Filomat 40:6 (2026), 1943–1967
https://doi.org/10.2298/FIL2606943K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation of functions of bounded derivatives by Legendre wavelet method

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Abstract. Since few last decades, a number of researchers have been working on wavelet methods and on approximation of functions with bounded derivatives in various function spaces. But they investigated the function $f \in L^2[0,1)$ whose either first, second, third or fourth derivative is bounded. Their results did not give any idea about a function whose fifth or higher order derivative is bounded. Therefore in present investigation we have taken a function $f \in L^2[0,1)$ whose fifth and m^{th} order derivative is bounded and found their error estimation by using Legendre wavelet method.

In this paper, we have established two new theorems on wavelet approximation of a function f with $0 < |f^v(t)| < \infty \quad \forall t \in [0,1]$. Four new estimates $E^{(1)}_{2^{t-1},0}$, $E^{(2)}_{2^{t-1},1}$, $E^{(3)}_{2^{k-1},2}$ and $E^{(4)}_{2^{k-1},M}$ of any function f on $[0,1)$ having bounded derivatives are calculated by Legendre Wavelet Method.

1. Introduction

In the start of 19^{th} century many researchers found the degree of approximations of functions belonging to various classes by using certain summability methods. Later on, the mathematician had started to use wavelet methods to find the error estimation of functions belonging to different spaces. Wavelet methods give more accurate results as compare to other existing techniques to find the degree of approximation of function of a given class. In recent years, several investigators have determined the degree of approximation of functions of bounded derivatives by using Legendre, Hermite and other wavelet methods. But they have used only lower order bounded derivatives in their investigations. In present scenario it becomes very important to use functions of higher order bounded derivatives for the advance study of approximation theory in prospective of wavelet analysis. Working in this direction Debnath [3], Mhaskar[2], Sablonniere[4], Lal & Kumar[[7],[8],[15],[9]], Lal et al.[10], Kumar[11], Kumar et al.[[12],[13],[14]] have studied the degree of approximation of bounded derivative functions belonging to different classes by using various wavelet method. The technique of Legendre wavelet method is also used by Lal & Rakesh[6] and Lal & Indra Bhan[5] to determine wavelet approximation of the function f with $0 < |f^{'}(t)| < \infty$, $0 < |f^{''}(t)| < \infty$ and $0 < |f^{'''}(t)| < \infty$, $0 < |f^{iv}(t)| < \infty \quad \forall t \in [0,1)$ respectively. Present investigation is the estimation of wavelet approximation of a function f whose fifth order derivative is bounded i.e. $0 < |f^v(t)| < \infty \quad \forall t \in [0,1)$. Thus,

2020 *Mathematics Subject Classification*. Primary 42C40; Secondary 65T60, 65L10.
Keywords. Legendre Wavelet, Legendre Polynomial, Wavelet approximation, Legendre Wavelet Series.
Received: 13 September 2024; Revised: 15 October 2025; Accepted: 24 December 2025
Communicated by Dragan S. Djordjević
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we could arrive at a significant observation regarding the comparative error estimation in the approximation of functions using Legendre wavelet methods.

2. Definitions and Preliminaries

2.1. Wavelet and Legendre Wavelet

The concept of wavelets, as a family of functions generated from translations and dilations of one function which is called "Mother wavelet", was first introduced by Jean Morlet et. al. in 1982. The mathematical representation of "Mother wavelet" is given by-

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a},\right) \qquad a, b \in \mathbb{R}, a \neq 0 \tag{1}$$

where a,b are scaling and translation parameter respectively and determines the location of wavelet.
Legendre wavelets are a type of wavelet function derived from Legendre polynomials. It is denoted by $\psi_{n,m}(t)$ and defined on the interval $[0, 1)$ by

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \text{if } \frac{\hat{n}-1}{2^k} \leq x < \frac{\hat{n}+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $k = 1, 2, 3\cdots$, $\quad \hat{n} = 2n-1$ and m is the order of Legendre polynomials. Few Legendre polynomials are given by-

$$L_0(t) = 1 \quad \& \quad L_1(t) = t, \quad L_2(t) = \frac{1}{2}(3t^2 - 1)$$

and recurrence formulae for Legendre polynomial is given by:

$$L_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)tL_m(t) - \left(\frac{m}{m+1}\right)L_{m-1}(t), \qquad m = 1, 2, 3, \cdots$$

In the Hilbert space $L^2[-1, 1]$ the set $\{L_m(t) : m = 1, 2, 3\cdots\}$ is a complete orthogonal set. On the interval $[-1, 1]$, the orthogonality of Legendre polynomials is defined as

$$\langle L_m(t), L_{m'}(t)\rangle = \int_{-1}^{1} L_m(t), \overline{L_{m'}(t)}dt = \begin{cases} \frac{2}{2m+1}, & \text{for } \quad m = m'; \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

2.2. Legendre Wavelet Series and Wavelet Approximation

The function $f(t) \in L^2[0, 1)$ can be expressed in the Legendre wavelet series as following:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)$$

where $c_{n,m} = \langle f, \psi_{n,m}\rangle = \int_0^t f(t)\psi_{n,m}(t)dt$. The $\left(2^{k-1}, M\right)^{th}$ partial sums of the Legendre wavelet series are given by:

$$S_{2^{k-1},M}(f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) = C^T\psi(t)$$

where C and $\psi(t)$ are matrices of order $2^{k-1}M \times 1$, represented by

$$C^T = \left[c_{1,0}, c_{1,1}, \cdots, c_{1,M}, c_{2,0}c_{2,1}, \cdots c_{2,M}, \cdots c_{2^{k-1},0}, \cdots c_{2^{k-1},M}\right]$$

and

$$\psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \cdots \psi_{1,M}, \psi_{2,0}, \psi_{2,1}, \cdots \psi_{2,M}, \psi_{2^{k-1},0}, \cdots \psi_{2^{k-1},M}\right]^T$$

Legendre wavelet approximation $E_{2^{k-1},M}(f)$ of any function $f(t) \in L^2[0,1)$ by $\left(2^{k-1}, M\right)^{th}$ partial sums $S_{2^{k-1},M}(f)$ of Legendre wavelet series is given by

$$E_{2^{k-1},M}(f) = min\|f - S_{2^{k-1},M}(f)\|_2 \tag{4}$$

where

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}$$

If $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty$, $M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of the function f of order $\left(2^{k-1}, M\right)$.(Zygmund[1],pp.115)

3. Main Theorems

In this paper we establish following theorems:

Theorem-3.1 Let a function $f \in L^2[0,1)$ such that its fifth derivative be bounded, i.e. $0 \le |f^v(t)| < \infty \quad \forall t \in [0,1)$. Then the approximations of f by Legendre wavelet is given by:

1. $E^{(1)}_{2^{k-1},0}(f) = \|f - \sum\limits_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}\|_2 = O\left(\frac{1}{2^k}\right)$
2. $E^{(2)}_{2^{k-1},1}(f) = \|f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{1} c_{n,m}\psi_{n,m}\|_2 = O\left(\frac{1}{2^{2k}}\right)$
3. $E^{(3)}_{2^{k-1},2}(f) = \|f - \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{2} c_{n,m}\psi_{n,m}\|_2 = O\left(\frac{1}{2^{3k}}\right)$

Theorem-3.2 Let a function $f \in L^2[0,1)$ such that its fifth derivative be bounded, i.e. $0 \le |f^v(t)| < \infty, \quad \forall t \in [0,1)$ where $f(t) = \sum\limits_{n=1}^{\infty} \sum\limits_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)$ and $(S_{2^{k-1},M}f)(t) = \sum\limits_{n=1}^{2^{k-1}} \sum\limits_{m=0}^{M} c_{n,m}\psi_{n,m}(t)$. Then the Legendre wavelet approximation $E^{(4)}_{2^{k-1},M}(f)$ of f by $(S_{2^{k-1},M}f)(t)$ is estimated as-

$$E^{(4)}_{2^{k-1},M}(f) = min\|f - S_{2^{k-1},M}(f)\|_2 = O\left(\frac{1}{(2M-7)^{\frac{9}{2}}.2^{5k}}\right); \quad \forall M \ge 4.$$

