\mathbb{C}$ have the conformable Laplace transform $\mathfrak{L}_\alpha\{g(t)\}(z)$. Then $g(t_0) = \lim_{z\to\infty} z\mathfrak{L}_\alpha\{g(t)\}(z)$, and $\lim_{t\to\infty} g(t) = \lim_{z\to 0} z\mathfrak{L}_\alpha\{g(t)\}(z)$ with $\lim_{t\to\infty}\mathbb{E}_{\ominus_\alpha z}(t,t_0)g(t) = 0$, provided the limit exists.*

Proof. We have

$$\mathfrak{L}_\alpha\{g^{(\alpha)}(t)\}(z) = z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0)$$

Now, as $z\to\infty$ on both sides, we get

$$\lim_{z\to\infty}\mathfrak{L}_\alpha\{g^{(\alpha)}\}(z) = \lim_{z\to\infty} z\mathfrak{L}_\alpha\{g(t)\}(z) - \lim_{z\to\infty} g(t_0).$$

Thus, $\lim_{z\to\infty} z\mathfrak{L}_\alpha\{g(t)\}(z) = g(t_0)$.

Again, for $z\to 0$, we get

$$\lim_{z\to 0}\mathfrak{L}_\alpha\{g^{\Delta}(t)\}(z) = \lim_{z\to 0} z\mathfrak{L}_\alpha\{g(t)\}(z) - \lim_{z\to 0} g(t_0);$$
$$\lim_{z\to 0}\int_{t_0}^{\infty} g^{\Delta}(t)\mathbb{E}_{\ominus_\alpha z}(\sigma(t),t_0)\Delta^\alpha t = \lim_{z\to 0} z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0);$$
$$\int_{t_0}^{\infty} g^{\Delta}(t)\Delta^\alpha t = \lim_{z\to 0} z\mathfrak{L}_\alpha\{g(t)\}(z) - g(t_0);$$
$$\lim_{t\to\infty} g(t) = \lim_{z\to 0} z\mathfrak{L}_\alpha\{g(t)\}(z).$$

□

4.1. Convolution

In this section, we discuss the concept of conformable convolution on time scales, and prove the convolution theorem for our proposed transform.

Definition 4.6 (Conformable convolution). *Let $g : \mathfrak{T} \to \mathbb{C}$ is rd-continuous function of t^α, and $h : \mathfrak{T} \to \mathbb{C}$ is piecewise rd-continuous and conformable exponential, then the conformable convolution of g and h denoted by $(g *_\alpha h)(t)$ is given by,*

$$\Big(g *_\alpha h\Big)(t) = \int_{t_0}^{t} g(t^\alpha, \sigma(\tau^\alpha))h(\tau)\Delta^\alpha\tau$$

Theorem 4.7 (Convolution). *Let $g : \mathfrak{T} \to \mathbb{C}$ is rd-continuous function of t^α, and $h : \mathfrak{T} \to \mathbb{C}$ is piecewise rd-continuous function having conformable Laplace transforms $\mathfrak{L}_\alpha\{g(t)\}(z)$ and $\mathfrak{L}_\alpha\{h(t)\}(z)$ respectively. Then,*

$$\mathfrak{L}_\alpha\{\Big(g *_\alpha h\Big)(t)\}(z) = \mathfrak{L}_\alpha\{g(t^\alpha)\}(z) \cdot \mathfrak{L}_\alpha\{h(t)\}(z)$$

Proof.

$$\begin{aligned}
\mathfrak{L}_\alpha\{\Big(g *_\alpha h\Big)(t)\}(z) &= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0)\Big[\Big(g *_\alpha h\Big)(t)\Big]\Delta^\alpha t \\
&= \int_{t_0}^{\infty} \left[\int_{t_0}^{t} g(t^\alpha, \sigma(\tau^\alpha))h(\tau)\Delta^\alpha\tau\right]\mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0)\Delta^\alpha t \\
&= \int_{t_0}^{\infty} h(\tau)\left[\int_{\sigma(\tau)}^{\infty} g(t^\alpha, \sigma(\tau^\alpha))\mathbb{E}_{\ominus_\alpha z}(\sigma(t), t_0)\Delta^\alpha t\right]\Delta^\alpha\tau \\
&= \int_{t_0}^{\infty} h(\tau)\mathfrak{L}_\alpha\Big\{\mathcal{H}_{\sigma(\tau)}(t)g(t^\alpha, \sigma(\tau^\alpha))\Big\}(z)\Delta^\alpha\tau \\
&= \int_{t_0}^{\infty} h(\tau)\Big[\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau), t_0)\mathfrak{L}_\alpha\{g(t^\alpha)\}(z)\Big]\Delta^\alpha\tau \\
&= \int_{t_0}^{\infty} \mathbb{E}_{\ominus_\alpha z}(\sigma(\tau), t_0)h(\tau)\Delta^\alpha\tau \ \mathfrak{L}_\alpha\{g(t^\alpha)\}(z) \\
&= \mathfrak{L}_\alpha\{g(t^\alpha)\}(z) \cdot \mathfrak{L}_\alpha\{h(t)\}(z).
\end{aligned}$$

□

5. Application

In this section, using conformable Laplace transform, we solve linear conformable dynamic equation.

Example 5.1. *Solve the following conformable dynamic equation.*

$$y^{(\alpha)}(t) = \lambda y(t) \tag{13}$$

with the initial condition $y(0) = y_0$.

Applying the conformable Laplace transform on both sides, and using Theorem 4.4, we get

$$\mathfrak{L}_\alpha\{y^{(\alpha)}(t)\}(z) = \mathfrak{L}_\alpha\{\lambda y(t)\}(z);$$

$$z\mathfrak{L}_\alpha\{y(t)\}(z) - y(0) = \lambda\mathfrak{L}_\alpha\{y(t)\}(z);$$

$$\mathfrak{L}_\alpha\{y(t)\}(z)(z - \lambda) = y(0);$$

$$\mathfrak{L}_\alpha\{y(t)\}(z) = y_0\frac{1}{z - \lambda}.$$

Applying the inverse conformable Laplace transform on both sides of the above equation, and using results from Subsection 3.3, we get

$$y(t) = y_0 \mathbb{E}_\lambda(t, 0).$$

Note that, for $\mathcal{T} = \mathbb{R}$, Equation (13) is linear conformable fractional differential equation $\mathcal{T}_\alpha\{y\}(t) = \lambda y(t)$ with the initial condition $y(0) = y_0$, and its solution is $y(t) = y_0 e^{\lambda \frac{t^\alpha}{\alpha}}$.

Conclusion

In this study, we generalized the conformable Laplace transform on time scales. The asymptotic decay properties of the generalized exponential function on time scales are discussed, and using them the conditions for the absolute convergence of the transform are given. Together with some fundamental properties, the concept of conformable convolution and the convolution theorem are given. The applicability of our proposed transform is shown by solving a linear conformable dynamic equation.

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