Filomat 40:5 (2026), 1713–1726
https://doi.org/10.2298/FIL2605713W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Chover's law of the iterated logarithm for weighted sums under sub-linear expectations

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Abstract. In this paper, the general Chover's law of the iterated logarithm (LIL) is established for weighted sums of a sequence of random variables under a sub-linear expectation space. Our results are very extensive versions which contain Chover's LIL for weighted sums of partial sums and moving sums and so on. As applications, several results on Chover's LIL for weighted sums of extended independence and identically distributed random variables have been generalized to the sub-linear expectation space context.

1. Introduction

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables having symmetric stable distribution with exponent $\alpha, 0 < \alpha < 2$, i.e.,

$$P(|X| > x) = \frac{c(x)l(x)}{x^\alpha} \quad \text{for} \quad x > 0,$$

where, $c(x) \geq 0$, $\lim\limits_{x\to\infty} c(x) = c > 0$, and $l(x) \geq 0$ is a slowly varying function.

Chover (1966 [4]) established

$$\limsup_{n\to\infty} \left|\frac{S_n}{n^{1/\alpha}}\right|^{\frac{1}{\log\log n}} = \mathrm{e}^{\frac{1}{\alpha}} \qquad \text{a.s..}$$

It is refereed as the Chover's law of the iterated logarithm (LIL). Many probability and statistics scholars have studied and obtained various versions of the Chover's LIL. Wichura (1974 [16]), Vasudeva (1984 [14]), Qi and Cheng (1996 [11]) obtained the Chover's LIL for partial sums of asymmetric stable distribution. Lu and Qi (2006 [7]), and Wang (2014 [15]) obtained respectively the Chover's LIL for trimmed sums and operator stable Lévy processes. Some Chover's LIL for weighted sums were obtained by Chen (2002 [2]), Peng and Qi (2003 [8]), Chen and Qi (2006 [3]), Cai (2007 [1]), Trapani (2014 [13]). Further, Peng and Qi

2020 *Mathematics Subject Classification*. Primary 60F15.
Keywords. Chover's law of the iterated logarithm; sub-linear expectation; weighted sums; extended independence.
Received: 30 December 2020; Accepted: 30 December 2025
Communicated by Dragan S. Djordjević
Supported by the National Natural Science Foundation of China (12561025).
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(2003 [8]), Chen and Qi (2006 [3]), Wu and Jiang (2010 [17]) obtained the Chover's laws of the k-iterated logarithm.

Recently, Wu and Jiang (2018 [18]) studied and extended the the Chover's laws of the k-iterated logarithm from sequences of probability space to sequences of the sub-linear expectations. In this paper, on the basis of Wu and Jiang (2018 [18]), we further study that the Chover's laws of the k-iterated logarithm for weighted sums also holds in sub-linear expected space.

In order to solve the problem of modeling in uncertainty problems, Peng (2006 [9]) proposed a sub-linear expectation space, which is a natural extension of the traditional probability space. Because the sub-linear expectation provides a very flexible framework for modeling sub-linear probability problems, it has been developed rapidly in recent ten years. Peng (2006 [9], 2008 [10]) constructed the basic framework, basic properties and central limit theorem, Zhang (2016 [20], [21]) established some inequalities about partial sums including the exponential inequalities, Rosenthal's inequalities, and Kolmogorov's strong law of larger numbers, Hartman-Wintner's law of iterated logarithm, Hu (2016 [6]) and Wu and Jiang (2018 [18]) further studied strong law of larger numbers in general form, Wu and Jiang (2018 [18]), Wu and Lu (2020 [19]) studied Chover's LIL, and so on.

However, the non additivity of sub-linear expectations and capacities brings some problems, which is essentially due to the fact that many traditional powerful tools and conventional methods for linear expectations and probabilities are no longer applicable. Therefore, it is much more complex and difficult to study limit theorem under sub-linear expectation than in traditional probability space.

The structure of this paper is as follows. In the next section, we give the basic concepts and properties of the sub-linear expectations. In Section 3, we present three lemmas, among them, Lemmas 3.4 and 3.5 are almost sure convergence theorems of partial sums and weighted sums in sub-linear expected space respectively, which play a key role in proving the main results. The Chover's LIL for weighted sums under sub-linear expectation is established in Section 4. In Section 5, we give two examples to show that the limits in Theorem 4.3 with the various forms.

2. Preliminaries

The general framework of the sub-linear expectation in a general function space was introduced by Peng (2006 [9], 2008 [10]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. *A function $\hat{\mathbb{E}} : \mathcal{H} \to [-\infty, \infty]$ is said to be a sub-linear expectation, if it satisfies for all $X, Y \in \mathcal{H}$,*
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X, \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. The conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ is defined by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \quad \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X, \quad \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c, \quad |\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}|X - Y| \quad \text{and} \quad \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Definition 2.2. *A function $V:\mathcal{F}\to[0,1]$ is called a capacity if*

$$V(\emptyset)=0,\ \ V(\Omega)=1\ \text{ and }\ V(A)\le V(B)\ \text{ for }\ \forall A\subseteq B, A,B\in\mathcal{F}.$$

It is called to be sub-additive if $V(A\bigcup B)\le V(A)+V(B)$ for all $A,B\in\mathcal{F}$. In the sub-linear space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, we denote a pair $(\mathbb{V},\nu)$ of capacities by

$$\mathbb{V}(A):=\inf\{\hat{\mathbb{E}}\xi; I(A)\le\xi,\xi\in\mathcal{H}\},\ \ \nu(A):=1-\mathbb{V}(A^c),\forall A\in\mathcal{F},$$

where A^c is the complement set of A.