4. Proofs:

Proof of Theorem-3.1

1-The error $e^{(0)}_n(t)$ between $f(t)$ and its expression over any sub interval is defined as-

$$e^{(0)}_n(t) = c_{n,0}\psi_{n,0}(t) - f(t), \qquad \forall t \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right), \qquad n = 1,2,3,\cdots 2^{k-1}.$$

$$\|e^{(0)}_n\|_2^2 = \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[e^{(0)}_n(t)\right]^2 dt$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[c_{n,0}\psi_{n,0}(t) - f(t)\right]^2 dt$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[c_{n,0}^2 \left(\psi_{n,0}(t)\right)^2 + \left(f(t)\right)^2 - 2c_{n,0}\psi_{n,0}(t)f(t)\right] dt$$

$$= c_{n,0}^2 \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(\psi_{n,0}(t)\right)^2 dt + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt - 2c_{n,0} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,0}(t)f(t)dt$$

$$= c_{n,0}^2 + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt - 2c_{n,0}c_{n,0}$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt - c_{n,0}^2 \tag{5}$$

Now consider

$$\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt = \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k} + h\right)\right]^2 dh, \qquad t = \frac{\hat{n}-1}{2^k} + h$$

$$= \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right) + hf'\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2}f''\left(\frac{\hat{n}-1}{2^k}\right)\right.$$
$$+ \frac{h^3}{6}f'''\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^4}{24}f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$
$$\left. + \frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k} + \theta h\right)\right]^2 dh, 0 < \theta < 1 \text{(By Taylor's expansion)}$$

$$\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt = \int_0^{\frac{1}{2^{k-1}}} \left[f\left(\frac{\hat{n}-1}{2^k}\right)\right]^2 dh + \int_0^{\frac{1}{2^{k-1}}} h^2\left[f'\left(\frac{\hat{n}-1}{2^k}\right)\right]^2 dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4}\left[f''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2 dh$$
$$+ \int_0^{\frac{1}{2^{k-1}}} \frac{h^6}{36}\left[f'''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2 dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^8}{576}\left[f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)\right]^2 dh$$
$$+ \int_0^{\frac{1}{2^{k-1}}} \left[\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k} + \theta h\right)\right]^2 dh + \int_0^{\frac{1}{2^{k-1}}} 2hf\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)dh$$
$$+ \int_0^{\frac{1}{2^{k-1}}} h^2 f\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)dh + \int_0^{\frac{1}{2^{k-1}}} \frac{h^3}{3}f\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^4}{12}f\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)dh+\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}h^3f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)dh+\int_0^{\frac{1}{2^{k-1}}}\frac{h^4}{3}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{12}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)dh+\int_0^{\frac{1}{2^{k-1}}}\frac{h^6}{60}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{6}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)dh+\int_0^{\frac{1}{2^{k-1}}}\frac{h^6}{24}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^7}{120}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh+\int_0^{\frac{1}{2^{k-1}}}\frac{h^7}{72}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^8}{360}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^9}{1440}f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}(f(t))^2\,dt=\frac{2}{2^k}\left[f\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{8}{3}\frac{1}{2^{3k}}\left[f'\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{8}{5}\frac{1}{2^{5k}}\left[f''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2$$

$$+\frac{32}{63}\frac{1}{2^{7k}}\left[f'''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{8}{81}\frac{1}{2^{9k}}\left[f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)\right]^2$$

$$+\int_0^{\frac{1}{2^{k-1}}}\left[\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right]^2dh+\frac{4}{2^{2k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{8}{3}\frac{1}{2^{3k}}f\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{4}{3}\frac{1}{2^{4k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{8}{15}\frac{1}{2^{5k}}f\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{60}f\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{4}{2^{4k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{32}{15}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{8}{9}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\int_0^{\frac{1}{2^{k-1}}}\frac{h^6}{60}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{16}{9}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{16}{21}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$
$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^7}{120}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh+\frac{4}{9}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$
$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^8}{360}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$
$$+\int_0^{\frac{1}{2^{k-1}}}\frac{h^9}{31440}f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \tag{6}$$

Now,

$$\begin{aligned}
c_{n,0} &= \langle f(t), \psi_{n,0}(t)\rangle \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)\psi_{n,0}(t)dt \\
&= 2^{\frac{k-1}{2}}\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)dt \\
&= 2^{\frac{k-1}{2}}\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f\left(\frac{\hat{n}-1}{2^k}+h\right)dh, \quad t=\frac{\hat{n}-1}{2^k}+h
\end{aligned}$$

$$c_{n,0} = 2^{\frac{k-1}{2}}\Bigg[\int_0^{\frac{1}{2^{k-1}}}\Bigg\{f\left(\frac{\hat{n}-1}{2^k}\right)+hf'\left(\frac{\hat{n}-1}{2^k}\right)+\frac{h^2}{2}f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{h^3}{6}f'''\left(\frac{\hat{n}-1}{2^k}\right)$$
$$+\frac{h^4}{24}f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\Bigg\}dh\Bigg] \quad 0<\theta<1$$
$$c_{n,0} = 2^{\frac{k-1}{2}}\Bigg[\frac{2}{2^k}f\left(\frac{\hat{n}-1}{2^k}\right)+\frac{2}{2^{2k}}f'\left(\frac{\hat{n}-1}{2^k}\right)+\frac{4}{3}\frac{1}{2^{3k}}f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{2}{3}\frac{1}{2^{4k}}f'''\left(\frac{\hat{n}-1}{2^k}\right)$$
$$+\frac{4}{15}\frac{1}{2^{5k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]$$

Next,

$$c_{n,0}^2 = \frac{2^k}{2}\Bigg[\frac{4}{2^{2k}}\left\{f\left(\frac{\hat{n}-1}{2^k}\right)\right\}^2+\frac{4}{2^{4k}}\left\{f'\left(\frac{\hat{n}-1}{2^k}\right)\right\}^2+\frac{16}{9}\frac{1}{2^{6k}}\left\{f''\left(\frac{\hat{n}-1}{2^k}\right)\right\}^2$$
$$+\frac{4}{9}\frac{1}{2^{8k}}\left\{f'''\left(\frac{\hat{n}-1}{2^k}\right)\right\}^2+\frac{16}{225}\frac{1}{2^{10k}}\left\{f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)\right\}^2$$

$$+\left\{\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right\}^2+\frac{8}{2^{3k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{16}{3}\frac{1}{2^{4k}}f\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{8}{3}\frac{1}{2^{5k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{16}{15}\frac{1}{2^{6k}}f\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{30}\frac{1}{2^k}\int_0^{\frac{1}{2^{k-1}}}h^5f\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{16}{3}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{8}{3}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{64}{15}\frac{1}{2^{7k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{30}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5f'\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{16}{9}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{32}{45}\frac{1}{2^{8k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{1}{45}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}}h^5f''\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{16}{45}\frac{1}{2^{9k}}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{90}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}}h^5f'''\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{1}{225}\frac{1}{2^{5k}}\int_0^{\frac{1}{2^{k-1}}}h^5f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]$$

$$c_{n,0}^2=\frac{2}{2^k}\left[f\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{2}{2^{3k}}\left[f'\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{8}{9}\frac{1}{2^{5k}}\left[f''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2$$

$$+\frac{2}{9}\frac{1}{2^{7k}}\left[f'''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{8}{225}\frac{1}{2^{9k}}\left[f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)\right]^2$$

$$+\frac{2^k}{2}\left[\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right]^2+\frac{4}{2^{2k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{8}{3}\frac{1}{2^{3k}}f\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{4}{3}\frac{1}{2^{4k}}f\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{8}{15}\frac{1}{2^{5k}}f\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{60}\int_0^{\frac{1}{2^{k-1}}}h^5f\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{8}{3}\frac{1}{2^{4k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{4}{3}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{32}{15}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{60}\frac{1}{2^k}\int_0^{\frac{1}{2^{k-1}}}h^5f'\left(\frac{\hat{n}-1}{2^k}\right)f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{8}{9}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{16}{45}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{1}{90}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5f''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{8}{45}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{180}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}}h^5f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{1}{450}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}}h^5f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh \tag{7}$$

From equation (5), (6) and (7), we have-

$$\|e_n^{(0)}\|_2^2=\frac{2}{3}\frac{1}{2^{3k}}\left[f'\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{32}{45}\frac{2}{2^{5k}}\left[f''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{2}{7}\frac{1}{2^{7k}}\left[f'''\left(\frac{\hat{n}-1}{2^k}\right)\right]^2$$

$$+\frac{128}{2025}\frac{1}{2^{9k}}\left[f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)\right]^2+\frac{1}{14400}\int_0^{\frac{1}{2^{k-1}}}h^{10}\left[f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)\right]^2dh$$

$$-\frac{2^k}{2}\left[\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right]^2+\frac{4}{3}\frac{1}{2^{4k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f''\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{4}{5}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)-\frac{56}{45}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{1}{60}\int_0^{\frac{1}{2^{k-1}}}h^6f'\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$-\frac{1}{60}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5f'\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{8}{9}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f'''\left(\frac{\hat{n}-1}{2^k}\right)+\frac{128}{315}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)$$

$$+\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^7f''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$-\frac{1}{90}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5f''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+\frac{4}{15}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}\right)+\frac{1}{360}\int_0^{\frac{1}{2^{k-1}}}h^8f'''\left(\frac{\hat{n}-1}{2^k}\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$- \frac{1}{180}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}} h^5 f'''\left(\frac{\hat{n}-1}{2^k}\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+ \frac{1}{1440}\int_0^{\frac{1}{2^{k-1}}} h^9 f^{iv}\left(\frac{\hat{n}-1}{2^k}\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$- \frac{1}{450}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}} h^5 f^{iv}\left(\frac{\hat{n}-1}{2^k}\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$\begin{aligned}||e_n^{(0)}||_2^2 &= I_1 + I_2 + I_3 + I_4 + I_5 - I_6 + I_7 + I_8 - I_9 + I_{10}\\ &\quad - I_{11} + I_{12} + I_{13} + I_{14} - I_{15} + I_{16} + I_{17} - I_{18} + I_{19} - I_{20}, \quad \text{say.}\end{aligned}$$

Since $|f'(t)| \le \zeta_1, |f''(t)| \le \zeta_2, |f'''(t)| \le \zeta_3, |f^{iv}(t)| \le \zeta_4, |f^v(t)| \le \zeta_5 \quad \forall x \in [0, 1)$
Therefore,

$|I_1| \le \frac{2}{3}\frac{1}{2^{3k}}\zeta_1^2, |I_2| \le \frac{32}{45}\frac{1}{2^{5k}}\zeta_2^2, |I_3| \le \frac{2}{7}\frac{1}{2^{7k}}\zeta_3^2, |I_4| \le \frac{128}{2025}\frac{1}{2^{9k}}\zeta_4^2,$

$|I_5| \le \frac{32}{2475}\frac{1}{2^{11k}}\zeta_5^2, |I_6| \le \frac{8}{2025}\frac{1}{2^{11k}}\zeta_5^2, |I_7| \le \frac{4}{3}\frac{1}{2^{4k}}\zeta_1\zeta_2, |I_8| \le \frac{4}{5}\frac{1}{2^{5k}}\zeta_1\zeta_3,$

$|I_9| \le \frac{56}{45}\frac{1}{2^{6k}}\zeta_1\zeta_4, |I_{10}| \le \frac{32}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5, |I_{11}| \le \frac{8}{15}\frac{1}{2^{7k}}\zeta_1\zeta_5, |I_{12}| \le \frac{8}{9}\frac{1}{2^{6k}}\zeta_2\zeta_3,$

$|I_{13}| \le \frac{128}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4, |I_{14}| \le \frac{4}{15}\frac{1}{2^{8k}}\zeta_2\zeta_5, |I_{15}| \le \frac{16}{135}\frac{1}{2^{8k}}\zeta_2\zeta_5, |I_{16}| \le \frac{4}{15}\frac{1}{2^{8k}}\zeta_3\zeta_4,$

$|I_{17}| \le \frac{64}{405}\frac{1}{2^{9k}}\zeta_3\zeta_5, |I_{18}| \le \frac{8}{135}\frac{1}{2^{9k}}\zeta_3\zeta_5, |I_{19}| \le \frac{16}{225}\frac{1}{2^{10k}}\zeta_4\zeta_5, |I_{20}| \le \frac{16}{675}\frac{1}{2^{10k}}\zeta_4\zeta_5$

Therefore,

$$\begin{aligned}||e_n^{(0)}||_2^2 &\le ||I_1|| + ||I_2|| + ||I_3|| + ||I_4|| + ||I_5|| + ||I_6|| + ||I_7|| + ||I_8|| + ||I_9|| + ||I_{10}|| + ||I_{11}||\\ &\quad + ||I_{12}|| + ||I_{13}|| + ||I_{14}|| + ||I_{15}|| + ||I_{16}|| + ||I_{17}|| + ||I_{18}|| + ||I_{19}|| + ||I_{20}||\\
||e_n^{(0)}||_2^2 &\le \frac{2}{3}\frac{1}{2^{3k}}\zeta_1^2 + \frac{32}{45}\frac{1}{2^{5k}}\zeta_2^2 + \frac{2}{7}\frac{1}{2^{7k}}\zeta_3^2 + \frac{128}{2025}\frac{1}{2^{9k}}\zeta_4^2 + \frac{32}{2475}\frac{1}{2^{11k}}\zeta_5^2 + \frac{8}{2025}\frac{1}{2^{11k}}\zeta_5^2\\ &\quad + \frac{4}{3}\frac{1}{2^{4k}}\zeta_1\zeta_2 + \frac{4}{5}\frac{1}{2^{5k}}\zeta_1\zeta_3 + \frac{56}{45}\frac{1}{2^{6k}}\zeta_1\zeta_4 + \frac{32}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{8}{15}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{8}{9}\frac{1}{2^{6k}}\zeta_2\zeta_3\\ &\quad + \frac{128}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4 + \frac{4}{15}\frac{1}{2^{8k}}\zeta_2\zeta_5 + \frac{16}{135}\frac{1}{2^{8k}}\zeta_2\zeta_5 + \frac{4}{15}\frac{1}{2^{8k}}\zeta_3\zeta_4 + \frac{64}{405}\frac{1}{2^{9k}}\zeta_3\zeta_5\\ &\quad + \frac{8}{135}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{16}{225}\frac{1}{2^{10k}}\zeta_4\zeta_5 + \frac{16}{675}\frac{1}{2^{10k}}\zeta_4\zeta_5\\
||e_n^{(0)}||_2^2 &= \frac{2}{3}\frac{1}{2^{3k}}\zeta_1^2 + \frac{32}{45}\frac{1}{2^{5k}}\zeta_2^2 + \frac{2}{7}\frac{1}{2^{7k}}\zeta_3^2 + \frac{128}{2025}\frac{1}{2^{9k}}\zeta_4^2 + \frac{376}{22275}\frac{1}{2^{11k}}\zeta_5^2 + \frac{4}{3}\frac{1}{2^{4k}}\zeta_1\zeta_2\\ &\quad + \frac{4}{5}\frac{1}{2^{5k}}\zeta_1\zeta_3 + \frac{56}{45}\frac{1}{2^{6k}}\zeta_1\zeta_4 + \frac{88}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{8}{9}\frac{1}{2^{6k}}\zeta_2\zeta_3 + \frac{128}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4\\ &\quad + \frac{52}{135}\frac{1}{2^{8k}}\zeta_2\zeta_5 + \frac{4}{15}\frac{1}{2^{8k}}\zeta_3\zeta_4 + \frac{88}{405}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{64}{675}\frac{1}{2^{10k}}\zeta_4\zeta_5\\
||e_n^{(0)}||_2^2 &< \frac{2}{2^{3k}}\left[\zeta_1^2 + \left(\frac{\zeta_2}{2^k}\right)^2 + \left(\frac{\zeta_3}{2^{2k}}\right)^2 + \left(\frac{\zeta_4}{2^{3k}}\right)^2 + \left(\frac{\zeta_5}{2^{4k}}\right)^2 + \frac{2\zeta_1\zeta_2}{2^k} + \frac{2\zeta_1\zeta_3}{2^{2k}} + \frac{2\zeta_1\zeta_4}{2^{3k}}\right.\\ &\quad \left. + \frac{2\zeta_1\zeta_5}{2^{4k}} + \frac{2\zeta_2\zeta_3}{2^{3k}} + \frac{2\zeta_2\zeta_4}{2^{4k}} + \frac{2\zeta_2\zeta_5}{2^{5k}} + \frac{2\zeta_3\zeta_4}{2^{5k}} + \frac{2\zeta_3\zeta_5}{2^{6k}} + \frac{2\zeta_4\zeta_5}{2^{7k}}\right]\\
||e_n^{(0)}||_2^2 &= \frac{2}{2^{3k}}\left[\zeta_1 + \frac{\zeta_2}{2^k} + \frac{\zeta_3}{2^{2k}} + \frac{\zeta_4}{2^{3k}} + \frac{\zeta_5}{2^{4k}}\right]^2\end{aligned}$$

$$= \frac{2\zeta^2}{2^{3k}}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}\right]^2, \qquad \zeta = max[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]$$