By definition of $\mathbb{V}$ and ν, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\nu(A)\le\mathbb{V}(A),\ \ \forall A\in\mathcal{F},$$

$$\hat{\mathbb{E}}f\le\mathbb{V}(A)\le\hat{\mathbb{E}}g,\hat{\varepsilon}f\le\nu(A)\le\hat{\varepsilon}g,\ \text{ if }\ f\le I(A)\le g,\ \ f,g\in\mathcal{H}.$$

Definition 2.3. *(i) A sub-linear expectation $\hat{\mathbb{E}}:\mathcal{H}\to\mathbb{R}$ is called to be countably sub-additive if it satisfies*

$$\hat{\mathbb{E}}(X)\le\sum_{n=1}^{\infty}\hat{\mathbb{E}}(X_n),\ \text{ whenever }\ X\le\sum_{n=1}^{\infty}X_n,\ \ X,X_n\in\mathcal{H},\ \ X\ge0,X_n\ge0,\ \ n\ge1.$$

It is called to be continuous if it satisfies

$$\hat{\mathbb{E}}(X_n)\uparrow\hat{\mathbb{E}}(X),\ \text{ if }\ 0\le X_n\uparrow X,\ \text{ and }\ \hat{\mathbb{E}}(X_n)\downarrow\hat{\mathbb{E}}(X),\ \text{ if }\ 0\le X_n\downarrow X,\ \text{ where }\ X,X_n\in\mathcal{H}.$$

(ii) A function $V:\mathcal{F}\to[0,1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty}A_n\right)\le\sum_{n=1}^{\infty}V(A_n),\ \ \forall A_n\in\mathcal{F}.$$

It is called to be continuous if it satisfies

$$V(A_n)\uparrow V(A),\ \text{ if }\ A_n\uparrow A,\ \text{ and }\ V(A_n)\downarrow V(A),\ \text{ if }\ A_n\downarrow A,\ \text{ where }\ A,A_n\in\mathcal{F}.$$

It is obvious that a continuous sub-additive capacity V (resp. a sub-linear expectation $\hat{\mathbb{E}}$) is countably sub-additive.

Definition 2.4. *(Peng (2006 [9], 2008 [10]), Zhang (2016 [20]))*
(i) (Identical distribution) Let X_1 and X_2 be two random variables defined respectively in sub-linear expectation spaces $(\Omega_1,\mathcal{H}_1,\hat{\mathbb{E}}_1)$ and $(\Omega_2,\mathcal{H}_2,\hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(X_1))=\hat{\mathbb{E}}_2(\varphi(X_2)),\ \ \forall\varphi\in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-expectations are finite. A sequence $\{X_n;n\ge1\}$ of random variables is said to be identically distributed if for each $i\ge1$, X_i and X_1 are identically distributed.
(ii) (Independence) In a sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, a random vector $\mathbf{Y}=(Y_1,\dots,Y_n)$, $Y_i\in\mathcal{H}$ is said to be independent to another random vector $\mathbf{X}=(X_1,\dots,X_m)$, $X_i\in\mathcal{H}$ under $\hat{\mathbb{E}}$ if for each test function $\varphi\in C_{l,Lip}(\mathbb{R}_m\times\mathbb{R}_n)$ we have $\hat{\mathbb{E}}(\varphi(\mathbf{X},\mathbf{Y}))=\hat{\mathbb{E}}[\hat{\mathbb{E}}(\varphi(\mathbf{x},\mathbf{Y}))|_{\mathbf{x}=\mathbf{X}}]$, whenever $\bar{\varphi}(\mathbf{x}):=\hat{\mathbb{E}}\left(|\varphi(\mathbf{x},\mathbf{Y})|\right)<\infty$ for all $\mathbf{x}$ and $\hat{\mathbb{E}}\left(|\bar{\varphi}(\mathbf{X})|\right)<\infty$.
(iii) (Independent random variables) A sequence of random variables $\{X_n;n\ge1\}$ is said to be independent, if X_{i+1} is independent to $(X_1,\dots,X_i)$ for each $i\ge1$.
(iv) (Extended independence) A sequence of random variables $\{X_n;n\ge1\}$ is said to be extended independent, if

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n}\varphi_i(X_i)\right)=\prod_{i=1}^{n}\hat{\mathbb{E}}(\varphi_i(X_i)),\ \ \forall\ \ n\ge2,\ \ \forall\ \ 0\le\varphi_i(x)\in C_{l,Lip}(\mathbb{R}).$$

It can be showed that the independence implies the extended independence, and if $\{X_n;n\ge1\}$ is a sequence of extended independent random variables and $f_1(x),f_2(x),\dots\in C_{l,Lip}(\mathbb{R})$, then $\{f_n(X_n);n\ge1\}$ is also a sequence of extended independent random variables.