Next,

$$\left(E^1_{2^{k-1},0}(f)\right)^2 = \int_0^1\left(\sum_{n=1}^{2^{k-1}} e_n^{(0)}(t)\right)^2 dt$$

$$= \int_0^1\sum_{n=1}^{2^{k-1}}\left(e_n^{(0)}(t)\right)^2 dt + 2\sum_{n=1}^{2^{k-1}}\sum_{n\neq n'}^{2^{k-1}}\int_0^1 e_n^{(0)}(t)e_{n'}^{(0)}(t)dt$$

$$= \sum_{n=1}^{2^{k-1}}\int_0^1\left(e_n^{(0)}(t)\right)^2 dt, 2^{nd} \text{ term vanished due to disjoint supports of } e_n \text{ and } e'_n.$$

$$= \sum_{n=1}^{2^{k-1}} \|e_n^{(0)}\|_2^2$$

$$\leq \sum_{n=1}^{2^{k-1}} \frac{2\zeta^2}{2^{3k}}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}\right]^2$$

$$= (2^{k-1})\frac{2\zeta^2}{2^{3k}}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}\right]^2$$

$$= \frac{\zeta^2}{2^{2k}}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}\right]^2$$

Then,

$$\left(E^1_{2^{k-1},0}(f)\right) \leq \frac{\zeta}{2^k}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}\right]$$

$$= \zeta\left[\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}+\frac{1}{2^{5k}}\right]$$

$$\leq \zeta\left[\frac{1}{2^k}+\frac{1}{2^k}+\frac{1}{2^k}+\frac{1}{2^k}+\frac{1}{2^k}\right]$$

$$= 5\zeta\left(\frac{1}{2^k}\right)$$

Hence, $\left(E^1_{2^{k-1},0}(f)\right) = O\left(\frac{1}{2^k}\right)$

2-The error $e_n^{(1)}(t)$ between $f(t)$ and its expression over any sub interval is defined as-

$$e_n^{(1)}(t) = c_{n,0}\psi_{n,0}(t) + c_{n,1}\psi_{n,1}(t) - f(t), \qquad t \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right), \qquad n = 1,2,3,\cdots 2^{k-1}.$$

$$\|e_n^{(1)}\|_2^2 = \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[e_n^{(1)}(t)\right]^2 dt$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[c_{n,0}\psi_{n,0}(t) + c_{n,1}\psi_{n,1}(t) - f(t)\right]^2 dt$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \Big[c_{n,0}^2 \left(\psi_{n,0}(t)\right)^2 + c_{n,1}^2 \left(\psi_{n,1}(t)\right)^2 + \left(f(t)\right)^2 + 2c_{n,0}c_{n,1}\psi_{n,0}(t)\psi_{n,1}(t)$$

$$- 2c_{n,1}\psi_{n,1}(t)f(t) - 2c_{n,0}\psi_{n,0}(t)f(t)\Big]dt$$

$$\|e_n^{(1)}\|_2^2 = c_{n,0}^2 \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(\psi_{n,0}(t)\right)^2 dt + c_{n,1}^2 \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(\psi_{n,1}(t)\right)^2 dt + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt$$

$$- 2c_{n,0}c_{n,1} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,0}(t)\psi_{n,1}(t)dt - 2c_{n,1} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,1}(t)f(t)dt - 2c_{n,0} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \psi_{n,0}(t)f(t)dt$$

$$\|e_n^{(1)}\|_2^2 = c_{n,0}^2 + c_{n,1}^2 + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt - 2c_{n,0}c_{n,0} - 2c_{n,1}c_{n,1}$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left(f(t)\right)^2 dt - c_{n,0}^2 - c_{n,1}^2 \tag{8}$$

Now,

$$c_{n,1} = \langle f(t), \psi_{n,1}(t) \rangle$$

$$= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)\psi_{n,1}(t)dt$$

$$= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)L_1(2^k t - \hat{n})dt$$

$$= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)(2^k t - \hat{n})dt$$

$$= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)(2^k t - 2n + 1)dt$$

$$= \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \int_{0}^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) \left\{ 2^k \left(\frac{\hat{n}-1}{2^k} + h\right) - 2n + 1 \right\} dh, \quad t = \frac{\hat{n}-1}{2^k} + h$$

$$\implies c_{n,1} = \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \Big[\int_{0}^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) \left\{ 2^k \left(\frac{\hat{n}-1}{2^k} + h\right) - 2n + 1 \right\} dh$$

$$+\int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k}+h\right)\left\{2^k\left(\frac{\hat{n}-1}{2^k}+h\right)-2n+1\right\}dh$$

$$+\frac{1}{2}\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k}+h\right)\left\{2^k\left(\frac{\hat{n}-1}{2^k}+h\right)-2n+1\right\}dh$$

$$+\frac{1}{6}\int_0^{\frac{1}{2^{k-1}}} h^3 f'''\left(\frac{\hat{n}-1}{2^k}+h\right)\left\{2^k\left(\frac{\hat{n}-1}{2^k}+h\right)-2n+1\right\}dh$$

$$+\frac{1}{24}\int_0^{\frac{1}{2^{k-1}}} h^4 f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)\left\{2^k\left(\frac{\hat{n}-1}{2^k}+h\right)-2n+1\right\}dh$$

$$+\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}} h^5 f^{v}\left(\frac{\hat{n}-1}{2^k}+h\right)\left\{2^k\left(\frac{\hat{n}-1}{2^k}+\theta h\right)-2n+1\right\}dh\Bigg]$$

$$c_{n,1} = \sqrt{\frac{3}{2}}2^{\frac{k}{2}}\Bigg[f\left(\frac{\hat{n}-1}{2^k}+h\right)\int_0^{\frac{1}{2^{k-1}}}(2^kh-1)dh + f'\left(\frac{\hat{n}-1}{2^k}+h\right)\int_0^{\frac{1}{2^{k-1}}}(2^kh-1)dh$$

$$+\frac{1}{2}f''\left(\frac{\hat{n}-1}{2^k}+h\right)\int_0^{\frac{1}{2^{k-1}}}h^2(2^kh-1)dh + \frac{1}{6}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)\int_0^{\frac{1}{2^{k-1}}}h^3(2^kh-1)dh$$

$$+\frac{1}{24}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)\int_0^{\frac{1}{2^{k-1}}}h^4(2^kh-1)dh + \frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]$$

$$c_{n,1} = \sqrt{\frac{3}{2}}2^{\frac{k}{2}}\Bigg[\frac{2}{3}\frac{1}{2^{2k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{2}{3}\frac{1}{2^{3k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{2}{5}\frac{1}{2^{4k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)$$

$$+\frac{8}{45}\frac{1}{2^{5k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]$$

$$c_{n,1}^2 = \frac{3}{2}2^k\Bigg[\frac{2}{3}\frac{1}{2^{2k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{2}{3}\frac{1}{2^{3k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{2}{5}\frac{1}{2^{4k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)$$

$$+\frac{8}{45}\frac{1}{2^{5k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]^2$$

$$c_{n,1}^2 = \frac{3}{2}2^k\Bigg[\frac{4}{9}\frac{1}{2^{4k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{4}{9}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{4}{25}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)^2$$

$$+\frac{64}{2025}\frac{1}{2^{10k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\left\{\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right\}^2$$

$$+\frac{8}{9}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{8}{15}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{32}{135}\frac{1}{2^{7k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{90}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$
$$+\frac{8}{15}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{32}{135}\frac{1}{2^{8k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{90}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$
$$+\frac{32}{225}\frac{1}{2^{9k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{150}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$
$$+\frac{2}{675}\frac{1}{2^{5k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\Bigg]$$

Therefore,

$$c_{n,1}^2=\frac{2}{3}\frac{1}{2^{3k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{2}{3}\frac{1}{2^{5k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{6}{25}\frac{1}{2^{7k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)^2$$
$$+\frac{32}{675}\frac{1}{2^{9k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{3}{2}2^k\left\{\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right\}^2$$
$$+\frac{4}{3}\frac{1}{2^{4k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{4}{5}\frac{1}{2^{5k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{16}{45}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{60}\frac{1}{2^{k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$
$$+\frac{4}{5}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{16}{45}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{60}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}\theta h\right)dh$$
$$+\frac{16}{75}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)$$
$$+\frac{1}{100}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}}h^5(2^kh-1)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}\theta h\right)dh$$

$$+\frac{1}{225}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}} h^5(2^kh-1)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}\theta h\right)dh \tag{9}$$

From equation (6), (7), (8) and (9), we have-

$$\begin{aligned}
\|e_n^{(1)}\|_2^2 = {} & \frac{2}{45}\frac{1}{2^{5k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{8}{175}\frac{1}{2^{7k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)^2+\frac{32}{2025}\frac{1}{2^{9k}}f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)^2\\
&+\frac{1}{14400}\int_0^{\frac{1}{2^{k-1}}} h^{10}\left\{f^{v}\left(\frac{\hat{n}-1}{2^k}\theta h\right)\right\}^2dh-\frac{2^k}{2}\left\{\int_0^{\frac{1}{2^{k-1}}}\frac{h^5}{120}f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right\}^2\\
&-\frac{3}{2}2^k\left\{\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}} h^5(2^kh-1)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\right\}^2-\frac{8}{5}\frac{1}{2^{6k}}f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)\\
&+\frac{1}{60}\int_0^{\frac{1}{2^{k-1}}} h^6f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&-\frac{1}{60}\frac{1}{2^k}\int_0^{\frac{1}{2^{k-1}}} h^5f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&-\frac{1}{60}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}} h^5(2^kh-1)f'\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&+\frac{4}{45}\frac{1}{2^{6k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)+\frac{16}{315}\frac{1}{2^{7k}}f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)\\
&+\frac{1}{120}\int_0^{\frac{1}{2^{k-1}}} h^7f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&-\frac{1}{90}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}} h^5f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&-\frac{1}{60}\frac{1}{2^{2k}}\int_0^{\frac{1}{2^{k-1}}} h^5(2^kh-1)f''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&+\frac{4}{75}\frac{1}{2^{8k}}f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right)\\
&+\frac{1}{360}\int_0^{\frac{1}{2^{k-1}}} h^8f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh\\
&-\frac{1}{100}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}} h^5(2^kh-1)f'''\left(\frac{\hat{n}-1}{2^k}+h\right)f^{v}\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh
\end{aligned}$$

$$- \frac{1}{180}\frac{1}{2^{3k}}\int_0^{\frac{1}{2^{k-1}}} h^5 f'''\left(\frac{\hat{n}-1}{2^k}+h\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$+ \frac{1}{1440}\int_0^{\frac{1}{2^{k-1}}} h^9 f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$- \frac{1}{450}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}} h^5 f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$- \frac{1}{225}\frac{1}{2^{4k}}\int_0^{\frac{1}{2^{k-1}}} h^5(2^k h-1) f^{iv}\left(\frac{\hat{n}-1}{2^k}+h\right) f^v\left(\frac{\hat{n}-1}{2^k}+\theta h\right)dh$$

$$\begin{aligned}||e_n^{(0)}||_2^2 &= I_1 + I_2 + I_3 + I_4 - I_5 - I_6 - I_7 + I_8 - I_9 - I_{10} + I_{11} \\ &+ I_{12} + I_{13} - I_{14} - I_{15} + I_{16} I_{17} - I_{18} - I_{19} + I_{20} - I_{21} - I_{22}, \quad \text{say.}\end{aligned}$$

Since $|f'(t)| \le \zeta_1, |f''(t)| \le \zeta_2, |f'''(t)| \le \zeta_3, |f^{iv}(t)| \le \zeta_4, |f^v(t)| \le \zeta_5 \quad \forall t \in [0,1)$
Therefore,

$|I_1| \le \frac{2}{45}\frac{1}{2^{5k}}\zeta_2^2$, $|I_2| \le \frac{8}{175}\frac{1}{2^{7k}}\zeta_3^2$, $|I_3| \le \frac{32}{2025}\frac{1}{2^{9k}}\zeta_4^2$, $|I_4| \le \frac{32}{2475}\frac{1}{2^{11k}}\zeta_5^2$,

$|I_5| \le \frac{8}{2025}\frac{1}{2^{11k}}\zeta_5^2$, $|I_6| \le \frac{8}{1323}\frac{1}{2^{11k}}\zeta_5^2$, $|I_7| \le \frac{8}{5}\frac{1}{2^{6k}}\zeta_1\zeta_4$, $|I_8| \le \frac{32}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5$,

$|I_9| \le \frac{8}{45}\frac{1}{2^{7k}}\zeta_1\zeta_5$, $|I_{10}| \le \frac{8}{63}\frac{1}{2^{7k}}\zeta_1\zeta_5$, $|I_{11}| \le \frac{4}{45}\frac{1}{2^{6k}}\zeta_2\zeta_3$, $|I_{12}| \le \frac{16}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4$,

$|I_{13}| \le \frac{4}{15}\frac{1}{2^{8k}}\zeta_2\zeta_5$, $|I_{14}| \le \frac{16}{135}\frac{1}{2^{8k}}\zeta_2\zeta_5$, $|I_{15}| \le \frac{8}{63}\frac{1}{2^{8k}}\zeta_2\zeta_5$, $|I_{16}| \le \frac{4}{75}\frac{1}{2^{8k}}\zeta_3\zeta_4$

$|I_{17}| \le \frac{64}{405}\frac{1}{2^{9k}}\zeta_3\zeta_5$, $|I_{18}| \le \frac{8}{105}\frac{1}{2^{9k}}\zeta_3\zeta_5$, $|I_{19}| \le \frac{8}{135}\frac{1}{2^{9k}}\zeta_3\zeta_5$, $|I_{20}| \le \frac{16}{225}\frac{1}{2^{10k}}\zeta_4\zeta_5$

$|I_{21}| \le \frac{16}{675}\frac{1}{2^{10k}}\zeta_4\zeta_5$, $|I_{22}| \le \frac{32}{945}\frac{1}{2^{10k}}\zeta_4\zeta_5$