In the following, let $\{X_n;n\ge1\}$ be a sequence of random variables in a sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, and $S_n=\sum_{i=1}^{n}X_i$. The symbol c stands for a generic positive constant which may differ from one place to another.

3. Lemmas

The proofs of our results are based on the following three lemmas.

Lemma 3.1. *(Qi and Cheng 1996 [11]). Suppose that $h(x)$ is a slowly varying function at infinity and $g(x)$ is a positive function with $\lim_{x\to\infty} g(x) = \infty$. Then, for any given $\delta > 0$, there exists an $x_0 > 0$ such that*

$$g^{-\delta}(x) \le \inf_{x\le y\le xg(x)} \frac{h(y)}{h(x)} \le \sup_{x\le y\le xg(x)} \frac{h(y)}{h(x)} \le g^{\delta}(x) \text{ for all } x > x_0.$$

It is noteworthy that in sub-linear expectation spaces, the almost sure convergence of random variable sequences differs from that in traditional probability spaces. As a preliminary, we first present the definition of almost sure convergence within this framework.

Definition 3.2. *For arbitrary event $A \in \mathcal{F}$, A is said almost surely V (denoted by A a.s. V), if $V(A^c) = 0$, where A^c is the complement set of A.*

In particular, a sequence of random variables $\{X_n; n \ge 1\}$ is said to converge to X almost surely V, denoted by $X_n \to X$ a.s. V as $n \to \infty$ if, $V(X_n \nrightarrow X) = 0$.

V can be replaced by $\mathbb{V}$ and ν respectively. By $\nu(A) \le \mathbb{V}(A)$ and $\nu(A) + \mathbb{V}(A^c) = 1$ for any $A \in \mathcal{F}$, it is obvious that $X_n \to X$ a.s. $\mathbb{V}$ implies $X_n \to X$ a.s. ν. However, we must point out that $X_n \to X$ a.s. ν does not imply $X_n \to X$ a.s. $\mathbb{V}$. Wu and Lu (2020 [19]) gave a counter example of this as follows.

Example 3.3. *(Wu and Lu, Example 3.3 (2020 [19])) Let X_n be independent G-normal random variables with $X_n \sim \mathcal{N}(0, [1/4^{2n}, 1])$ in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. $\hat{\mathbb{E}}$ and $\mathbb{V}$ are continuous. Then $X_n \to 0$ a.s. ν; but not $X_n \to 0$ a.s.$\mathbb{V}$.*

This essential distinction renders the study of almost sure convergence in sub-linear expectation spaces considerably more complex and challenging. Consequently, to ensure that the sequence of truncated random variables preserves extended independence, a modification of the indicator function via functions from $C_{l,Lip}$ is required.

For $0 < \mu < 1$, let $g(x) \in C_{l,Lip}(\mathbb{R})$ be a non-increasing function such that $0 \le g(x) \le 1$ for all x and $g(x) = 1$ if $x \le \mu$, $g(x) = 0$ if $x > 1$. For any $i \le n$, let

$$X_i(a_n) = X_i g\left(\frac{|X_i|}{a_n}\right). \tag{1}$$

To establish a theory of almost sure convergence in sub-linear expectation spaces, the convergence criteria for partial sums and weighted sums are fundamental. The following two lemmas provide the almost sure convergence criteria for these two types of sums, respectively. They not only lay the critical groundwork for subsequent arguments but also serve as the core analytical tools for proving the main results of this paper.

Lemma 3.4. *(Wu and Jiang 2018 [18], Theorem 3.3) Assume that $\{X_n; n \ge 1\}$ is a sequence of extended independent and identically distributed (e.i.i.d.) random variables, $\mathbb{V}$ is continuous, and $\{a_n; n \ge 1\}$ is a sequence of positive numbers with $a_n/n^{1/\beta} \uparrow$ for some $\beta \in (0, 2)$ and $\sup_{n\ge1} a_{2n}/a_n < \infty$.*

(i) *If*

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_1| > a_n) < \infty, \tag{2}$$

then

$$\limsup_{n\to\infty}\frac{S_n-c_n}{a_n}\leq 0 \quad \text{a.s. } \mathbb{V}, \tag{3}$$

where $c_n = n\hat{\mathbb{E}}(X_1(a_n))$ *and* $X_1(a_n)$ *defined by (1), in particular,* $c_n = 0$ *if* $\frac{a_n}{n}\uparrow\infty$ *and* $c_n = n\hat{\mathbb{E}}X_1$ *if* $\frac{n}{a_n}\uparrow, \hat{\mathbb{E}}|X_1| < \infty, \hat{\mathbb{E}}$ *is countably sub-additive. And*

$$\liminf_{n\to\infty}\frac{S_n-\tilde{c}_n}{a_n}\geq 0 \quad \text{a.s. } \mathbb{V}, \tag{4}$$

where $\tilde{c}_n = n\hat{\varepsilon}(X_1(a_n))$*, in particular,* $\tilde{c}_n = 0$ *if* $\frac{a_n}{n}\uparrow\infty$ *and* $\tilde{c}_n = n\hat{\varepsilon}X_1$ *if* $\frac{n}{a_n}\uparrow, \hat{\mathbb{E}}|X_1| < \infty, \hat{\mathbb{E}}$ *is countably sub-additive.*

(ii) *If*

$$\sum_{n=1}^{\infty}\mathbb{V}(|X_1| > a_n) = \infty, \tag{5}$$

then

$$\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n} = \infty \quad \text{a.s. } \nu$$

for every sequence $\{d_n; n\geq 1\}$ *such that*

$$\limsup_{n\to\infty}\frac{|d_n-d_{n-1}|}{a_n} = c < \infty. \tag{6}$$

Lemma 3.5. *Under the conditions of Lemma 3.4, and one of the following two conditions holds.*

$$\frac{a_n}{n}\uparrow\infty, \tag{7}$$

$$\frac{n}{a_n}\uparrow,\ \ \hat{\mathbb{E}}|X_1| < \infty,\ \ \hat{\mathbb{E}}X_1 = \hat{\varepsilon}X_1,\ \ \text{and}\ \ \hat{\mathbb{E}}\ \ \text{is}\ \ \text{countably}\ \ \text{sub}-\text{additive}. \tag{8}$$

Let $\{a_{n,k}; 1\leq k\leq n, n\geq 1\}$ *be an array of weights satisfying*

$$\sup_{n\geq 1}\left(\sum_{k=1}^{n-1}|a_{n,k}-a_{n,k+1}| + |a_{n,n}|\right) < \infty, \quad \text{and}\ \ a := \liminf_{n\to\infty}|a_{n,n}| > 0. \tag{9}$$

(i) *If (2) holds, then*

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_{n,k}(X_k-b)}{a_n} = 0 \quad \text{a.s. } \mathbb{V}, \tag{10}$$

where $b = 0$*, if (7) holds, and* $b = \hat{\mathbb{E}}X_1$ *if (8) holds.*

(ii) *If (5) holds, then*

$$\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n} = \infty \quad \text{a.s. } \nu \tag{11}$$

for every sequence $\{d_n; n\geq 1\}$ *satisfying (6).*

Remark 3.6. *Lemmas 3.4 and 3.5 themselves are of great significance. They are powerful tools to study the almost sure convergence of partial sums and weighted sums of sub-linear expectation spaces respectively.*

Proof of Lemma 3.5. If (2) holds, we get from (3) and (4) in Lemma 3.4

$$\lim_{n\to\infty}\frac{S_n - C_n}{a_n} = 0 \quad \text{a.s. } \mathbb{V}, \tag{12}$$

where $C_n = 0$ if (7) holds, and $C_n = n\hat{\mathbb{E}}X_1$ if (8) holds. Note the fact that $a_n > 0$ is a non-decreasing function of n, we can get that (12) implies

$$\lim_{n\to\infty}\frac{\max\limits_{1\le k\le n}|S_k - C_k|}{a_n} = 0 \quad \text{a.s. } \mathbb{V}, \tag{13}$$

Set $S_0 = 0$, by (9)

$$\begin{aligned}\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right| &= \left|\sum_{k=1}^{n} a_{n,k}((S_k - S_{k-1}) - (C_k - C_{k-1}))\right| \\ &= \left|\sum_{k=1}^{n-1}(a_{n,k} - a_{n,k+1})(S_k - C_k) + a_{n,n}(S_n - C_n)\right| \\ &\le c\max_{1\le k\le n}|S_k - C_k|.\end{aligned}$$

This and (13) imply (10).
(ii) If (5) holds, then by P264 in Wu and Jiang (2018 [18]), we get

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|X_n|}{a_n} > M\right) = 1, \quad \text{for any } M > 0.$$

Combination condition (9), i.e., $\liminf\limits_{n\to\infty}|a_{n,n}| = a > 0$, it follows

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|a_{n,n}X_n|}{a_n} > M\right) \ge \mathbb{V}\left(\limsup_{n\to\infty}\frac{|X_n|}{a_n} > M/a\right) = 1. \tag{14}$$

Hence, for any $\{d_n; n \ge 1\}$ which satisfies (6),

$$\begin{aligned}\limsup_{n\to\infty}\frac{|a_{n,n}X_n|}{a_n} &= \limsup_{n\to\infty}\frac{\left|\left(\sum\limits_{k=1}^{n} a_{n,k}X_k - d_n\right) - \left(\sum\limits_{k=1}^{n-1} a_{n,k}X_k - d_{n-1}\right) + (d_n - d_{n-1})\right|}{a_n} \\ &\le 2\limsup_{n\to\infty}\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}X_k - d_n\right|}{a_n} + \limsup_{n\to\infty}\frac{|d_n - d_{n-1}|}{a_n} \\ &= 2\limsup_{n\to\infty}\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}X_k - d_n\right|}{a_n} + c.\end{aligned}$$