Therefore,

$$
\begin{aligned}
\|e_n^{(0)}\|_2^2 &\le \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\| + \|I_5\| + \|I_6\| + \|I_7\| + \|I_8\| + \|I_9\| + \|I_{10}\| + \|I_{11}\| + \|I_{12}\| \\
&\quad + \|I_{13}\| + \|I_{14}\| + \|I_{15}\| + \|I_{16}\| + \|I_{17}\| + \|I_{18}\| + \|I_{19}\| + \|I_{20}\| + \|I_{21}\| + \|I_{22}\| \\
\|e_n^{(0)}\|_2^2 &\le \frac{2}{45}\frac{1}{2^{5k}}\zeta_2^2 + \frac{8}{175}\frac{1}{2^{7k}}\zeta_3^2 + \frac{32}{2025}\frac{1}{2^{9k}}\zeta_4^2 + \frac{32}{2475}\frac{1}{2^{11k}}\zeta_5^2 + \frac{8}{2025}\frac{1}{2^{11k}}\zeta_5^2 + \frac{8}{1323}\frac{1}{2^{11k}}\zeta_5^2 + \frac{8}{5}\frac{1}{2^{6k}}\zeta_1\zeta_4 \\
&\quad + \frac{32}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{8}{45}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{8}{63}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{4}{45}\frac{1}{2^{6k}}\zeta_2\zeta_3 + \frac{16}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4 + \frac{4}{15}\frac{1}{2^{8k}}\zeta_2\zeta_5 + \frac{16}{135}\frac{1}{2^{8k}}\zeta_2\zeta_5 \\
&\quad + \frac{8}{63}\frac{1}{2^{8k}}\zeta_2\zeta_5 + \frac{4}{75}\frac{1}{2^{8k}}\zeta_3\zeta_4 + \frac{64}{405}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{8}{105}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{8}{135}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{16}{225}\frac{1}{2^{10k}}\zeta_4\zeta_5 \\
&\quad + \frac{16}{675}\frac{1}{2^{10k}}\zeta_4\zeta_5 + \frac{32}{945}\frac{1}{2^{10k}}\zeta_4\zeta_5 \\
\|e_n^{(0)}\|_2^2 &\le \frac{2}{45}\frac{1}{2^{5k}}\zeta_2^2 + \frac{8}{175}\frac{1}{2^{7k}}\zeta_3^2 + \frac{32}{2025}\frac{1}{2^{9k}}\zeta_4^2 + \frac{25024}{1091475}\frac{1}{2^{11k}}\zeta_5^2 + \frac{8}{5}\frac{1}{2^{6k}}\zeta_1\zeta_4 + \frac{64}{105}\frac{1}{2^{7k}}\zeta_1\zeta_5 + \frac{4}{45}\frac{1}{2^{6k}}\zeta_2\zeta_3 \\
&\quad + \frac{16}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4 + \frac{484}{945}\frac{1}{2^{7k}}\zeta_2\zeta_5 + \frac{4}{75}\frac{1}{2^{8k}}\zeta_3\zeta_4 + \frac{832}{2835}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{608}{4725}\frac{1}{2^{10k}}\zeta_4\zeta_5 \\
\|e_n^{(0)}\|_2^2 &\approx \frac{2}{45}\frac{1}{2^{5k}}\zeta_2^2 + \frac{8}{175}\frac{1}{2^{7k}}\zeta_3^2 + \frac{32}{2025}\frac{1}{2^{9k}}\zeta_4^2 + \frac{25024}{1091475}\frac{1}{2^{11k}}\zeta_5^2 + \frac{4}{45}\frac{1}{2^{6k}}\zeta_2\zeta_3 + \frac{16}{315}\frac{1}{2^{7k}}\zeta_2\zeta_4 \\
&\quad + \frac{484}{945}\frac{1}{2^{7k}}\zeta_2\zeta_5 + \frac{4}{75}\frac{1}{2^{8k}}\zeta_3\zeta_4 + \frac{832}{2835}\frac{1}{2^{9k}}\zeta_3\zeta_5 + \frac{608}{4725}\frac{1}{2^{10k}}\zeta_4\zeta_5 \\
\|e_n^{(0)}\|_2^2 &< \frac{2}{2^{5k}}\left[\zeta_2^2 + \left(\frac{\zeta_3}{2^k}\right)^2 + \left(\frac{\zeta_4}{2^{2k}}\right)^2 + \left(\frac{\zeta_5}{2^{3k}}\right)^2 + \frac{2\zeta_2\zeta_3}{2^k} + \frac{2\zeta_2\zeta_4}{2^{2k}} + \frac{2\zeta_2\zeta_5}{2^{3k}} + \frac{2\zeta_3\zeta_4}{2^{3k}} + \frac{2\zeta_3\zeta_5}{2^{4k}} + \frac{2\zeta_4\zeta_5}{2^{5k}}\right] \\
&= \frac{2}{2^{5k}}\left[\zeta_2 + \frac{\zeta_3}{2^k} + \frac{\zeta_4}{2^{2k}} + \frac{\zeta_5}{2^{3k}}\right]^2 \\
&= \frac{2\zeta^2}{2^{5k}}\left[1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}}\right]^2 \qquad \zeta = max[\zeta_2, \zeta_3, \zeta_4, \zeta_5]
\end{aligned}
$$

Next,

$$
\begin{aligned}
\left(E^2_{2^{k-1},1}(f)\right)^2 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} e_n^{(1)}(t)\right)^2 dt \\
&= \int_0^1 \sum_{n=1}^{2^{k-1}} \left(e_n^{(1)}(t)\right)^2 dt + 2\sum_{n=1}^{2^{k-1}}\sum_{n\neq n'}^{2^{k-1}} \int_0^1 e_n^{(1)}(t)e_{n'}^{(1)}(t)dt \\
&= \sum_{n=1}^{2^{k-1}} \int_0^1 \left(e_n^{(1)}(t)\right)^2 dt, \quad 2^{nd} \text{ term vanished due to disjoint supports of } e_n \text{ and } e_n'. \\
&= \sum_{n=1}^{2^{k-1}} \|e_n^{(1)}\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
\left(E^2_{2^{k-1},1}(f)\right)^2 &\le \sum_{n=1}^{2^{k-1}} \frac{2\zeta^2}{2^{5k}}\left[1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}}\right]^2 \\
&= \left(2^{k-1}\right)\frac{2\zeta^2}{2^{5k}}\left[1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}}\right]^2 \\
&= \frac{\zeta^2}{2^{4k}}\left[1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}}\right]^2
\end{aligned}
$$

$$\begin{aligned}\left(E^2_{2^{k-1},1}(f)\right) &\leq \frac{\zeta}{2^{2k}}\left[1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}\right]\\ &= \zeta\left[\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}+\frac{1}{2^{5k}}\right]\\ &\leq \zeta\left[\frac{1}{2^{2k}}+\frac{1}{2^{2k}}+\frac{1}{2^{2k}}+\frac{1}{2^{2k}}\right]\\ &= 4\zeta\left(\frac{1}{2^{2k}}\right)\end{aligned}$$

Hence, $\left(E^2_{2^{k-1},1}(f)\right) = O\left(\frac{1}{2^{2k}}\right)$

Similarly we can prove the following result-

$$E^{(3)}_{2^{k-1},2}(f) = \|f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{1} c_{n,m}\psi_{n,m}\|_2 = O\left(\frac{1}{2^{3k}}\right)$$

Proof of Theorem-3.2

$0 \leq |f^{v}(t)| < \zeta_5, \forall t \in [0,1)$

$$\begin{aligned}c_{n,m} &= \int_0^1 f(t)\psi_{n,m}(t)dt\\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(t)\sqrt{\frac{2m+1}{2}}2^{\frac{k}{2}}L_m\left(2^kt-\hat{n}\right)dt\\ &= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+1}{2^k}\right)L_m(t)dt\\ &= \sqrt{\frac{2m+1}{2^{k+1}}}\int_{-1}^{1} f\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m+1}(t)-L_{m-1}(t)\right)}{2m+1}dt\\ &= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}}\\ &\times\left[\left\{f'\left(\frac{\hat{n}+1}{2^k}\right)\left(L_{m+1}(t)-L_{m-1}(t)\right)\right\}_{-1}^{1} - \int_{-1}^{1}\frac{1}{2^k}f'\left(\frac{\hat{n}+1}{2^k}\right)\left(L_{m+1}(t)-L_{m-1}(t)\right)dt\right]\end{aligned}$$

$$\begin{aligned}c_{n,m} &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f'\left(\frac{\hat{n}+1}{2^k}\right)\left(L_{m-1}(t)-L_{m+1}(t)\right)dt\right]\\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f'\left(\frac{\hat{n}+1}{2^k}\right)L_{m-1}(t)dt - \int_{-1}^{1} f'\left(\frac{\hat{n}+1}{2^k}\right)L_{m+1}(t)dt\right]\\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\int_{-1}^{1} f'\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_m(t)-L_{m-2}(t)\right)}{(2m-1)}dt\right.\end{aligned}$$