From the arbitrariness of M, this and (14) follow that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}X_k - d_n\right|}{a_n} > M_1\right) = 1, \quad \forall\, M_1 > 0.$$

Therefore, from the continuity of $\mathbb{V}$

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n}=\infty\right)=\mathbb{V}\left(\forall m\geq 1,\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n}>m\right)$$
$$=\mathbb{V}\left(\bigcap_{m=1}^{\infty}\left(\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n}>m\right)\right)=\lim_{m\to\infty}\mathbb{V}\left(\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n}>m\right)=1.$$

That is

$$\limsup_{n\to\infty}\frac{\left|\sum_{k=1}^{n}a_{n,k}X_k-d_n\right|}{a_n}=\infty\quad \text{a.s.}\quad \nu$$

Thus (11) is proved.

4. Chover's law of the iterated logarithm

In order to generalize the theory of stable distributions to sub-linear expectation spaces, an appropriate extension of the concept for exponents $\alpha\in(0,2)$ is needed. In this spirit, we introduce the following definition, which adapts the classical probabilistic definition to this non-additive setting.

Definition 4.1. *A sequence of identically distributed random variables defined in a sub-linear expected space $\{X_n;n\geq 1\}$ is called to be a stable distribution with exponent $\alpha\in(0,2)$, if*

$$\mathbb{V}(|X_1|>x)=\frac{c(x)l(x)}{x^{\alpha}}\quad \text{for}\quad x>0, \tag{15}$$

where, $c(x)\geq 0$, $\lim_{x\to\infty}c(x)=c>0$, and $l(x)\geq 0$ is a slowly varying function.

From Seneta (1976 [12]), $l(x)$ is a slowly varying function if and only if

$$l(x)=c_1(x)\exp\left\{\int_1^x\frac{f(u)}{u}\mathrm{d}u\right\},x>0,$$

where $c_1(x)\geq 0$, $\lim_{x\to\infty}c_1(x)=c_1>0$, and $\lim_{x\to\infty}f(x)=0$.

Set $\lg_0 x=x$ and denote $\lg_j x=\ln\{\max(\mathrm{e},\lg_{j-1}x)\}$ for $j\geq 1$, $\lg x=\lg_1 x$. We also assume $k\geq 0$ is fixed integer. For convenience, the product $\prod_{j=1}^{k}(\cdot)$ is defined as 1 if $k<1$. $A\sim B$ denotes $A/B\to 1$.

Recently, under sub-linear expectation, for a sequence of e.i.i.d. random variables satisfying (15), Wu and Jiang (2018 [18]) proved that Chover's LIL is as follows: there exist some constants $A_n\in\mathbb{R}$, $B_n>0$ such that

$$\limsup_{n\to\infty}\left|\frac{S_n-A_n}{B_n}\right|^{\frac{1}{\lg_{k+2}n}}=\mathrm{e}^{\frac{1}{\alpha}}\quad \text{a.s.}\quad \nu. \tag{16}$$

The central contribution of this work is to establish that the result given in (16) holds for generalized weighted sums. This encompasses various important structures, including but not limited to weighted sums of partial sums and moving sums.

Set $G(x) = \mathbb{V}(|X_1| \geq x)$, and define

$$B(x) = \inf\{y; G(y) \leq 1/x\} \text{ for } x > 0.$$

From (15), $G(x)$ is a regularly varying function with index $-\alpha$ at infinity. Hence, from De Haan (1970 [5]), $B(x)$ is a regularly varying function with index $1/\alpha$ at infinity, and by Karamata's representation,

$$B(x) = x^{1/\alpha} c(x) \exp\left\{\int_1^x \frac{b_1(u)}{u} \mathrm{d}u\right\} := x^{1/\alpha} l_1(x), \tag{17}$$

where $l_1(x)$ is a slowly varying function, $\lim_{x\to\infty} c(x) = c \in (0, \infty)$ and $\lim_{x\to\infty} b_1(x) = 0$.

Theorem 4.2. *Assume that $\{X_n; n \geq 1\}$ is a sequence of e.i.i.d. random variables, $\mathbb{V}$ is continuous, and (15) holds with some $\alpha \in (0, 2)$. For $\alpha \geq 1$, suppose that $\hat{\mathbb{E}}$ is countably sub-additive and $\hat{\mathbb{E}}X_1 = \hat{\varepsilon}X_1$, for $\alpha = 1$, further suppose that $f_n l_1(n) \downarrow$. Let $\{f_n\}$ be a sequence of positive nondecreasing numbers and $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be an array of weights satisfying (9). Set*

$$\limsup_{n\to\infty} \frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B(nf_n)} \begin{cases} = 0 \text{ a.s. } \mathbb{V}, \\ = \infty \text{ a.s. } \nu \end{cases} \tag{18}$$

where $b = 0$ for $0 < \alpha < 1$, and $b = \hat{\mathbb{E}}X_1$ for $1 \leq \alpha < 2$.

$$\sum_{n=1}^{\infty} \mathbb{V}\left(|X_1| > B(nf_n)\right) \begin{cases} < \infty, \\ = \infty. \end{cases} \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{1}{nf_n} \begin{cases} < \infty, \\ = \infty. \end{cases} \tag{20}$$

Then (19) is equivalent to (20), and (19) implies (18).

Theorem 4.3. *Under the conditions of Theorem 4.2,*

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B\left(n \prod_{j=1}^{k} \lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \leq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \mathbb{V}, \tag{21}$$

and

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B\left(n \prod_{j=1}^{k} \lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \geq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu. \tag{22}$$

This implies

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B\left(n \prod_{j=1}^{k} \lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu,$$

where b is defined by Theorem 4.2.

Remark 4.4. *Take $a_{n,k} = 1$ in Theorems 4.2 and 4.3 respectively, then Theorems 4.2 and 4.3 become Theorems 4.1 and 4.2 of Wu and Jiang (2018 [18]). Therefore, Theorems 4.1 and 4.2 of Wu and Jiang (2018 [18]) are the special cases of Theorems 4.2 and 4.3 respectively in this paper.*

Remark 4.5. *Theorem 4.3 is a extensive results, we can obtain various forms LIL for weighted sum by taking different forms of weights $a_{n,k}$. In particular, let h be a bounded variation function on $[0,1]$ with $h(1) > 0$. Then both $a_{n,k} := h(k/n)$ and $a_{n,k} := h(1) - h((k-1)/n) = \sum_{i=k}^{n}(h(i/n) - h((i-1)/n))$ satisfy the condition (7). Therefore, there are the following Corollaries 4.6 and 4.7.*

Corollary 4.6. *Let $\{X_n; n \geq 1\}$ be a sequence of e.i.i.d. random variables. Assume that $\mathbb{V}$ is continuous, and (15) holds for some $\alpha \in (0,2)$. For $\alpha \geq 1$, further suppose that $\hat{\mathbb{E}}$ is countably sub-additive and that $\hat{\mathbb{E}}X_1 = \hat{\varepsilon}X_1$. Let h be a function of bounded variation on $[0,1]$ with $h(1) > 0$. Then*

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n} h\left(\frac{k}{n}\right)(X_k - b)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \leq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \mathbb{V},$$

and

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n} h\left(\frac{k}{n}\right)(X_k - b)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \geq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu.$$

This implies

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n} h\left(\frac{k}{n}\right)(X_k - b)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu,$$

where b is defined by Theorem 4.2.

Setting $a_{n,k} := h(1) - h((k-1)/n) = \sum_{i=k}^{n}(h(i/n) - h((i-1)/n))$ in Theorem 4.3, by $\sum_{k=1}^{n} a_{n,k}(X_k - b) = \sum_{k=1}^{n}\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right)(S_k - kb)$, we can immediately obtain the Chover's LIL of weighted sums of partial sums as follows.

Corollary 4.7. *Under the conditions of Corollary 4.6,*

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n}\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right)(S_k - kb)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \leq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \mathbb{V},$$

and

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n}\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right)(S_k - kb)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \geq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu.$$

This implies

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{k=1}^{n}\left(h\left(\frac{k}{n}\right)-h\left(\frac{k-1}{n}\right)\right)(S_k-kb)\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu,$$

where b is defined by Theorem 4.2.

As another application of Theorem 4.3, we derive the following Chover's LIL for moving sums $Y_i := \sum_{k=1}^{i} b_{i-k+1}(X_k-b)$. Taking the weight function such that $a_{n,k} := \sum_{i=k}^{n} b_{i-k+1}$ in Theorem 4.3, by $\sum_{k=1}^{n} a_{n,k}(X_k-b) = \sum_{i=1}^{n} Y_i$, yields the following corollary.

Corollary 4.8. *Let $\{X_n; n \geq 1\}$ be a sequence of e.i.i.d. random variables. Assume that $\mathbb{V}$ is continuous, and (15) holds for some $\alpha \in (0,2)$. For $\alpha \geq 1$, further suppose that $\hat{\mathbb{E}}$ is countably sub-additive and that $\hat{\mathbb{E}}X_1 = \hat{\varepsilon}X_1$. Let $Y_i := \sum_{k=1}^{i} b_{i-k+1}(X_k - b)$, where $\{b_n; n \geq 1\}$ satisfies $\sum_{n=1}^{\infty}|b_n| < \infty$ and $b_1 \neq 0$. Then*

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{i=1}^{n} Y_i\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \leq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \mathbb{V},$$

and

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{i=1}^{n} Y_i\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} \geq \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu.$$

This implies

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{i=1}^{n} Y_i\right|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s. } \nu,$$

where b is defined by Theorem 4.2.