$$- \int_{-1}^{1} f'\left(\frac{\hat{n}+1}{2^k}\right) \frac{d\left(L_{m+2}(t) - L_m(t)\right)}{(2m+3)} dt \Big]$$

$$c_{n,m} = \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \Big[\int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{(L_{m+2}(t) - L_m(t))}{(2m+3)} dt$$

$$- \int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{(L_m(t) - L_{m-2}(t))}{(2m-1)} dt \Big]$$

$$c_{n,m} = \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)} \left[\int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{d\left(L_{m+3}(t) - L_{m+1}(t)\right)}{(2m+5)} dt \right]$$

$$- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)} \left[\int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{d\left(L_{m+1}(t) - L_{m-1}(t)\right)}{(2m+1)} dt \right]$$

$$+ \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)} \left[\int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{d\left(L_{m-1}(t) - L_{m-3}(t)\right)}{(2m-3)} dt \right]$$

$$- \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)} \left[\int_{-1}^{1} f''\left(\frac{\hat{n}+1}{2^k}\right) \frac{d\left(L_{m+1}(t) - L_{m-1}(t)\right)}{(2m+1)} dt \right]$$

$$c_{n,m} = \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m+1}(t) dt \right]$$

$$- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+5)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m+3}(t) dt \right]$$

$$+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m+1}(t) dt \right]$$

$$- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m+3)(2m+1)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m-1}(t) dt \right]$$

$$- \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m-1}(t) dt \right]$$

$$+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m-3)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m-3}(t) dt \right]$$

$$+ \left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{(2m-1)(2m+1)} \left[\int_{-1}^{1} f'''\left(\frac{\hat{n}+1}{2^k}\right) L_{m+1}(t) dt \right]$$

$$-\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)L_{m-1}(t)dt\right]$$

$$c_{n,m}=\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+5)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m+2}(t)-L_m(t)\right)}{(2m+3)}dt\right]$$

$$-\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+5)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m+4}(t)-L_{m+2}(t)\right)}{(2m+7)}dt\right]$$

$$+\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+1)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m+2}(t)-L_m(t)\right)}{(2m+3)}dt\right]$$

$$-\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+1)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_m(t)-L_{m-2}(t)\right)}{(2m-1)}dt\right]$$

$$-\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-3)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_m(t)-L_{m-2}(t)\right)}{(2m-1)}dt\right]$$

$$+\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-3)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m-2}(t)-L_{m-4}(t)\right)}{(2m-5)}dt\right]$$

$$+\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_{m+2}(t)-L_m(t)\right)}{(2m+3)}dt\right]$$

$$-\left(\frac{1}{2^{7k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)}\left[\int_{-1}^{1}f'''\left(\frac{\hat{n}+1}{2^k}\right)\frac{d\left(L_m(t)-L_{m-2}(t)\right)}{(2m-1)}dt\right]$$

$$c_{n,m}=\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+3)(2m+5)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_m(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+3)(2m+5)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m+2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+5)(2m+7)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m+4}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+3)(2m+5)(2m+7)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m+2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m+3)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_m(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m+1)(2m+3)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m+2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m-2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-1)(2m-3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-1)(2m-3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m-2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-3)(2m-5)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m-4}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-3)(2m-5)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m-2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m+1)(2m+3)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m+2}(t)dt\right]$$

$$+\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-1)(2m+1)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m}(t)dt\right]$$

$$-\left(\frac{1}{2^{9k+1}(2m+1)}\right)^{\frac{1}{2}}\frac{1}{(2m-1)(2m-1)(2m+1)}\left[\int_{-1}^{1}f^{iv}\left(\frac{\hat{n}+1}{2^k}\right)L_{m-2}(t)dt\right]$$

$$c_{n,m}=\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}f^{v}\left(\frac{\hat{n}+1}{2^k}\right)\Bigg[\frac{L_{m+5}(t)}{(2m+3)(2m+5)(2m+7)(2m+9)}$$

$$+L_{m+3}(t)\Bigg\{\frac{1}{(2m+3)(2m+5)(2m+3)(2m+5)}+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+9)}$$

$$+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+5)}+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+5)}$$

$$+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+5)}\Bigg\}-L_{m+1}(t)\Bigg\{\frac{1}{(2m+3)(2m+5)(2m+3)(2m+1)}$$

$$+\frac{1}{(2m+3)(2m+5)(2m+3)(2m+5)}+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+5)}$$

$$+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+1)}+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+5)}$$
$$+\frac{1}{(2m+3)(2m+1)(2m-1)(2m+1)}+\frac{1}{(2m-1)(2m-3)(2m-1)(2m+1)}$$
$$+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+1)}+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+5)}$$
$$+\frac{1}{(2m-1)(2m+1)(2m-1)(2m+1)}\bigg\}+L_{m-1}(t)\bigg\{\frac{1}{(2m+3)(2m+5)(2m+3)(2m+1)}$$
$$+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+1)}+\frac{1}{(2m+3)(2m+1)(2m-1)(2m+1)}$$
$$+\frac{1}{(2m+3)(2m+1)(2m-1)(2m-5)}+\frac{1}{(2m-1)(2m-3)(2m-1)(2m+1)}$$
$$+\frac{1}{(2m-1)(2m-3)(2m-1)(2m-3)}+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-3)}$$
$$+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+1)}+\frac{1}{(2m-1)(2m+1)(2m-1)(2m+1)}$$
$$+\frac{1}{(2m-1)(2m+1)(2m-1)(2m-3)}\bigg\}-L_{m-3}(t)\bigg\{\frac{1}{(2m+3)(2m+1)(2m-1)(2m-5)}$$
$$+\frac{1}{(2m-1)(2m-3)(2m-1)(2m-3)}+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-7)}$$
$$+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-3)}+\frac{1}{(2m-1)(2m+1)(2m-1)(2m-3)}\bigg\}$$
$$+\frac{L_{m-5}(t)}{(2m-1)(2m-3)(2m-5)(2m-7)}\bigg]dt$$

$$c_{n,m}=\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}f^{v}\left(\frac{\hat{n}+1}{2^{k}}\right)\Big[\tau_1+\tau_2-\tau_3+\tau_4-\tau_5+\tau_6\Big]dt$$

$$|c_{n,m}|\leq\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\left|f^{v}\left(\frac{\hat{n}+1}{2^{k}}\right)\right||\tau_1+\tau_2-\tau_3+\tau_4-\tau_5+\tau_6|\,dt$$

$$\implies|c_{n,m}|\leq\zeta_5\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\Big[\int_{-1}^{1}|\tau_1|dt+\int_{-1}^{1}|\tau_2|dt+\int_{-1}^{1}|\tau_3|dt+\int_{-1}^{1}|\tau_4|dt$$
$$+\int_{-1}^{1}|\tau_5|dt+\int_{-1}^{1}|\tau_6|dt\Big]\tag{10}$$

where,

$$\tau_1=\frac{L_{m+5}(t)}{(2m+3)(2m+5)(2m+7)(2m+9)}$$
$$\tau_2=L_{m+3}(t)\bigg[\frac{1}{(2m+3)(2m+5)(2m+3)(2m+5)}+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+9)}$$
$$+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+5)}+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+5)}$$
$$+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+5)}\bigg]$$

$$
\begin{aligned}
\implies \tau_2 &\le L_{m+3}(t)\Big[\frac{6}{(2m-1)(2m+1)(2m+3)(2m+5)}\Big]\\
\tau_3 &= L_{m+1}(t)\Big[\frac{1}{(2m+3)(2m+5)(2m+3)(2m+1)}+\frac{1}{(2m+3)(2m+5)(2m+3)(2m+5)}\\
&+\frac{1}{(2m+3)(2m+5)(2m+7)(2m+5)}+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+1)}\\
&+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+5)}+\frac{1}{(2m+3)(2m+1)(2m-1)(2m+1)}\\
&+\frac{1}{(2m-1)(2m-3)(2m-1)(2m+1)}+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+1)}\\
&+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+5)}+\frac{1}{(2m-1)(2m+1)(2m-1)(2m+1)}\Big]\\
\implies \tau_3 &\le L_{m+1}(t)\Big[\frac{10}{(2m-1)(2m-3)(2m-1)(2m+1)}\Big]\\
\tau_4 &= L_{m-1}(t)\Big[\frac{1}{(2m+3)(2m+5)(2m+3)(2m+1)}+\frac{1}{(2m+3)(2m+1)(2m+3)(2m+1)}\\
&+\frac{1}{(2m+3)(2m+1)(2m-1)(2m+1)}+\frac{1}{(2m+3)(2m+1)(2m-1)(2m-5)}\\
&+\frac{1}{(2m-1)(2m-3)(2m-1)(2m+1)}+\frac{1}{(2m-1)(2m-3)(2m-1)(2m-3)}\\
&+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-3)}+\frac{1}{(2m-1)(2m+1)(2m+3)(2m+1)}\\
&+\frac{1}{(2m-1)(2m+1)(2m-1)(2m+1)}+\frac{1}{(2m-1)(2m+1)(2m-1)(2m-3)}\Big]\\
\implies \tau_4 &\le L_{m-1}(t)\Big[\frac{10}{(2m-1)(2m-3)(2m-5)(2m-3)}\Big]\\
\tau_5 &= L_{m-3}(t)\Big[\frac{1}{(2m+3)(2m+1)(2m-1)(2m-5)}+\frac{1}{(2m-1)(2m-3)(2m-1)(2m-3)}\\
&+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-7)}+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-3)}\\
&+\frac{1}{(2m-1)(2m+1)(2m-1)(2m-3)}\Big]\\
\implies \tau_5 &\le L_{m-3}(t)\Big[\frac{5}{(2m-1)(2m-3)(2m-5)(2m-7)}\Big]\\
\tau_6 &= \frac{L_{m-5}(t)}{(2m-1)(2m-3)(2m-5)(2m-7)}
\end{aligned}
$$

Now consider,

$$
\begin{aligned}
\int_{-1}^{1}|\tau_1(t)|^2dt &= \int_{-1}^{1}1.|\tau_1(t)|^2dt\\
&\le\left(\int_{-1}^{1}1^2dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|\tau_1(t)|^2dt\right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \sqrt{2}\left(\int_{-1}^{1} \frac{L_{m+5}^2(t)}{(2m+3)^2(2m+5)^2(2m+7)^2(2m+9)^2}dt\right)^{\frac{1}{2}}$$

$$= \sqrt{2}\left(\int_{-1}^{1} \frac{2}{(2m+3)^2(2m+5)^2(2m+7)^2(2m+9)^2(2m+11)}dt\right)^{\frac{1}{2}}$$

$$= \frac{2\sqrt{2}\times 1}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)^{\frac{1}{2}}} \tag{11}$$

Similarly,

$$\int_{-1}^{1} |\tau_2(t)|^2 dt = \frac{2\sqrt{2}\times 6}{(2m-1)(2m+1)(2m+3)(2m+5)(2m+7)^{\frac{1}{2}}} \tag{12}$$

$$\int_{-1}^{1} |\tau_3(t)|^2 dt = \frac{2\sqrt{2}\times 10}{(2m-1)(2m-3)(2m-1)(2m+1)(2m+3)^{\frac{1}{2}}} \tag{13}$$

$$\int_{-1}^{1} |\tau_4(t)|^2 dt = \frac{2\sqrt{2}\times 10}{(2m-1)(2m-3)(2m-5)(2m-3)(2m-1)^{\frac{1}{2}}} \tag{14}$$

$$\int_{-1}^{1} |\tau_5(t)|^2 dt = \frac{2\sqrt{2}\times 5}{(2m-1)(2m-3)(2m-1)(2m-7)(2m-5)^{\frac{1}{2}}} \tag{15}$$

$$\int_{-1}^{1} |\tau_6(t)|^2 dt = \frac{2\sqrt{2}}{(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)^{\frac{1}{2}}} \tag{16}$$

From equation (10), (11), (12), (13), (14), (15) and (16), we have-

$$|c_{n,m}| \leq 2\sqrt{2}\zeta_5\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{1}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)^{\frac{1}{2}}}\right.$$
$$+\frac{6}{(2m-1)(2m+1)(2m+3)(2m+5)(2m+7)^{\frac{1}{2}}}$$
$$+\frac{10}{(2m-1)(2m-3)(2m-1)(2m+1)(2m+3)^{\frac{1}{2}}}$$
$$+\frac{10}{(2m-1)(2m-3)(2m-5)(2m-3)(2m-1)^{\frac{1}{2}}}$$
$$+\frac{5}{(2m-1)(2m-3)(2m-1)(2m-7)(2m-5)^{\frac{1}{2}}}$$
$$\left.+\frac{1}{(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)^{\frac{1}{2}}}\right]$$

$$|c_{n,m}| \leq 2\sqrt{2}\zeta_5\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{33}{(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)^{\frac{1}{2}}}\right]$$

$$\leq 66\sqrt{2}\zeta_5\left(\frac{1}{2^{11k+1}(2m+1)}\right)^{\frac{1}{2}}\left[\frac{1}{(2m-9)^{\frac{9}{2}}}\right]$$

$$\implies |c_{n,m}| \leq \frac{66\sqrt{2}\zeta_5}{2^{\frac{11k+1}{2}}(2m-9)^5}, \qquad \forall m \geq 5 \tag{17}$$

Next,

$$\begin{aligned} S_{2^{k-1},M}(f)(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \\ f(t) - S_{2^{k-1},M}(f)(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) + \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(t) \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - S_{2^{k-1},M}(f)\|_2^2 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(t) \right)^2 dt \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2, \quad \text{(by orthogonality property of } \psi_{n,m}) \\ &\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\frac{66\sqrt{2}\zeta_5}{2^{\frac{11k+1}{2}}(2m-9)^5} \right)^2 \\ &= 8712\zeta_5^2 \times 2^{k-1} \frac{1}{2^{11k+1}} \sum_{m=M+1}^{\infty} \frac{1}{(2m-9)^{10}} \\ &= 8712\zeta_5^2 \times \frac{1}{2^{10k+2}} \int_{M+1}^{\infty} \frac{1}{(2m-9)^{10}} dm \\ &= \frac{2178\zeta_5^2}{2^{10k}} \left[\frac{(2m-9)^{-9}}{-9} \right]_{M+1}^{\infty} \\ &= \frac{2178\zeta_5^2}{2^{10k}} \left[\frac{1}{9\{2(M+1)-9\}^2} \right] \\ &= \frac{2178\zeta_5^2}{2^{10k}} . \frac{1}{9(2M-7)^9} \end{aligned}$$

Therefore,

$$\begin{aligned} E^{(4)}_{2^{k-1},M}(f) &\leq \frac{11\sqrt{2}\zeta_5}{2^{5k}(2M-7)^{\frac{9}{2}}} \\ \implies E^{(4)}_{2^{k-1},M}(f) &= O\left(\frac{1}{(2M-7)^{\frac{9}{2}} . 2^{5k}} \right), \quad \forall M \geq 4. \end{aligned}$$

5. Conclusion

Present work is devoted to find the approximation of functions of bounded derivatives by using Legendre wavelet method. *Theorem 3.1* gives the estimation of degree of approximation of functions $f \in L^2[0,1)$ of which fifth derivative is bounded. Taking $m = 0, 1$ and 2 the error estimation $E^{(1)}_{2^{k-1},0}(f)$, $E^{(2)}_{2^{k-1},1}(f)$ and $E^{(3)}_{2^{k-1},2}(f)$ have been determined by using Legendre wavelet technique. *Theorem 3.2* is the generalization of the results of *Theorem 3.1*. The estimation $E^{(4)}_{2^{k-1},M}(f) = O\left(\frac{1}{(2M-7)^{\frac{9}{2}}.2^{5k}}\right), \quad \forall M \geq 4$ reduces to the results of Lal & Rakesh[6] and Lal & Indra Bhan[5] by taking various values of k and M.

Acknowledgement

The authors are sincerely thankful to the anonymous reviewer(s) for their valuable suggestions and comments for the improvements of the manuscript, which significantly clarify the quality of the paper.

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