Proof of Theorem 4.2. Without loss of generality, we may assume that f_n is a slowly varying function. We first prove that (19) ⇔ (20). Using the properties of regular variation (see Seneta (1976 [12]), we have

$$G(B(x)) \sim x^{-1},\ x \to \infty. \tag{23}$$

From (15) and (23),

$$\mathbb{V}(|X_1| > B(nf_n)) = G(B(nf_n)) \sim \frac{1}{nf_n}.$$

Hence, (19) is equivalent (20).

Now, we prove that (19)⇒(18).

Set now $l_2(x) = c\exp\left\{\int_1^x \frac{b_1(u)}{u}\mathrm{d}u\right\}$, where $c = \lim_{x\to\infty} c(x), c(x)$ defined by (17), and $b(x) = x^{1/\alpha} l_2(x)$. By (17), it is easy to see that

$$B(x) \sim b(x), \quad x \to \infty.$$

Therefore, $B(nf_n)$ in (18)~(19) can be replaced by $b(nf_n)$. Let $a_n = b(nf_n)$, from the proof process of Theorem 4.1 in Wu and Jiang (2018 [18]), the sequence $\{a_n\}$ satisfies the conditions of Lemma 3.4.

From the properties of slowly varying functions: if $l(x)$ is a slowly varying function, then for any $\beta > 0$, there exists a increasing function $\varphi(x)$ such that $l(x)x^\beta \sim \varphi(x)$ and $\lim_{x\to\infty} l(x)x^\beta = \infty$. Therefore, without loss of generality, we can suppose $l(x)x^\beta \uparrow \infty$ for any $\beta > 0$. This implies

$\frac{a_n}{n} \sim n^{1/\alpha-1} f_n^{1/\alpha} l_1(n) \uparrow \infty$, for $0 < \alpha < 1$.

$\frac{n}{a_n} \sim \frac{n^{1-1/\alpha}}{f_n^{1/\alpha} l_1(n)} \uparrow$, for $\alpha > 1$.

By assumption, $\frac{n}{a_n} \sim \frac{1}{f_n l_1(n)} \uparrow$, for $\alpha = 1$.

Therefore, from Lemma 3.5 and assumptions of the Theorem 4.2, we obtain that (19) implies (18).

Proof of Theorem 4.3. For $\forall\varepsilon > 0$, let $f_n = \prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n$. By (15) and (23), we obtain

$$\sum_{n=1}^\infty \mathbb{V}\Big(|X_1| > B(nf_n)\Big) = \sum_{n=1}^\infty G\Big(B(nf_n)\Big) \sim \sum_{n=1}^\infty \frac{1}{n\prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n} < \infty.$$

Applying Theorem 4.2, we get

$$\limsup_{n\to\infty} \frac{\left|\sum_{k=1}^n a_{n,k}(X_k - b)\right|}{B(nf_n)} = 0 \quad \text{a.s. } \mathbb{V}.$$

Thus, let $\delta = \frac{\varepsilon}{(1+\varepsilon)\alpha} > 0$ in Lemma 3.1, for sufficiently large n, by combining (17) we have

$$\begin{aligned}
\frac{\left|\sum_{k=1}^n a_{n,k}(X_k - b)\right|}{B\Big(n\prod_{j=1}^k \lg_j n\Big)} &= \frac{\left|\sum_{k=1}^n a_{n,k}(X_k - b)\right|}{B(nf_n)} \frac{B\Big(n\prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n\Big)}{B\Big(n\prod_{j=1}^k \lg_j n\Big)} \le \frac{B\Big(n\prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n\Big)}{B\Big(n\prod_{j=1}^k \lg_j n\Big)} \\
&= \frac{\Big(n\prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n\Big)^{1/\alpha} l_1\Big(n\prod_{j=1}^k \lg_j n \lg_{k+1}^{1+\varepsilon} n\Big)}{\Big(n\prod_{j=1}^k \lg_j n\Big)^{1/\alpha} l_1\Big(n\prod_{j=1}^k \lg_j n\Big)} \\
&\le \Big(\lg_{k+1} n\Big)^{(1+2\varepsilon)/\alpha} \quad \text{a.s. } \mathbb{V}.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^n a_{n,k}(X_k - b)\right|}{B\Big(n\prod_{j=1}^k \lg_j n\Big)}\right)^{\frac{1}{\lg_{k+2} n}} \le \mathrm{e}^{\frac{1+2\varepsilon}{\alpha}} \quad \text{a.s. } \mathbb{V}. \tag{24}$$

Similarly, for $\varepsilon \in (0, 1/2)$, let $f_n = \prod_{j=1}^{k} \lg_j n \lg_{k+1}^{1-\varepsilon} n$, then

$$\sum_{n=1}^{\infty} \mathbb{V}\Big(|X_1| > B(nf_n)\Big) \sim \sum_{n=1}^{\infty} \frac{1}{n \prod\limits_{j=1}^{k} \lg_j n \lg_{k+1}^{1-\varepsilon} n} = \infty.$$

It then follows from Theorem 4.2 that

$$\limsup_{n\to\infty} \frac{\left|\sum\limits_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B(nf_n)} = \infty \quad \text{a.s. } \nu.$$

Hence, we conclude the existence of infinitely many indices n for which

$$\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B(nf_n)} \geq 1 \quad \text{a.s. } \nu.$$

Let $\delta = \dfrac{\varepsilon}{(1-\varepsilon)\alpha} > 0$ in Lemma 3.1, for these n, combing (17), we obtain

$$\begin{aligned}
\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B\Big(n \prod\limits_{j=1}^{k} \lg_j n\Big)} &= \frac{\left|\sum\limits_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B(nf_n)} \frac{B\Big(n \prod\limits_{j=1}^{k} \lg_j n \lg_{k+1}^{1-\varepsilon} n\Big)}{B\Big(n \prod\limits_{j=1}^{k} \lg_j n\Big)} \\
&\geq \frac{\Big(n \prod\limits_{j=1}^{k} \lg_j n \lg_{k+1}^{1-\varepsilon} n\Big)^{1/\alpha} l_1\Big(n \prod\limits_{j=1}^{k} \lg_j n \lg_{k+1}^{1-\varepsilon} n\Big)}{\Big(n \prod\limits_{j=1}^{k} \lg_j n\Big)^{1/\alpha} l_1\Big(n \prod\limits_{j=1}^{k} \lg_j n\Big)} \\
&\geq \Big(\lg_{k+1} n\Big)^{\frac{1-2\varepsilon}{\alpha}} \quad \text{a.s. } \nu.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \left(\frac{\left|\sum\limits_{k=1}^{n} a_{n,k}(X_k - b)\right|}{B\Big(n \prod\limits_{j=1}^{k} \lg_j n\Big)} \right)^{\frac{1}{\lg_{k+2} n}} \geq \mathrm{e}^{\frac{1-2\varepsilon}{\alpha}} \quad \text{a.s. } \nu. \tag{25}$$

Since $\varepsilon \in (0, 1/2)$ is arbitrary, we get (21) and (22) from (24) and (25).

5. Examples

This section presents two examples demonstrating the variety of forms that the limits in Theorem 4.3 may assume.

Example 5.1. *Let X_1 be such that it satisfies*

$$G(x) = \mathbb{V}(|X_1| > x) = \frac{\alpha^\beta (\lg x)^\beta}{x^\alpha}, \quad \alpha \in (0,2), \quad \beta \in \mathbb{R}.$$

A direct verification shows that

$$B(x) = x^{\frac{1}{\alpha}} (\lg x)^{\frac{\beta}{\alpha}}.$$

This implies that (23) holds and

$$B\left(n \prod_{j=1}^{k} \lg_j n\right) \sim n^{\frac{1}{\alpha}} (\lg n)^{\frac{\beta}{\alpha}} \left(\prod_{j=1}^{k} \lg_j n\right)^{\frac{1}{\alpha}}.$$

Hence, by Theorem 4.3,

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{n^{\frac{1}{\alpha}} (\lg n)^{\frac{\beta}{\alpha}} \left(\prod_{j=1}^{k} \lg_j n\right)^{\frac{1}{\alpha}}} \right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}}, \quad \text{a.s.} \quad \nu,$$

where $a_{n,k}$ and b are defined by Theorem 4.3.

In particular, let $k = 0$, then

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{n^{\frac{1}{\alpha}} (\lg n)^{\frac{\beta}{\alpha}}} \right)^{\frac{1}{\lg\lg n}} = \mathrm{e}^{\frac{1}{\alpha}}, \quad \text{a.s.} \quad \nu$$

or

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{n^{\frac{1}{\alpha}}} \right)^{\frac{1}{\lg\lg n}} = \mathrm{e}^{\frac{1+\beta}{\alpha}}, \quad \text{a.s.} \quad \nu.$$

Example 5.2. *Let X_1 be such that it satisfies*

$$G(x) = \mathbb{V}(|X_1| > x) \sim \frac{\exp(\beta (\lg x)^r)}{x^\alpha}, \quad r \in (0,1), \quad \beta \in \mathbb{R}, \quad \alpha \in (0,2).$$

It is easy to check

$$B(x) = x^{\frac{1}{\alpha}} \exp\left(\frac{\beta (\lg x)^r}{\alpha^{r+1}}\right).$$

This implies that (23) holds and

$$B\left(n \prod_{j=1}^{k} \lg_j n\right) \sim n^{\frac{1}{\alpha}} \left(\prod_{j=1}^{k} \lg_j n\right)^{\frac{1}{\alpha}} \exp\left(\frac{\beta (\lg n)^r}{\alpha^{r+1}}\right).$$

Hence, by Theorem 4.3,

$$\limsup_{n\to\infty} \left(\frac{\left|\sum_{k=1}^{n} a_{n,k}(X_k - b)\right|}{n^{\frac{1}{\alpha}} \left(\prod_{j=1}^{k} \lg_j n\right)^{\frac{1}{\alpha}} \exp(\frac{\beta (\lg n)^r}{\alpha^{r+1}})} \right)^{\frac{1}{\lg_{k+2} n}} = \mathrm{e}^{\frac{1}{\alpha}}, \quad \text{a.s.} \quad \nu.$$

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